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Optimal Design of Life Testing Plans under Type II Hybrid Censoring Scheme with Fixed and Random Sample Sizes

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Abstract Hybrid censoring is a combination of Type I and Type II censoring schemes, which is divided into two types, Type I and Type II hybrid censoring schemes. One practical problem in the discussion of censoring is choosing the best censoring scheme. To this end, different criteria can be considered. One of the most important criteria is the cost of the experiment. In this article, considering a cost function as an optimization criterion in Type II hybrid censoring, the optimal censoring scheme is determined. Here, the sample size is considered as a fixed value as well as a random variable from the power series distribution, and the optimal scheme of censoring is determined so that the cost function does not exceed a pre-determined value. Numerical computation as well as a simulation study are presented for illustrating the results. One data set is finally analyzed for real life applications.

Keywords Proportional Hazard Rate Model, Type II Hybrid Censoring, Optimization Problem, Power Series Distribution

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1. Introduction

Suppose an absolutely continuous random variable X has a cumulative distribution function (cdf) as

$$F_\lambda(x) = 1 - \left[\bar{F}_0(x)\right]^\lambda, \qquad \lambda > 0, \;\; -\infty \le a < x < b \le \infty, \tag{1}$$

in which $\bar{F}_0(\cdot) = 1 - F_0(\cdot)$ and $F_0(\cdot)$ is an arbitrary continuous cdf, which is independent of the parameter λ and $F_0(a) = 0$ and $F_0(b) = 1$. The family of distributions given in (1) includes several life distributions such as one-parameter exponential, Lomax, Pareto, Rayleigh, Burr Type XII distributions and so on. The above mentioned model is well-known in the lifetime experiments as *Proportional hazard rate* (PHR) model with the underlying distribution $F_0(\cdot)$ (see [22]). Several authors have studied different problems related to PHR model. See, for example, [3], [4], [23] [28], [27] and [19] among the others.

In many lifetime tests, the experimenter is unable to observe the complete survival times of units under the test. For example, individuals in a clinical trial may drop out of the study, or the study may have to be terminated early for different reasons such as lack of time or funds. Also, in an industrial experiment, units may break accidently. Data obtained in these situations are called censored data. Censoring can be done in different ways. The most famous censoring methods are Type I and Type II censoring schemes. The test continues until the pre-fixed period τ in Type I censoring method. In Type II censoring, the experiment continues until the m-th failure is observed, when m is a pre-fixed value.

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For the first time, [21] introduced a design in a lifetime test that it terminates in time $T = \min(X_{m:n}, \tau)$, when the values of τ and m are pre-determined. This concept is known as Type I hybrid censoring scheme. The drawback of this method is that we might only have a very small number of failed units. As a result, a new censoring method known as Type II hybrid censoring was introduced by [14]. With this approach, the experiment ends at the time $T = \max(X_{m:n}, \tau)$. This scheme has the advantage of guaranteeing at least m failure times.

The problem of choosing the best sample size or the best censoring scheme is one of discussions that has been researched by numerous authors using various criteria. For example, the ideal sample size in Type I hybrid censoring scheme was discovered in [20] by taking into account a cost function and the exponential distribution. The best sample size for Type II censoring has been established in [24] and [25], by considering a cost function based on the time of test and the number of units under test. The papers by [11], [12] and [13] have investigated the best censoring scheme in the progressively Type II censoring by minimizing the variance and the covariance matrix of the best linear unbiased estimators for a location-scale family of distributions which includes exponential, uniform and Pareto distributions. [29] obtained the optimal progressively censoring scheme from the exponential distribution by considering Pitman closeness criterion. It was shown that the optimal progressively censoring scheme was the Type II censoring. In the research of [26], after obtaining the maximum likelihood estimator for the parameters of the Birnbaum-Saunders distribution based on the progressively Type II censored order statistics, the researchers compared two different censoring schemes and determined the optimal scheme. The best censoring scheme in Type I hybrid censoring was studied by [8] by minimizing a cost function associated with the experiment. Weibull distribution was considered as the data lifetime distribution. [9] developed an algorithm for choosing the optimal censoring scheme under Type II progressive censoring for a location-scale family of distributions by considering cost criterion. [1] discussed the problem of Bayesian predicting future observations from an exponential distribution based on an observed sample, when the information sample size was fixed as well as a random variable. Some distributions for the information sample size were considered and then for each case they found the optimal parameter of distribution such that the point predictor of a future order statistic had minimum mean squared prediction error when the total cost of experiment was bounded. The ideal sample size in Type II censoring was derived in [15] by developing a new cost function as a generalization of the cost functions proposed by [24] and [25], which considers the reliability in addition to the test duration. In [10] a multi-criteria-based optimization problem was considered in the context of hybrid censored life-testing experiment. Both the variance and the cost factors are optimized simultaneously. [5] considered the proportional hazard rate model and progressively Type II censoring with random sample size. The degenerate, binomial, and Poisson distributions were considered for the distribution of the random sample size, and the parameter of these distributions was established such that the cost of the experiment was bounded. The problem of finding the best sample size in Type I hybrid censoring scheme was studied by [6], when the sample size was a random variable from a truncated binomial distribution. For selecting the optimum progressive censoring scheme from the Unit-Lindley distribution, several optimality criteria were offered by [2] using both non-Bayesian and Bayesian methods. For more works we can refer the reader to [7], [17] and [18].

The purpose of this article is to identify the acceptable values for the censoring scheme in Type II hybrid censoring, $\mathcal{S} = (n, m, \tau)$, from the PHR model, based on a cost criterion. Three cases are considered as follows: (i) The values of m and τ are fixed and the aim is to find the acceptable values for n such that the cost function is less than a pre-fixed value, B^*; (ii) τ and n are fixed values and we seek the acceptable values for m in which the cost function does not exceed a pre-fixed value, B^*; (iii) The values of m and n are known and we investigate the admissible values for τ. After that, we consider the sample size is a random variable, say N. In this case, since N is a random variable we find the acceptable values for the parameter of its distributions so that the cost function is bounded. Power series distribution, which includes several distributions such as geometric, logarithmic and zero-truncated Poisson distributions, is considered for the distribution of sample size.

A detailed overview of the paper is as below. In Section 2, we present the cost function which plays an essential role in this study. Then assuming the sample size as a fixed value, the problem of finding the optimal censoring scheme is studied. Section 3 contains a similar problem when the sample size is a random variable. A simulation study and a real example are carried out in Section 4 and Section 5 for illustrating the theoretical results. In Section 6, the conclusion is presented.

2. Description of model when $N = n$ is fixed

Assume that a random sample $\tilde{X} = (X_1, \ldots, X_n)'$ of n units, from the PHR model given in (1), has been tested. Here, let the corresponding order statistics be $X_{1:n} \leq \cdots \leq X_{n:n}$ as well. Assuming Type II hybrid censoring, we denote $\mathcal{D}$ and $T = max(X_{m:n}, \tau)$ as the number of failures and the duration of the test, respectively, where τ and m are the pre-fixed values. Clearly, $\mathcal{D}$ and T are both random variables. In the following, we describe computational methodologies to determine the optimal hybrid censoring scheme $\mathcal{S} = (n, m, \tau)$. Towards this end, we first introduce the following expected cost function.

$$\mathcal{C}(n, m, \tau) = C_0 + C_n n + C_d \mathbb{E}(\mathcal{D}) + C_t \mathbb{E}(T), \tag{2}$$

where C_0, C_n, C_d and C_t are the sampling set-up cost or any other related cost involved in sampling, the cost per unit, the cost per unit of failed item, and the cost per unit of duration of life-testing, respectively.

On the one hand, for Type II hybrid censoring there are three cases as

$$\begin{cases} \text{Case I}: & \{x_{1:n}, \cdots, x_{m:n}\}, \text{ if } \tau < x_{m:n}, \\ \text{Case II}: & \{x_{1:n}, \cdots, x_{\mathcal{D}:n}\}, \text{ if } x_{m:n} < \cdots < x_{\mathcal{D}:n} < \tau < x_{\mathcal{D}+1:n}, \ m \leq \mathcal{D} < n, \\ \text{Case III}: & \{x_{1:n}, \cdots, x_{n:n}\}, \text{ if } x_{n:n} < \tau, \end{cases}$$

where $x_{1:n} \leq \cdots \leq x_{n:n}$ are the observed values of $X_{1:n} \leq \cdots \leq X_{n:n}$. Clearly, for Case I, $T = max(x_{m:n}, \tau) = x_{m:n}$, so m failures took place. For Case II, $T = \tau$ which leads to j failures, $j = m, \ldots, n-1$ and for Case III, the number of failures is n. So, we have

$$\begin{aligned} P(\mathcal{D} = m) &= \sum_{j=0}^{m} \binom{n}{j} (F_X(\tau))^j \left(\bar{F}_X(\tau)\right)^{n-j}, \\ P(\mathcal{D} = j) &= \binom{n}{j} (F_X(\tau))^j \left(\bar{F}_X(\tau)\right)^{n-j}, \ \ j = m+1, \cdots, n. \end{aligned}$$

It follows that

$$\mathbb{E}(\mathcal{D}) = m \sum_{j=0}^{m} \binom{n}{j} (F_X(\tau))^j \left(\bar{F}_X(\tau)\right)^{n-j} + \sum_{j=m+1}^{n} j \binom{n}{j} (F_X(\tau))^j \left(\bar{F}_X(\tau)\right)^{n-j}.$$

Using the binomial expansion for $(F_X(\tau))^j = \left(1 - \bar{F}_X(\tau)\right)^j$ leads to

$$\begin{aligned} \mathbb{E}(\mathcal{D}) &= m \sum_{j=0}^{m} \sum_{k=0}^{j} \binom{n}{j} \binom{j}{k} (-1)^k \left(\bar{F}_X(\tau)\right)^{n-j+k} \\ &\quad + \sum_{j=m+1}^{n} \sum_{k=0}^{j} j \binom{n}{j} \binom{j}{k} (-1)^k \left(\bar{F}_X(\tau)\right)^{n-j+k} \\ &= m \sum_{j=0}^{m} \sum_{k=0}^{j} \binom{n}{n-j, k, j-k} (-1)^k \left(\bar{F}_0(\tau)\right)^{(n-j+k)\lambda} \\ &\quad + \sum_{j=m+1}^{n} \sum_{k=0}^{j} j \binom{n}{n-j, k, j-k} (-1)^k \left(\bar{F}_0(\tau)\right)^{(n-j+k)\lambda}, \end{aligned}$$

where the last equality is derived from (1) and

$$\binom{a}{b, c, d} = \frac{a!}{b!c!d!}, \quad \text{with} \quad a = b + c + d.$$

On the other hand, we can write

$$\begin{aligned}\mathbb{E}(T) &= E(\max(X_{m:n},\tau)) \\ &= \tau.P(X_{m:n}<\tau)+\int_{\tau}^{\infty} x f_{X_{m:n}}(x)dx \\ &= \tau.F_{X_{m:n}}(\tau)+\int_{\tau}^{\infty} x f_{X_{m:n}}(x)dx, \qquad (3)\end{aligned}$$

where $f_{X_{m:n}}(\cdot)$ and $F_{X_{m:n}}(\cdot)$ are the probability density function (pdf) and cdf of $X_{m:n}$, respectively. Integrating by parts leads to

$$\begin{aligned}\mathbb{E}(T) &= \tau.F_{X_{m:n}}(\tau)+\tau.\bar{F}_{X_{m:n}}(\tau)+\int_{\tau}^{\infty} \bar{F}_{X_{m:n}}(x)dx \\ &= \tau+\int_{\tau}^{\infty} \bar{F}_{X_{m:n}}(x)dx \\ &= \tau+\sum_{j=0}^{m-1}\binom{n}{j}\int_{\tau}^{\infty}(F_X(x))^j\left(\bar{F}_X(x)\right)^{n-j}dx \\ &= \tau+\sum_{j=0}^{m-1}\sum_{k=0}^{j}\binom{n}{n-j,k,j-k}(-1)^k\int_{\tau}^{\infty}\left(\bar{F}_0(x)\right)^{(n-j+k)\lambda}dx, \qquad (4)\end{aligned}$$

where the last equality is obtained by using the binomial expansion for $(F_X(x))^j=\left(1-\bar{F}_X(x)\right)^j$ and Equation (1).

As a special case, we consider the one-parameter exponential distribution with cdf given by

$$F_\lambda(x)=1-exp\{-\lambda x\}, \qquad \lambda>0,\ \ x\geq 0. \qquad (5)$$

Combining Equations (3), (4) and (5), we get

$$\begin{aligned}\mathbb{E}(\mathcal{D}) &= m\sum_{j=0}^{m}\sum_{k=0}^{j}\binom{n}{n-j,k,j-k}(-1)^k\exp\{-(n-j+k)\lambda\tau\} \\ &\quad+\sum_{j=m+1}^{n}\sum_{k=0}^{j} j\binom{n}{n-j,k,j-k}(-1)^k\exp\{-(n-j+k)\lambda\tau\}, \qquad (6)\end{aligned}$$

and

$$\mathbb{E}(T)=\tau+\sum_{j=0}^{m-1}\sum_{k=0}^{j}\binom{n}{n-j,k,j-k}(-1)^k\frac{\exp\{-\lambda(n-j+k)\tau\}}{\lambda(n-j+k)}. \qquad (7)$$

As an another special case, we consider the Rayleigh distribution with cdf given by

$$F_\lambda(x)=1-exp\{-\lambda x^2\}, \qquad \lambda>0,\ \ x\geq 0. \qquad (8)$$

From Equations (3), (4) and (8), we find

$$\begin{aligned}\mathbb{E}(\mathcal{D}) &= m\sum_{j=0}^{m}\sum_{k=0}^{j}\binom{n}{n-j,k,j-k}(-1)^k\exp\{-(n-j+k)\lambda\tau^2\} \\ &\quad+\sum_{j=m+1}^{n}\sum_{k=0}^{j} j\binom{n}{n-j,k,j-k}(-1)^k\exp\{-(n-j+k)\lambda\tau^2\}, \qquad (9)\end{aligned}$$

and

$$\begin{aligned}\mathbb{E}(T) &= \tau + \sum_{j=0}^{m-1}\sum_{k=0}^{j}\binom{n}{n-j,k,j-k}(-1)^k\int_{\tau}^{\infty}\exp\{-(n-j+k)\lambda x^2\}dx \\ &= \tau + \sum_{j=0}^{m-1}\sum_{k=0}^{j}\binom{n}{n-j,k,j-k}(-1)^k\sqrt{\frac{\pi}{(n-j+k)\lambda}}\left\{1-\Phi(2(n-j+k)\lambda)\right\}, \qquad (10)\end{aligned}$$

since

$$\begin{aligned}\int_{\tau}^{\infty}\exp\{-(n-j+k)\lambda x^2\}dx &= \sqrt{\frac{\pi}{(n-j+k)\lambda}}\int_{2(n-j+k)\lambda}^{\infty}\frac{1}{\sqrt{2\pi}}\exp\{\frac{-z^2}{2}\}dz \\ &= \sqrt{\frac{\pi}{(n-j+k)\lambda}}\left\{1-\Phi(2(n-j+k)\lambda)\right\},\end{aligned}$$

in which $z=\sqrt{2(n-j+k)\lambda}x$ and $\Phi(\cdot)$ denotes the cdf of the standard normal distribution.

Table 1 gives the values of $\mathcal{C}(n,m,\tau)$ based on Equations (2), (6) and (7) for different choices of n, m and τ, when $\lambda=1$, $C_0=10$, $C_n=15$, $C_d=10$ and $C_t=20$ and for the exponential distribution. From Table 1, by an empirical evidence, we have the following points:

- The values of $\mathcal{C}(n,m,\tau)$ increase along with the values of m and τ increase, for the fixed values of n. In fact, it was expected because increasing the values of m and τ means that we will have more failed items.
- As we expected, the values of $\mathcal{C}(n,m,\tau)$ increase with increasing the values of n, when all other parameters are kept fixed.
- For a large value of τ it is observed that the values of $\mathcal{C}(n,m,\tau)$ are quite close to the each others by changing the values of m.

Table 1. The values of $\mathcal{C}(n,m,\tau)$ for different choices for τ, m and n, when $\lambda=1$, $C_0=10$, $C_n=15$, $C_d=10$, $C_t=20$.

n	m/τ	0.5	1	2	5	10
5	1	115.8227	136.7004	168.2339	234.6631	334.9977
	2	121.0477	137.6698	168.2563	234.6631	334.9977
	3	132.6107	142.0625	168.5901	234.6632	334.9977
	4	150.9684	154.1704	171.3587	234.6721	334.9977
	5	180.6854	181.1426	186.8590	235.6693	335.0045
10	1	209.4278	243.2126	286.4665	359.3262	459.9955
	3	212.2700	243.3084	286.4665	359.3262	459.9955
	5	226.0559	245.3200	286.4779	359.3262	459.9955
	7	252.1275	257.3809	287.1028	359.3262	459.9955
	10	318.5795	318.6075	320.4975	361.3274	460.0091
20	1	398.6944	456.4241	522.9329	608.6524	709.9909
	5	399.5995	456.4254	522.9329	608.6524	709.9909
	10	425.6855	457.7994	522.9331	608.6524	709.9909
	15	486.2923	488.1658	523.7313	608.6524	709.9909
	20	581.9548	581.9550	582.2218	612.6112	710.0182
40	1	777.5869	882.8482	995.8659	1107.3000	1210.0000
	10	777.9146	882.8483	995.8659	1107.3000	1210.0000
	20	825.1373	883.4463	995.8659	1107.3000	1210.0000
	30	936.9096	937.8128	996.1301	1107.3000	1210.0000
	40	1095.4000	1095.6000	1095.6000	1115.1000	1210.0000

One can see that the expected cost function defined in (2) is a nonlinear function in decision variables $\mathcal{S} = (n, m, \tau)$, where n and m are integers and τ is continuous. From practical considerations, the budget is limited. So, it is desirable for the experimenter to obtain the values of n, m and τ in which the cost function is less than a pre-fixed value for the budget, say B^*. This difficult optimization problem cannot be solved analytically and numerical computations should be utilized.

In some practical situations, some values of $\mathcal{S} = (n, m, \tau)$ may be known to the experimenter, and the decision is finding the optimal value for the unknown values. So, we consider three cases as follows.

(i) The values of m and τ are known: Let m and τ be some known values and the aim is finding the optimal value for n, say n_{opt}, such that $\mathcal{C}(n, m, \tau) \leq B^*$. Table 2 presents the values of n_{opt} for different choices of m and τ, with $\lambda = 1$, $C_0 = 10$, $C_n = 15$, $C_d = 10$, $C_t = 20$ and $B^* = 300,\ 400$, when the underlying distribution is the exponential distribution. From Table 2 one can observe that for the large values of m there is no n_{opt}, which satisfies the condition $\mathcal{C}(n, m, \tau) \leq B^*$. These cases are indicated by a dash $(-)$ in Table 2. For the other cases, the smallest value for n_{opt} is m and the largest one decreases as τ increases. From Table 2, we find that n_{opt} is not unique. Figure 1 shows the values of $\mathcal{C}(n, m, \tau)$ with respect to n for different choices of m and τ with $\lambda = 1$, $C_0 = 10$, $C_n = 15$, $C_d = 10$, $C_t = 20$ and $B^* = 300,\ 400$. Figure 1 confirms the results in Table 2.

(ii) The values of n and τ are known: Let n and τ be known values and the aim is finding the optimal value for m, say m_{opt}, in which $\mathcal{C}(n, m, \tau) \leq B^*$. Table 3 shows the values of m_{opt} for different choices of n and τ with $\lambda = 1$, $C_0 = 10$, $C_n = 15$, $C_d = 10$, $C_t = 20$ and $B^* = 300,\ 400$, for the exponential distribution. From Table 3 we find that for the large values of n, the values of m_{opt} do not exist. These are displayed by dash. For the other cases, the largest values for m_{opt} are increasing in n. Figure 2 also confirms this result.

(iii) The values of m and n are known: Suppose that m and n are known values and the problem is finding the optimal values for τ, say τ_{opt}, so that $\mathcal{C}(n, m, \tau) \leq B^*$. Table 4 presents the values of τ_{opt} for different choices of m and n, when $\lambda = 1$, $C_0 = 10$, $C_n = 15$, $C_d = 10$, $C_t = 20$ and $B^* = 300,\ 400$, when the underlying distribution is the exponential distribution. Figure 3 also shows the values of $\mathcal{C}(n, m, \tau)$ with respect to τ for different choices of n and m, with $\lambda = 1$, $C_0 = 10$, $C_n = 15$, $C_d = 10$, $C_t = 20$ and $B^* = 300,\ 400$. From Table 4 and Figure 3 we conclude that for the large values of m and n there is no τ_{opt}.

3. Results when N is random

Let $X_{1:n} \leq \cdots \leq X_{N:N}$ be the order statistics associated with a sample of size N from the PHR model given in (1). Independently, let N be a non-negative integer-valued random variable from the power series distribution truncated at point m with probability mass function (pmf) as

$$P(N = n) = \frac{a(n)\theta^n}{b(\theta)}, \quad n = m, m+1, \cdots, \tag{11}$$

where $b(\theta) = \sum_{n=m}^{\infty} a(n)\theta^n$ is positive, finite and differentiable and $a(n) > 0$ depends only on n. Here, we denote the Type II hybrid censoring scheme as $\mathcal{S}' = (\theta, m, \tau)$. For finding the best values for $\mathcal{S}' = (\theta, m, \tau)$, we consider the modified expected cost function as

$$\mathcal{C}(\theta, m, \tau) = C_0 + C_n \mathbb{E}_N(N) + C_d \mathbb{E}_N\left(\mathbb{E}(\mathcal{D}|N = n)\right) + C_t \mathbb{E}_N\left(\mathbb{E}(T|N = n)\right),$$

in which the values C_0, C_n, C_d and C_t are defined as in (2). Form (11), it is easy to show that

$$\mathbb{E}_N(N) = \theta \frac{b'(\theta)}{b(\theta)},$$

Table 2. The values of n_{opt} for different choices of m and τ with $\lambda = 1$, $C_0 = 10$, $C_n = 15$, $C_d = 10$, $C_t = 20$ and $B^* = 300,\ 400$.

			$B^* = 300$		
$m/\ \tau$	0.5	1	2	5	10
1	$\{1, 2, \cdots, 14\}$	$\{1, 2, \cdots, 12\}$	$\{1, 2, \cdots, 10\}$	$\{1, 2, \cdots, 7\}$	$\{1, 2, 3\}$
2	$\{2, 3, \cdots, 14\}$	$\{2, 3, \cdots, 12\}$	$\{2, 3, \cdots, 10\}$	$\{2, 3, \cdots, 7\}$	$\{2, 3\}$
3	$\{3, 4, \cdots, 14\}$	$\{3, 4, \cdots, 12\}$	$\{3, 4, \cdots, 10\}$	$\{3, 4, \cdots, 7\}$	$\{3\}$
4	$\{4, 5, \cdots, 14\}$	$\{4, 5, \cdots, 12\}$	$\{4, 5, \cdots, 10\}$	$\{4, 5, \cdots, 7\}$	–
5	$\{5, 6, \cdots, 14\}$	$\{5, 6, \cdots, 12\}$	$\{5, 6, \cdots, 10\}$	$\{5, 6, 7\}$	–
7	$\{7, 8, \cdots, 13\}$	$\{7, 8, \cdots, 12\}$	$\{7, 8, 9, 10\}$	$\{7\}$	–
10	–	–	–	–	–
15	–	–	–	–	–
20	–	–	–	–	–
			$B^* = 400$		
$m/\ \tau$	0.5	1	2	5	10
1	$\{1, 2, \cdots, 20\}$	$\{1, 2, \cdots, 17\}$	$\{1, 2, \cdots, 14\}$	$\{1, 2, \cdots, 11\}$	$\{1, 2, \cdots, 7\}$
2	$\{2, 3, \cdots, 20\}$	$\{2, 3, \cdots, 17\}$	$\{2, 3, \cdots, 14\}$	$\{2, 3, \cdots, 11\}$	$\{2, 3, \cdots, 7\}$
3	$\{3, 4, \cdots, 20\}$	$\{3, 4, \cdots, 17\}$	$\{3, 4, \cdots, 14\}$	$\{3, 4, \cdots, 11\}$	$\{3, 4, \cdots, 7\}$
4	$\{4, 5, \cdots, 20\}$	$\{4, 5, \cdots, 17\}$	$\{4, 5, \cdots, 14\}$	$\{4, 5, \cdots, 11\}$	$\{4, 5, 6, 7\}$
5	$\{5, 6, \cdots, 20\}$	$\{5, 6, \cdots, 17\}$	$\{5, 6, \cdots, 14\}$	$\{5, 6, \cdots, 11\}$	$\{5, 6, 7\}$
7	$\{7, 8, \cdots, 19\}$	$\{7, 8, \cdots, 17\}$	$\{7, 8, \cdots, 14\}$	$\{7, 8, \cdots, 11\}$	$\{7\}$
10	$\{10, 11, \cdots, 18\}$	$\{10, 11, \cdots, 17\}$	$\{10, 11, \cdots, 14\}$	$\{10, 11\}$	–
15	–	–	–	–	–
20	–	–	–	–	–

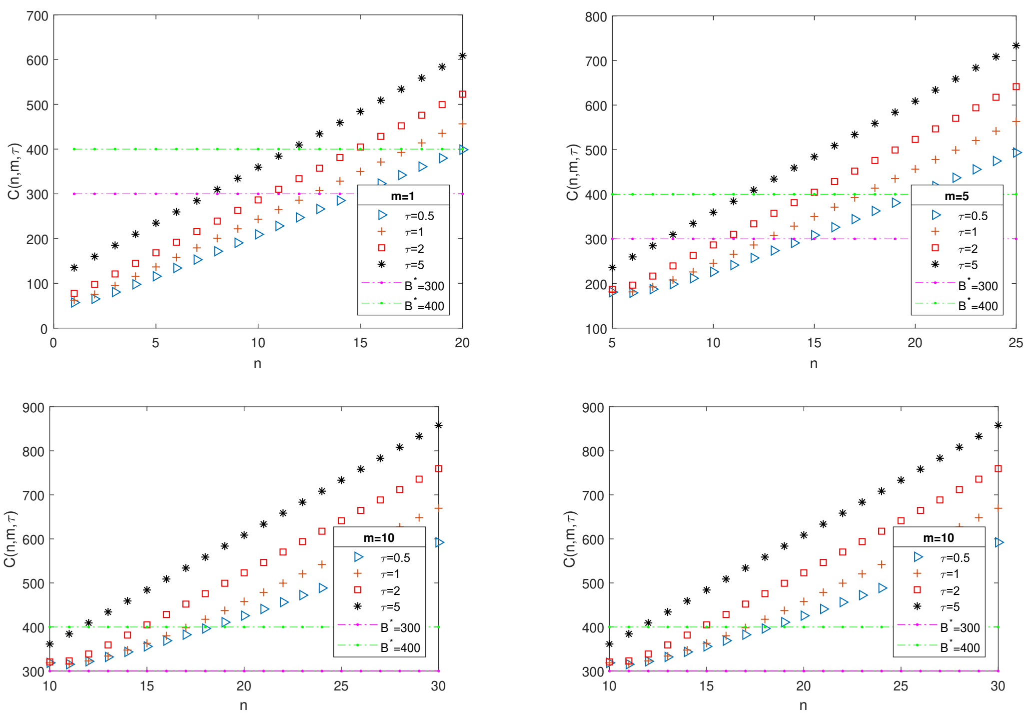

Figure 1. The plots of $\mathcal{C}(n, m, \tau)$ with respect to n for different choices of m and τ, with $\lambda = 1$, $C_0 = 10$, $C_n = 15$, $C_d = 10$, $C_t = 20$ and $B^* = 300,\ 400$.

where $b'(\theta) = \frac{\partial}{\partial\theta} b(\theta)$. The optimal values for $\mathcal{S}' = (\theta, m, \tau)$ are ones that the condition $\mathcal{C}(\theta, m, \tau) \leq B^*$ is satisfied, where B^* is a pre-fixed value.

Table 3. The values of m_{opt} for different choices of n and τ, when $\lambda = 1$, $C_0 = 10$, $C_n = 15$, $C_d = 10$, $C_t = 20$ and $B^* = 300,\ 400$.

		$B^* = 300$			
$n/\ \tau$	0.5	1	2	5	10
5	$\{1, 2 \cdots, 5\}$	$\{1, 2, \cdots, 5\}$	$\{1, 2, \cdots, 5\}$	$\{1, 2, \cdots, 5\}$	$\{1, 2, \cdots, 5\}$
10	$\{1, 2 \cdots, 9\}$	$\{1, 2, \cdots, 9\}$	$\{1, 2, \cdots, 9\}$	–	–
20	–	–	–	–	–
30	–	–	–	–	–
		$B^* = 400$			
$n/\ \tau$	0.5	1	2	5	10
5	$\{1, 2 \cdots, 5\}$	$\{1, 2, \cdots, 5\}$	$\{1, 2, \cdots, 5\}$	$\{1, 2, \cdots, 5\}$	$\{1, 2, \cdots, 5\}$
10	$\{1, 2 \cdots, 10\}$	$\{1, 2, \cdots, 10\}$	$\{1, 2, \cdots, 10\}$	$\{1, 2, \cdots, 10\}$	–
20	$\{1, 2 \cdots, 15\}$	–	–	–	–
30	–	–	–	–	–

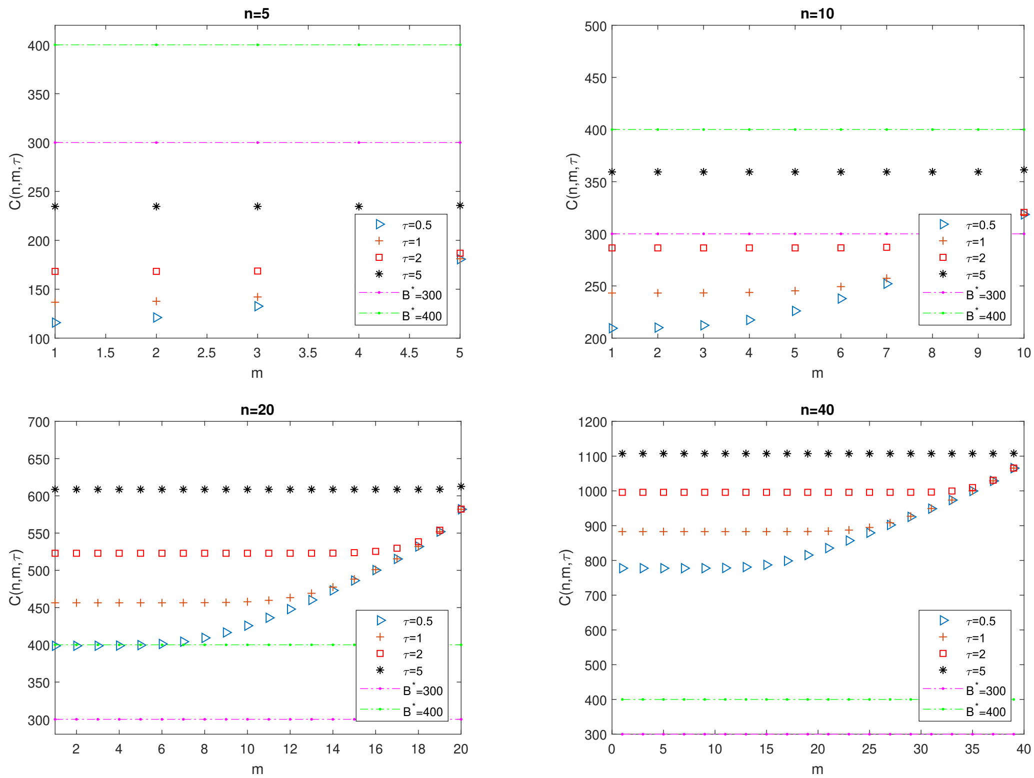

Figure 2. The plots of $\mathcal{C}(n, m, \tau)$ with respect to m for different choices of n and τ, with $\lambda = 1$, $C_0 = 10$, $C_n = 15$, $C_d = 10$, $C_t = 20$ and $B^* = 300,\ 400$.

As a special case, let N be a geometric random variable truncated at point m, say $Ge(1-\theta; m)$, i.e.,

$$P(N = n) = (1-\theta)\theta^{n-m}, \quad n = m, m+1, \cdots,\ m \geq 1, \quad 0 \leq \theta \leq 1.$$

Then, we have

$$\mathbb{E}_N(N) = m + \frac{\theta}{1-\theta},$$

Table 4. The values of τ_{opt} for different choices of n and m, with $\lambda = 1$, $C_0 = 10$, $C_n = 15$, $C_d = 10$, $C_t = 20$ and $B^* = 300,\ 400$.

		$B^* = 300$		
$m/\ n$	5	10	20	50
1	$(0, 8.2507]$	$(0, 2.4415]$	–	–
2	$(0, 8.2507]$	$(0, 2.4415]$	–	–
3	$(0, 8.2507]$	$(0, 2.4415]$	–	–
4	$(0, 8.2507]$	$(0, 2.4415]$	–	–
5	$(0, 8.2417]$	$(0, 2.4415]$	–	–
7	–	$(0, 2.4359]$	–	–
10	–	–	–	–
15	–	–	–	–
20	–	–	–	–
		$B^* = 400$		
$m/\ n$	5	10	20	50
1	$(0, 13.25]$	$(0, 7.0045]$	$(0, 0.5113]$	–
2	$(0, 13.25]$	$(0, 7.0045]$	$(0, 0.5112]$	–
3	$(0, 13.25]$	$(0, 7.0045]$	$(0, 0.5108]$	–
4	$(0, 13.25]$	$(0, 7.0045]$	$(0, 0.5091]$	–
5	$(0, 13.25]$	$(0, 7.0045]$	$(0, 0.5035]$	–
7	–	$(0, 7.0045]$	–	–
10	–	$(0, 6.9908]$	–	–
15	–	–	–	–
20	–	–	–	–

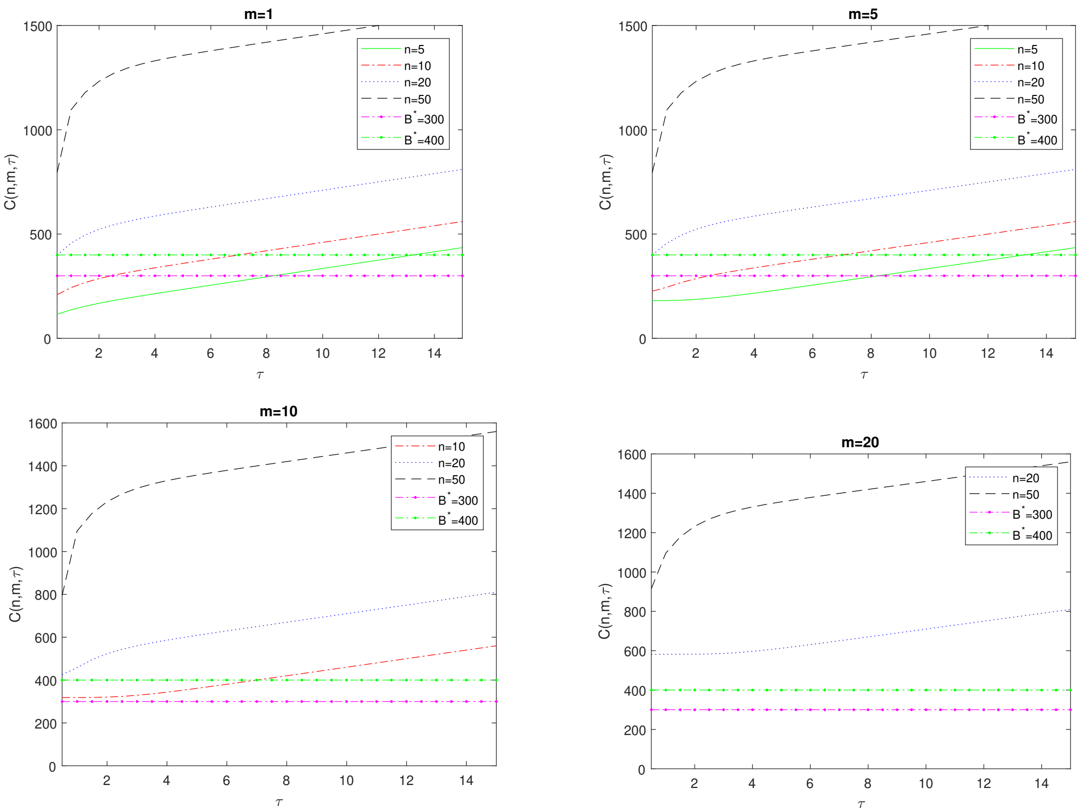

Figure 3. The plots of $\mathcal{C}(n, m, \tau)$ with respect to τ for different choices of n and m, when $\lambda = 1$, $C_0 = 10$, $C_n = 15$, $C_d = 10$, $C_t = 20$ and $B^* = 300,\ 400$.

which is increasing in θ. Also, we have

$$\mathbb{E}_N\left(\mathbb{E}(\mathcal{D}|N=n)\right) = \sum_{n=m}^{\infty} \mathbb{E}(\mathcal{D}|N=n)(1-\theta)\theta^{n-m},$$

and

$$\mathbb{E}_N\left(\mathbb{E}(T|N=n)\right) = \sum_{n=m}^{\infty} \mathbb{E}(T|N=n)(1-\theta)\theta^{n-m},$$

where for the one-parameter exponential distribution, the values of $\mathbb{E}(\mathcal{D}|N=n)$ and $\mathbb{E}(T|N=n)$ are defined in (6) and (7), respectively. Also, for the Rayleigh distribution, the values of $\mathbb{E}(\mathcal{D}|N=n)$ and $\mathbb{E}(T|N=n)$ are as in (9) and (10), respectively.

In Table 5, the values of the expected cost function $\mathcal{C}(\theta, m, \tau)$ for different choices for τ, m and θ are reported, when $\lambda = 1$, $C_0 = 10$, $C_n = 15$, $C_d = 10$, $C_t = 20$ and for the exponential distribution. From Table 5 one can find that the values of $\mathcal{C}(\theta, m, \tau)$ are increasing in τ, m and θ, when all other parameters are held fixed.

Table 5. The values of $\mathcal{C}(\theta, m, \tau)$ for different choices for τ, m and θ, when $\lambda = 1$, $C_0 = 10$, $C_n = 15$, $C_d = 10$, $C_t = 20$.

θ	$m/\ \tau$	0.5	1	2	5	10
0.1	1	58.1125	63.8743	79.9614	137.8849	237.7785
	5	180.6228	181.2749	188.0349	238.3402	337.7816
	10	318.3393	318.3925	320.9277	363.9020	462.7854
	15	451.0620	451.0670	452.0992	489.4559	587.7893
	20	581.6195	581.6200	582.0642	615.0020	712.7932
0.3	1	61.6123	68.8108	86.7026	145.7597	245.7146
	5	181.4112	182.7890	192.4148	246.0557	345.7166
	10	318.6295	318.8246	323.4720	371.4228	470.7191
	15	451.1825	451.2127	453.5619	496.7867	595.7216
	20	581.6504	581.6553	582.8861	622.1475	720.7241
0.5	1	69.4491	78.9617	99.4204	159.9665	259.9998
	5	185.1188	188.2641	202.4600	260.1022	360.0007
	10	321.4202	322.1998	331.0861	385.2718	485.0018
	15	453.6861	453.8849	459.4768	510.4413	610.003
	20	584.0043	584.0563	587.6158	635.6107	735.0041
0.7	1	91.0147	105.1093	130.1237	193.1696	293.3322
	5	199.4734	207.7199	230.1220	293.1439	393.3320
	10	333.3665	337.1002	355.4984	418.1130	518.3318
	15	464.9162	466.5467	481.1879	543.0834	643.3313
	20	594.9159	595.6148	607.0999	668.0541	768.3299
0.9	1	217.8255	243.4582	285.2366	357.2771	457.6402
	5	313.3383	335.2969	380.8779	456.0044	556.4043
	10	441.1638	452.2169	500.0994	578.5829	678.9130
	15	576.3930	570.4430	618.4472	699.6453	799.6951
	20	716.0770	688.7908	735.0432	818.1396	917.5532

In the following, it is tried to find the optimal value for $\mathcal{S}' = (\theta, m, \tau)$ in such a way that $\mathcal{C}(\theta, m, \tau) \leq B^*$, where B^* is a pre-fixed value. Similar to Section 2, we consider three cases.

(i) The values of m and τ are known: If m and τ are some known values and the aim is finding the optimal value for θ, say θ_{opt}, such that $\mathcal{C}(\theta, m, \tau) \leq B^*$. Table 6 presents the values of θ_{opt} for different choices of m and τ, with $\lambda = 1$, $C_0 = 10$, $C_n = 15$, $C_d = 10$, $C_t = 20$ and $B^* = 300,\ 400$, when the underlying distribution is the exponential distribution. From Table 6 we observe that, when θ_{opt} exists, the upper values are decreasing in m and τ.

(ii) The values of θ and τ are known: If θ and τ are known then the aim is finding the optimal value for m, say m_{opt}, such that $\mathcal{C}(\theta, m, \tau) \leq B^*$. Table 7 shows the values of m_{opt} for different choices of θ and T for the exponential distribution, with $\lambda = 1$, $C_0 = 10$, $C_n = 15$, $C_d = 10$, $C_t = 20$ and $B^* = 300,\ 400$. It can be concluded from Table 7 that the largest value for m_{opt} is a decreasing function of θ and τ.

(iii) The values of m and θ are known: Let m and θ be known values and the aim is finding the optimal value for τ, say τ_{opt}, such that $\mathcal{C}(\theta, m, \tau) \leq B^*$. Table 8 presents the values of τ_{opt} for different choices of m and θ, when the underlying distribution is the exponential distribution, with $\lambda = 1$, $C_0 = 10$, $C_n = 15$, $C_d = 10$, $C_t = 20$ and $B^* = 300,\ 400$. The results in Table 8 show that the upper values of τ_{opt} are decreasing in m and θ.

Figures 4-6 confirm the results of Tables 6-8.

Table 6. The values of θ_{opt} for different choices of m and τ, with $\lambda = 1$, $C_0 = 10$, $C_n = 15$, $C_d = 10$, $C_t = 20$ and $B^* = 300,\ 400$.

			$B^* = 300$		
$m/\ \tau$	0.5	1	2	5	10
1	$[0, 1]$	$[0, 1]$	$[0, 1]$	$[0, 0.8533]$	$[0, 0.716]$
2	$[0, 1]$	$[0, 1]$	$[0, 0.8291]$	$[0, 0.833]$	$[0, 0.612]$
3	$[0, 1]$	$[0, 1]$	$[0, 0.8712]$	$[0, 0.8126]$	$[0, 0.372]$
4	$[0, 1]$	$[0, 0.8878]$	$[0, 0.8501]$	$[0, 0.7768]$	–
5	$[0, 0.8853]$	$[0, 0.8639]$	$[0, 0.8287]$	$[0, 0.7165]$	–
7	$[0, 0.8283]$	$[0, 0.8134]$	$[0, 0.7542]$	$[0, 0.3655]$	–
10	–	–	–	–	–
15	–	–	–	–	–
20	–	–	–	–	–
			$B^* = 400$		
$m/\ \tau$	0.5	1	2	5	10
1	$[0, 1]$	$[0, 1]$	$[0, 1]$	$[0, 1]$	$[0, 0.853]$
2	$[0, 1]$	$[0, 1]$	$[0, 1]$	$[0, 1]$	$[0, 0.8327]$
3	$[0, 1]$	$[0, 1]$	$[0, 1]$	$[0, 0.8945]$	$[0, 0.8123]$
4	$[0, 1]$	$[0, 1]$	$[0, 1]$	$[0, 0.8742]$	$[0, 0.776]$
5	$[0, 1]$	$[0, 1]$	$[0, 1]$	$[0, 0.8539]$	$[0, 0.716]$
7	$[0, 1]$	$[0, 1]$	$[0, 0.8744]$	$[0, 0.8128]$	$[0, 0.372]$
10	$[0, 0.8531]$	$[0, 0.8417]$	$[0, 0.8084]$	$[0, 0.612]$	–
15	–	–	–	–	–
20	–	–	–	–	–

Table 7. The values of m_{opt} for different choices of θ and τ, with $\lambda = 1$, $C_0 = 10$, $C_n = 15$, $C_d = 10$, $C_t = 20$ and $B^* = 300,\ 400$.

			$B^* = 300$		
$\theta/\ \tau$	0.5	1	2	5	10
0.1	$\{1, 2 \cdots, 9\}$	$\{1, 2, \cdots, 9\}$	$\{1, 2, \cdots, 9\}$	$\{1, 2, \cdots, 7\}$	$\{1, 2, 3\}$
0.3	$\{1, 2 \cdots, 9\}$	$\{1, 2, \cdots, 9\}$	$\{1, 2, \cdots, 9\}$	$\{1, 2, \cdots, 7\}$	$\{1, 2, 3\}$
0.5	$\{1, 2 \cdots, 9\}$	$\{1, 2, \cdots, 9\}$	$\{1, 2, \cdots, 8\}$	$\{1, 2, \cdots, 6\}$	$\{1, 2\}$
0.7	$\{1, 2 \cdots, 8\}$	$\{1, 2, \cdots, 8\}$	$\{1, 2, \cdots, 7\}$	$\{1, 2, \cdots, 5\}$	$\{1\}$
0.9	$\{1, 2, 3, 4\}$	$\{1, 2, 3\}$	$\{1\}$	–	–
			$B^* = 400$		
$\theta/\ \tau$	0.5	1	2	5	10
0.1	$\{1, 2 \cdots, 13\}$	$\{1, 2, \cdots, 13\}$	$\{1, 2, \cdots, 13\}$	$\{1, 2, \cdots, 11\}$	$\{1, 2, \cdots, 7\}$
0.3	$\{1, 2 \cdots, 13\}$	$\{1, 2, \cdots, 13\}$	$\{1, 2, \cdots, 12\}$	$\{1, 2, \cdots, 11\}$	$\{1, 2, \cdots, 7\}$
0.5	$\{1, 2 \cdots, 12\}$	$\{1, 2, \cdots, 12\}$	$\{1, 2, \cdots, 12\}$	$\{1, 2, \cdots, 10\}$	$\{1, 2, \cdots, 6\}$
0.7	$\{1, 2 \cdots, 12\}$	$\{1, 2, \cdots, 12\}$	$\{1, 2, \cdots, 11\}$	$\{1, 2, \cdots, 9\}$	$\{1, 2, \cdots, 5\}$
0.9	$\{1, 2 \cdots, 8\}$	$\{1, 2, \cdots, 7\}$	$\{1, 2, \cdots, 5\}$	$\{1, 2\}$	–

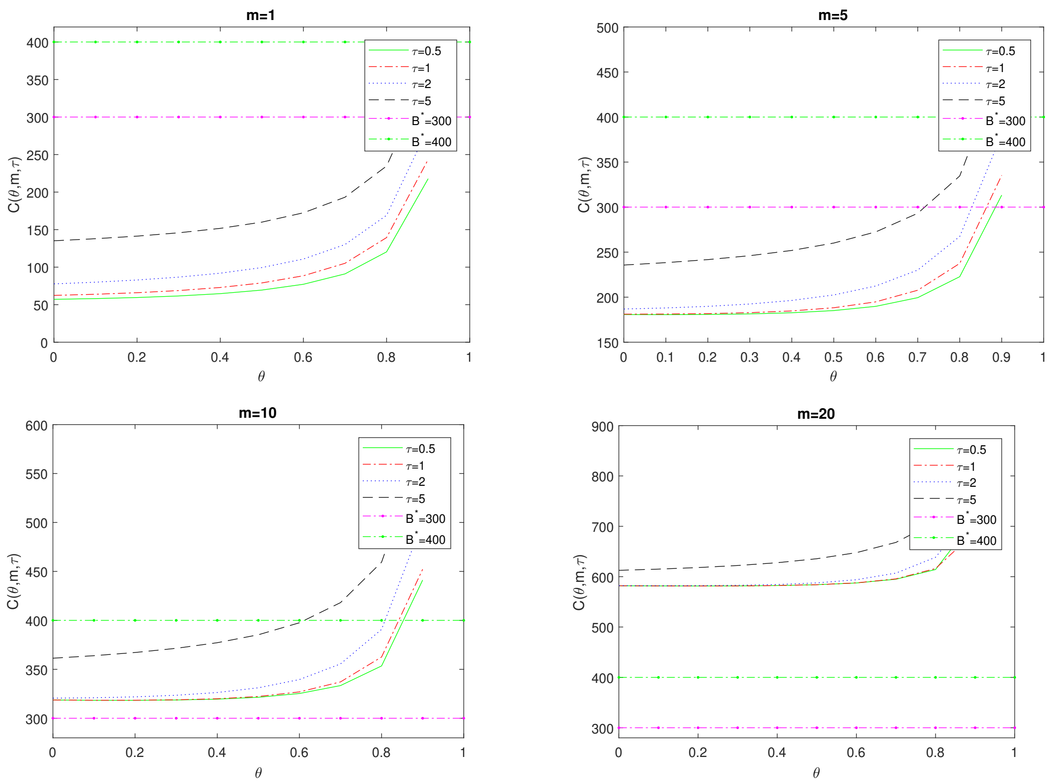

Figure 4. The plots of $\mathcal{C}(\theta, m, \tau)$ with respect to θ for different choices of m, τ with $\lambda = 1$, $C_0 = 10$, $C_n = 15$, $C_d = 10$, $C_t = 20$ and $B^* = 300,\ 400$.

Table 8. The values of τ_{opt} for different choices of θ and m, with $\lambda = 1$, $C_0 = 10$, $C_n = 15$, $C_d = 10$, $C_t = 20$ and $B^* = 300,\ 400$.

			$B^* = 300$		
$m/\ \theta$	0.1	0.3	0.5	0.7	0.9
1	$(0, 13.1111]$	$(0, 12.7143]$	$(0, 12]$	$(0, 10.3334]$	$(0, 2.4915]$
2	$(0, 11.8611]$	$(0, 11.4643]$	$(0, 10.75]$	$(0, 9.0835]$	$(0, 1.7558]$
3	$(0, 10.611]$	$(0, 10.2142]$	$(0, 9.5]$	$(0, 7.8339]$	$(0, 1.2196]$
4	$(0, 9.3608]$	$(0, 8.964]$	$(0, 8.2499]$	$(0, 6.5852]$	$(0, 0.7398]$
5	$(0, 8.1098]$	$(0, 7.7131]$	$(0, 6.9993]$	$(0, 5.3403]$	–
7	$(0, 5.5888]$	$(0, 5.1936]$	$(0, 4.4856]$	$(0, 2.9089]$	–
10	–	–	–	–	–
15	–	–	–	–	–
20	–	–	–	–	–
			$B^* = 400$		
$m/\ \theta$	0.1	0.3	0.5	0.7	0.9
1	$(0, 18.1111]$	$(0, 17.7143]$	$(0, 17]$	$(0, 15.3333]$	$(0, 7.1138]$
2	$(0, 16.8611]$	$(0, 16.4643]$	$(0, 15.75]$	$(0, 14.5833]$	$(0, 5.8829]$
3	$(0, 15.6111]$	$(0, 15.2143]$	$(0, 14.5]$	$(0, 12.8333]$	$(0, 4.6793]$
4	$(0, 14.3611]$	$(0, 13.9643]$	$(0, 13.25]$	$(0, 11.5833]$	$(0, 3.5538]$
5	$(0, 13.1111]$	$(0, 12.7143]$	$(0, 12]$	$(0, 10.3334]$	$(0, 2.604]$
7	$(0, 10.611]$	$(0, 10.2141]$	$(0, 9.4999]$	$(0, 7.834]$	$(0, 1.3484]$
10	$(0, 6.852]$	$(0, 6.4559]$	$(0, 5.7434]$	$(0, 4.1098]$	–
15	–	–	–	–	–
20	–	–	–	–	–

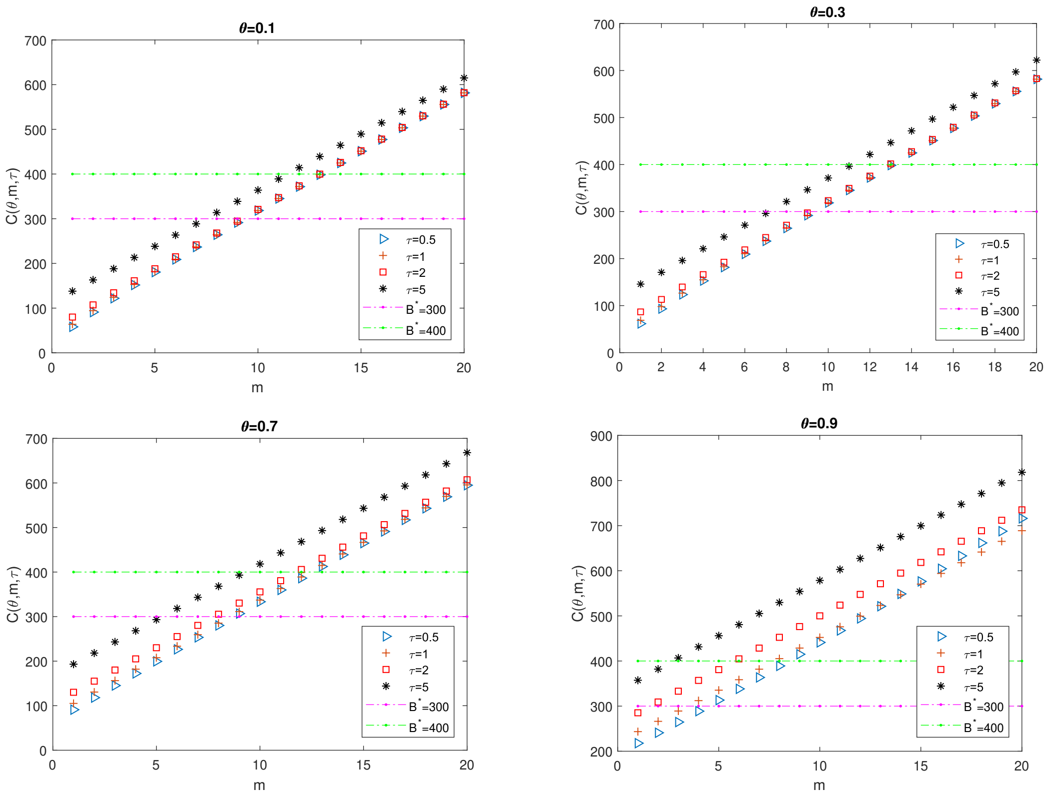

Figure 5. The plots of $\mathcal{C}(\theta, m, \tau)$ with respect to m for different choices of θ and τ, with $\lambda = 1$, $C_0 = 10$, $C_n = 15$, $C_d = 10$, $C_t = 20$ and $B^* = 300,\ 400$.

4. Simulation Study

In this section, we run a simulation study to assess the performances of the results in Sections 2 and 3. The following algorithm has been applied for this purpose:

Fixed Step-I Set the values of m, τ, λ, B^*, C_0, C_n, C_d and C_t;

Fixed Step-II Select n_{opt} such that the condition $\mathcal{C}(n, m, \tau) \leq B^*$ is satisfied.

Fixed Step-III Generate n_{opt} iid random variables $X_1, \ldots, X_{n_{opt}}$ from the one-parameter exponential distribution (or the Rayleigh distribution) with parameter λ. Then sort them as $X_{1:n_{opt}} \leq \cdots \leq X_{n_{opt}:n_{opt}}$.

Fixed Step-IV Set $T^* = \tau$ and $D^* = \sum_{j=1}^{n_{opt}} \mathbb{I}(X_{j:n_{opt}} \leq \tau)$ if $\tau > X_{m:n_{opt}}$, else set $T^* = X_{m:n_{opt}}$ and $D^* = m$, where $\mathbb{I}(\cdot)$ denotes the indicator function.

Fixed Step-V Repeat Steps III-IV for $K = 10^4$ times and let $D^*(i)$ and $T^*(i)$ be the associated results from Steps III-IV in the i-th iteration, $i = 1, \ldots, K$. Then, calculate the average cost function ($A(\mathcal{C})$) by using

$$A(\mathcal{C}) = C_0 + C_n n_{opt} + \frac{C_d}{K} \sum_{i=1}^{K} D^*(i) + \frac{C_t}{K} \sum_{i=1}^{K} T^*(i).$$

When N is a random variable the following steps can be utilized.

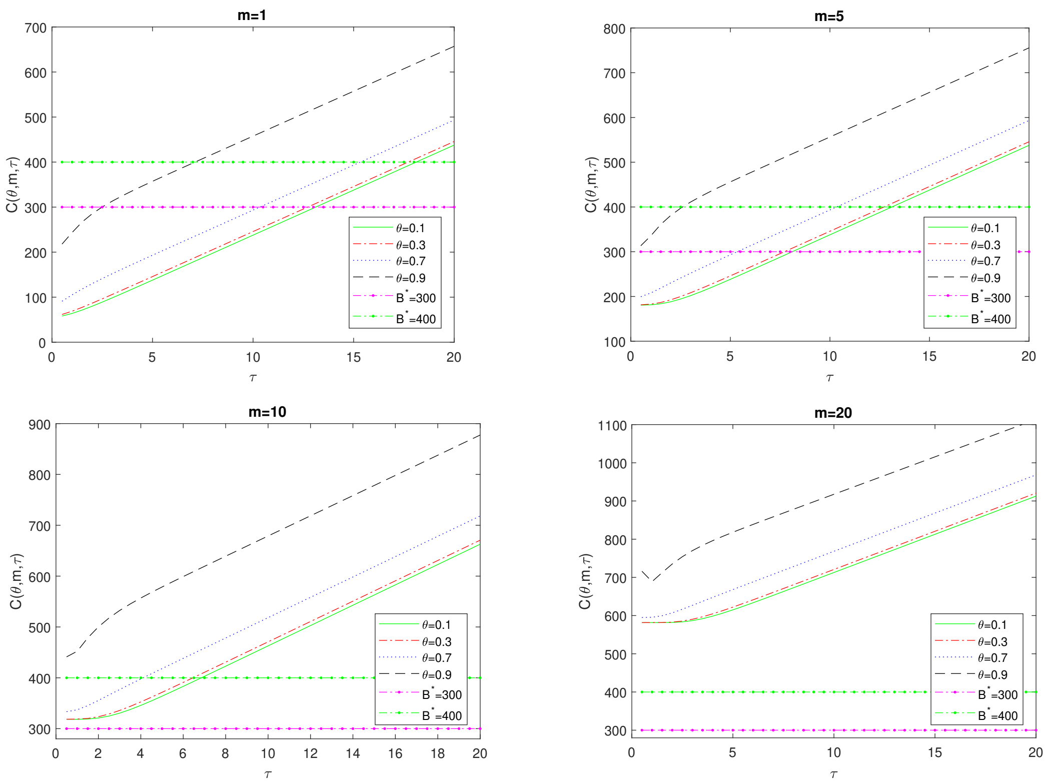

Figure 6. The plots of $\mathcal{C}(\theta, m, \tau)$ with respect to τ for different choices of θ and m, with $\lambda = 1$, $C_0 = 10$, $C_n = 15$, $C_d = 10$, $C_t = 20$ and $B^* = 300,\ 400$.

Random Step-I Set the values of m, τ, λ, B^*, C_0, C_n, C_d and C_t;

Random Step-II Choose θ_{opt} such that the condition $\mathcal{C}(\theta, m, \tau) \leq B^*$ is satisfied.

Random Step-III Generate N from the distribution $Ge(1-\theta_{opt}; m)$, say N_{opt}.

Random Step-IV Generate N_{opt} iid random variables $X_1, \ldots, X_{N_{opt}}$ from the one-parameter exponential distribution (or the Rayleigh distribution) with parameter λ. Then sort them as $X_{1:N_{opt}} \leq \cdots \leq X_{N_{opt}:N_{opt}}$.

Random Step-V Set $T^* = \tau$ and $D^* = \sum_{j=1}^{N_{opt}} \mathbb{I}(X_{j:N_{opt}} \leq \tau)$ if $\tau > X_{m:N_{opt}}$, else set $T^* = X_{m:N_{opt}}$ and $D^* = m$.

Random Step-VI Repeat Steps III-V for $K = 10^4$ times and let $N_{opt}(i)$, $D^*(i)$ and $T^*(i)$ be the associated results from Steps III-V in the i-th iteration, $i = 1, \ldots, K$. Then, calculate the average cost function $(A(\mathcal{C}))$ by using

$$A(\mathcal{C}) = C_0 + \frac{C_n}{K}\sum_{i=1}^{K} N_{opt}(i) + \frac{C_d}{K}\sum_{i=1}^{K} D^*(i) + \frac{C_t}{K}\sum_{i=1}^{K} T^*(i).$$

Note that one can used a similar algorithm for the case that m and τ are unknown.

Based on the algorithm mentioned above and the results in Tables 2 and 6, we have computed the values of $A(\mathcal{C})$ with different choices of m and τ for $\lambda = 1$, $C_0 = 10$, $C_n = 15$, $C_d = 10$, $C_t = 20$ and $B^* = 300,\ 400$, when the underlying distribution is the one-parameter exponential distribution. All the obtained results are reported in Tables 9 and 10. For Tables 9 and 10, we have selected the largest values of n_{opt} and θ_{opt} from Tables 2 and 6, respectively.

We mention that the results reported in Tables 9 and 10 are based on 10000 Monte Carlo simulations. As expected, from Tables 9 and 10, it is observed that in all cases $A(\mathcal{C}) \leq B^*$. It should be noted that all the computations in this paper have been done using MATLAB software.

Table 9. The values of $\left(n_{opt},\ A(\mathcal{C})\right)$ with different choices for τ ans m, with $\lambda = 1$, $C_0 = 10$, $C_n = 15$, $C_d = 10$, $C_t = 20$ and $B^* = 300,\ 400$.

		$B^* = 300$		
$m/\ \tau$	0.5	1	2	5	10
1	(14, 285.3109)	(12, 285.4750)	(10, 286.5730)	(7, 284.5170)	(3, 285.0000)
2	(14, 285.2361)	(12, 285.5633)	(10, 286.4400)	(7, 284.5360)	(3, 285.0000)
3	(14, 285.4739)	(12, 285.6901)	(10, 286.3360)	(7, 284.5340)	(3, 285.0008)
4	(14, 285.4467)	(12, 285.7504)	(10, 286.5650)	(7, 284.5310)	–
5	(14, 286.1908)	(12, 286.1443)	(10, 286.5629)	(7, 284.5620)	–
7	(13, 271.3001)	(12, 287.0798)	(10, 286.5691)	(7, 285.5026)	–
			$B^* = 400$		
$m/\ \tau$	0.5	1	2	5	10
1	(20, 398.8310)	(17, 392.5160)	(14, 380.7930)	(11, 384.2310)	(7, 384.9970)
2	(20, 398.1490)	(17, 392.8290)	(14, 380.9660)	(11, 384.2860)	(7, 384.9950)
3	(20, 320.0078)	(17, 392.6670)	(14, 381.0870)	(11, 384.2250)	(7, 384.9980)
4	(20, 398.6816)	(17, 392.2990)	(14, 380.9420)	(11, 384.2350)	(7, 384.9960)
5	(20, 398.9210)	(17, 392.5087)	(14, 381.0160)	(11, 384.2660)	(7, 384.9980)
7	(19, 380.8620)	(17, 392.9783)	(14, 380.8695)	(11, 384.2750)	(7, 384.9985)
10	(18, 295.7160)	(17, 393.4466)	(14, 381.0131)	(11, 384.3036)	–

Table 10. The values of $\left(\theta_{opt},\ A(\mathcal{C})\right)$ with different choices for τ ans m, with $\lambda = 1$, $C_0 = 10$, $C_n = 15$, $C_d = 10$, $C_t = 20$ and $B^* = 300,\ 400$.

			$B^* = 300$		
$m/\ \tau$	0.5	1	2	5	10
1	(1, 47.0263)	(1, 58.8653)	(1, 76.4278)	(0.8533, 135.6563)	(0.716, 279.1150)
2	(1, 70.3165)	(1, 86.1068)	(0.8291, 102.6413)	(0.8330, 160.2569)	(0.612, 274.2575)
3	(1, 91.4828)	(1, 93.6468)	(0.8712, 102.7995)	(0.8126, 185.2749)	(0.372, 285.8161)
4	(1, 111.7250)	(0.8878, 112.4577)	(0.8501, 119.8576)	(0.7768, 210.2737)	–
5	(0.8853, 130.8408)	(0.8639, 131.2783)	(0.8287, 180.0497)	(0.7165, 247.1009)	–
7	(0.8283, 166.9904)	(0.8134, 167.1755)	(0.7542, 231.4201)	(0.3655, 285.4988)	–
			$B^* = 400$		
$m/\ \tau$	0.5	1	2	5	10
1	(1, 49.4513)	(1, 58.7262)	(1, 76.2857)	(1, 135.0425)	(0.853, 235.5475)
2	(1, 78.5252)	(1, 73.1624)	(1, 102.5558)	(1, 160.1193)	(0.8327, 260.1529)
3	(1, 95.3286)	(1, 93.7784)	(1, 102.4749)	(0.8945, 185.1718)	(0.8123, 285.0574)
4	(1, 111.2424)	(1, 112.5845)	(1, 119.6949)	(0.8742, 210.2526)	(0.776, 310.0225)
5	(1, 134.2864)	(1, 131.3073)	(1, 136.8813)	(0.8539, 235.3200)	(0.716, 335.0102)
7	(1, 190.4949)	(1, 166.8974)	(0.8744, 170.5173)	(0.8128, 285.516)	(0, 0.372, 385.0086)
10	(0.8531, 242.0660)	(0.8417, 218.3278)	(0.8084, 306.9402)	(0.6120, 360.6855)	–

5. Real data example

Here, one real data set is analyzed to illustrate the use of our results. The data from [16], which are 36 failure times of 500 MW generators recorded over the period of 6 years. Observations of this data set are given as

$$0.058,\ 0.070,\ 0.090,\ 0.105,\ 0.113,\ 0.121,\ 0.153,\ 0.159,\ 0.224,$$

$$0.421,\ 0.570,\ 0.596,\ 0.618,\ 0.834,\ 1.019,\ 1.104,\ 1.497,\ 2.027,$$

$$2.234,\ 2.372,\ 2.433,\ 2.505,\ 2.690,\ 2.877,\ 2.879,\ 3.166,\ 3.455,$$

$$3.551,\ 4.378,\ 4.872,\ 5.085,\ 5.272,\ 5.341,\ 8.952,\ 9.188,\ 11.399.$$

This data set was used by [10] as the exponential random variables with $\hat{\lambda} = 0.389$ as the MLE for the parameter. [10] considered the values $C_0 = 100$, $C_n = 2$, $C_d = 5$ and $C_t = 25$, with the budget constraint $B^* = 500$. Here, we consider the same values. Table 11 gives the maximum value of m_{opt} and $\mathcal{C}(n, m_{opt}, \tau)$ for $n = 36$ and different choices of τ. The maximum values of τ_{opt} and $\mathcal{C}(n, m, \tau_{opt})$ for $n = 36$ and $m = 10,\ 30$ are reported in Table 12. From Tables 11 and 12 it can be seen that for all cases, the condition $\mathcal{C}(n, m, \tau) \leq B^*$ is held.

Table 11. The values of m_{opt} and $\mathcal{C}(n, m_{opt}, \tau)$ for $n = 36$ and different choices of τ, when $\lambda = 0.389$, $C_0 = 100$, $C_n = 2$, $C_d = 5$, $C_t = 25$ and $B^* = 500$.

τ	m_{opt}	$x_{m_{opt}:n}$	T	D	$\mathcal{C}(n, m_{opt}, \tau)$
1	33	5.341	5.341	33	470.525
5	33	5.341	5.341	33	470.525

Table 12. The values of τ_{opt} and $\mathcal{C}(n, m, \tau_{opt})$ for $n = 36$ and different choices of m, when $\lambda = 0.389$, $C_0 = 100$, $C_n = 2$, $C_d = 5$, $C_t = 25$ and $B^* = 500$.

m	τ_{opt}	$x_{m:n}$	T	D	$\mathcal{C}(n, m, \tau_{opt})$
10	6.4955	0.421	6.4955	33	499.3875
30	6.4811	4.872	6.4811	33	499.0275

6. Conclusion

In this paper, the optimal scheme in Type II hybrid censoring from the PHR model is obtained. Two cases as fixed and random sample sizes are considered in this paper. For each case, a cost function is introduced and the optimal scheme is determined in such a way that the cost function does not exceed a pre-fixed value. For the random case, since the distribution of the sample size depends on the parameter θ, the optimal value for θ is the aim. Although in this paper we have mainly addressed the optimal scheme, one can consider the determination of $\mathcal{S}(n, m, \tau)$ (or $\mathcal{S}'(\theta, m, \tau)$) simultaneously using any criterion.

Appendix

In order to perform the calculations in the paper, various codes have been used and run by MATLAB software. For brevity, only one procedure which have been used in Table 2 is presented here. For other procedures one can contact with authors.

```
function output=EN(m,theta)
output=m+(theta/(1-theta));
end
***************************
function output= EET(m,lambda,T,theta)
s=0;
for n=m:100
s=s+ET(m,n,lambda,T)*(1-theta)*((theta)^ (n-m) );
output=s;
end
******************************
function output= EEdelta(m,lambda,T,theta)
s=0;
for n = m:100
```

```
s=s+Edelta(m,n,lambda,T)*(1-theta)*(theta ˆ (n-m));
output=s;
end
***************************
lambda=1;
c1=10;
c2=15;
c3=10;
c4=20;
disp('************ T= 0.5 T=1 T=2 T=5 T=10 ************');
for theta=[0.1, 0.3, 0.5, 0.7, 0.9]
for m=[1, 5, 10, 15 , 20]
array=[];
for T=[0.5,1,2,5,10]
result=c1+c2*EN(m,theta)+c3*EEdelta(m,lambda,T,theta)+c4*EET(m,lambda,T,theta);
array=[array result];
end
disp(sprintf(' theta= %1.1f, m= %d: %3.4f %3.4f %3.4f %3.4f %3.4f',theta,m,array));
end
end
```


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No potential conflict of interest was reported by the authors.

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