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Levinson Parallel Algorithm: A Finite-Dimensional Approach with an Infinite-Dimensional Perspective

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Abstract A normalization of the generators of the defect spaces of an isometry is obtained, a version of the Levinson algorithm for Toeplitz block matrices in the infinite-dimensional case is built. Additionally, a factorization of the inverse of the Toeplitz matrix by blocks is obtained. Under this methodology, the obtained recurrences in the infinite dimensional case coincide with the case of the finite dimension, and an autoregressive linear filter to estimate stationary second-order stochastic processes is obtained, usually, the area extension in statistics, applications to spectral estimation, analysis of functional data and prediction problems among other applications is required. The parallelized algorithm for computing multiplications and inverses of block matrices is developed using the Pthreads POSIX library. Two real examples of the literature is illustrated, the parameters of a VAR(1) model and an autoregressive process of order 5 (AR (5)) are estimated. The predicted values in each case are obtained. The estimated quality of the parallelized algorithm is validated, the T^{RC} test as a measure of goodness of fit is used, negligible estimation errors are shown. The performance of the parallel algorithm by the acceleration and efficiency factors is measured, an increase of 8% in speed with respect to the sequential version and the most efficient for $P = 2$ threads are shown.

Keywords Levinson parallel algorithm, Block Toeplitz matrix, Autoregressive linear filter, Partial autocorrelation matrices.

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1. Introduction

The Levinson algorithm ([19]) is a very important result, signal prediction and processing theory is widely used (cf. [6], [13], [17], [20] and [25]) . In the infinite dimensional framework, linear prediction problems have applications in the context of continuous time processes, additionally, large amounts of time series is used (cf. [2], [3]). A first matrix version of the Levinson algorithm in ([33]) is obtained, and later in ([35]). Matrix versions of this algorithm are also discussed, in ([10], [11], [12], [14], [26] and [27]). The algorithm to orthogonal polynomials in the unit calculation is also related ([14], [16]). Additionally, that to estimate Schur parameters in the theory of analytical functions is used (cf. [29] and [30]), the partial autocorrelation coefficients are identified (cf. [2], [8], [10] and [26]), and the coefficients of reflection in geophysics (cf.[9], [15]). An extension to the case of operator values is found ([15]), the matrix structure of the unit operator from the Naimark dilatation theorem is obtained. In this way, the succession of choice in Levinson recurrences is obtained. Also, in ([21], [22], [23] and [24]) different approaches to the algorithm are presented, certain normalization processes of the generators of the defect spaces of an isometry

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are used. Then, The located recurrences in a geometric frame similar to the one studied in ([1]) is allowed, but, the recurrences for square roots of operators have difficulty calculated.

The main contribution of this work is related to the normalisation of the generators of the defect spaces of an isometry ([23]). In addition, a version of Levinson's recurrences is developed which does not require the use of square roots of operators. Furthermore, a factorisation of the inverse of the Toeplitz matrix by blocks is obtained ([11], [14]), and a parallelized algorithm for computing multiplications and inverses of block matrices is developed using the Pthreads POSIX library. An iterative inverse through the recurrences of Levinson is obtained. The key idea is to construct a sequence of operators $\{R_k\}_{k=0}^{p}$ for the Levinson recurrences in a separable Hilbert space $\mathcal{G}$. For example, in the finite-dimensional case, a finite number of covariances of a multivariate process can be shown. More specifically, the sequence of operators $\{R_k\}_{k=0}^{p}$ in a Hilbert space $\mathcal{H}_p$ is constructed and a surjective isometry is obtained:

$$V_p : \mathcal{D}_p \to \mathcal{R}_p \tag{1}$$

where $\mathcal{D}_p$, $\mathcal{R}_p$ are two closed subspaces $\mathcal{H}_p$. In this context, a normalization of the generators of the defect spaces of the isometry V_p is proposed, i.e.,

$$\mathcal{N}_p = \mathcal{H}_p \ominus \mathcal{D}_p,\ \mathcal{M}_p = \mathcal{H}_p \ominus \mathcal{R}_p \tag{2}$$

to a version of the operator values of the Levinson algorithm is obtained. The methodology of orthogonal decompositions does not differentiate, the finite dimensional case from the infinite dimensional case. Also, in the finite dimensional case, the mathematical results in the work as a contribution to the prediction theory can be interpreted, specifically, stationary second-order multivariate stochastic processes. More specifically, an estimator for a recursively multivariate autoregressive linear filter is obtained, and simultaneously, the covariance variance matrix of the error is recursively calculated. Finally, the parameters from Levinson recurrences are obtained, that as the partial autocorrelation coefficients is interpreted. The characterization of those parameters for the development of new spectral estimation techniques is used (cf. [7] and [27]).

A methodology in this work is proposed, the methodology from the results known is differed, (cf. [4], [5], [7], [10], [11], [12], [14], [15], [18], [19], [26], [33] and [35]). Another contribution of the work, the recursive algorithm to solve problems of extension in statistics, analysis of functional data, and prediction problems can be used in an infinite dimensional framework.

The rest of the article follows: In Section 2, the Preliminary Section and the used notation are established. In Section 3, the theoretical aspects are developed, the efficient normalization of the Levinson algorithm in the infinite dimensional case is obtained. In the Section 4 an application in the finite dimensional case is shown. In Section 5 the implementation of the algorithm is shown. In Section 6 an application through two real examples is presented, the prediction capacity is shown, finally, in Section 7 the discussions and the conclusions are established.

2. Preliminaries

First, the notation is introduced. The sets of natural and integers numbers by $\mathbf{N}$ and $\mathbf{Z}$ are denoted. The symbols $\mathbf{R}$ and $\mathbf{C}$ to denote the set of real and complex numbers are used; $\mathbf{D}$ is used, denote the open unitary disk in the complex plane,

$$\mathbf{D} := \{z \in \mathbf{C} : |z| < 1\} \tag{3}$$

The unit calculation, the border $\mathbf{D}$ by $\mathbf{T}$ is denoted. e_k is defined

$$e_k(\zeta) := \zeta^k, \quad \zeta \in \mathbf{T}, \quad k \in \mathbf{Z} \tag{4}$$

As usual, the set of all bounded linear operators acting from the Hilbert space $\mathcal{H}$ to the same space as $\mathcal{L}(\mathcal{H})$ is denoted. The symbol 1, the scalar unit and the operator identity is denoted, that is depending on the context. For the matrix case $1_{q\times q}$ for the identity matrix is used. The symbol A^*, an attached operator and the transpose conjugate matrix is denoted.

Given a succession of bounded linear operators $\{R_k\}_{k=0}^{\infty}$ in a Hilbert space separable $\mathcal{G}$ is defined, where $R_{-n} = R_n^*$ for all $n \in \mathbf{N}$, and that is said, that is strictly defined positive if

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\langle R_{m-n}h_n, h_m\rangle_{\mathcal{G}} > 0 \tag{5}$$

for each non-zero succession $\{h_k\}_{k=0}^{\infty} \subset \mathcal{G}$ with finite support. Rewriting (5) in terms of matrices, the sequence $\{R_k\}_{k=0}^{\infty}$ is strictly defined positive, if the bounded operator

$$T_p : \mathcal{G}^{p+1} \to \mathcal{G}^{p+1} \tag{6}$$

is defined by

$$T_p = \begin{pmatrix} R_0 & R_1^* & \cdots & R_p^* \\ R_1 & R_0 & \cdots & R_{p-1}^* \\ \vdots & \vdots & \ddots & \vdots \\ R_p & R_{p-1} & \cdots & R_0 \end{pmatrix} \tag{7}$$

that is strictly positive for everything $p \in \mathbf{N}$.
That is said that the succession of the first $p+1$ terms $\{R_k\}_{k=0}^{p}$ of the succession $\{R_k\}_{k=0}^{\infty}$ is strictly defined positive if T_p is positive.
For each $p \in \mathbf{N}$, is denoted by $T_p^{-1} = \{[T_p^{-1}]_{ij}\}_{i,j=0,1,\cdots,p}$, the inverse of T_p. As a result of the T_p operators being strictly positive, then T_p^{-1} is the inverse. That follows, the operators $[T_p^{-1}]_{00}$ and $[T_p^{-1}]_{pp}$ are strictly positive, hence, those are operators of compression of T_p^{-1} to suitable subspaces of $\mathcal{G}^{p+1}$.
The methodology in this work is explained. Assuming, the succession $\{R_k\}_{k=0}^{p}$ is strictly defined positive, a Hilbert space is built $\mathcal{H}_p$, and an overjective isometry is defined

$$V_p : \mathcal{D}_p \to \mathcal{R}_p \tag{8}$$

where $\mathcal{D}_p$ and $\mathcal{R}_p$ of $\mathcal{H}_p$ are certain subspaces. Finally, the defect spaces are obtained

$$\mathcal{N}_p = \mathcal{H}_p \ominus \mathcal{D}_p \tag{9}$$

and

$$\mathcal{M}_p = \mathcal{H}_p \ominus \mathcal{R}_p \tag{10}$$

In addition, that is defined

$$\mathcal{E}_p = \left\{ \phi = \sum_{k=0}^{p} e_k \xi_k : \xi_k \in \mathcal{G}, k = 1, \cdots, p \right\} \tag{11}$$

as the set of trigonometric polynomials in $\mathbf{T}$ to values in a Hilbert space. Now, an internal product in $\mathcal{E}_p$ is defined

$$\begin{aligned} \left\langle \sum_{n=0}^{p} e_n f_n, \sum_{m=0}^{p} e_m g_m \right\rangle_p &= \sum_{m=0}^{p}\sum_{n=0}^{p}\langle R_{m-n}f_n, g_m\rangle_{\mathcal{G}} \\ &= \left\langle T_p \begin{pmatrix} f_0 \\ \vdots \\ f_p \end{pmatrix}, \begin{pmatrix} g_0 \\ \vdots \\ g_p \end{pmatrix} \right\rangle_{\mathcal{G}^{p+1}} . \end{aligned} \tag{12}$$

The space $(\mathcal{E}_p, \langle , \rangle_p)$ is a Hilbert space. In effect, that result is concluded from the operator

$$I_p : (\mathcal{E}_p, \langle ., . \rangle_p) \to \mathcal{G}^{p+1} \tag{13}$$

that is defined by

$$I_p\left(\sum e_k h_k\right) = (h_0, h_1 \cdots, h_p) \tag{14}$$

that is a bounded and invertible operator.
Let

$$\mathcal{D}_p = \left\{ \sum_{k=0}^{p-1} e_k \xi_k : \xi_j \in \mathcal{G}, j = 0, 1, \ldots, p-1 \right\} \tag{15}$$

$$\mathcal{R}_p = \left\{ \sum_{k=1}^{p} e_k \xi_j : \xi_j \in \mathcal{G}, k = 1, \ldots, p \right\} \tag{16}$$

and the application $V_p : \mathcal{D}_p \to \mathcal{R}_p$ is defined as the linear extension of $V_p(e_k\xi) = e_{k+1}\xi$. V_p is an overjective isometry, that is easy demonstrated. Note, that V_p is an isometric extension of V_{p-1}, that is $V_p|_{\mathcal{D}_{p-1}} = V_{p-1}$. Now in the usual way, that is defined

$$L^2_{\mathcal{G}} = \left\{ f : \mathbf{T} \to \mathcal{G} | f \text{ is measured} \quad \text{and} \quad \frac{1}{2\pi} \int_0^{2\pi} \|f(e^{it})\|^2_{\mathcal{G}} dt < \infty \right\} \tag{17}$$

Where, $L^2_{\mathcal{G}}$ is a Hilbert space under the internal product

$$\langle f, g \rangle_{L^2_{\mathcal{G}}} = \frac{1}{2\pi} \int_0^{2\pi} \langle f(e^{it}), g(e^{it}) \rangle_{\mathcal{G}} dt \tag{18}$$

For any k, $\mathcal{G}_k$ the subspace of $L^2_{\mathcal{G}}$ is denoted as $\mathcal{G}_k$ by the functions of the shape $e_k a$ $(a \in \mathcal{G})$ is generated. Of ([28]) is known that $\mathcal{G}_i \perp \mathcal{G}_j$ for $i \neq j$, and even more, that is contained

$$L^2_{\mathcal{G}} = \bigoplus_{-\infty}^{\infty} \mathcal{G}_k \tag{19}$$

and

$$\|a\|_{\mathcal{G}} = \|e_k a\|_{L^2_{\mathcal{G}}} \tag{20}$$

Moreover, the application is defined

$$\Gamma_p : \left(\mathcal{E}_p(\mathcal{G}), \langle ., . \rangle_p \right) \to (\mathcal{E}_p, \langle ., . \rangle_{L^2_{\mathcal{G}}} \Big) \tag{21}$$

through equality

$$\Gamma_p(f) = \sum_{k=0}^{p} e_k \sum_{s=0}^{p} R_{k-s} f_s, \quad f \in \mathcal{E}_p. \tag{22}$$

Then, consider the operator

$$J_p : (\mathcal{E}_p(\mathcal{G}), \langle ., . \rangle_{L^2_{\mathcal{G}}}) \to \mathcal{G}^{p+1} \tag{23}$$

is defined, by

$$J_p(\sum_{k=0}^{p} e_k h_k) = (h_0, h_1 \cdots, h_p) \tag{24}$$

Note, that operator is an isometric isomorphism.
The operator Γ_p in terms of operators I_p, J_p and T_p can be written,

$$\Gamma_p = J^*_p T_p I_p. \tag{25}$$

Hence, Γ_p is bicontinuous. Of ([23]) the operator Γ_p verifies

$$\langle \Gamma_p f, g \rangle_{L^2_{\mathcal{G}}} = \langle f, g \rangle_p \tag{26}$$

and, the default spaces of V_p,

$$\mathcal{N}_p = \mathcal{E}_p \ominus \mathcal{D}_p \tag{27}$$

and

$$\mathcal{M}_p = \mathcal{E}_p \ominus \mathcal{R}_p \tag{28}$$

by elements of the form $\Gamma_p^{-1}(e_p x)$ and $\Gamma_p^{-1}(e_0 x),\ x \in \mathcal{G}$ are generated, respectively. Even more

$$\begin{aligned} \widetilde{M}_p(e^{it})x &= \Gamma_p^{-1}(e_0 x) = \left(e_0[T_p^{-1}]_{00} + \cdots + e_p[T_p^{-1}]_{p0}\right)x, \\ \widetilde{N}_p(e^{it})x &= \Gamma_p^{-1}(e_p x) = \left(e_0[T_p^{-1}]_{0p} + \cdots + e_p[T_p^{-1}]_{pp}\right)x. \end{aligned} \tag{29}$$

From those operators, the normalized operators $M_p(e^{it})$ and $N_p(e^{it})$ are defined, that is through of the expressions

$$M_p(e^{it})x = \widetilde{M}_p(e^{it})[T_p^{-1}]_{00}^{-1/2}x, \quad N_p(e^{it})x = \widetilde{N}_p(e^{it})[T_p^{-1}]_{pp}^{-1/2}x. \tag{30}$$

To simplify notation, M_p and N_p instead of $M_p(e^{it})$ and $N_p(e^{it})$ is written, respectively. Furthermore, the polynomial operators M_p and N_p are denoted

$$M_p = e_0 M_{p,0} + \cdots + e_p M_{p,p} \tag{31}$$

and

$$N_p = e_0 N_{p,0} + \cdots + e_p N_{p,p} \tag{32}$$

where $N_{p,p} = [T_p^{-1}]_{pp}^{1/2}$ and $M_{p,0} = [T_p^{-1}]_{00}^{1/2}$.
Consider, the new normalizations

$$\overline{N}_p = N_p[T_p^{-1}]_{pp}^{-1/2} \tag{33}$$

and

$$\overline{M}_p = M_p[T_p^{-1}]_{00}^{-1/2}. \tag{34}$$

Notice, the new trigonometric polynomials $\overline{N}_p$ and $\overline{M}_p$ are defined as

$$\begin{aligned} \overline{N}_p &= e_0\overline{N}_{p,0} + \cdots + e_{p-1}\overline{N}_{p,p-1} + e_p 1, \\ \overline{M}_p &= e_0 1 + \cdots + e_{p-1}\overline{M}_{p,p-1} + e_p\overline{M}_{p,p}. \end{aligned} \tag{35}$$

Notice, of (29), (35) and

$$T_p \begin{pmatrix} [T_p^{-1}]_{00} \\ \vdots \\ [T_p^{-1}]_{p0} \end{pmatrix} = \begin{pmatrix} 1 \\ \vdots \\ 0 \end{pmatrix} \quad \text{and} \quad T_p \begin{pmatrix} [T_p^{-1}]_{0p} \\ \vdots \\ [T_p^{-1}]_{pp} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 1 \end{pmatrix} \tag{36}$$

that follows

$$T_p \begin{pmatrix} 1 \\ \overline{M}_{p,1} \\ \vdots \\ \overline{M}_{p,p} \end{pmatrix} = \begin{pmatrix} \sigma_{0p}^2 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad \text{and} \quad T_p \begin{pmatrix} \overline{N}_{p,0} \\ \vdots \\ \overline{N}_{p,p-1} \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \sigma_{pp}^2 \end{pmatrix} \tag{37}$$

where $\sigma_{0p}^2 = [T_p^{-1}]_{00}^{-1}$ and $\sigma_{pp}^2 = [T_p^{-1}]_{pp}^{-1}$. Moreover, from (37) is obtained

$$\sigma_{0p}^2 = R_0 + R_{-1}\overline{M}_{p,1} + \cdots + R_{-p}\overline{M}_{p,p} \tag{38}$$

and

$$\sigma_{pp}^2 = R_0 + R_1\overline{N}_{p,p-1} + \cdots + R_p\overline{N}_{p,0} \tag{39}$$

The trigonometric polynomials by the classic normal equations (38) and (39) are found (cf. [15], [26] and [36]). In the rest of the work, the notation is used: $\sigma_{0p}^{-2} = (\sigma_{0p}^2)^{-1}$, $\sigma_{pp}^{-2} = (\sigma_{pp}^2)^{-1}$, $\sigma_{0p} = (\sigma_{0p}^2)^{1/2}$, $\sigma_{pp} = (\sigma_{pp}^2)^{1/2}$, $\sigma_{0p}^{-1} = (\sigma_{0p}^{-2})^{1/2}$ and $\sigma_{pp} = (\sigma_{pp}^{-2})^{1/2}$. The polynomials $\overline{N}_p$ and $\overline{M}_p$ are rewritten as

$$\overline{N}_p = N_p \sigma_{pp} \tag{40}$$

and

$$\overline{M}_p = M_p \sigma_{0p} \tag{41}$$

3. Theoretical aspects of the Levinson algorithm in the infinite dimensional case

In this section, the proposed normalization in the previous section for a version of the Levinson algorithm [23] is obtained. The obtained version can be efficiently implemented, the square roots of bounded linear operators are not calculated. The similar parameters to the finite dimensional case are used, the parameters by partial autocorrelation coefficients are called, the parameters in the statistical analysis of functional data is used.
The trigonometric polynomials N_p and M_p for the recurrences of Levinson is verified, [23]

$$\begin{aligned} N_p &= (\zeta N_{p-1} - M_{p-1}\Lambda_p)\,\sigma_{p-1,p-1}\sigma_{pp}^{-1}, \quad N_0 = 1 \\ M_p &= \left(M_{p-1} - \zeta N_{p-1}\Lambda_p^*\right)\sigma_{0,p-1}\sigma_{0p}^{-1}, \quad M_0 = 1. \end{aligned} \tag{42}$$

where

$$\sigma_{pp}^2 = R_0 - A_p^* \alpha_{p-1}\alpha_{p-1}^* A_p = R_0 - A_p^* \beta_{p-1}\beta_{p-1}^* A_p, \tag{43}$$

$$\sigma_{p0}^2 = R_0 - C_p \alpha_{p-1}\alpha_{p-1}^* C_p^* = R_0 - C_p \beta_{p-1}\beta_{p-1}^* C_p^* \tag{44}$$

and the matrices α_p and β_p are given

$$\alpha_p = \begin{pmatrix} M_{p,0} & \cdots & M_{1,0} & M_{0,0} \\ M_{p,1} & \cdots & M_{1,1} & 0 \\ \vdots & \vdots & \vdots & \vdots \\ M_{p,p} & \cdots & 0 & 0 \end{pmatrix} \quad \textit{and} \quad \beta_p = \begin{pmatrix} N_{0,0} & N_{1,0} & \cdots & N_{p,0} \\ 0 & N_{1,1} & \cdots & N_{p,1} \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & N_{p,p} \end{pmatrix} \tag{45}$$

and where A_p^* and C_p are defined

$$A_p^* = [R_p, R_{p-1}, \cdots, R_1] \tag{46}$$

and

$$C_p = [R_{-1}, R_{-2}, \cdots, R_{-p}]. \tag{47}$$

Moreover, the following property for the operator $\Lambda_p : \mathcal{G} \to \mathcal{G}$ is verified (cf. [23]):

$$\Lambda_p = \sum_{m=1}^{p} \sum_{n=0}^{p-1} M_{p-1,n}^* R_{n-m} N_{p-1,m-1} \tag{48}$$

and additionally $\|\Lambda_p\| = \cos\left(\angle_{\mathcal{M}_{p-1}}^{V_p \mathcal{N}_{p-1}}\right)$ and $\Lambda_1 = R_1$.
Now, the matrices $\overline{\alpha}_p$ and $\overline{\beta}_p$ are defined

$$\overline{\alpha}_p = \begin{pmatrix} \overline{M}_{p,0} & \cdots & \overline{M}_{1,0} & \overline{M}_{0,0} \\ \overline{M}_{p,1} & \cdots & \overline{M}_{1,1} & 0 \\ \vdots & \vdots & \vdots & \vdots \\ \overline{M}_{p,p} & \cdots & 0 & 0 \end{pmatrix} \quad \textit{and} \quad \overline{\beta}_p = \begin{pmatrix} \overline{N}_{0,0} & \overline{N}_{1,0} & \cdots & \overline{N}_{p,0} \\ 0 & \overline{N}_{1,1} & \cdots & \overline{N}_{p,1} \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & \overline{N}_{p,p} \end{pmatrix} \tag{49}$$

and the matrices $\gamma_{0p}^{1/2}$ and $\gamma_{pp}^{1/2}$ are defined

$$\gamma_{0p}^{1/2} = \begin{pmatrix} \sigma_{0p}^{-1} & \cdots & 0 & 0 \\ 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & \sigma_{00}^{-1} \end{pmatrix} \quad \textit{and} \quad \gamma_{pp}^{1/2} = \begin{pmatrix} \sigma_{00}^{-1} & \cdots & 0 & 0 \\ 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & \sigma_{pp}^{-1} \end{pmatrix}. \tag{50}$$

Note

$$\alpha_p = \overline{\alpha}_p \gamma_{0p}^{1/2} \qquad \textit{and} \qquad \beta_p = \overline{\beta}_p \gamma_{pp}^{1/2}. \tag{51}$$

In addition, $\overline{\Lambda}_p$ is defined

$$\overline{\Lambda}_p = \sigma_{0,p-1}^{-1} \Lambda_p \sigma_{p-1,p-1}. \tag{52}$$

Proposition 1
Let, the matrices $\overline{\alpha}_{p-1}$ and $\overline{\beta}_{p-1}$ in the equations (49) are defined

$$\sigma_{pp}^2 = R_0 - A_p^* \overline{\alpha}_{p-1} \gamma_{0,p-1} \overline{\alpha}_{p-1}^* A_p = R_0 - A_p^* \overline{\beta}_{p-1} \gamma_{p-1,p-1} \overline{\beta}_{p-1}^* A_p \tag{53}$$

and

$$\sigma_{0p}^2 = R_0 - C_p \overline{\alpha}_{p-1} \gamma_{0,p-1} \overline{\alpha}_{p-1}^* C_p^* = R_0 - C_p \overline{\beta}_{p-1} \gamma_{p-1,p-1} \overline{\beta}_{p-1}^* C_p^*. \tag{54}$$

Even more,

$$\overline{\Lambda}_p = \sigma_{0,p-1}^{-2} \sum_{m=1}^{p} \sum_{n=0}^{p-1} \overline{M}_{p-1,n}^* R_{n-m} \overline{N}_{p-1,m-1}. \tag{55}$$

Proof. The first two formulas from (43), (44) and (51) are obtained. The last formula is obtained

$$\overline{\Lambda}_p = \sigma_{0,p-1}^{-1} \Lambda_p \sigma_{p-1}$$

$$= \sigma_{0,p-1}^{-1} \sum_{m=1}^{p} \sum_{n=0}^{p-1} M_{p-1,n}^* R_{n-m} N_{p-1,m-1} \sigma_{p-1}$$

$$= \sigma_{0,p-1}^{-2} \sum_{m=1}^{p} \sum_{n=0}^{p-1} \overline{M}_{p-1,n}^* R_{n-m} \overline{N}_{p-1,m-1}$$

that shows the result.
That operator can be written

$$\overline{\Lambda}_p = \sigma_{0,p-1}^{-2} \left[1, \overline{M}_{p-1,1}^*, \cdots, \overline{M}_{p-1,p-1}^*\right] \begin{pmatrix} R_{-1} & R_{-2} & \cdots & R_{-p} \\ R_0 & R_{-1} & \cdots & R_{-(p-1)} \\ \vdots & \vdots & \cdots & \vdots \\ R_{p-2} & R_{p-3} & \cdots & R_{-1} \end{pmatrix} \begin{bmatrix} \overline{N}_{p-1,0} \\ \overline{N}_{p-1,1} \\ \vdots \\ 1 \end{bmatrix} \tag{56}$$

The inverse of the Toeplitz matrix by blocks can be factored, in [23] is shown,

$$T_p^{-1} = \alpha_p \alpha_p^* = \beta_p \beta_p^* \tag{57}$$

In terms of the new normalization, the factorization is rewritten

$$T_p^{-1} = \overline{\alpha}_p \gamma_{0p} \overline{\alpha}_p^* = \overline{\beta}_p \gamma_{pp} \overline{\beta}_p^* \tag{58}$$

The Levinson algorithm to a rapid factorization of the inverse of the Toeplitz matrix is obtained.
Now, the equivalent recurrences are obtained, the calculation of square roots of operators is not required.

Theorem 1
For each $p \in \mathbf{N}$ trigonometric polynomial operators $\overline{N}_p$ and $\overline{M}_p$, the recurrences are verified

$$\overline{N}_p = \zeta \overline{N}_{p-1} - \overline{M}_{p-1}\overline{\Lambda}_p,$$
$$\overline{M}_p = \overline{M}_{p-1} - \zeta \overline{N}_{p-1}\widetilde{\Lambda}_p,$$

where

$$\widetilde{\Lambda}_p = \sigma^{-2}_{p-1,p-1}\overline{\Lambda}^*_p \sigma^2_{0,p-1} \tag{59}$$

and initial conditions $\overline{N}_0 = 1$, $\overline{M}_0 = 1$, $\overline{\Lambda}_1 = R_0^{-1}R_{-1}$ and $\widetilde{\Lambda}_1 = R_0^{-1}R_1$.

Proof. From (42) is obtained

$$\begin{aligned} N_p\sigma_{pp} &= \zeta N_{p-1}\sigma_{p-1,p-1} - M_{p-1}\sigma_{0,p-1}\left(\sigma^{-1}_{0,p-1}\Lambda_p\sigma_{p-1,p-1}\right), \\ M_p\sigma_{0p} &= M_{p-1}\sigma_{0,p-1} - \zeta N_{p-1}\sigma_{p-1,p-1}\left(\sigma^{-1}_{p-1,p-1}\Lambda^*_p\sigma_{0,p-1}\right), \end{aligned} \tag{60}$$

thereby

$$\begin{aligned} \overline{N}_p &= \zeta \overline{N}_{p-1} - \overline{M}_{p-1}\overline{\Lambda}_p, \\ \overline{M}_p &= \overline{M}_{p-1} - \zeta \overline{N}_{p-1}\left(\sigma^{-1}_{p-1,p-1}\Lambda^*_p\sigma_{0,p-1}\right). \end{aligned} \tag{61}$$

Now, $\overline{\Lambda}^*_p = \sigma_{p-1,p-1}\Lambda^*_p\sigma^{-1}_{0,p-1}$ of $\overline{\Lambda}_p = \sigma^{-1}_{0,p-1}\Lambda_p\sigma_{p-1,p-1}$ is obtained. Therefore

$$\sigma^{-2}_{p-1,p-1}\overline{\Lambda}^*_p\sigma^2_{0,p-1} = \sigma^{-1}_{p-1,p-1}\Lambda^*_p\sigma_{0,p-1} = \widetilde{\Lambda}_p. \tag{62}$$

Finally, $\overline{\Lambda}_1 = R_0^{-1}R_{-1}$ of (56) is obtained, and therefore $\widetilde{\Lambda}_1 = R_0^{-1}R_1$.
The following result to computational recurrences is allowed:

Corollary 1
For $k = 2, \cdots, p$,

$$\overline{N}_{k,k} = \overline{M}_{k,0} = 1, \quad \overline{N}_{k,0} = -\overline{\Lambda}_k, \quad \overline{M}_{k,k} = -\widetilde{\Lambda}_k, \tag{63}$$

and for $i = 1, \cdots, k-1$

$$\begin{aligned} \overline{N}_{k,i} &= \overline{N}_{k-1,i-1} - \overline{M}_{k-1,i}\overline{\Lambda}_k, \\ \overline{M}_{k,i} &= \overline{M}_{k-1,i} - \overline{N}_{k-1,i-1}\widetilde{\Lambda}_k, \end{aligned} \tag{64}$$

with initial conditions

- $\overline{M}_{0,0} = \overline{M}_{1,0} = \overline{N}_{0,0} = \overline{N}_{1,1} = 1$,
- $\overline{N}_{1,0} = -\overline{\Lambda}_1$,
- $\overline{M}_{1,1} = -\widetilde{\Lambda}_1$,
- $\overline{\Lambda}_1 = R_0^{-1}R_{-1}$,
- $\widetilde{\Lambda}_1 = R_0^{-1}R_1$.

Proof. Of the theorem 1 is obtained

$$\begin{aligned} \left[\ \overline{N}_{k,0}, \overline{N}_{k,1}, \cdots, \overline{N}_{k,k-1}, 1\ \right] &= \left[\ 0, \overline{N}_{k-1,0}, \cdots, \overline{N}_{k-1,k-2}, 1\ \right] \\ &- \left[\ \overline{\Lambda}_k, \overline{M}_{k-1,1}\overline{\Lambda}_k, \cdots, \overline{M}_{k-1,k-1}\overline{\Lambda}_k, 0\ \right]; \\ \left[\ 1, \overline{M}_{k,1}, \cdots, \overline{M}_{k,k-1}, \overline{M}_{k,k}\ \right] &= \left[\ 1, \overline{M}_{k-1,1}, \cdots, \overline{M}_{k-1,k-1}, 0\ \right] \\ &- \left[\ 0, \overline{N}_{k-1,0}\widetilde{\Lambda}_k, \cdots, \overline{N}_{k-1,k-2}\widetilde{\Lambda}_k, \widetilde{\Lambda}_k\ \right] \end{aligned} \tag{65}$$

the result is obtained.

4. Applications in the finite dimensional case

Let $X = \{X_t\}_{t\in\mathbf{Z}}$ a stochastic process q-varied of the second order and stationary with zero mean and covariance function $R_k, k = 0, \pm 1, \cdots$, is given

$$R_k = E(X_{t+k}X_t^*) = \frac{1}{2\pi}\int_{\mathbf{T}} e^{ik\theta}W(e^{i\theta})d\theta \tag{66}$$

where E is the mathematical expectation and W is the spectral density of the process.
The space by the X process is generated

$$H^X = \overline{Lin\{\sum A_t X_t : A_t \in \mathbf{C}^{q\times q}\}} \tag{67}$$

and space $L^2(W)$ is defined

$$L^2(W) = \{\Phi : \mathbf{T} \to \mathbf{C}^{q\times q} : \|\Phi\|^2 = \frac{1}{2\pi}\int_{\mathbf{T}} Trace(\Phi W \Phi^*)d\theta < \infty\} \tag{68}$$

An internal matrix product in the space can be introduced

$$(\Phi, \Psi)_W = \frac{1}{2\pi}\int_{\mathbf{T}} \Phi W \Phi^* d\theta \tag{69}$$

On the other hand,

$$X = \left\{X_t = \left(X_t^1, \cdots, X_t^q\right)\right\}_{t\in\mathbf{Z}} \tag{70}$$

the Gramians of the process is defined

$$[X_t, X_s] = \left\{E(X_t^i \overline{X_s^j})\right\}_{i,j=1,\cdots,q} \tag{71}$$

Note, $R_k = [X_k, X_0]$.
The application from (66) is obtained

$$\sum A_t X_t \to \sum A_t e_t \tag{72}$$

is an isometric isomorphism between H^X and $L^2(W)$.
The following result from the expression of the covariance is obtained

$$\left[\sum_{j\in J} A_j X_j, \sum_{k\in K} B_k X_k\right] = \left(\sum_{j\in J} A_j e_j, \sum_{k\in K} B_k e_k\right)_W \tag{73}$$

where J and K are two finite sets of indices.
The following result is a direct consequence of the previous equality and the proposition 1. $\sigma^2_{0,p-1}\overline{\Lambda}_p$ as the partial autocorrelation coefficients between X_0 and X_p is interpreted.

Proposition 2
For each $p \in \mathbf{N}$, the following formula is validated

$$\sigma^2_{0,p-1}\overline{\Lambda}_p = \left(\sum_{m=0}^{p-1} \overline{M}^*_{p-1,m} e_m, \sum_{n=0}^{p-1} \overline{N}^*_{p-1,n} e_{n+1}\right)_W = \left[\sum_{m=0}^{p-1} \overline{M}^*_{p-1,m} X_m, \sum_{n=0}^{p-1} \overline{N}^*_{p-1,n} X_{n+1}\right].$$

Now, the estimation of a multivariate autoregressive filter is considered. A multivariate input time series $\{\mathbf{Z}_t\}$ in a multivariate output time series $\{\mathbf{X}_t\}$ is transformed, the rule is followed

$$\mathbf{X}_t + \overline{N}^*_{l,l-1}\mathbf{X}_{t-1} + \cdots + \overline{N}^*_{l,0}\mathbf{X}_{t-l} = \mathbf{Z}_t, \quad t = 0, 1, \cdots \tag{74}$$

Let $\{\mathbf{X}_t = (X_t^1, \cdots, X_t^q)^*\}$ a time series q-varied second-order centered stationary with covariance, such that

$$R_k = E(\mathbf{X}_{t+k}\mathbf{X}_t^*) = E(\mathbf{X}_k\mathbf{X}_0^*) = [\mathbf{X}_k, \mathbf{X}_0] \tag{75}$$

The time series to $\{\mathbf{Z}_t\}$ is supposed

$$E(\mathbf{Z}_t) = 0 \quad \text{and} \quad [\mathbf{Z}_t, \mathbf{Z}_s] = \boldsymbol{\sigma}^2\delta_{t-s}. \tag{76}$$

Now, of (74) and (37),

$$\boldsymbol{\sigma}^2 = \sum_{k=0}^{l} \overline{N}_{l,k}^*[\mathbf{X}_{t-l+k}, \mathbf{X}_t] = \sum_{k=0}^{l} \overline{N}_{l,k}^* R_{k-l} = \sigma_{ll}^2 \tag{77}$$

in as much as

$$[\mathbf{X}_t + \overline{N}_{l,l-1}^*\mathbf{X}_{t-1} + \cdots + \overline{N}_{l,0}^*\mathbf{X}_{t-l}, \overline{N}_{l,l-1}^*\mathbf{X}_{t-1} + \cdots + \overline{N}_{l,0}^*\mathbf{X}_{t-l}]$$

$$= [\overline{N}_{l,0}^*, \cdots, \overline{N}_{l,l-1}^*, 1] \begin{pmatrix} R_0 & \cdots & R_{-l} \\ \vdots & \cdots & \vdots \\ R_l & \cdots & R_0 \end{pmatrix} \begin{bmatrix} \overline{N}_{l,0} \\ \vdots \\ \overline{N}_{l,l-1} \\ 0 \end{bmatrix} = [0, \cdots, \sigma_{ll}^2] \begin{bmatrix} \overline{N}_{l,0} \\ \vdots \\ \overline{N}_{l,l-1} \\ 0 \end{bmatrix} = 0$$

The multivariate filter and the variance of the time series $\{\mathbf{Z}_t\}$ from the recurrences of Levinson are estimated.

5. Efficient version of Levinson parallel algorithm

Suppose, a sample $\mathbf{x}_1, \cdots, \mathbf{x}_p, \mathbf{x}_{p+1}$ from a time series q-varied centered stationary is taken. The covariance function is estimated

$$\widehat{R}_k = \begin{cases} \frac{1}{p+1}\sum_{t=1}^{p+1-k} \mathbf{x}_{t+k}\mathbf{x}_t & \text{for } k = 0, 1, \cdots, p \\ \frac{1}{p+1}\sum_{t=-k+1}^{p+1} \mathbf{x}_{t+k}\mathbf{x}_t & \text{for } k = -p, \cdots, -1 \end{cases}$$

The notation for the estimator is simplified σ_{kk}^{-2}, σ_{0k}^{-2}, σ_{kk}^2, σ_{0k}^2, $\overline{\Lambda}_k$, $\widetilde{\Lambda}_k$, $\overline{\alpha}_k$, A_k^*, C_k instead of $\widehat{\sigma_{kk}^{-2}}$, $\widehat{\sigma_{0k}^{-2}}$, $\widehat{\sigma_{kk}^2}$, $\widehat{\sigma_{0k}^2}$, $\widehat{\overline{\Lambda}}_k$, $\widehat{\widetilde{\Lambda}}_k$ $\widehat{\overline{\alpha}}_k$, $\widehat{A}_k^*$, $\widehat{C}_k$, respectively. The Levinson's recurrences to adjust an autoregression in an increasing way. The Levinson's recurrences to adjust an autoregression in an increasing way is made, i.e., an autoregressive order process $l < p+1$, $AR(l)$ is adjusted.The order l of autoregression is estimated, the criterion of minimize is used

$$AICC = -2\ln L(\overline{N}_{l,0}^*, \cdots, \overline{N}_{l,l-1}^*, \sigma_{ll}^2) + 2(lq^2+1)pq/(pq - lq^2 - 2) \tag{78}$$

where L is the likelihood.
The normalized version methodology of the Levinson algorithm is illustrated, the data is [5]. One of the important results in the investigation is obtained, an algorithm to estimate a multivariate autoregressive process is obtained, that is defined

$$\gamma_{0k} = \begin{pmatrix} \sigma_{0k}^{-2} & \cdots & 0 & 0 \\ 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & \sigma_{00}^{-2} \end{pmatrix}.$$

Note: The square root of the matrix is defined (50).

Efficient version of Levinson algorithm

- **Step 1.** Enter the data:
  - Read the data $\mathbf{x}_1, \cdots, \mathbf{x}_p, \mathbf{x}_{p+1}$,
  - Analyze the stationarity, if is required, the data can be transformed
  - Estimate the covariance $\widehat{R}_0, \widehat{R}_1, \cdots, \widehat{R}_p$ (only covariance matrices are needed $\widehat{R}_0, \widehat{R}_1, \cdots, \widehat{R}_l$ so, that can be estimated).
- **Step 2.** Initialize the input parameters:
  - $\overline{M}_{0,0} = \overline{M}_{1,0} = \overline{N}_{0,0} = \overline{N}_{1,1} = 1_{q\times q}$, $\overline{N}_{1,0} = -\widehat{R}_0^{-1}\widehat{R}_{-1}$, $\overline{M}_{1,1} = -\widehat{R}_0^{-1}\widehat{R}_1$, $\overline{\Lambda}_1 = \widehat{R}_0^{-1}\widehat{R}_{-1}$, $\widetilde{\Lambda}_1 = \widehat{R}_0^{-1}\widehat{R}_1$, $\gamma_{00} = \sigma_{00}^{-2} = \widehat{R}_0^{-1}$ and $\overline{\alpha}_0 = 1_{q\times q}$,
  - If the order of the autoregressive process is $l = 1$ calculate $\sigma_{11}^2 = \widehat{R}_0 - \widehat{R}_1\widehat{R}_0^{-1}\widehat{R}_{-1}$ then finish the algorithm with an output filter $\overline{N}_{1,0} = -\widehat{R}_0^{-1}\widehat{R}_1$ and a prediction error matrix σ_{11}^2.
- **Step 3. For** $k = 2, \cdots, l+1$
  - Construct the matrix $A_{k-1}^* = [\widehat{R}_{k-1}, \cdots, \widehat{R}_1]$ and the conjugate matrix A_{k-1},
  - Construct the matrix $C_{k-1} = [\widehat{R}_1^*, \cdots, \widehat{R}_{k-1}^*]$ and the conjugate matrix C_{k-1}^*,
  - Construct the matrix $\overline{\alpha}_{k-2}$ as in (49) and the conjugated transpose $\overline{\alpha}_{k-2}^*$,
  - Construct the matrix $\gamma_{0,k-2}$ with (50).
- **Step 3.1.** Create Threads
  - For each block matrix multiplication, P-threads are created,
  - P-threads execute the block matrix multiplication algorithm as a function,
  - Obtain $P_{k-2} = \overline{\alpha}_{k-2}\gamma_{0,k-2}\overline{\alpha}_{k-2}^*$,
  - Obtain $\sigma_{0,k-1}^2 = \widehat{R}_0 - C_{k-1}P_{k-2}C_{k-1}^*$,
  - Obtain $\sigma_{0,k-1}^{-2} = \left(\sigma_{0,k-1}^2\right)^{-1}$,
  - Obtain $\sigma_{k-1,k-1}^2 = \widehat{R}_0 - A_{k-1}^*P_{k-2}A_{k-1}$,
  - Obtain $\sigma_{k-1,k-1}^{-2} = \left(\sigma_{k-1,k-1}^2\right)^{-1}$,
- **Step 3.1.1.** If $k \leq l$
  - Obtain $\overline{\Lambda}_k = \sigma_{0,k-1}^{-2}\sum_{m=1}^{k}\sum_{n=0}^{k-1}\overline{M}_{k-1,n}^* R_{n-m}\overline{N}_{k-1,m-1}$. using matrix multiplication given in (56),
  - Obtain $\Lambda_k^{partial} = \sigma_{0,k-1}^2\overline{\Lambda}_k$,
  - Calculate $\widetilde{\Lambda}_k = \sigma_{k-1,k-1}^{-2}\overline{\Lambda}_k^*\sigma_{0,k-1}^2$,
  - Set
$$\overline{N}_{k,k} = \overline{M}_{k,0} = 1_{q\times q}, \quad \overline{N}_{k,0} = -\overline{\Lambda}_k, \quad \overline{M}_{k,k} = -\widetilde{\Lambda}_k,$$

  Set $i = 1, \cdots, k-1$
  - Set
$$\overline{N}_{k,k-i} = \overline{N}_{k-i-1,k-i} - \overline{M}_{k-1,k-i}\overline{\Lambda}_k,$$
$$\overline{M}_{k,k-i} = \overline{M}_{k-1,k-i} - \overline{N}_{k-1,k-i-1}\widetilde{\Lambda}_k,$$
- **Step 4.** Output:
  - The estimated coefficients $\overline{N}_{l,l-1}, \cdots, \overline{N}_{l,0}, \overline{M}_{l,l}, \cdots, \overline{M}_{l,1}$, the variances of prediction errors forward and backward σ_{ll}^2 and σ_{0l}^2 respectively, and the partial autocorrelation coefficients $\Lambda_1^{parcial}, \cdots, \Lambda_l^{parcial}$.
  - Calculate the execution time of the total program.
  - Calculate the criteria for model validation.
  - Calculate the efficiency and speedup factors for the P threads.

The implementation of the algorithm is illustrated, a time series of stationary second-order multivariate stochastic processes is used, the manipulation of large dimensions in covariance matrices is required, for this reason, P-threads is implemented for handling the parallel version about inverse matrix and matrix multiplication; especially, the inverse factorization of the Toeplitz matrix of blocks is calculated. If data series is massive then the matrix multiplication and inverse matrix can be very complicated, that is the reason, a parallel implementation in the matrix multiplication and inverse matrix of the version of the Levinson algorithm is proposed, that was another contribution of the work. Parallel computing with the objective of reducing time is used, computationally expensive problems are solved, the problems with time and spatial complexity are concerned, [31] and [32]. In particular in the algorithm, the parallelization of the matrix multiplication is proposed, specifically, the calculation of σ^2_{0l}, σ^2_{ll}, (POSIX is necessary to calculate those inverses σ^{-2}_{0l}, σ^{-2}_{ll}), $\overline{\Lambda}_l$, and $\widetilde{\Lambda}_l$. The POSIX (Pthread) thread library of the ANSI **C** programming environment, an interface to create and interact with the threads is provided, that run separately within a program, the individual matrix multiplication in small batches of independently data is performed.

6. Results

In the following, an application of the methodology is shown through two well-known examples in the literature.

Example 1
The information is obtained, a simulation of 1000 observations of a VAR(1) model is based

$$x_{1t} = 0.7x_{1t-1} + 0.2x_{2t-1} + \epsilon_{1t}, \quad x_{2t} = 0.2x_{1t-1} + 0.7x_{2t-1} + \epsilon_{2t} \tag{79}$$

where ϵ_{1t} and ϵ_{2t} are errors with Gaussian distribution $N(\mu, \Sigma)$.
The model in matrix form

$$\begin{bmatrix} x_{1t} \\ x_{2t} \end{bmatrix} = \begin{pmatrix} 0.7 & 0.2 \\ 0.2 & 0.7 \end{pmatrix} \begin{bmatrix} x_{1t-1} \\ x_{2t-1} \end{bmatrix} + \begin{bmatrix} \epsilon_{1t} \\ \epsilon_{2t} \end{bmatrix} \tag{80}$$

where

$$\Pi_1 = \begin{pmatrix} 0.7 & 0.2 \\ 0.2 & 0.7 \end{pmatrix} \qquad \mu = \begin{bmatrix} 1 \\ 5 \end{bmatrix} \qquad \Sigma = \begin{pmatrix} 1 & 0.5 \\ 0.5 & 1 \end{pmatrix} \tag{81}$$

The parameters of the VAR(1) model with the proposed algorithm are estimated

$$\widehat{\Phi}_{11} = \begin{pmatrix} -0.716904 & -0.192663 \\ -0.190657 & -0.720195 \end{pmatrix} \tag{82}$$

The variances of prediction errors forward and backward

$$\sigma^2_{11} = \begin{pmatrix} 1.354971 & -0.739258 \\ -0.739258 & 1.485544 \end{pmatrix} \quad , \quad \sigma^2_{01} = \begin{pmatrix} 1.306668 & -0.659502 \\ -0.659502 & 1.406564 \end{pmatrix} \tag{83}$$

and partial autocorrelation coefficients

$$\Lambda_2^{partial} = \begin{pmatrix} -0.021336 & -0.003619 \\ -0.013424 & -0.063632 \end{pmatrix} \tag{84}$$

In Table (1) the estimates for the data are shown, the estimation of six observations by the VAR (1) model is

presented, the algorithm for the first four simulated observations is well fitted.

Table 1. Data real and predictions of model VAR(1) by the proposed algorithm are shown.

Y	Y_{t1}	*prediction*	Y_{t2}	prediction
992	0.9113	0.8395	4.7072	4.7830
993	0.7333	0.9113	4.9311	4.7072
994	1.0136	0.7333	4.4093	4.9311
995	1.1683	1.0136	4.3109	4.4093
996	-1.7974	1.1683	2.6171	4.3109
997	-3.3086	0.5698	1.9862	4.8428

Table 2. Internal temperature and predictions with the AR(5) model by the proposed algorithm are generated.

T	*Internal temperature*	*prediction*
1001	25.3700	25.0522
1002	24.8100	22.5688
1003	24.2600	25.4424
1004	23.6900	23.6065
1005	23.0900	21.8244
1006	22.5100	22.8673

Example 2
The information at the DGF weather station is obtained, the temperature variation for the months of January 2015 until October 2015, the data from the web site http://infomet.dgf.uchile.cl/OBSERVACIONES/observaciones.html can be obtained. The internal temperature of the weather station $X_{t1}, t = 1, \ldots, 1000$ and the external temperature (ambient temperature) of the weather station $X_{t2}, t = 1, \ldots, 1000$.

The parameters of an autoregressive model of order $l = 5$ with the proposed algorithm are estimated

$$\widehat{\Phi}_{51} = \begin{pmatrix} 9.144482 & 5.407727 \\ -9.538830 & -5.633143 \end{pmatrix} \quad , \quad \widehat{\Phi}_{52} = \begin{pmatrix} -12.036434 & -7.260416 \\ 0.294659 & 0.175459 \end{pmatrix}$$

$$\widehat{\Phi}_{53} = \begin{pmatrix} 9.648738 & 5.851278 \\ 1.102902 & 0.680174 \end{pmatrix} \quad , \quad \widehat{\Phi}_{54} = \begin{pmatrix} -3.758930 & -3.993432 \\ 0.972126 & 0.650071 \end{pmatrix}$$

$$\widehat{\Phi}_{55} = \begin{pmatrix} -0.995541 & 0.895917 \\ 0.822545 & 0.744335 \end{pmatrix}$$

The variances of prediction errors forward and backward

$$\sigma^2_{ll} = \begin{pmatrix} -0.5796179 & 0.8374511 \\ 0.8838184 & -1.1885251 \end{pmatrix} \quad , \quad \sigma^2_{0l} = \begin{pmatrix} -0.3347509 & 0.7757447 \\ 0.7147060 & -1.2169395 \end{pmatrix}$$

The partial autocorrelation coefficients is defined

$$\Lambda_5^{partial} = \begin{pmatrix} 0.2535453 & -0.1503487 \\ -0.2272904 & -1.345889 \end{pmatrix}$$

Assuming, the fitted AR(5) model is the true model for $\mathbf{X}_t = (X_{t1}, X_{t2})'$, the one- and two-step ahead predictors of $X_{1001}, X_{1002}, \ldots, X_{1006}$ in Tables (2) and (3) are shown. The estimation of six temperature measurements is presented, the first three measurements of internal temperature and air temperature by the proposed algorithm are well adjusted. To validate the estimation quality of the algorithm, the T^{RC} statistical test [34] as a measure of

Table 3. Air temperature and predictions with the AR(5) model by the proposed algorithm are generated.

T	*Air temperature*	*prediction*
1001	21.2700	27.2549
1002	20.8700	24.1049
1003	20.2500	23.4520
1004	19.4900	22.2563
1005	19.0200	26.8286
1006	18.6100	26.9572

Table 4. The T^{RC} for the AR(2) model is estimated.

Measures	$\mathbf{Y}_{992}, \ldots, \mathbf{Y}_{997}$
T^{RC}	0.0189

goodness of fit is used. In the case, that is interesting to know if any model, $k = 1, 2, \ldots, m$, are better than the benchmark in terms of expected loss. Let

$$d_{k,t} = L\left(\xi_t, \delta_{0,t-h}\right) - L\left(\xi_t, \delta_{k,t-h}\right) \quad , \quad k = 1, 2, \ldots, m \tag{85}$$

where $d_{k,t}$ denotes the performance of model k relative to the benchmark at time t, and those variables in the relative yield vector performances are stacked, $d_t = (d_{1,t}, \ldots, d_{m,t})$. Provided that $\mu = \mathbf{E}(d_t)$ is well defined, the null hypothesis of interest is formulated

$$H_0 : \mu \leq 0 \quad , \quad \textit{and our maintained hypothesis is} \quad \mu \in R^m \tag{86}$$

Under the assumption that model k is better than the benchmark if and only if $\mathbf{E}(d_{k,t}) > 0$, the exclusive focus on the properties of d_t and abstract entirely from all aspects that relate to the construction of the $\delta - variables$ (where δ is a finite set of possible decision rules). Thus d_t , $t = 1, \ldots, n$, is de facto viewed as our data.
In [34], the RC from the test statistic is built

$$T_n^{RC} = \max\left(\sqrt{n}\bar{d}_1, \ldots, \sqrt{n}\bar{d}_m\right) \tag{87}$$

where

$$\bar{d} = \frac{1}{n}\sum_{i=1}^{n} d_t \tag{88}$$

and an asymptotic null hypothesis of distribution is based

$$n^{\frac{1}{2}}\bar{d} \sim N_m\left(0, \hat{\Sigma}\right) \tag{89}$$

where

$$\mathbf{E}\left(\hat{\Sigma}\right) = \Sigma \tag{90}$$

Table (4), the results of the prediction evaluation through the T^{RC} statistic are presented, a low error for the predicted values with respect to true values is shown. To measure the execution time, the function `gettimeofday` is used in the library sys\time:h. The 169467 μs execution time of the sequential algorithm is shown for 1000 data from the bivariate series of the example 1. Table (5) the execution time of the Levinson parallel algorithm are shown, the speedup and efficiency factors for the parallel algorithm for different numbers of threads are presented.
When the number of pthreads greater than 4, the efficiency is not increased, i.e., not all the threads are executing useful work. Perhaps, the loss of efficiency is due to the increased cost of communication among processors, also, the delays in the communications and synchronizations with the non-parallelizable processes. The ideal case P = 2

Table 5. The speedup and efficiency factors of the Levison parallelized algorithm for different numbers of threads.

Pthread	*Time (μs)*	*speedup*	*efficiency*
P = 2	156002	1.086312996	0.543156498
P = 3	258954	0.654428972	0.218142991
P = 4	267358	0.633857973	0.158464493
P = 8	455937	0.37168951	0.046461189
P = 16	873991	0.193900166	0.01211876
P = 32	1601456	0.105820578	0.003306893

Table 6. The T^{RC} statistic for Air temperature and predictions with the AR(5) model.

Measures	$\mathbf{Y}_{1001}, \ldots, \mathbf{Y}_{1006}$
T^{RC}	0.4495

Table 7. The speedup and efficiency factors of the Levison parallelized algorithm for different numbers of threads.

Pthread	*Time (μs)*	*speedup*	*efficiency*
P = 2	392342	1.051671756	0.525835878
P = 3	535435	0.770616415	0.256872138
P = 4	605314	0.68165448	0.17041362
P = 8	1169730	0.352743796	0.044092974
P = 16	1340776	0.307743426	0.019233964
P = 32	1914501	0.215520911	0.006735028

can be used, a 8% increase in speed-up with respect to the sequential algorithm is provided.
The results of the prediction evaluation are measured, the T^{RC} statistic in the Table (6) is presented. A low error for the predicted values with respect to the values true by the considered series is shown. The execution time 412615 μs of the sequential algorithm for 1000 bivariate series data from the example 2 is shown. In the Table (7), the execution time of the Levinson parallel algorithm is shown, and the speedup and efficiency factors for the parallel algorithm by different numbers of threads are observed.
When the number of pthreads greater than 4, the efficiency is not increased, i.e., not all the threads are executing useful work. Perhaps, the loss of efficiency is due to the increased cost of communication among processors, also, the delays in the communications and synchronizations with the non-parallelizable processes. The ideal case P = 2 can be used, a 5% increase in speed-up with respect to the sequential algorithm is provided.

7. Discussion and Conclusions

In this work, we have obtained a version of the Levinson algorithm through techniques of operator theory. The introduction of a normalization technique for the generators of defect spaces associated with an isometry has enabled the development of a version of the Levinson algorithm tailored for infinite-dimensional Toeplitz block matrices. This framework facilitates the estimation of stationary second-order stochastic processes, particularly in the context of autoregressive linear filters. The implications of this work extend to spectral estimation, functional data analysis, and prediction tasks.

To evaluate the performance and applicability of the algorithm, we have implemented a parallelized version utilizing the Pthreads POSIX library and tested it on two real-world examples: estimating parameters for a Vector Autoregression (VAR)(1) model and an autoregressive process of order 5 (AR(5)). The accuracy of the parallelized algorithm has been assessed using the T^{RC} goodness-of-fit test, which demonstrates negligible estimation errors.

The study provides a analysis of performance metrics, including speedup and efficiency factors. Our results reveal that the parallel algorithm achieves an optimal scenario with a speed increase of 8% when using 2 threads compared to the sequential algorithm. However, as the number of threads increases beyond 4, the study acknowledges a decrease in efficiency attributed to communication overhead and synchronization delays.

The analysis of the algorithm's performance has revealed several weaknesses that should be addressed in future work. Firstly, the algorithm exhibits scalability limitations, as its efficiency decreases when the number of threads increases. Although it demonstrates good performance with a small number of threads, this limitation may restrict its applicability in high-performance computing environments. To overcome this challenge, future research should focus on proposing strategies to improve scalability, such as exploring load balancing techniques or communication optimizations. By addressing these scalability limitations, the algorithm's suitability for high-performance computing environments can be significantly improved, ensuring efficient processing of large-scale data.

Secondly, a weakness of the presented study is the lack of a comparative analysis with existing parallel algorithms or optimization strategies for similar problems. While the algorithm's performance and accuracy have been demonstrated, a direct comparison with state-of-the-art methods would provide a clearer understanding of its novelty and competitive edge. Therefore, it is recommended to incorporate a comparative analysis section in future work. This section should benchmark the proposed algorithm against existing parallel algorithms or optimization strategies in terms of computational efficiency and accuracy. By conducting such a comparative analysis, researchers can assess the algorithm's competitive advantages and highlight its unique contributions to the field, further strengthening its credibility and impact.

To address the identified weaknesses and further improve the algorithm's performance, several areas of future work can be pursued. Firstly, comprehensive comparative studies should be conducted to establish the effectiveness of the proposed algorithm. These studies would involve benchmarking the algorithm against state-of-the-art methods, specifically focusing on computational efficiency and accuracy. By comparing the proposed algorithm with existing parallel algorithms or optimization strategies, researchers can gain insights into its relative strengths and weaknesses, thereby establishing its competitive advantages and highlighting its novelty. Incorporating a comparative analysis section would provide a more comprehensive evaluation of the algorithm's performance and strengthen its position in the field.

Furthermore, future work should prioritize the investigation and development of scalability strategies. The observed scalability limitations indicate the need for solutions that can enhance the algorithm's performance with an increasing number of threads. Exploring load balancing techniques and communication optimizations would be crucial in improving scalability. By addressing these limitations, the algorithm's applicability in high-performance computing environments can be significantly expanded, enabling efficient processing of larger datasets. It is recommended to discuss potential strategies and approaches in the future work section, emphasizing the importance of overcoming scalability challenges.

Lastly, it is recommended to provide a more detailed description of the implementation details related to parallelization. Researchers should clarify the strategies employed to manage communication overhead and any optimizations utilized in the usage of the Pthreads library. This level of transparency and technical clarity will enable readers to better understand the intricacies of the parallelization process and facilitate the replication and implementation of the algorithm in future research. Ensuring transparency in implementation details would strengthen the integrity of the study and allow for a more comprehensive evaluation of its methodology.

By addressing these weaknesses and incorporating the recommended future work, the analysis and discussion of the results section will be strengthened. This will provide a more comprehensive evaluation of the algorithm's performance, scalability, and competitiveness, ultimately contributing to its further development and broader impact in the field.

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