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Temporal Regularity of Stochastic Differential Equations Driven by G-Brownian Motion

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Abstract This paper is devoted to studying the temporal regularity of the solutions of stochastic differential equations driven by G-Brownian motion (G-SDEs) under global Lipschitz and linear growth conditions. In addition, a numerical simulation of a particular G-SDE is provided.

Keywords G-Brownian motion, G-expectation, G-stochastic differential equation, Temporal regularity, Numerical simulation

AMS 2010 subject classifications 93E20, 60H07, 60H10, 60H30

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1. Introduction

Stochastic (partial) differential equations play a prominent role in modelling many phenomena in applied sciences. There is a huge number of works in the literature that are concerned with this type of equations, see for a short list [1, 2, 3, 4, 5].

Recently, significant progress in applied Mathematics has occurred through the emergence of the so-called G-stochastic analysis, by virtue of the new results of the pioneer S. Peng [22]. The reason behind this, was the need to model the uncertainty or the ambiguity, due to incomplete information about the parameters, namely, when the noise is big. Aspects of model ambiguity such as volatility uncertainty have been studied by Peng ([18, 20, 21]), who has introduced a sublinear G-expectation with a process called G-Brownian motion. Denis and Martini [11] have also suggested a structure based on quasi-sure analysis from abstract potential theory to construct a similar structure, by using a tight family $\mathcal{P}$ of possibly mutually singular probability measures. This theory developed quickly due to the great interest of many researchers, which led to the publication of many articles investigating stochastic differential equations driven by G-Brownian motion (G-SDEs) and their applications (see e.g. [6, 8, 9, 12, 14, 23, 24, 25] and the references therein). Also, a few works concerned with their qualitative properties (see, for instance [7, 16, 17]).

Systems subject to stochastic influences often require control strategies or optimisation techniques. Understanding the temporal regularity of solutions allows for more accurate modelling of real-world phenomena where random fluctuations play a crucial role and aids in designing effective control mechanisms and optimising system performance while accounting for random fluctuations. For instance, in finance, where stochastic processes are widely used to model asset prices, interest rates, and other financial variables, understanding the temporal

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regularity is substantial. It helps in assessing risk and making the right decisions in the context of uncertain market conditions.

Motivated by the above discussion, in this paper we deal with the class of stochastic differential equations driven by G-Brownian motion $B = (B_t)_{0\leq t\leq T}$, on a particular sublinear expectation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}}, \mathbb{F}^{\mathcal{P}})$, where $\mathbb{F}^{\mathcal{P}}$ is the universal filtration. More precisely, we aim to study the temporal regularity of the solutions of the following system

$$\begin{cases} dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t + \gamma(t, X_t)d\langle B\rangle_t, t \in (0, T] \\ X_0 = x, \end{cases} \tag{1.1}$$

where $x \in \mathbb{R}^n$ is the initial condition, $\langle B\rangle$ is the quadratic variation process of B, and the coefficients

$$b : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n,$$

$$\sigma : [0, T] \times \mathbb{R}^n \to \mathbb{R}^{n\times d},$$

and

$$\gamma : [0, T] \times \mathbb{R}^n \to (\mathbb{R}^{d^2})^{\times n},$$

are given deterministic functions for $n, d \in \mathbb{N}^*$.

The paper is laid out where in Section 2, we give some notions and preliminaries about G-expectation theory, G-Brownian motion, and the related stochastic calculus. Section 3, is devoted to the temporal regularity of the solution of problem (1.1). Section 4, is dedicated to the numerical analysis.

2. Preliminaries

In this section, we recall some notions and results about stochastic calculus, that needed in the G-framework, mainly based on the references [10, 11, 19, 20, 21, 26, 27].

2.1. *G-expectation and G-Brownian motion*

Let $\Omega := \{\omega \in C(\mathbb{R}_+, \mathbb{R}^d) : \omega(0) = 0\}$, equipped with the topology of uniform convergence on compact intervals, $\mathcal{B}(\Omega)$ the associated Borel σ-algebra, $\Omega_t := \{w_{.\wedge t} : w \in \Omega\}$, B the canonical process and $\mathbb{P}_0$ be the Wiener measure on Ω. Let $\mathbb{F} := \mathbb{F}^B = \{\mathcal{F}_t\}_{t\geq 0}$ be the raw filtration generated by B, which is only left-continuous. Further, consider the right-limit filtration $\mathbb{F}^+ := \{\mathcal{F}_t^+, t \geq 0\}$, where $\mathcal{F}_t^+ := \mathcal{F}_{t+} := \cap_{s>t}\mathcal{F}_s$.

Given a probability measure $\mathbb{P}$ on $(\Omega, \mathcal{B}(\Omega))$, we consider the right-continuous $\mathbb{P}$-completed filtrations $\mathcal{F}_t^{\mathbb{P}} := \mathcal{F}_t^+ \vee \mathcal{N}^{\mathbb{P}}(\mathcal{F}_t^+)$ and $\hat{\mathcal{F}}_t^{\mathbb{P}} := \mathcal{F}_t^+ \vee \mathcal{N}^{\mathbb{P}}(\mathcal{F}_\infty)$, where the $\mathbb{P}$-negligible set $\mathcal{N}^{\mathbb{P}}(\mathcal{G})$ on a σ-algebra $\mathcal{G}$ is defined as

$$\mathcal{N}^{\mathbb{P}}(\mathcal{G}) := \{D \subset \Omega : \text{there exists } \widetilde{D} \in \mathcal{G} \text{ such that } D \subset \widetilde{D} \text{ and } \mathbb{P}[\widetilde{D}] = 0\}.$$

Lemma 2.1 ([26])
Let $\mathbb{P}$ be an arbitrary probability measure on $(\Omega, \mathcal{F}_\infty)$. For every $\hat{\mathcal{F}}_t^{\mathbb{P}}$-measurable random variable $\hat{\xi}$, there exists a $\mathbb{P}$-a.s unique $\mathcal{F}_t$-measurable random variable ξ such that $\xi = \hat{\xi}$, $\mathbb{P}$-$a.s.$

For every $\hat{\mathbb{F}}^{\mathbb{P}}$-progressively measurable process $\hat{X}$, there exists a unique $\mathbb{F}$-progressively measurable process X such that $X = \hat{X}$, $dt \times \mathbb{P}$-$a.s.$ Moreover, if $\hat{X}$ is $\mathbb{P}$-almost surely continuous, then X can be chosen to $\mathbb{P}$-almost surely continuous. The G-expectation is defined by Peng in [18, 20, 21] through the nonlinear heat equation in the following sense. A d-dimensional random vector X is said to be G-normally distributed under the G-expectation $\hat{\mathbb{E}}[.]$, and denoted by $X \rightsquigarrow N(0, \Sigma)$, if for each bounded and Lipschitz continuous function φ on $\mathbb{R}^d$, $\varphi \in Lip(\mathbb{R}^d)$, the function u defined by

$$u(t, x) := \hat{\mathbb{E}}[\varphi(x + \sqrt{t}X)], t \geq 0, \quad x \in \mathbb{R}^d,$$

is the unique bounded Lipschitz continuous viscosity solution of the following parabolic equation

$$\begin{cases} \frac{\partial u}{\partial t} - G(D^2u) = 0, \quad (t, x) \in (0, +\infty) \times \mathbb{R}^d \\ u(0, x) = \varphi(x) \end{cases},$$

where $D^2u = (\partial^2_{x_ix_j}u)_{1\leq i,j\leq d}$ is the Hessian matrix of u and the nonlinear operator G is defined by

$$G(A) := \frac{1}{2}\sup_{\gamma\in\Gamma}\{tr(\gamma\gamma^* A)\}, \quad \gamma \in \mathbb{R}^{d\times d}, \tag{2.1}$$

where A is a $d \times d$ symmetric matrix and Γ is a given non empty, bounded and closed subset of $\mathbb{R}^{d\times d}$, γ^* denotes the transpose of γ and

$$\Sigma := \{\gamma\gamma^*,\ \gamma \in \Gamma\}.$$

Peng [20, 21] has showed that the G-expectation $\hat{\mathbb{E}} : \mathcal{H} := Lip(\mathbb{R}^d) \longrightarrow \mathbb{R}$ is a consistent sublinear expectation on the lattice $\mathcal{H}$ of real functions i.e., it satisfies:

(i) **Sub-additivity:** For all $X, Y \in \mathcal{H}$, $\hat{\mathbb{E}}[X + Y] \leq \hat{\mathbb{E}}[X] + \hat{\mathbb{E}}[Y]$,
(ii) **Monotonicity:** For all $X, Y \in \mathcal{H}$, $X \geq Y \Rightarrow \hat{\mathbb{E}}[X] \geq \hat{\mathbb{E}}[Y]$,
(iii) **Constant preserving:** For all $c \in \mathbb{R}$, $\hat{\mathbb{E}}[c] = c$,
(v) **Positive homogeneity:** For all $\lambda \geq 0$, $X \in \mathcal{H}$, $\hat{\mathbb{E}}[\lambda X] = \lambda\hat{\mathbb{E}}[X]$.

Now, let $Lip(\Omega)$ be the set of random variables of the form $\xi := \varphi(B_{t_1}, B_{t_2}, ..., B_{t_n})$ for some bounded Lipschitz continuous function φ on $\mathbb{R}^{d\times n}$ and $0 \leq t_1 \leq t_2 \leq ... \leq t_n \leq T$. The coordinate process $(B_t,\ t \geq 0)$ is called G-Brownian motion whenever B_1 is G-normally distributed under $\hat{\mathbb{E}}[.]$ and for each $s, t \geq 0$ and $t_1, t_2, ..., t_n \in [0, t]$ we have

$$\hat{\mathbb{E}}[\varphi(B_{t_1}, ..., B_{t_n}, B_{t+s} - B_t)] = \hat{\mathbb{E}}[\psi(B_{t_1}, ..., B_{t_n})],$$

where $\psi(x_1, ..., x_n) = \hat{\mathbb{E}}[\varphi(x_1, ..., x_n, \sqrt{s}B_1)]$. This property implies that the increments of the G-Brownian motion are independent and that $B_{t+s} - B_t$ and B_s are identically $N(0, s\Sigma)$-distributed.

A remarkable result of Peng [20, 21] is that if $\mathcal{H}$ is a lattice of real functions on Ω such that $Lip(\Omega) \subset \mathcal{H}$, then the G-expectation $\hat{\mathbb{E}} :\ \mathcal{H} \longrightarrow \mathbb{R}$ is a consistent sublinear expectation.

For $p \in [0, +\infty)$, we denote by $\mathbb{L}^p_G(\Omega)$ the closure of $Lip(\Omega)$ under the Banach norm

$$\|X\|^p_{\mathbb{L}^p_G(\Omega)} := \hat{\mathbb{E}}[|X|^p].$$

For each $t \geq 0$, let $L^0(\Omega_t)$ be the set of $\mathcal{F}_t$-measurable functions. We set

$$Lip(\Omega_t) := Lip(\Omega) \cap L^0(\Omega_t),\ \ \mathbb{L}^p_G(\Omega_t) := \mathbb{L}^p_G(\Omega) \cap L^0(\Omega_t).$$

2.2. *G-stochastic integral*

For $p \in [0, +\infty)$. Let $M^{0,p}_G(0, T)$ be the space of $\mathbb{F}$-progressively measurable, $\mathbb{R}^d$-valued elementary processes of the form

$$\eta(t) = \sum_{i=0}^{n-1}\eta_i 1_{[t_i,t_{i+1})}(s),$$

where $0 = t_0 < t_1 < \cdots < t_n = T,\ n \geq 1$ and $\eta_i \in Lip(\Omega_{t_i})$. Let $M^p_G(0, T)$ be the closure of $M^{0,p}_G(0, T)$ under the norm

$$\|\eta\|^p_{M^p_G(0,T)} := \hat{\mathbb{E}}[\int_0^T |\eta\,(t)\,|^p ds].$$

For each $\eta \in M^{0,2}_G(0, T)$, the G-stochastic integral is defined pointwisely by

$$I_t(\eta) = \int_0^t \eta\,(s)\,d_G B_s := \sum_{j=0}^{n-1}\eta_j(B_{t\wedge t_{j+1}} - B_{t\wedge t_j}),$$

with $I(\eta) := I_T(\eta)$, the mapping $I:\ M_G^{0,2}(0,T) \to \mathbb{L}_G^2(\Omega_t)$ is continuous and thus can be continuously extended to $M_G^2(0,T)$.

The quadratic variation process of G-Brownian motion can be formulated in $\mathbb{L}_G^2(\Omega_t)$ by the continuous $d\times d$-symmetric-matrix-valued process defined by

$$\langle B\rangle_t^G := B_t \otimes B_t - 2\int_0^t B_s \otimes d_G B_s, \tag{2.2}$$

whose diagonal is constituted of non-decreasing processes. Here, for $a, b \in \mathbb{R}^d$, the $d\times d$-symmetric matrix $a\otimes b$ is defined by $(a\otimes b)x = (a\cdot x)b$ for $x\in\mathbb{R}^d$, where " $\cdot$ " denotes the scalar product in $\mathbb{R}^d$.

Define the mapping $\mathcal{J} : M_G^{0,1}(0,T) \mapsto \mathbb{L}_G^1(\Omega_T)$:

$$\mathcal{J} = \int_0^T \eta(t)\, d\langle B\rangle_t^G := \sum_{j=0}^{n-1} \eta_j(\langle B\rangle_{t_{j+1}}^G - \langle B\rangle_{t_j}^G).$$

Then $\mathcal{J}$ can be uniquely extended to $\mathcal{Q} : M_G^1(0,T) \to \mathbb{L}_G^1(\Omega_T)$, where

$$\mathcal{Q} := \int_0^T \eta(t)\, d\langle B\rangle_t^G, \quad \eta\in M_G^1(0,T).$$

Now, we have the following "isometry" (formulated for the case $d=1$, for simplicity).

Lemma 2.2 ([21])
Assume $d=1$ and let $\eta\in M_G^2(0,T)$, then we have

$$\hat{\mathbb{E}}\left[\left(\int_0^T \eta(s)\, d_G B_s\right)^2\right] = \hat{\mathbb{E}}\left[\int_0^T \eta(s)^2\, d\langle B\rangle_s^G\right].$$

2.3. *A dual representation of G-expectation*

Denis et al. [10, 11] have proved the following dual representation of the G-expectation in terms of a weakly compact (tight) family $\mathcal{P}$ of possibly mutually singular probability measures on $(\Omega, \mathcal{B}(\Omega))$. Moreover, they have given an explicit constructions of $\mathcal{P}$. This duality expresses the G-expectation as a robust expectation with respect to $\mathcal{P}$. Soner et al. [26, 27] have performed an in-depth analysis of such a construction and its consequences on the G-stochastic analysis, in particular the question of aggregation of processes.

Proposition 2.3 ([10, 11])
There exists a family of weakly compact probability measures $\mathcal{P}$ on $(\Omega, \mathcal{B}(\Omega))$ such that for each $\xi\in\mathbb{L}_G^1(\Omega)$

$$\hat{\mathbb{E}}[\xi] = \sup_{\mathbb{P}\in\mathcal{P}} \mathbb{E}^{\mathbb{P}}[\xi] \tag{2.3}$$

Moreover, the set function

$$C(A) := \sup_{\mathbb{P}\in\mathcal{P}} \mathbb{P}(A), \ \ A\in\mathcal{B}(\Omega),$$

defines a regular Choquet capacity.

This leads us to the following ([11, 26]).

Definition 2.4
A set $A\in\mathcal{B}(\Omega)$ is called polar if $C(A)=0$, or equivalent if $\mathbb{P}(A)=0$ for all $\mathbb{P}\in\mathcal{P}$. We say that a property holds $\mathcal{P}$-quasi-surely (q.s.) if it holds almost-surely for all $\mathbb{P}\in\mathcal{P}$, i.e. outside a polar set. A probability measure $\mathbb{P}$ is called absolutely continuous with respect to $\mathcal{P}$ if $\mathbb{P}(A)=0$ for all $A\in\mathcal{N}_{\mathcal{P}}$.

Denote by $\mathcal{N}_{\mathcal{P}} := \bigcap_{\mathbb{P}\in\mathcal{P}} \mathcal{N}^{\mathbb{P}}(\mathcal{F}_\infty)$ the $\mathcal{P}$-polar sets. We shall use the following universal filtration $\mathbb{F}^{\mathcal{P}}$ for the possibly mutually singular probability measures $\{\mathbb{P}, \mathbb{P}\in\mathcal{P}\}$ (cf. [27]).

$$\mathbb{F}^{\mathcal{P}} := \{\hat{\mathcal{F}}_t^{\mathcal{P}}\}_{t\geq 0}, \text{ where } \hat{\mathcal{F}}_t^{\mathcal{P}} := \bigcap_{\mathbb{P}\in\mathcal{P}} (\mathcal{F}_t^{\mathbb{P}} \vee \mathcal{N}_{\mathcal{P}}) \text{ for } t\geq 0. \tag{2.4}$$

The dual formulation of the G-expectation suggests the following aspect of aggregation.

Lemma 2.5 ([26])
Let $\eta\in M_G^2(0,T)$. Then, η is Itô-integrable for every $\mathbb{P}\in\mathcal{P}$. Moreover, for every $t\in[0,T]$,

$$\int_0^t \eta(s)\, d_G B_s = \int_0^t \eta(s)\, dB_s,\ \mathbb{P}\text{-}a.s. \text{ for every } \mathbb{P}\in\mathcal{P} \tag{2.5}$$

where the right hand side is the usual Itô integral. Consequently, the quadratic variation process $\langle B\rangle^G$ defined in (2.2) coincide with the usual quadratic variation process quasi-surely.

In the rest of this paper, we will omit the letter G from both the G-stochastic integral and the G-quadratic variation.

Lemma 2.6 ([10])
If $(\mathbb{P}_n)_{n=1}^\infty \subset \mathcal{P}$ converges weakly to $\mathbb{P}\in\mathcal{P}$. Then for each $\xi\in\mathbb{L}_G^1(\Omega_T)$, $\mathbb{E}^{\mathbb{P}_n}[\xi]\to\mathbb{E}^{\mathbb{P}}[\xi]$.

Considering the properties of the quadratic variation process $\langle B\rangle$ in the G-framework and the dual formulation of the G-expectation, we have the following Burkholder-Davis-Gundy-type estimates.

Lemma 2.7 ([13])
For each $p\geq 2$ and $\eta\in M_G^p(0,T)$, there exists a constant C_p depends only on p and T such that

$$\begin{aligned}\hat{\mathbb{E}}\left[\sup_{s\leq u\leq t}\left|\int_s^u \eta(r)\, dB_r\right|^p\right] &\leq C_p\hat{\mathbb{E}}\left[\left(\int_s^t |\eta(r)|^2 dr\right)^{p/2}\right] \\ &\leq C_p|t-s|^{\frac{p}{2}-1}\int_s^t \hat{\mathbb{E}}[|\eta(r)|^p]dr\end{aligned} \tag{2.6}$$

If $\bar{\sigma}$ is a positive constant such that $\frac{d\langle B\rangle_t}{dt}\leq\bar{\sigma}$ quasi-surely, then, for each $p\geq 1$ and $\eta\in M_G^p(0,T)$,

$$\hat{\mathbb{E}}\left[\sup_{s\leq u\leq t}\left|\int_s^u \eta(r)\, d\langle B\rangle_r\right|^p\right] \leq \bar{\sigma}^p|t-s|^{p-1}\int_s^t \hat{\mathbb{E}}[|\eta(r)|^p]dr \tag{2.7}$$

3. Temporal regularity of the solution

Let $T>0$ be fixed, $\{B_t, t\in[0,T]\}$ be one dimensional G-Brownian motion and let $(\Omega,\mathcal{H},\hat{\mathbb{E}},\mathbb{F}^{\mathcal{P}})$ be a sublinear expectation space, we associate the universal filtration $\mathbb{F}^{\mathcal{P}}$:

$$\mathbb{F}^{\mathcal{P}} := \{\hat{\mathcal{F}}_t^{\mathcal{P}}\}_{t\geq 0},$$

where

$$\hat{\mathcal{F}}_t^{\mathcal{P}} := \bigcap_{\mathbb{P}\in\mathcal{P}} (\mathcal{F}_t^{\mathbb{P}}\vee\mathcal{N}_{\mathcal{P}}) \text{ for } t\geq 0.$$

Consider the following system written in integral form,

$$X(t) = x + \int_0^t b(s, X(s))ds + \int_0^t \sigma(s, X(s))dB_s + \int_0^t \gamma(s, X(s))d\langle B\rangle_s, \text{ for } t \in [0,T]. \tag{3.1}$$

To ensure the well-posedness of problem (3.1), we consider the following assumptions:

(A1): The functions $b(.,x)$, $\gamma(.,x)$ and $\sigma(.,x)$ are belong to $M_G^2(0,T)$ for every $x \in \mathbb{R}^n$.
(A2): The functions b, σ and γ are bounded and Lipschitz continuous with respect to x uniformly in time, also we suppose that $\gamma(t,x)$ is a symmetric $d \times d$ matrix with each element.

Theorem 3.1 ([19])
Under the above assumptions, the G-SDE (3.1) admits a unique solution $X \in M_G^2(0,T)$, that satisfies

$$\hat{\mathbb{E}}\left(\sup_{t\in[0,T]} |X(t)|^2\right) < \infty.$$

Also, we need the following proposition.

Proposition 3.2 ([16])
Let $p > 0$ and let $X := (X(t))_{t\in[0,T]}$ be the solution of the G-SDE (3.1). Under the global Lipschitz condition imposed on the functions b, σ and γ, the solution X satisfies for every $t \in [0,T]$ the following

$$\hat{\mathbb{E}}[\left(\sup_{s\in[0,t]} |X(s)|^p\right) \leq C < \infty \tag{3.2}$$

where C is a positive constant depends on p, x, t and the Lipschitz constant.

Lemma 3.3
Let $n \in \mathbb{N}^*$, $p \geq 2$ and $a_i \geq 0$, $i \in \{1, ..., n\}$. We have

$$\left(\sum_{i=1}^n a_i\right)^p \leq n^{p-1} \sum_{i=1}^n a_i^p \tag{3.3}$$

In order to prove the temporal regularity of the solution of the G-SDE (3.1), we require that the functions b, σ and γ satisfy the linear growth condition with respect to the second argument uniformly in t, i.e.
Assumption $\mathcal{H}$. There exists a constant $c > 0$ such that for all $y \in \mathbb{R}^n$ and $t \in [0,T]$, we have

$$|b(t,y)| + |\sigma(t,y)| + |\gamma(t,y)| \leq c|y|. \tag{3.4}$$

Definition 3.4
Let $p > 0$ and $T > 0$ be fixed. The solution $X = (X(t))_{t\in[0,T]}$ of the G-SDE (3.1), is said to be time Hölder continous with exponent $\theta \in (0,1)$, if there exists a positive constant $\mathcal{C}$ such that

$$\hat{\mathbb{E}}\left(|X(t) - X(s)|^p\right) \leq \mathcal{C}|t-s|^{p\theta}$$

for all $t, s \in [0,T]$.

The main result of this paper reads as follows.

Theorem 3.5
Let $p \geq 2$ and $T > 0$ be a fixed time. Under the hypotheses **(A1), (A2)** and **Assumption** $\mathcal{H}$, the solution X of the G-SDE (3.1) is time Hölder continous with exponent $\frac{1}{2}$, i.e. for all $0 \leq s < t \leq T$, we have

$$\hat{\mathbb{E}}\left(|X(t) - X(s)|^p\right) \leq \mathcal{C}|t-s|^{\frac{p}{2}},$$

where $\mathcal{C} = 3^{p-1}c^p C(T^{\frac{p}{2}} + \bar{\sigma}^p T^{\frac{p}{2}} + C_p)$.

Proof
Let $p \geq 2$, $T > 0$, $t, s \in [0, T]$ and X be the solution of the problem. Without loss the generality, we assume that $s < t$. From Eq.(3.1), the basic inequality (3.3) and the sub-addivity of the G-expectation, we have

$$\begin{aligned}\hat{\mathbb{E}}\left(|X(t) - X(s)|^p\right) &\leq 3^{p-1}\hat{\mathbb{E}}\left(|\int_s^t b(r, X(r))dr|^p\right) \\ +3^{p-1}\hat{\mathbb{E}}\left(|\int_s^t \sigma(r, X(r))dB_r|^p\right) &+ 3^{p-1}\hat{\mathbb{E}}\left(|\int_s^t \gamma(r, X(r))d\langle B\rangle_r|^p\right) \\ &= 3^{p-1}(I_1 + I_2 + I_3). \end{aligned} \tag{3.5}$$

We estimate I_1, by using of Hölder's inequality, **Assumption** $\mathcal{H}$ and Proposition 3.2, we can write

$$\begin{aligned} I_1 &= \hat{\mathbb{E}}\left(|\int_s^t b(r, X(r))dr|^p\right) \\ &\leq (t-s)^{p-1}\int_s^t \hat{\mathbb{E}}\left(|b(r, X(r))|^p\right)dr \\ &\leq c^p(t-s)^{p-1}\int_s^t \hat{\mathbb{E}}\left(|X(r)|^p\right)dr \\ &\leq c^p(t-s)^{p-1}\int_s^t \hat{\mathbb{E}}\left(\sup_{0\leq r\leq T}|X(r)|^p\right)dr \\ &\leq c^p C(t-s)^p. \end{aligned} \tag{3.6}$$

Next, to estimate I_2, we first use Burkholder-Davis-Gundy inequality in Lemma 2.7 as follows

$$I_2 = \hat{\mathbb{E}}\left(|\int_s^t \sigma(r, X(r))dB_r|^p\right) \leq C_p(t-s)^{\frac{p}{2}-1}\int_s^t \hat{\mathbb{E}}|\sigma(r, X(r))|^p dr.$$

By the **Assumption** $\mathcal{H}$ and Proposition 3.2 we obtain

$$\begin{aligned} I_2 &\leq C_p c^p(t-s)^{\frac{p}{2}-1}\int_s^t \hat{\mathbb{E}}\left(|X(r)|^p\right)dr \\ &\leq C_p c^p(t-s)^{\frac{p}{2}-1}\int_s^t \hat{\mathbb{E}}\left(\sup_{0\leq r\leq T}|X(r)|^p\right)dr \\ &\leq C_p c^p C(t-s)^{\frac{p}{2}}. \end{aligned} \tag{3.7}$$

Similarly, we estimate I_3,

$$\begin{aligned} I_3 &= \hat{\mathbb{E}}\left(|\int_s^t \gamma(r, X(r))d\langle B\rangle_r|^p\right) \\ &\leq \bar{\sigma}^p(t-s)^{p-1}\int_s^t \hat{\mathbb{E}}\left(|\gamma(r, X(r))|^p\right)dr \\ &\leq \bar{\sigma}^p c^p C(t-s)^p. \end{aligned} \tag{3.8}$$

By substituting the estimates (3.6), (3.7) and (3.8) into (3.5), we obtain the desired result which is

$$\hat{\mathbb{E}}\left(|X(t) - X(s)|^p\right) \leq \mathcal{C}(t-s)^{\frac{p}{2}} \tag{3.9}$$

$\square$

4. Simulation under uncertainty

In this section, we focus on the numerical simulation of a particular G-SDE. Precisely, we introduce an example in which we will apply the local linearization method consists in approximating locally the drift of the stochastic differential equation with a linear function, which has been proposed in the context of stochastic differential equations by Ozaki, especially in the general case in which the drift is allowed to depend on the time variable and also the diffusion coefficient can vary in the Shoji-Ozaki method, for more details see [15].

Consider the following G-SDE:

$$\left\{\begin{array}{c} dX(t) = \left(\alpha_0 + \alpha_1 X(t) + \alpha_2 X(t)^2 + \alpha_3 X(t)^3\right) dt + \sigma\sqrt{X(t)}dB_t \\ -\frac{1}{2}\sigma^2 d\langle B\rangle_t, \;\; t \in (0,T], \\ X(0) = x, \end{array}\right. \tag{4.1}$$

Since this equation does not have a constant diffusion coefficient, we apply the Lamperti transform see [15] for more details to obtain the process Y that has a unitary diffusion coefficient given by

$$F(x) = \frac{1}{\sigma}\int_0^x \frac{1}{\sqrt{u}}du = \frac{2}{\sigma}\sqrt{x},$$

and its inverse

$$F^{-1}(y) = \left(\frac{\sigma}{2}y\right)^2.$$

We apply the Shoji-Ozaki method for the particular choice of $\alpha_0 = 6$, $\alpha_1 = -11$, $\alpha_2 = 6$, $\alpha_3 = -1$, and $\sigma = 1$, hence the process Y satisfies

$$\left\{\begin{array}{l} dY(t) = \frac{23 - 11Y(t)^2 + \frac{3}{2}Y(t)^4 - \frac{1}{2^4}Y(t)^6}{2Y(t)}dt + dB_t - \frac{1}{2}\sigma^2 d\langle B\rangle_t, \;\; t \in (0,T], \\ Y(0) = 2\sqrt{X(0)}. \end{array}\right. \tag{4.2}$$

Now, basing on the numerical simulation for the G-Brownian motion, which has been achieved by Yang and Zhao in [28], we simulate the solution of the G-SDE (4.2), where its classical version (i.e., SDE) has been already studied by Iacus (see [15] page 91).

To do this, we first need to simulate the G-normal distribution, the G-Brownian motion and the G-quadratic variation process. Then, for $N \in \mathbb{N}^*$ be fixed, the set of points $(t_n = \frac{n.T}{N})_{n=\overline{0,N}}$, and $h = t_{n+1} - t_n$, we use Euler-Maruyama scheme to obtain the following:

$$\left\{\begin{array}{c} Y_0 = Y(0), \\ Y_{n+1} = Y_n + \left(\frac{23 - 11Y_n^2 + \frac{3}{2}Y_n^4 - \frac{1}{2^4}Y_n^6}{2Y_n}\right)h + \left(B_{t_{n+1}} - B_{t_n}\right) - \frac{1}{2}\sigma^2\left(\langle B\rangle_{t_{n+1}} - \langle B\rangle_{t_n}\right). \end{array}\right.$$

Since Y_n appears in the denominator, then we require that the initial condition not be zero. Figure 1, represents the trajectories of the approximate solution of G-SDE (4.2) where the volatility of the G-Brownian motion ranges within the interval $[0.4, 0.9]$. In Figure 2, we represent the trajectories of the approximate solution with volatility interval $[0.4, 0.5]$.

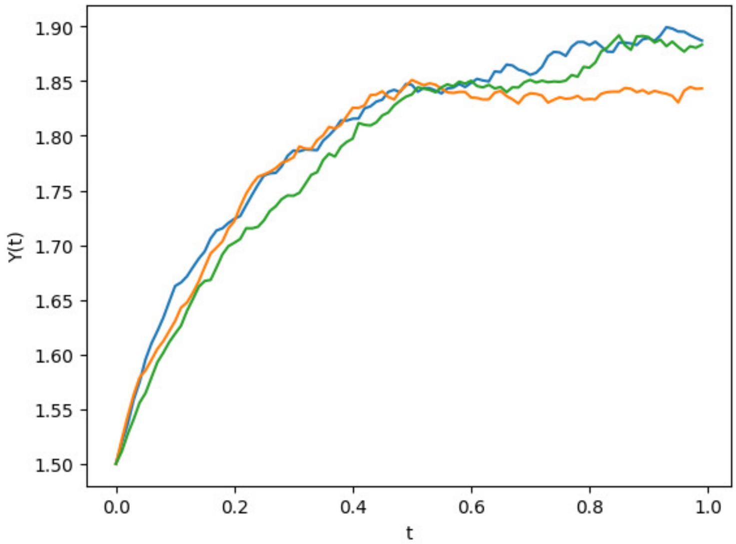

Figure 1. Approximate solution of G-SDE (4.2) with $\underline{\sigma}^2$=0.4 and $\overline{\sigma}^2$=0.9, and initial condition $Y(0) = 3/2$.

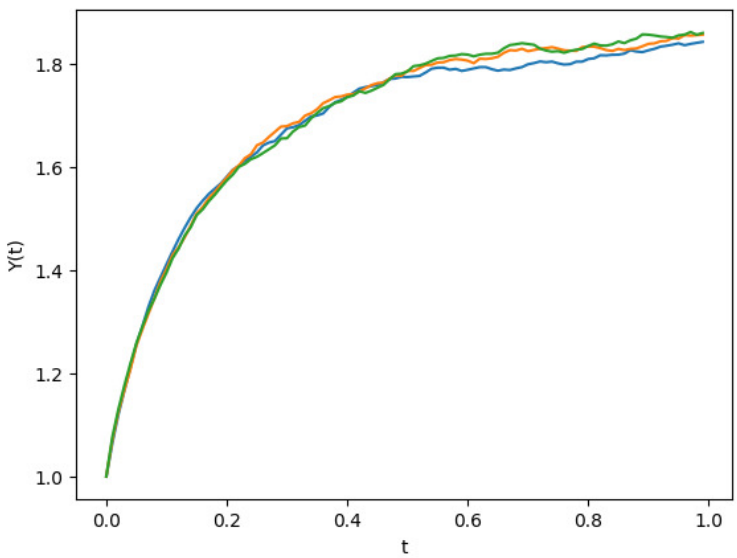

Figure 2. Approximate solution of G-SDE (4.2) with $\underline{\sigma}^2$=0.4, $\overline{\sigma}^2$=0.5, and initial condition $Y(0) = 1$.

Figure 3, showed the trajectories of the approximate solutions of both $Y(t)$ and $X(t)$. Note that, we can easily obtain $X(t)$ from $Y(t)$ by the relation

$$X(t) = \left(\frac{Y(t)}{2}\right)^2.$$

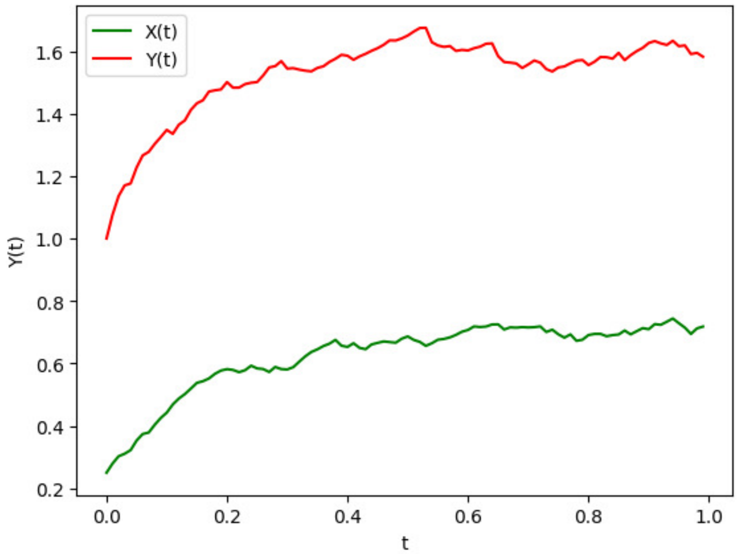

Figure 3. Approximate solution of G-SDE of (4.2) versus the approximate solution of (4.1) with $\underline{\sigma}^2$=0.4, $\overline{\sigma}^2$=0.5, and initial condition $Y(0) = 1$.

Conclusion

In this article, we investigated the temporal regularity of solutions of stochastic systems in the context of G-Brownian motion, which provides valuable insights into the behaviour of the solutions under uncertainty and enhances the understanding of the evolution of dynamical systems and how random fluctuations can influence them over time. We have imposed global Lipschitz and linear growth conditions to ensure certain mathematical constraints, such as the existence and uniqueness of the solutions.

On the other hand, this paper contributes to the development of more robust mathematical models for real-world phenomena and refining predictions, particularly when we face probability uncertainty, which is described by the Brownian motion. Also, we have established the numerical simulations of a G-SDE, which facilitates the implementation of other more complex systems subject to random influences in various scientific and engineering domains.

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