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On Probabilistic Cooperative Search Model to Detect a Lost Target in N-Disjoint Areas

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Abstract This paper presents a new probabilistic coordinated search technique for finding a randomly located target in n-disjoint known regions using n-searchers. Each region contains one searcher. The searchers use advanced technology to communicate with each other. This paper aims to obtain the candidate utility function, namely the expected value of the time for detecting the target, and minimize this expected value considering a restricted amount of time. We present a particular case when the target has a multinomial distribution, which is essential for searching for a valuable target that is missing at sea or lost in a wilderness area.

Keywords Coordinated search technique; nonlinear optimisation problem; multinomial distribution; probabilistic method; strategic planning

AMS 2010 subject classifications 60K30, 90B40

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1. Introduction

High-rise building collapses are caused by disastrous earthquakes or floods that obliterate the landscape and land. These occur suddenly and violently in numerous nations worldwide, resulting in significant material and human losses. As a result, search and rescue teams are working around the clock to extract people from the debris, a difficult task that calls for a lot of bravery. It appears that mice are prepared to be paid by humans and offer their services to humanity, suggesting that this supernatural work is no longer just performed by people. In order to locate survivors in difficult-to-reach areas, there are severe efforts and suggestions to train rats or try to control their movement by utilizing electronics inside the debris. This concept is based on giving these tiny mice a high-tech backpack that can be used to regulate their movement and outfitted with a camera to look for survivors among the debris and in difficult-to-reach areas. This backpack will also have a tracking gadget and a microphone that can be used in all directions to aid communication between searchers and victims, in addition to the camera. The mice were chosen because they have a sense of adventure, are small, and smell very well, which makes them perfect for finding objects in small areas. The secret to search and rescue is that they are curious and want to investigate.
There are numerous ways that these mice ought to take in order to discover the target rapidly and precisely. Hence, a perfect look arrangement ought to be created that suits the geological nature, which increases the likelihood of identifying the target and decreases the discovery taking toll. The coordinated search method is considered one of the leading proposed look strategies, with the look range partitioned into particular cells. The coordinated

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search method for a randomly located target encompasses an exceptional significance in our lives due to its great applicability, such as looking for a bomb within a plane. It is an imperative technique to discover a randomly found target within the plane. A target is found someplace within the plane, concurring with a known likelihood dispersion. In this procedure, the searchers coordinate their search for a misplaced target with the objective of minimizing the anticipated time until one of them is rejoined with the target at a few concurred assembly points. The assembly point is the point where the searchers check back after each period of looking to discover, on the off chance, whether one of the searchers has found the target or not. In an earlier work, Thomas [1] illustrated this technique with the example of two parents searching for a lost child. The child is equally likely to be anywhere on the circumference of a circle of a given radius. The searchers start at the center and meet back there after each period of searching. More generally, Thomas' model is one where the time required for the searchers to meet each other is independent of the positions they are searching. Thomas finds that with optimal play, the first search before returning to check with the partner is the longest one, and each subsequent search period is shorter. He also finds that the number of planned returns to the center is finite and is a decreasing function of the radius (i.e., the penalty for returning). In earlier work, Reyniers [2] studied this technique when the target is hidden on an interval and has a known bounded symmetric distribution. Reyniers found a general optimality condition and used it to solve the problem when the target has uniform, triangular, or truncated exponential distributions. In addition, Reyniers [3] studied this technique for symmetric continuous distributed targets on the line. Reyniers gave necessary and sufficient conditions for the existence of an optimal search strategy when the target density is continuous and decreasing. Recently, this problem has been studied on the line when the target has asymmetric distribution by Teamah et al. [4]. However, Teamah et al. have the optimal search strategy to find the asymmetrically distributed targets; they also show that some previous studies are special cases from their results.

Furthermore, interesting search strategies that give the minimum expected value of the cost of detecting the lost target have been studied. In the case of a randomly located target, many authors, such as El-Hadidy et al. [5, 6, 12, 13, 38, 39] discussed this problem in the plane when the located target has symmetric and asymmetric distributions and with less information available to the searchers. In these papers, the authors desired to minimize the expected time for detecting the target. More recently, El-Hadidy [7] introduced a new search strategy in the plane by dividing the plane into identical cells, and the searcher moves along a spiral with a line segment curve. He found the optimal value of the arcs that the searcher should do and found the target with minimum cost. Also, Beltagy and El-Hadidy [8] introduced more analysis for another new search strategy, where the searcher moves along a parabolic spiral curve. They are shown that the distance between the target position and the searcher starting point depends on the number of revolutions, where the complete revolution is made when $t = 2\pi$.

In the case of the randomly moving target in the plane, Mohamed and El-Hadidy [9] discussed the existence of the search strategy that finds the conditionally deterministic target motion. Also, they are shown the existence of the optimal search strategy and found the necessary conditions that make the expected value of the first meeting time finite. When the target starts its motion as a parabolic spiral from a random point in the plane, Mohamed and El-Hadidy [10] studied this problem such that no time information about the target's position is available to the searcher. The searcher starts its motion from the origin. They formulated a search model and found the conditions that make this search strategy finite. For more interesting search strategies that find the randomly moving target, see El-Hadidy et al. [14–17, 19–37]. In this paper, we extend the theory of the coordinated search to n-regions by using n-searchers. We aim to minimize the expected value of detecting the target. In addition, we solve this problem when the target has a multinomial distribution.

This paper is organized as follows. In Section 2, we formulate the problem. The optimal strategy is studied in Section 3. A special case when the target has a multinomial distribution is discussed in Section 4. Finally, the paper concludes with a discussion of the results and directions for future research.

2. Problem Description

The stationary target is likely to be at any point in the plane. The plane is divided into $n-$regions, and we have n-searchers coordinate their search for the target from time to time and connect to the center of the plane (their agreed meeting point) to check if one of the searchers $S_i,\ i=1,2,\ldots,n$, has found the target. In this paper, the agreed meeting point is the point where the searchers send a message after each period of searching to find out if one of the searchers has already found or met the lost target. Suppose that the search process takes time Ψ. Each searcher S_i would take a time T_{ij} to search its region i at time step j. The searchers $S_i,\ i=1,2,\ldots,n$, search some points (places) on its regions at the same time before sending a message to the meeting point. The probability of detecting the target on the region i at time step j is P_{ij}. If the target is at a location that is searched, it will be found with certainty. The searcher S_i takes time r_{ij} to translate from one point to another and send a message to the center.

A typical strategy is described by K, the number of times (say returns) that $S_i,\ i=1,2,\ldots,n$, plan to send a message to the meeting point to check if the target has been found, and by P_{ij} is the probability of detecting the target for the points by $S_i,\ i=1,2,\ldots,n$, between two different points (places) planned to send a message to the meeting point. Note that, $\sum\limits_{i=1}^{n}\sum\limits_{j=1}^{K}P_{ij}=1$.

Since the events of detecting the target are mutually exclusive and independent events, then if the target has been detected within the first time in step one, then the probability of detecting the target is $P_{11}+P_{21}+P_{31}+\ldots+P_{n1}$. And, if the target has been detected in the second time step, then the probability of detecting the target is $\left(\sum_{i=1}^{n}P_{i2}\right)\left(1-\sum_{j=1}^{n}P_{j1}\right)$, also in third time step, the probability of detecting the target is given by $\left(\sum_{i=1}^{n}P_{i3}\right)\left(1-\sum_{j=1}^{2}\sum_{i=1}^{n}P_{ij}\right)$, etc.

Let E_K be the expected value of the time until once $S_i, i=1,2,...,n$, and the targets are reunited under the strategy, then we have

$$\begin{aligned}
E_K = \; & \left(\sup_{1\le i\le n}T_{i1}+\sup_{1\le i\le n}r_{i1}\right)\sum_{\ell=1}^{n}P_{\ell 1}+\left(\sum_{j=1}^{2}\sum_{\ell=1}^{n}P_{\ell 2}\sup_{1\le i\le n}T_{i2}\right)\left(1-\sum_{m=1}^{n}P_{m1}\right)+\\
& \left(\sum_{j=1}^{3}\sum_{\ell=1}^{n}P_{\ell 3}\sup_{1\le i\le n}T_{i3}\right)\left(1-\sum_{j=1}^{2}\sum_{m=1}^{n}P_{mj}\right)+\ldots+\left(\sum_{j=1}^{K}\sum_{\ell=1}^{n}P_{\ell K}\sup_{1\le i\le n}T_{iK}\right)\left(1-\sum_{j=1}^{K-1}\sum_{m=1}^{n}P_{mj}\right)+\\
& \sum_{j=1}^{2}\sum_{\ell=1}^{n}P_{\ell 2}\sup_{1\le i\le n}r_{ij}+\sum_{j=1}^{3}\sum_{\ell=1}^{n}P_{\ell 3}\sup_{1\le i\le n}r_{ij}+\ldots+\sum_{j=1}^{K}\sum_{\ell=1}^{n}P_{\ell K}\sup_{1\le i\le n}r_{ij},\\
= \; & \sum_{j=1}^{K}\left[\left(\sum_{i=1}^{n}P_{ij}\right)\left[\left(\sum_{h=1}^{j}\sup_{1\le i\le n}T_{ih}\right)\left(1-\sum_{q=1}^{j-1}\sum_{i=1}^{n}P_{iq}\right)+\left(\sum_{h=1}^{j}\sup_{1\le i\le n}r_{ih}\right)\right]\right].
\end{aligned}$$

Therefore,

$$E_K=\sum_{j=1}^{K}\left[\left(\sum_{i=1}^{n}P_{ij}\right)\left[\left(\sum_{h=1}^{j}\sup_{1\le i\le n}T_{ih}\right)\left(\sum_{q=j}^{K}\sum_{i=1}^{n}P_{iq}\right)+\left(\sum_{h=1}^{j}\sup_{1\le i\le n}r_{ih}\right)\right]\right]. \tag{1}$$

Assuming no false detection from the searchers, then the cumulative probability of detection is given by $\sum\limits_{q=j}^{K}\sum\limits_{i=1}^{n}P_{iq}$. The goal of a searching strategy could either be to maximize the chances of finding the target given a restricted amount of time by maximizing $\sum\limits_{q=j}^{K}\sum\limits_{i=1}^{n}P_{iq}$ over the time horizon or to minimize the expected time to find the target by minimizing E_K.

3. Optimal strategy with minimum expected search time and maximum probability

On the region i and at time step h, we let:

$$\Psi_{ih} \leq \sup_{1\leq i\leq n} r_{ih} + \sup_{1\leq i\leq n} T_{ih}. \tag{2}$$

We aim to minimize E_K and maximize the probability of detecting the target. Therefore, the lowest expected search time is to obtain the minimum value of $\sup_{1\leq i\leq n} r_{ih}$ and $\sup_{1\leq i\leq n} T_{ih}$ that minimizes E_K (i.e. $E = \inf(E_K)$). The best strategy is to solve the following minimax nonlinear optimization problem **(MNLP1)**:

MNLP1:

$$\min_{\sup_{1\leq i\leq n} r_{ih},\ \sup_{1\leq i\leq n} T_{ih}} \max_{P_{ij}} \sum_{j=1}^{K} \left[\left(\sum_{i=1}^{n} P_{ij} \right) \left[\left(\sum_{h=1}^{j} \sup_{1\leq i\leq n} T_{ih} \right) \left(\sum_{q=j}^{K} \sum_{i=1}^{n} P_{iq} \right) + \left(\sum_{h=1}^{j} \sup_{1\leq i\leq n} r_{ih} \right) \right] \right],$$

subjected to $\sum_{i=1}^{n}\sum_{j=1}^{K} P_{ij} = 1,\ 0 \leq \sup_{1\leq i\leq n} r_{ih} \leq \sup_{1\leq i\leq n} T_{ih},\ \sup_{1\leq i\leq n} r_{ih} + \sup_{1\leq i\leq n} T_{ih} \leq \Psi_{ih}, \sum_{j=1}^{K} \Psi_{ij} \leq \Psi,$
$0 \leq P_{ij} \leq 1,\ \sup_{1\leq i\leq n} T_{ih} > 0,\ i = 1, 2, ..., n,\ h = 1, 2, ..., j,\ j = 1, 2, ..., K.$

From (1), it is clear that E_K is convex on compact region then it has a minimum P_{ij}. Using Lagrangian multiplier method as follows:

$$\frac{\partial}{\partial P_{ij}} \left(P_{ij} \left[\left(\sum_{h=1}^{j} \sup_{1\leq i\leq n} T_{ih} \right) \left(P_{ij} + \sum_{q=j+1}^{K} \sum_{i=1}^{n} P_{iq} \right) + \sum_{h=1}^{j} \sup_{1\leq i\leq n} r_{ih} \right] + \lambda P_{ij} \right) = 0, \quad \lambda \geq 0. \tag{3}$$

Then the minimum value of P_{ij} is given by,

$$P_{ij} = \frac{\left(\sum_{h=1}^{j} \sup_{1\leq i\leq n} T_{ih} \right) \left(\sum_{q=j+1}^{K} \sum_{i=1}^{n} P_{iq} \right) - \sum_{h=1}^{j} \sup_{1\leq i\leq n} r_{ih} - \lambda}{2 \sum_{h=1}^{j} \sup_{1\leq i\leq n} T_{ih}}, \tag{4}$$

which leads to a maximum value of the cumulative probability of detection, that is

$$1 - P_{ij} = \frac{2 \left(\sum_{h=1}^{j} \sup_{1\leq i\leq n} T_{ih} \right) \left(1 - \sum_{q=j+1}^{K} \sum_{i=1}^{n} P_{iq} \right) + \sum_{h=1}^{j} \sup_{1\leq i\leq n} r_{ih} + \lambda}{2 \sum_{h=1}^{j} \sup_{1\leq i\leq n} T_{ih}}. \tag{5}$$

By substituting from (5) in (1), then **MNLP1** will equivalent to **MNLP2**,

MNLP2:

$$\min_{\sup_{1\leq i\leq n} r_{ih},\ \sup_{1\leq i\leq n} T_{ih}} \sum_{j=1}^{K} \left[\left(\sum_{i=1}^{n} \frac{2 \sum_{h=1}^{j} \sup_{1\leq i\leq n} T_{ih} + \sum_{h=1}^{j} \sup_{1\leq i\leq n} r_{ih} - \lambda}{2 \sum_{h=1}^{j} \sup_{1\leq i\leq n} T_{ih}} \right) \left[\left(\sum_{h=1}^{j} \sup_{1\leq i\leq n} T_{ih} \right) \times \right. \right.$$

$$\left. \left. \left(\frac{2 \sum_{h=1}^{j} \sup_{1\leq i\leq n} T_{ih} + \sum_{h=1}^{j} \sup_{1\leq i\leq n} r_{ih} - \lambda}{2 \sum_{h=1}^{j} \sup_{1\leq i\leq n} T_{ih}} + \sum_{q=j+1}^{K} \sum_{i=1}^{n} P_{iq} \right) + \left(\sum_{h=1}^{j} \sup_{1\leq i\leq n} r_{ih} \right) \right] \right],$$

subject to:

$$0 \leq \sup_{1\leq i\leq n} r_{ih} \leq \sup_{1\leq i\leq n} T_{ih},\ \sup_{1\leq i\leq n} r_{ih} + \sup_{1\leq i\leq n} T_{ih} \leq \Psi_{ih}, \sum_{j=1}^{K} \Psi_{ij} \leq \Psi,$$

$$\lambda > \sum_{h=1}^{j} \sup_{1\leq i\leq n} r_{ih},\ 0 \leq \frac{2\sum\limits_{h=1}^{j} \sup\limits_{1\leq i\leq n} T_{ih} + \sum\limits_{h=1}^{j} \sup\limits_{1\leq i\leq n} r_{ih} - \lambda}{2\sum\limits_{h=1}^{j} \sup\limits_{1\leq i\leq n} T_{ih}} \leq 1,$$

$$\sup_{1\leq i\leq n} T_{ih} > 0,\ i = 1, 2, ..., n,,\ h = 1, 2, ..., j,\ \ j = 1, 2, ..., K.$$

Choosing certain weights $\delta_i, i = 1, 2, ..., n$ such that $\sum\limits_{i=1}^{n} \delta_i = 1$, then **MNLP2** takes the form:

MNLP3:

$$\min_{\sup\limits_{1\leq i\leq n} r_{ih},\ \sup\limits_{1\leq i\leq n} T_{ih}} \quad \sum_{i=1}^{n} \delta_i W_i,$$

$$\text{subject to } \sup_{1\leq i\leq n} r_{ih} - \sup_{1\leq i\leq n} T_{ih} \leq 0,\ \sup_{1\leq i\leq n} r_{ih} + \sup_{1\leq i\leq n} T_{ih} - \Psi_{ih} \leq 0, \sum_{j=1}^{K} \Psi_{ij} - \Psi \leq 0,$$

$$\lambda - \sum_{h=1}^{j} \sup_{1\leq i\leq n} r_{ih} < 0,\ \frac{2\sum\limits_{h=1}^{j} \sup\limits_{1\leq i\leq n} T_{ih} + \sum\limits_{h=1}^{j} \sup\limits_{1\leq i\leq n} r_{ih} - \lambda}{2\sum\limits_{h=1}^{j} \sup\limits_{1\leq i\leq n} T_{ih}} \leq 0,\quad -\sup_{1\leq i\leq n} T_{ih} < 0,$$

$$\delta_i \geq 0,\ \ \lambda > 0,\ i = 1, 2, ..., n,\ \ h = 1, 2, ..., j,\ \ j = 1, 2, ..., K,$$

where

$$W_i = \sum_{j=1}^{K} \left[\left(\sum_{i=1}^{n} \frac{2\sum\limits_{h=1}^{j} \sup\limits_{1\leq i\leq n} T_{ih} + \sum\limits_{h=1}^{j} \sup\limits_{1\leq i\leq n} r_{ih} - \lambda}{2\sum\limits_{h=1}^{j} \sup\limits_{1\leq i\leq n} T_{ih}} \right) \left[\left(\sum_{h=1}^{j} \sup_{1\leq i\leq n} T_{ih} \right) \times \right.\right.$$

$$\left.\left. \left(\frac{2\sum\limits_{h=1}^{j} \sup\limits_{1\leq i\leq n} T_{ih} + \sum\limits_{h=1}^{j} \sup\limits_{1\leq i\leq n} r_{ih} - \lambda}{2\sum\limits_{h=1}^{j} \sup\limits_{1\leq i\leq n} T_{ih}} + \sum_{q=j+1}^{K} \sum_{i=1}^{n} P_{iq} \right) + \left(\sum_{h=1}^{j} \sup_{1\leq i\leq n} r_{ih} \right) \right]\right].$$

From the Kuhn–Tucker conditions, we have

$$\sum_{i=1}^{n} \delta_i \sum_{j=1}^{K} \left[\left(\sum_{i=1}^{n} \frac{2j\sum\limits_{h=1}^{j} \left(\sup\limits_{1\leq i\leq n} T_{ih} - \sup\limits_{1\leq i\leq n} r_{ih} \right) - \lambda j}{4\left(\sum\limits_{h=1}^{j} \sup\limits_{1\leq i\leq n} T_{ih} \right)^2} \right) \left(\sum_{h=1}^{j} \left(\sup_{1\leq i\leq n} T_{ih} + \sup_{1\leq i\leq n} r_{ih} \right) - \frac{\lambda}{2} \right.\right.$$

$$+ \left(\sum_{h=1}^{j} \sup_{1\leq i\leq n} T_{ih} \right) \left(\sum_{q=j+1}^{K} \sum_{i=1}^{n} P_{iq} \right) + \left(\sum_{h=1}^{j} \sup_{1\leq i\leq n} r_{ih} \right)$$

$$+\left(\sum_{i=1}^{n}\frac{\sum_{h=1}^{j}\left(2\sup_{1\le i\le n}T_{ih}+\sup_{1\le i\le n}r_{ih}\right)-\lambda}{2\sum_{h=1}^{j}\sup_{1\le i\le n}T_{ih}}\right)\left(j\left(1+\sum_{q=j+1}^{K}\sum_{i=1}^{n}P_{iq}\right)+\left(\sum_{h=1}^{j}\sup_{1\le i\le n}r_{ih}\right)\right)\Bigg]$$

$$+U_1(-1)+U_2(1)+U_5\left(\frac{j\left(2\sum_{h=1}^{j}\sup_{1\le i\le n}T_{ih}-\left(\sum_{h=1}^{j}\left(2\sup_{1\le i\le n}T_{ih}+\sup_{1\le i\le n}r_{ih}\right)-\lambda\right)\right)}{2\left(\sum_{h=1}^{j}\sup_{1\le i\le n}T_{ih}\right)^2}\right)+U_6(-1)=0, \quad (6)$$

$$\sum_{i=1}^{n}\delta_i\sum_{j=1}^{K}\left[\left(\sum_{i=1}^{n}\frac{j}{2\sum_{h=1}^{j}\sup_{1\le i\le n}T_{ih}}\right)\left(\sum_{h=1}^{j}\left(\sup_{1\le i\le n}T_{ih}+\sup_{1\le i\le n}r_{ih}\right)-\frac{\lambda}{2}\right.\right.$$

$$\left.+\left(\sum_{h=1}^{j}\sup_{1\le i\le n}T_{ih}\right)\left(\sum_{q=j+1}^{K}\sum_{i=1}^{n}P_{iq}\right)+\left(\sum_{h=1}^{j}\sup_{1\le i\le n}r_{ih}\right)\right)$$

$$\left.+2j\left(\sum_{i=1}^{n}\frac{\sum_{h=1}^{j}\left(2\sup_{1\le i\le n}T_{ih}+\sup_{1\le i\le n}r_{ih}\right)-\lambda}{2\sum_{h=1}^{j}\sup_{1\le i\le n}T_{ih}}\right)\right]+U_1(1)+U_2(1)+U_4(-1)+U_5\left(\frac{j}{2\sum_{h=1}^{j}\sup_{1\le i\le n}T_{ih}}\right)=0, \quad (7)$$

$$U_1\left\{\sup_{1\le i\le n}r_{ih}-\sup_{1\le i\le n}T_{ih}\right\}=0, \quad (8)$$

$$U_2\left\{\sup_{1\le i\le n}r_{ih}+\sup_{1\le i\le n}T_{ih}-\Psi_{ih}\right\}=0, \quad (9)$$

$$U_3\left\{\sum_{j=1}^{K}\Psi_{ij}-\Psi\right\}=0, \quad (10)$$

$$U_4\left\{\lambda-\sum_{h=1}^{j}\sup_{1\le i\le n}r_{ih}\right\}=0, \quad (11)$$

$$U_5\left\{\frac{2\sum_{h=1}^{j}\sup_{1\le i\le n}T_{ih}+\sum_{h=1}^{j}\sup_{1\le i\le n}r_{ih}-\lambda}{2\sum_{h=1}^{j}\sup_{1\le i\le n}T_{ih}}\right\}=0, \quad (12)$$

$$U_6\left\{-\sup_{1\le i\le n}T_{ih}\right\}=0, \quad (13)$$

$$\forall\ i=1,2,\ldots,n,\ \ j=1,2,\ldots,K.$$

Solving Equations (6)-(12), and using assumption (2) the optimal values of $\sup_{1\le i\le n}T_{ih}$ and $\sup_{1\le i\le n}r_{ih}$ that give the minimum value of E_K is given from solving the following equation:

$$\sum_{i=1}^{n}\delta_i\sum_{j=1}^{K}\left[\left(\sum_{i=1}^{n}\frac{2j\sum_{h=1}^{j}\left(\sup_{1\le i\le n}T_{ih}-\sup_{1\le i\le n}r_{ih}\right)-\lambda j}{4\left(\sum_{h=1}^{j}\sup_{1\le i\le n}T_{ih}\right)^2}\right)\left[\sum_{h=1}^{j}\left(\sup_{1\le i\le n}T_{ih}+\sup_{1\le i\le n}r_{ih}\right)-\frac{\lambda}{2}\right.\right.$$

$$+\left(\sum_{h=1}^{j}\sup_{1\le i\le n}T_{ih}\right)\left(\sum_{q=j+1}^{K}\sum_{i=1}^{n}P_{iq}\right)+\left(\sum_{h=1}^{j}\sup_{1\le i\le n}r_{ih}\right)\Bigg]$$

$$-\left(\sum_{i=1}^{n}\frac{j}{2\sum\limits_{h=1}^{j}\sup\limits_{1\le i\le n}T_{ih}}\right)\left(2\sum_{h=1}^{j}\sup_{1\le i\le n}r_{ih}-2\lambda\right)-2j\left(\sum_{i=1}^{n}\left(\frac{\sum\limits_{h=1}^{j}\sup\limits_{1\le i\le n}r_{ih}-\lambda}{2\sum\limits_{h=1}^{j}\sup\limits_{1\le i\le n}T_{ih}}\right)-3n\right)$$

$$+\left(n+\sum_{i=1}^{n}\frac{\sum\limits_{h=1}^{j}\sup\limits_{1\le i\le n}r_{ih}-\lambda}{2\sum\limits_{h=1}^{j}\sup\limits_{1\le i\le n}T_{ih}}\right)\left(j\left(1+\sum_{q=j+1}^{K}\sum_{i=1}^{n}P_{iq}\right)+\left(\sum_{h=1}^{j}\sup_{1\le i\le n}r_{ih}\right)\right)\Bigg]=0. \tag{14}$$

From (14) one can find that the times $\{\sup\limits_{1\le i\le n}T^*_{ih},\ \sup\limits_{1\le i\le n}r^*_{ih}\}\in W_i(\sup\limits_{1\le i\le n}T_{ih},\ \sup\limits_{1\le i\le n}r_{ih})$ are minimum times that give the optimal search plan if there does not exist any other times $\{\sup\limits_{1\le i\le n}T_{ih},\ \sup\limits_{1\le i\le n}r_{ih}\}\in W_i(\sup\limits_{1\le i\le n}T_{ih},\ \sup\limits_{1\le i\le n}r_{ih})$ such that $W_i(\sup\limits_{1\le i\le n}T^*_{ih},\ \sup\limits_{1\le i\le n}r^*_{ih})\le W_i(\sup\limits_{1\le i\le n}T_{ih},\ \sup\limits_{1\le i\le n}r_{ih}), i=1,2,...,n$, with strict inequality holding for at least one i.

In fact, n action plans with the same cumulative probability of detection do not have to find the target simultaneously. Ideally, we get the uniformly optimal plan that gives the maximum probability of detection and the minimum time to translate from one point to another in the region i.

4. An application in the case of multinomial distribution

An application in another famous accident occurred in June 2009; an Airbus 330-200 with 228 passengers and Air France Flight 447 crew disappeared over the South Atlantic during a night flight from Rio de Janeiro to Paris. An international air and surface search effort recovered the first wreckage on June 6th, five and one-half days after the accident. More than 1000 pieces of the aircraft and 50 bodies were recovered and their positions logged. A French submarine, as well as French and American research teams, searched acoustically for the underwater locator beacons on each of the two flight recorders (black boxes) for 30 days from June 10th to July 10th, 2009, with no results. Stone et al. [11] described this problem and analyzed the results of this analysis. The analysis shows that all impact points are contained within a 20-nautical mile radius circle from where the emergency began. The results of this analysis are represented by a second distribution, which is circular normal with the center at the last known position. of multinomial distribution

To find the victims of this disaster as soon as possible, we divide the disaster region to small $n-$regions. We have $n-$searchers coordinate their search for the target from time to time and connect to the point at which the emergency began to check if one of the searchers $S_i, i=1,2,\ldots,n$, has found the target. If we consider the search strategy as an experiment with n possible outcomes in time step j, that are mutually exclusive and exhaustive, say $A_{1j}, A_{2j},\ldots,A_{nj}$ and $P(A_{ij})=P_{ij}$, $i=1,2,\ldots,n$, $j=1,2,\ldots,K$ and $\sum\limits_{i=1}^{n}\sum\limits_{j=1}^{K}P_{ij}=1$. In addition, the search strategy consists of n repeated and independent multinomial trials (time steps) with parameters $P_{1j},\ P_{2j},\ldots,\ P_{nj}$. Let X_{ij} be the random variables denoting the number of outcomes in time step j which result in A_{ij}, then $(X_{1j},\ X_{2j},\ldots,\ X_{nj})$ follows the multinomial distribution and hence at step j the target location has a joint probability function,

$$f_{X_{1j}\,X_{2j}\ldots X_{nj}}(x_{1j},x_{2j},\ldots,x_{nj})=\frac{j!}{x_{1j}!\,x_{2j}!\ldots x_{nj}!}P_{1j}^{x_{1j}}P_{2j}^{x_{2j}}\ldots P_{nj}^{x_{nj}}, \tag{15}$$

for $x_{ij} = 0, 1, 2, \ldots, j,\ i = 1, 2, \ldots, n$, such that $\sum\limits_{i=1}^{n} x_{ij} = j$. Therefore, for the case of K-returns the target has joint probability function

$$\sum_{j=1}^{K} f_{X_{1j}\ X_{2j} \ldots X_{nj}}(x_{1j}, x_{2j}, \ldots, x_{nj}) = \sum_{j=1}^{K} \left(\frac{j!}{x_{1j}!\ x_{2j}! \ldots x_{nj}!} P_{1j}^{x_{1j}} P_{2j}^{x_{2j}} \ldots\ P_{nj}^{x_{nj}} \right). \tag{16}$$

Therefore, by substituting from (16) in (1) we have,

$$E_K = \sum_{j=1}^{K} \Big[f_{X_{1j}\ X_{2j} \ldots X_{nj}}(x_{1j}, x_{2j}, \ldots, x_{nj}) \times$$

$$\left(\left(\sum_{h=1}^{j} \sup_{1 \le i \le n} T_{ih} \right) \left(\sum_{q=j}^{K} f_{X_{1q}\ X_{2q} \ldots X_{nq}}(x_{1q}, x_{2q}, \ldots, x_{nq}) \right) + \left(\sum_{h=1}^{j} \sup_{1 \le i \le n} r_{ih} \right) \right) \Big]. \tag{17}$$

Now we get the following minimax non linear optimization problem **(MNLP1)**:

MNLP4:

$$\min_{\sup\limits_{1\le i\le n} r_{ih},\ \sup\limits_{1\le i\le n} T_{ih}} \max_{P_{ij}} \sum_{j=1}^{K} \left[\left(\frac{j!}{x_{1j}! \ldots\ x_{nj}!} P_{1j}^{x_{1j}} \ldots P_{nj}^{x_{nj}} \right) \left[\left(\sum_{h=1}^{j} \sup_{1 \le i \le n} T_{ih} \right) \left(\left[\sum_{q=j}^{K} \left(\frac{q!}{x_{1q}! \ldots\ x_{nq}!} P_{1q}^{x_{1q}} \ldots P_{nq}^{x_{nq}} \right) \right] \right) \right. \right.$$

$$\left. \left. + \left(\sum_{h=1}^{j} \sup_{1 \le i \le n} r_{ih} \right) \right] \right],$$

$$\text{subject to } \sum_{j=1}^{K} \left(\frac{j!}{x_{1j}! \ldots\ x_{nj}!} P_{1j}^{x_{1j}} P_{2j}^{x_{2j}} \ldots\ P_{nj}^{x_{nj}} \right) = 1,\ 0 \le \sup_{1 \le i \le n} r_{ih} \le \sup_{1 \le i \le n} T_{ih},\ \sum_{j=1}^{K} \Psi_{ij} \le \Psi,$$

$$\sup_{1 \le i \le n} r_{ih} + \sup_{1 \le i \le n} T_{ih} \le \Psi_{ih}, 0 \le \left(\frac{j!}{x_{1j}! \ldots\ x_{nj}!} P_{1j}^{x_{1j}} P_{2j}^{x_{2j}} \ldots\ P_{nj}^{x_{nj}} \right) \le 1,$$

$$\sup_{1 \le i \le n} T_{ih} > 0,\ \ = 1, 2, \ldots, n,\ j = 1, 2, \ldots, K,\ h = 1, 2, \ldots, j.$$

Leads to, the maximum probability of detecting the target is

$$\left(\sum_{q=j+1}^{K} \left(\frac{q!}{x_{1q}! \ldots x_{nq}!} P_{1q}^{x_{1q}} \ldots P_{nq}^{x_{nq}} \right) \right) + \frac{\sum\limits_{h=1}^{j} \sup\limits_{1 \le i \le n} r_{ih} + \lambda}{2 \sum\limits_{h=1}^{j} \sup\limits_{1 \le i \le n} T_{ih}}. \tag{18}$$

Also, by substituting with $\sum\limits_{i=1}^{n} P_{iq} = \frac{q!}{x_{1q}! \ldots x_{nq}!} P_{1q}^{x_{1q}} \ldots P_{nq}^{x_{nq}}$ in (14) then we get the optimal values of $\sup\limits_{1 \le i \le n} T_{ih}$ and $\sup\limits_{1 \le i \le n} r_{ih}$ that give the minimum value of E_K .

5. Discussion and conclusions

The coordinated search technique for a lost target in n-regions using n-searchers has been presented. We found the expected value of the time E_K until once $S_i, i = 1, 2, ..., n,$ and the target are reunited. The optimal strategy with minimum expected search time and maximum probability has been illustrated. We get the optimal value of E_K and P_{ij}, assuming the multinomial distributed estimates of the target's position.

In future research, it seems that the proposed model will be extendible to the multiple searchers case by considering the combinations of various lost targets.

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