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Bayesian Estimation of the Odd Lindley Exponentiated Exponential Distribution: Applications in-Reliability

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Abstract In this work, we investigate the estimation of the unknown parameters and the reliability characteristics of the Odd Lindley Exponentiated Exponential distribution. The Bayes estimators and corresponding risks are derived using various loss functions with complete data and a gamma prior distribution. A simulation study was carried out to calculate all the results. We used Pitman's closeness criterion and the integrated mean squared error to compare the performance of the Bayesian and maximum likelihood estimators. Finally, we illustrate our techniques by analysing a real-life data set.

Keywords Reliability, Estimation, Bayesian analysis, Simulations, Monte Carlo Methods.

AMS 2010 subject classifications 62E99,60E05

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1. Introduction

This work is based on the estimation of the Odd Lindley Exponentiated Exponential model introduced by Goual et al (2019) [10] where they studied the validation of the OLEE by a modified goodness-of-fit test and its applications to censor and complete data. They found that the proposed lifetime model is better than the exponential exponential, Moment exponential, Log Burr Hatke exponential and the two parameter odd Lindley exponential models. So the new lifetime model is a good alternative to these models in modeling failure time data. This model is introduced after many works among them Gupta and Kundu (2001) [18] who studied the Exponentiated exponential family: an alternative to gamma and Weibull distributions and Korkmaz and Yousof (2017) [16] who studied the one-parameter odd Lindley exponential model: mathematical properties and applications.

The probability density function (PDF) (1), cumulative distribution function (CDF) (2) reliability function (RF) (3) and hazard rate function (HRF) (4) are respectively given by:

$$f_{(\alpha,\theta,\lambda)}(x)=\frac{\alpha^2\theta\lambda\exp(-\lambda x)\left(1-\exp(-\lambda x)\right)^{\theta-1}}{(1+\alpha)\left(1-\left(1-\exp(-\lambda x)\right)^{\theta}\right)^{3}}\exp\left(-\alpha\frac{\left(1-\exp(-\lambda x)\right)^{\theta}}{1-\left(1-\exp(-\lambda x)\right)^{\theta}}\right) \tag{1}$$

$$F_{(\alpha,\theta,\lambda)}(x)=1-\frac{\alpha+\left(1-\left(1-\exp(-\lambda x)\right)^{\theta}\right)}{(1+\alpha)\left(1-\left(1-\exp(-\lambda x)\right)^{\theta}\right)}\exp\left(-\alpha\frac{\left(1-\exp(-\lambda x)\right)^{\theta}}{1-\left(1-\exp(-\lambda x)\right)^{\theta}}\right) \tag{2}$$

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$$R_{(\alpha,\lambda,\theta)}\left(x\right)=\frac{\alpha+\left(1-\left(1-\exp\left(-\lambda x\right)\right)^{\theta}\right)}{\left(1+\alpha\right)\left(1-\left(1-\exp\left(-\lambda x\right)\right)^{\theta}\right)}\exp\left(-\alpha\frac{\left(1-\exp\left(-\lambda x\right)\right)^{\theta}}{1-\left(1-\exp\left(-\lambda x\right)\right)^{\theta}}\right) \tag{3}$$

$$h_{(\alpha,\lambda,\theta)}(x)=\frac{\alpha^{2}\theta\lambda\exp\left(-\lambda x\right)\left(1-\exp\left(-\lambda x\right)\right)^{\theta-1}}{\left(1-\left(1-\exp\left(-\lambda x\right)\right)^{\theta}\right)^{2}\left(\alpha+\left(1-\left(1-\exp\left(-\lambda x\right)\right)^{\theta}\right)\right)} \tag{4}$$

Where $\alpha>0$, $\lambda>0$, $\theta>0$ and $x\geq 0$.

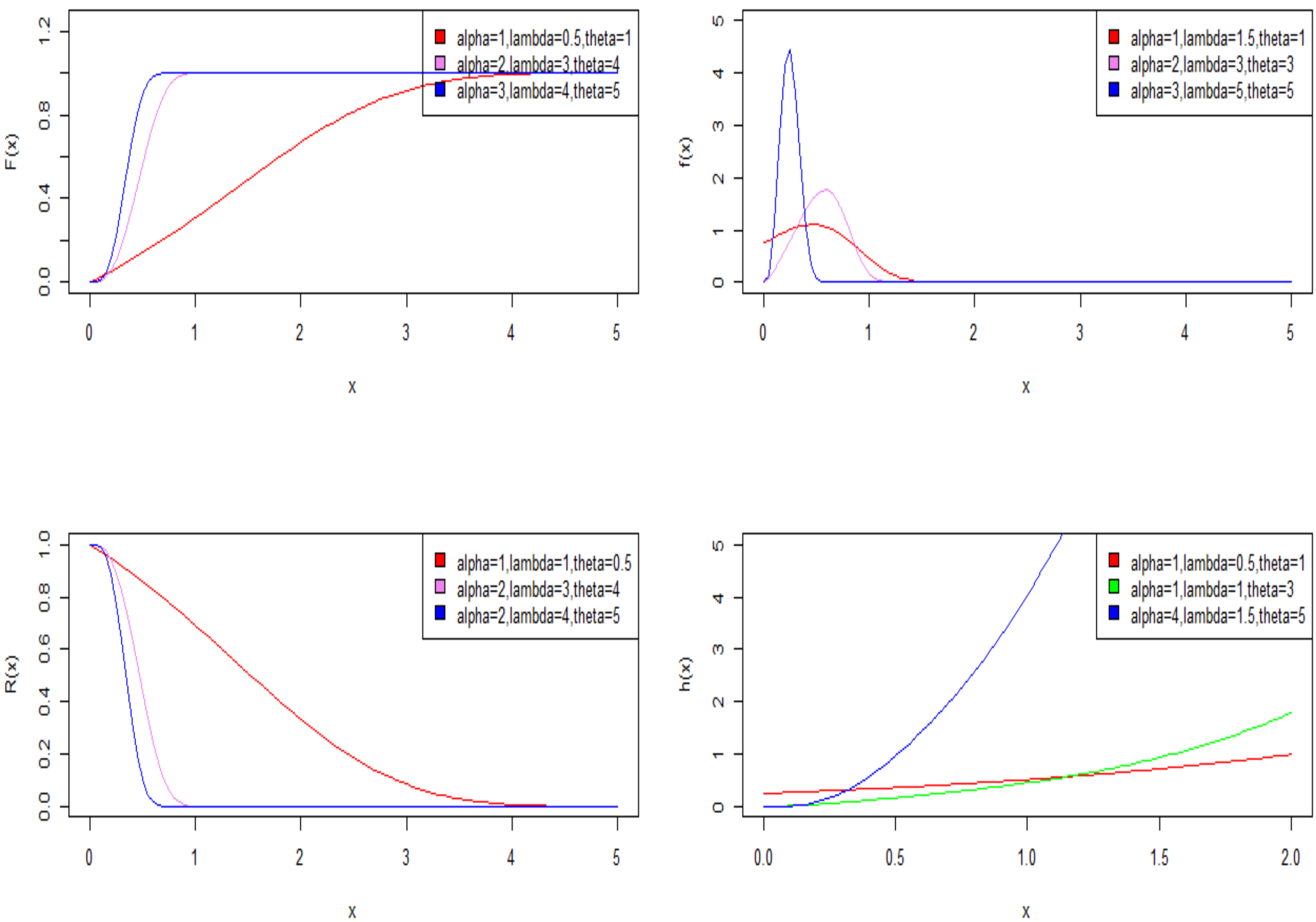

Figure 1. The CDF, PDF, SF and HRF of OLEE distribution.

The article is outlined as follows: in Section 2, we present the maximum likelihood estimation of the model. In Section 3, we discuss the Bayesian approaches under two loss functions: the symmetric one is the quadratic loss function and the asymmetric is the Linex assumes that the parameters are independent and gamma prior distribution. In Section 4, the estimators of the three parameters and the reliability characteristics are obtained by

the Markov chain Monte-Carlo Metropolis-Hastings sampling algorithm.
The different estimates are compared using the Pitman closeness criterion and the integrated mean squared error ”IMSE” in Section 5. The application to a real data set is considered in Section 6. Finally, the last section offers some concluding remarks. We give the next definition and theorem.

2. Maximum Likelihood Estimation

Let $(x_1, ..., x_n)$ a random sample of size n from the OLEE(α,λ,θ) distribution, the likelihood function is given by

$$L\left(\underline{x}, \alpha, \theta, \lambda\right) = \frac{\alpha^{2n}\theta^n\lambda^n}{(1+\alpha)^n} \exp\left(-\lambda\sum_{i=1}^{n} x_i - \sum_{i=1}^{n} \alpha\frac{(1-\exp(-\lambda x_i))^\theta}{1-(1-\exp(-\lambda x_i))^\theta}\right) \prod_{i=1}^{n} \frac{(1-\exp(-\lambda x_i))^{\theta-1}}{\left(1-(1-\exp(-\lambda x_i))^\theta\right)^3} \quad (5)$$

For $\alpha > 0$, $\lambda > 0$, $\theta > 0$.
The log-likelihood function

$$\begin{aligned}\ln(L(\underline{x}, \alpha, \theta, \lambda t)) &= n\ln(\frac{\alpha^2\theta\lambda}{1+\alpha}) + \sum_{i=1}^{n}(-\lambda x_i - \alpha(\frac{\phi_\lambda(x_i)^\theta}{1-\phi_\lambda(x_i)^\theta}) + (\theta-1)\ln(\phi_\lambda(x_i)) \\ &\quad -3\ln(1-\phi_\lambda(x_i)^\theta))\end{aligned} \quad (6)$$

Where $\phi_\lambda(x) = 1 - \exp(-\lambda x)$
The estimators of the parameters are obtained by solving the following non-linear system:

$$\begin{cases} -\sum_{i=1}^{n}\frac{\phi_\lambda(x_i)^\theta}{1-\phi_\lambda(x_i)^\theta} - \frac{n}{\alpha+1} + \frac{2n}{\alpha} = 0 \\ \frac{n}{\theta} + \sum_{i=1}^{n}\ln(\phi_\lambda(x_i)) + \sum_{i=1}^{n}\frac{(3-\alpha)\ln(\phi_\lambda(x_i))(\phi_\lambda(x_i)^\theta)}{1-\phi_\lambda(x_i)^\theta} - \sum_{i=1}^{n}\frac{\alpha\ln(\phi_\lambda(x_i))(\phi_\lambda(x_i))^{2\theta}}{(1-\phi_\lambda(x_i)^\theta)^2} = 0 \\ \frac{n}{\lambda} - \sum_{i=1}^{n} x_i + \sum_{i=1}^{n}\frac{(\theta-1)(\phi_\lambda(x_i))'}{\phi_\lambda(x_i)} + \sum_{i=1}^{n}\frac{(3-\alpha)\theta(\phi_\lambda(x_i))'(\phi_\lambda(x_i))^{\theta-1}}{1-\phi_\lambda(x_i)^\theta} - \sum_{i=1}^{n}\frac{\alpha\theta(\phi_\lambda(x_i))'(\phi_\lambda(x_i))^{2\theta-1}}{(1-\phi_\lambda(x_i)^\theta)^2} = 0 \end{cases} \quad (7)$$

Here, there is no analytical solution. So, we use numerical methods to obtain the corresponding approximate maximum likelihood estimates[19].
Then, the reliability characteristics are:

$$\hat{R}_{(\hat{\alpha},\hat{\lambda},\hat{\theta})}(x)_{MLE} = \frac{\hat{\alpha}+\left(1-\left(\phi_{\hat{\lambda}}(x)\right)^{\hat{\theta}}\right)}{(1+\hat{\alpha})\left(1-\left(\phi_{\hat{\lambda}}(x)\right)^{\hat{\theta}}\right)} \exp\left(-\hat{\alpha}\frac{\left(\phi_{\hat{\lambda}}(x)\right)^{\hat{\theta}}}{1-\left(\phi_{\hat{\lambda}}(x)\right)^{\hat{\theta}}}\right), x \geq 0 \quad (8)$$

And

$$\hat{h}_{(\hat{\alpha},\hat{\lambda},\hat{\theta})}(x)_{MLE} = \frac{\hat{\alpha}^2\hat{\theta}\hat{\lambda}\exp\left(-\hat{\lambda}x\right)\left(\phi_{\hat{\lambda}}(x)\right)^{\hat{\theta}-1}}{\left(1-\left(\phi_{\hat{\lambda}}(x)\right)^{\hat{\theta}}\right)^2\left(\hat{\alpha}+(1-\left(\phi_{\hat{\lambda}}(x)\right)^{\hat{\theta}})\right)}, x \geq 0 \quad (9)$$

3. Bayesian Estimation

In this part, we give the different Bayesian estimators with complete data using quadratic and Linex loss functions. The posterior risks are calculated for each parameter estimator.

3.1. The posterior density

For the following prior density with parameters α, λ and θ.

$$\pi(\alpha, \lambda, \theta) \propto \alpha^{b-1}\lambda^{c-1}\theta^{d-1}\exp(-h\alpha - m\lambda - l\theta), b, c, d, h, m, l > 0 \quad (10)$$

We find the last posterior density:
$\pi(\alpha,\lambda,\theta/\underline{x}) = \frac{L(\alpha,\lambda,\theta/\underline{x})\Pi(\alpha,\lambda,\theta)}{\int_\alpha\int_\lambda\int_\theta L(\alpha,\lambda,\theta/\underline{x})\Pi(\alpha,\lambda,\theta)d\alpha d\lambda d\theta}$
With

$$K^{-1} = \int_\alpha\int_\lambda\int_\theta L(\alpha,\lambda,\theta/\underline{x})\pi(\alpha,\lambda,\theta)d\alpha d\lambda d\theta$$

3.2. The Bayes estimators

Loss Functions

1. Legendre (1805) and Gauss (1810)[1]proposed this Quadratic loss function $L(\phi,\delta) = (\phi-\delta)^2$, which is the most commonly used functions in the literature. The posterior mean

$$E_\pi(./x)(|\phi|) = \int_\phi L(\phi,\delta)\pi(\delta)d\phi$$

is then the Bayesian estimator of ϕ .

2. The Linex loss function is represented by

$$L(\phi,\delta) = \exp(a(\delta-\phi)) - a(\delta-\phi) - 1$$

where $a \neq 0$ The Bayes estimator [12],[2] that corresponds is

$$\hat{\phi}_L = -\frac{1}{a}\ln E_\pi(\exp(-a\phi))$$

For the quadratic loss function, the Bayesian estimators of $\alpha>0$, $\lambda>0$ and $\theta>0$ are:

$$\hat{\alpha}_{BQ} = K^{-1}\int_\alpha\int_\lambda\int_\theta \frac{\alpha^{2n+b}\lambda^{n+c-1}\theta^{n+d-1}}{(1+\alpha)^n}\exp(-\lambda(\sum_{i=1}^n x_i+m) - h\alpha - l\theta - \sum_{i=1}^n \frac{\alpha(\phi_\lambda(x_i))^\theta}{1-(\phi_\lambda(x_i))^\theta}) \prod_{i=1}^n \frac{(\phi_\lambda(x_i))^{\theta-1}}{(1-(\phi_\lambda(x_i))^\theta)^3}\mathrm{d}\alpha\mathrm{d}\lambda\mathrm{d}\theta \quad (11)$$

$$\hat{\lambda}_{BQ} = K^{-1}\int_\alpha\int_\lambda\int_\theta \frac{\alpha^{2n+b-1}\lambda^{n+c}\theta^{n+d-1}}{(1+\alpha)^n}\exp(-\lambda(\sum_{i=1}^n x_i+m) - h\alpha - l\theta - \sum_{i=1}^n \frac{\alpha(\phi_\lambda(x_i))^\theta}{1-(\phi_\lambda(x_i))^\theta}) \prod_{i=1}^n \frac{(\phi_\lambda(x_i))^{\theta-1}}{(1-(\phi_\lambda(x_i))^\theta)^3}\mathrm{d}\alpha\mathrm{d}\lambda\mathrm{d}\theta \quad (12)$$

$$\hat{\theta}_{BQ} = K^{-1}\int_\alpha\int_\lambda\int_\theta \frac{\alpha^{2n+b-1}\lambda^{n+c-1}\theta^{n+d}}{(1+\alpha)^n}\exp(-\lambda(\sum_{i=1}^n x_i+m) - h\alpha - l\theta - \sum_{i=1}^n \frac{\alpha(\phi_\lambda(x_i))^\theta}{1-(\phi_\lambda(x_i))^\theta}) \prod_{i=1}^n \frac{(\phi_\lambda(x_i))^{\theta-1}}{\left(1-(\phi_\lambda(x_i))^\theta\right)^3}\mathrm{d}\alpha\mathrm{d}\lambda\mathrm{d}\theta \quad (13)$$

The posterior risks are:

$$PR(\hat{\alpha}_{QB}) = E_\pi(\alpha^2) - 2\hat{\alpha}_{QB}E_\pi(\alpha) + \hat{\alpha}_{QB}^2 E_\pi(\alpha)$$

$$PR(\hat{\lambda}_{QB}) = E_\pi(\lambda^2) - 2\hat{\lambda}_{QB}E_\pi(\lambda) + \hat{\lambda}_{QB}^2 E_\pi(\lambda)$$

$$PR(\hat{\theta}_{QB}) = E_\pi(\theta^2) - 2\hat{\theta}_{QB}E_\pi(\theta) + \hat{\theta}_{QB}^2 E_\pi(\theta)$$

The Bayesian estimators of the reliability and the hazard rate function of the quadratic loss function are given by:

$$\begin{aligned}\hat{R}(x)_{_{BQ}} &= K^{^{-1}}\int_\alpha\int_\lambda\int_\theta \frac{\alpha^{2n+b-1}\lambda^{n+c-1}\theta^{n+d-1}R(x)}{(1+\alpha)^n} \\ &\exp\left(-\lambda(\sum_{i=1}^n x_i+m)-h\alpha-l\theta-\sum_{i=1}^n \frac{\alpha(\phi_\lambda(x_i)^\theta}{1-(\phi_\lambda(x_i))^\theta}\right) \\ &\prod_{i=1}^n \frac{(\phi_\lambda(x_i))^{\theta-1}}{\left(1-(\phi_\lambda(x_i))^\theta\right)^3}\mathrm{d}\alpha\mathrm{d}\lambda\mathrm{d}\theta \end{aligned} \tag{14}$$

$$\begin{aligned}\hat{h}(x)_{_{BQ}} &= K^{^{-1}}\int_\alpha\int_\lambda\int_\theta \frac{\alpha^{2n+b-1}\lambda^{n+c-1}\theta^{n+d-1}h(x)}{(1+\alpha)^n} \\ &\exp\left(-\lambda(\sum_{i=1}^n x_i+m)-h\alpha-l\theta-\sum_{i=1}^n \frac{\alpha(\phi_\lambda(x_i))^\theta}{1-(\phi_\lambda(x_i))^\theta}\right) \\ &\prod_{i=1}^n \frac{(\phi_\lambda(x_i))^{\theta-1}}{\left(1-(\phi_\lambda(x_i))^\theta\right)^3}\mathrm{d}\alpha\mathrm{d}\lambda\mathrm{d}\theta \end{aligned} \tag{15}$$

For Linex loss function, the Bayesian estimators of $\alpha > 0$,$\lambda > 0$ and $\theta > 0$ are:

$$\begin{aligned}\hat{\alpha}_{BL} &= -\frac{1}{a}\ln[K^{-1}\int_\alpha\int_\lambda\int_\theta \frac{\alpha^{2n+b-1}\lambda^{n+c-1}\theta^{n+d-1}}{(1+\alpha)^n} \\ &\exp\left(-\lambda(\sum_{i=1}^n x_i+m)-(h+a)\,\alpha-l\theta-\alpha\sum_{i=1}^n \frac{(\phi_\lambda(x_i))^\theta}{1-(\phi_\lambda(x_i))^\theta}\right) \\ &\prod_{i=1}^n \frac{(\phi_\lambda(x_i))^{\theta-1}}{\left(1-(\phi_\lambda(x_i))^\theta\right)^3}\mathrm{d}\alpha\mathrm{d}\lambda\mathrm{d}\theta] \end{aligned} \tag{16}$$

$$\begin{aligned}\hat{\lambda}_{BL} &= -\frac{1}{a}\ln[K^{-1}\int_\alpha\int_\lambda\int_\theta \frac{\alpha^{2n+b-1}\lambda^{n+c-1}\theta^{n+d-1}}{(1+\alpha)^n} \\ &\exp\left(-\lambda(\sum_{i=1}^n x_i+m+a)-h\alpha-l\theta-\sum_{i=1}^n \frac{\alpha(\phi_\lambda(x_i))^\theta}{1-(\phi_\lambda(x_i))^\theta}\right) \\ &\prod_{i=1}^n \frac{(\phi_\lambda(x_i))^{\theta-1}}{\left(1-(\phi_\lambda(x_i))^\theta\right)^3}\mathrm{d}\alpha\mathrm{d}\lambda\mathrm{d}\theta] \end{aligned} \tag{17}$$

$$\begin{aligned}\hat{\theta}_{BL} &= -\frac{1}{a}\ln[K^{-1}\int_\alpha\int_\lambda\int_\theta \frac{\alpha^{2n+b-1}\lambda^{n+c-1}\theta^{n+d-1}}{(1+\alpha)^n} \\ &\exp\left(-\lambda(\sum_{i=1}^n x_i-m)-h\alpha-(l+a)\,\theta-\sum_{i=1}^n \frac{\alpha(\phi_\lambda(x_i))^\theta}{1-(\phi_\lambda(x_i))^\theta}\right) \\ &\prod_{i=1}^n \frac{(\phi_\lambda(x_i))^{\theta-1}}{\left(1-(\phi_\lambda(x_i))^\theta\right)^3}\mathrm{d}\alpha\mathrm{d}\lambda\mathrm{d}\theta] \end{aligned} \tag{18}$$

The posterior risks are:

$$PR(\hat{\alpha}_{BL}) = a(\hat{\alpha}_{BQ} - \hat{\alpha}_{BL})$$
$$PR(\hat{\lambda}_{BL}) = a(\hat{\lambda}_{BQ} - \hat{\lambda}_{BL})$$
$$PR(\hat{\theta}_{BL}) = a(\hat{\theta}_{BQ} - \hat{\theta}_{BL})$$

The Bayesian estimators of the reliability and the hazard rate function of Linex loss function are:

$$\begin{aligned} \hat{R}(x)_{BL} &= \frac{-1}{a}\ln[K^{-1}\int_{\alpha}\int_{\lambda}\int_{\theta}\frac{\alpha^{2n+b-1}\lambda^{n+c-1}\theta^{n+d-1}}{(1+\alpha)^n} \\ &\exp\left(-\lambda(\sum_{i=1}^{n}x_i+m)-aR(x)-h\alpha-l\theta-\sum_{i=1}^{n}\frac{\alpha(\phi_\lambda(x_i))^\theta}{1-(\phi_\lambda(x_i))^\theta}\right) \\ &\prod_{i=1}^{n}\frac{(\phi_\lambda(x_i))^{\theta-1}}{\left(1-(\phi_\lambda(x_i))^\theta\right)^3}\mathrm{d}\alpha\mathrm{d}\lambda\mathrm{d}\theta] \end{aligned} \tag{19}$$

$$\begin{aligned} \hat{h}(x)_{BL} &= \frac{-1}{a}\ln[K^{-1}\int_{\alpha}\int_{\lambda}\int_{\theta}\frac{\alpha^{2n+b-1}\lambda^{n+c-1}\theta^{n+d-1}}{(1+\alpha)^n} \\ &\exp\left(-\lambda(\sum_{i=1}^{n}x_i+m)-h\alpha-l\theta-ah(x)-\sum_{i=1}^{n}\frac{\alpha(\phi_\lambda(x_i))^\theta}{1-(\phi_\lambda(x_i))^\theta}\right) \\ &\prod_{i=1}^{n}\frac{(\phi_\lambda(x_i))^{\theta-1}}{\left(1-(\phi_\lambda(x_i))^\theta\right)^3}\mathrm{d}\alpha\mathrm{d}\lambda\mathrm{d}\theta] \end{aligned} \tag{20}$$

4. The Metropolis-Hastings Algorithm

The Bayesian estimators given cannot be computed analytically, that's why we propose an MCMC procedure to approximate them.
The Metropolis-Hastings algorithm [[17],[5]], which is associated with a target density π, necessitates the selection of a conditional density q, also known as a proposal or candidate kernel. The transition between the value of the Markov chain $X(t)$ at time t and its value at time $t+1$ is accomplished through the following transition step.
Algorithm Metropolis-Hastings:
Given $X^{(t)} = x^{(t)}$

1. Generate $Y_t \sim q(x, y)$
2. Take $X^{(t+1)} = \begin{cases} Y_t \text{ with probability } q(x^{(t)}, Y_t) \\ x^{(t)} \text{ with probability } 1 - q(x^{(t)}, Y_t) \end{cases}$
   Where
   $$q(x, y) = min\left(1, \frac{\pi(y)q(x, y)}{\pi(x)q(y, x)}\right)$$

The trace plot, a time series plot that illustrates the realizations of the Markov chain at each iteration against the iteration numbers, is one of the extensively used graphical tools for MCMC convergence diagnostics. The trace plot indicates bits to indicate delayed convergence, which means the MCMC chain is stuck in some portion of the state space. In contrast, if the trace plot resembles a hairy caterpillar, it indicates that the MCMC method is efficient.

The Markov chain [13] is executed for 10000 iterations with the suggested parameters (in Section 5). The trace plots and density plots of the Markov chain samples are shown in the figure. For the three parameters, we note that the trace plot blends well. Furthermore, we can see from the trace plot that no burn-in is required, and our decision appears to be an acceptable starting point.

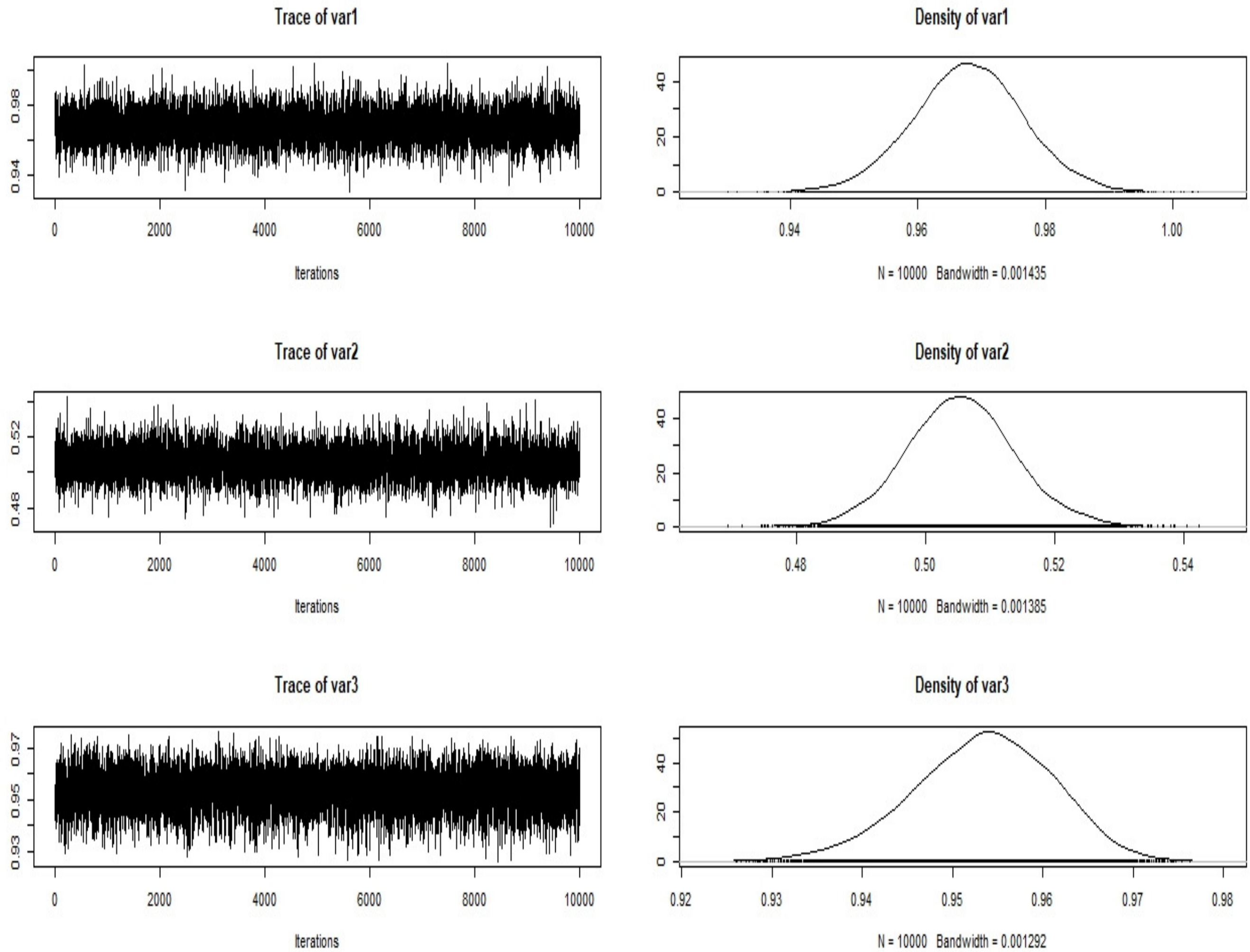

Figure 2. Plots of the MCMC trace and density.

5. Simulations study

In this section, we have a Monte Carlo simulation study using different sample sizes: 10, 30 and 50. We generated 10000 samples from the OLEE(1,0.5,1). The hyper-parameters are $b = 5, c = 7$ and $d = 6$, $l = 0.001, m = 0.002, h = 0.003$. For $x = 1.5$, $R(x) = 0.5100349$ and $h(x) = 0.3395893$.

Table 1. Estimated parameter values and the reliability characteristics with Maximum Likelihood and quadratic error (in brackets).

n	α	λ	θ	$R(x)$	$h(x)$
10	1.0822759	0.4947097	0.9467962	0.4483764	0.3834711
	(6.769326e − 07)	(2.798731e − 09)	(2.830642e − 07)	(3.801772e − 07)	(1.925606e − 07)
30	1.1120305	0.4590154	0.9699686	0.489312	0.3580764
	(1.255083e − 06)	(1.67974e − 07)	(9.018842e − 08)	(4.294359e − 08)	(3.417727e − 08)
50	1.1000632	0.4703721	0.9645665	0.4783602	0.3647678
	(1.001265e − 06)	(8.77813e − 08)	(1.255532e − 07)	(1.003282e − 07)	(6.339533e − 08)

Table 2. Bayesian estimation under Quadratic loss function and PR (in brackets).

n	α	λ	θ	$R(x)$	$h(x)$
10	0.9653196	0.5022732	0.9653196	0.4987497	0.3352815
	(1.202727e − 07)	(5.167592e − 10)	(2.303424e − 07)	(1.273542e − 08)	(1.855791e − 09)
30	0.9664603	0.5038764	0.9524761	0.4966746	0.3370249
	(1.124911e − 07)	(1.502648e − 09)	(2.258521e − 07)	(1.784978e − 08)	(6.576347e − 10)
50	0.9678314	0.5054845	0.9532078	0.4946228	0.338831
	(1.034819e − 07)	(3.007974e − 09)	(2.18951e − 07)	(2.37531e − 08)	(5.750503e − 11)

Table 3. Bayesian estimation under Linex loss function and PR (in brackets).

n	parameter	Linex (a)			
		$a=-0.5$	$a=0.5$	$a=-1$	$a=1$
10	α	0.9520192	0.9649570	0.9650321	0.9648707
		$(1.201331e-07)$	$(1.228015e-07)$	$(1.222757e-07)$	$(1.234067e-07)$
	λ	0.5022931	0.5031456	0.5032177	0.5030205
		$(5.258102e-10)$	$(2.468643e-05)$	$(2.467926e-05)$	$(2.469887e-05)$
	θ	0.9653398	0.9522974	0.9524344	0.9523551
		$(2.302155e-07)$	$(2.275535e-07)$	$(2.262486e-07)$	$(2.270032e-07)$
	$R(x)$	0.4987251	0.4981335	0.498092	0.498346
		$(1.2791e-08)$	$(1.712616e-05)$	$(1.712272e-05)$	$(1.714376e-05)$
	$h(x)$	0.3353043	0.3357708	0.335837	0.3356173
		$(1.836187e-09)$	$(2.321348e-05)$	$(2.32071e-05)$	$(2.322827e-05)$
30	α	0.9664799	0.9661772	0.9663137	0.9662440
		$(1.123597e-07)$	$(1.143983e-07)$	$(1.134765e-07)$	$(1.139468e-07)$
	λ	0.5038955	0.5050546	0.5049795	0.5048338
		$(1.517492e-09)$	$(2.449709e-05)$	$(2.450453e-05)$	$(2.451895e-05)$
	θ	0.9524899	0.9531147	0.9529576	0.9528284
		$(2.25721e-07)$	$(2.198229e-07)$	$(2.21299e-07)$	$(2.225159e-07)$
	$R(x)$	0.4966514	0.4958738	0.4958013	0.4959251
		$(1.791181e-08)$	$(1.693965e-05)$	$(1.693368e-05)$	$(1.694387e-05)$
	$h(x)$	0.3370468	0.3377334	0.3377661	0.3376402
		$(6.464518e-10)$	$(2.302475e-05)$	$(2.302161e-05)$	$(2.303369e-05)$
50	α	0.9678506	0.9678046	0.9679779	0.9678273
		$(1.033584e-07)$	$(1.036547e-07)$	$(1.025416e-07)$	$(1.035084e-07)$
	λ	0.5055026	0.5056606	0.5067134	0.5068733
		$(3.027861e-09)$	$(3.204229e-09)$	$(2.433317e-05)$	$(2.431739e-05)$
	θ	0.9532225	0.9533514	0.9542460	0.9540359
		$(2.188135e-07)$	$(2.176096e-07)$	$(2.093428e-07)$	$(2.112698e-07)$
	$R(x)$	0.4946014	0.4945219	0.4937712	0.4935554
		$(2.381903e-08)$	$(2.406517e-08)$	$(1.676701e-05)$	$(1.674934e-05)$
	$h(x)$	0.3388518	0.3389347	0.3397016	0.3398056
		$(5.439828e-11)$	$(4.286151e-11)$	$(2.283625e-05)$	$(2.282631e-05)$

Discussion For the different values of $n = 10$, the best estimators of λ, $h(x)$ and $R(x)$ are obtained for $a = -0.5$. However, for θ and α, the estimates are almost equivalent in terms of the posterior risk.

6. Comparison between estimation methods

In this section, we compare the best Bayesian estimators above with MLE. For this, we propose to use the following two Criteria, the Pitman closeness criterion (Pitman (1937) and Jozani (2012)) [[7],[15]]and the integrated mean square error defined as follows.

1. According to the Pitman closeness criterion, an estimator δ_1 of a parameter performs better than an estimator δ_2 .

$$P_\delta\left(|\delta_1-\delta|<|\delta_2-\delta|\right)>\frac{1}{2}$$

2. The integrated mean square error is equal to

$$IMSE=\frac{1}{N}\sum_{k=1}^{N}\left(\delta_k-\delta\right)^2$$

An estimator δ_k

The first comparison between the classical estimators and the Bayesian one by using the Pitman closeness criterion is presented in table 4. We remark that the Bayesian estimators are better than the classic one.

Table 4. Comparison of the methods using the Pitman closeness criterion.

n		Quadratic	Linex$(a=-0.5)$
10	α	1.0000	1.0000
	λ	0.2418	0.2426
	θ	0.6683	0.6728
	$R(x)$	1.0000	1.0000
	$h(x)$	1.0000	1.0000
30	α	1.0000	1.0000
	λ	0.9998	0.9998
	θ	0.8589	0.8622
	$R(x)$	0.9951	0.9951
	$h(x)$	0.9999	0.9999
50	α	1.0000	1.0000
	λ	0.9998	0.9998
	θ	0.9342	0.9353
	$R(x)$	0.9969	0.9969
	$h(x)$	1.0000	1.0000

The second comparison is done using the integrated mean squared error. The following table contains the results. We remark that the Bayesian estimators are better than the classic one.

Table 5. Comparison of the methods using the IMSE.

n		MLE	Quadratic	Linex $(a = -0.5)$
10	α	$1.147302e-06$	$1.197123e-07$	$1.193045e-07$
	λ	$8.391364e-08$	$9.087909e-09$	$9.110775e-09$
	θ	$4.715616e-07$	$2.305556e-07$	$2.297191e-07$
	$R(x)$	$7.429486e-07$	$2.929291e-08$	$2.930238e-08$
	$h(x)$	$3.954084e-07$	$7.289941e-09$	$7.268815e-09$
30	α	$1.147302e-06$	$1.197123e-07$	$1.193045e-07$
	λ	$8.391364e-08$	$9.087909e-09$	$9.110775e-09$
	θ	$4.715616e-07$	$2.305556e-07$	$2.297191e-07$
	$R(x)$	$7.429486e-07$	$2.929291e-08$	$2.930238e-08$
	$h(x)$	$3.954084e-07$	$7.289941e-09$	$7.268815e-09$
50	α	$1.163917e-06$	$1.197873e-07$	$1.193775e-07$
	λ	$8.344384e-08$	$9.353396e-09$	$9.374588e-09$
	θ	$4.825232e-07$	$2.317138e-07$	$2.308777e-07$
	$R(x)$	$7.600464e-07$	$2.994713e-08$	$2.995526e-08$
	$h(x)$	$4.051625e-07$	$7.523594e-09$	$7.501531e-09$

7. Real data analysis

This part presents in detail a real application where we take the relaxation periods of 20 analgesic patients (Gross and Clark (1975)) [3] and are as follows:
1.1, 1.4, 1.3, 1.7, 1.9, 1.8, 1.6, 2.2, 1.7, 2.7, 4.1, 1.8, 1.5, 1.2, 1.4, 3, 1.7, 2.3, 1.6, 2.
Then, we compare the fit of the studied model with classical models such as Lindley, Generalized Lindley and Odd Lindley Weibull for(λ, $\beta > 0$).
Lindley Distribution, L(λ) given by:

$$F(x,\lambda) = 1 - \frac{(1+\lambda+\lambda x)}{1+\lambda}\exp(-\lambda x), x \geq 0$$

Generalized Lindley Distribution[4], GL(β,λ)is:

$$F(x,\beta,\lambda) = 1 - \left(1 + \frac{\lambda\beta x}{\beta+\lambda}\exp(-\lambda x)\right), x \geq 0$$

Exponential Distribution, E(λ) Presented by:

$$F(x,\lambda) = 1 - exp(-\lambda x), x \geq 0$$

We consider four goodness of fit measures like Akaike information criterion ”AIC”[[20],[9]], the corrected Akaike information criterion ”CAIC”, Bayesian information criterion ”BIC” and Hannan Quinn information criterion ”HQIC” given by:

$$AIC = -2LL + 2p$$

$$CAIC = -2LL + 2p\frac{n}{n-p-1}$$

$$BIC = -2LL + p\log(n)$$

And

$$HAIC = -2LL + 2p\log(\log(n))$$

Where LL is the log-likelihood, n is the size of the sample and p the number of parameters. The table below contains the computations with R software.

Table 6. Parameter MLE estimates using data.

n	α	β	λ	θ
Lindley(λ)			0.8161184	
			$(4.676941e-05)$	
Generalized Lindley(β,λ)		6.223813	0.575961	
		(0.002001703)	(0.002707966)	
Exponential(λ)			0.5263157	
			$(2.243768e-05)$	
OLEE(α,λ,θ)	1.0590418		0.5161234	0.9926819
	$(3.485934e-07)$		$(2.59964e-08)$	$(5.355459e-09)$

Table 7. The different statistics using data.

Model	AIC	BIC	CAIC	HAIC	-LL	K.S	p-value
Lindley(λ)	62.4991	63.49483	62.72132	62.69348	30.24955	0.39108	0.004407
Generalized Lindley(β,λ)	110.3514	112.3429	110.7402	111.0573	53.17571	0.43826	0.0009213
Exponential(α,β)	67.67416	68.66989	67.89638	67.86853	32.83708	0.43951	0.0008817
OLEE(α,λ,θ)	53.10838	56.09558	54.60838	53.69151	23.55419	0.38345	0.005583

We remark that the OLEE model has the smallest values of all statistics comparing with the other tested models. In this case, we can say that the OLEE is the best model for the data set.

We apply the proposed methods to these data, the results are listed in Table 8.

Table 8. Results using real data.

n		MLE	Quadratic	Linex $(a = -0.5)$
20	α	1.0590418	0.9667411	0.9667320
		$(3.485939e-07)$	$(1.106759e-07)$	$(1.106157e-07)$
	λ	0.5161234	0.5016447	0.5016398
		$(2.599625e-08)$	$(2.688846e-10)$	$(2.704893e-10)$
	θ	0.9926819	0.9693560	0.9693470
		$(5.355388e-09)$	$(9.396084e-08)$	$(9.390556e-08)$
	$R(x)$	0.4604129	0.5081812	0.5081762
		$(2.462337e-07)$	$(3.43596e-10)$	$(3.454472e-10)$
	$h(x)$	0.3823691	0.3319762	0.3319826
		$(1.830105e-07)$	$(5.795958e-09)$	$(5.786338e-09)$

Table 9. Pitman criterion comparison under real data.

n		Quadratic	Linex£$(a = -0.5)$
20	α	1	1
	λ	0.9992	0.992
	θ	0.0007	0.0007
	$R(x)$	1	1
	$h(x)$	1	1

8. Conclusion

This article presents the study of a continuous distribution called ”Odd Lindley exponented exponential”. The unknown parameters and the characteristics of this model have been estimated using the maximum likelihood and Bayesian methods. The computations of these estimates are obtained by Monte Carlo methods, where we have noticed that Bayesian estimators (Linex where $a = -0.5$) performed better than the classic ones. The proposed distribution has been applied to a real data set.

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