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Overlap Analysis in Progressive Hybrid Censoring: A Focus on Adaptive Type-II and Lomax Distribution

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Abstract This article explores the adaptive type-II progressive hybrid censoring scheme, introduced by Ng et al. (2009), which is used to make inferences about three measures of overlap: Matusita's measure (ρ), Morisita's measure (λ), and Weitzman's measure (Δ) for two Lomax distributions with different parameters. The article derives the bias and variance of these overlap measures' estimators. If sample sizes are limited, the precision or bias of these estimators is difficult to determine because there are no closed-form expressions for their variances and exact sampling distributions, so Monte Carlo simulations are used. Also, confidence intervals for these measures are constructed using both the bootstrap method and Taylor approximation.
To demonstrate the practical significance of the proposed estimators, an illustrative application is provided by analyzing real data.

Keywords Bootstrap method; Matusita's measure; Morisita's measure; Weitzman's measure; adaptive type-II progressive hybrid censoring

AMS 2010 subject classifications 62E10, 62N01, 62N02, 62G30

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1. Introduction

Life-testing experiments pose challenges in controlling test duration and conserving experimental units while ensuring efficient estimation. Censoring techniques offer a solution by removing active units and stopping the experiment before all units fail. Progressive censoring is crucial, as it involves removing units at predetermined or random time points during the experiment, accounting for potential losses or removals.

Over the years, progressive censoring has been extensively studied, with models falling into two categories: progressive Type-I censoring, concluding the experiment at predefined times, and progressive Type-II censoring, ending after a predetermined number of failures, both approaches provide flexibility by allowing unit removal at non-terminal times.

Progressive Type-I censoring involves fixed durations at specific time points, potentially resulting in few or no observed failures for units with long lifetimes. In contrast, progressive Type-II censoring, although flexible, may lead to extended test durations when units have extended lifetimes, which is considered a drawback.

Kundu and Joarder (2006) introduced two progressive hybrid censoring schemes, offering alternatives to traditional progressive Type-II censoring by ending experiments at a certain time T. These schemes adapt to the data, allowing fewer than m observations in Type-I hybrid censoring or extended testing in Type-II hybrid censoring.

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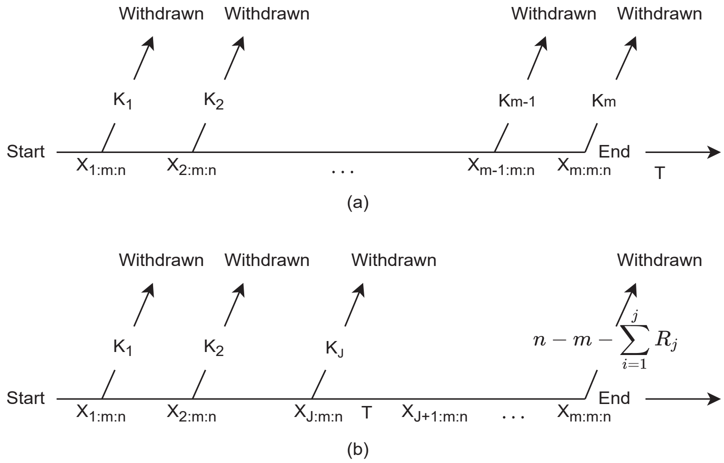

Figure 1. Adaptive type-II progressive hybrid censoring model as proposed by Ng et al. (2009). (a) Experiment ends before time T. (b) Experiment ends after time T.

During real-life experiments, it is imperative to acknowledge that a fixed censoring scheme may not always be a practical approach. Any intentional or unintentional alternation during the experiment can significantly impact the results. However, Ng et al. (2009) have introduced a new model (depicted in Figure (1) that allows the censoring scheme to be changed as required during the experiment. This model is called adaptive type-II progressive hybrid censoring (Adaptive-IIPH), in which a threshold time T switches between the original and modified schemes.

Assume there are n units in a life-testing experiment, and the effective sample size $m(< n)$ is predetermined, along with the censoring scheme $(K_1, K_2, \ldots, K_m)$; however, the values of some of the K_i may change as the experiment progresses. Assuming the experimenter has provided an ideal total test time T. If the m-th failure occurs before time T (see Figure 1(a)), the experiment proceeds similarly to type-II progressive censoring. It halts at time X_m with the pre-fixed censoring scheme $(K_1, K_2, \ldots, K_m)$. Otherwise, if the experimental time has passed T, but the number of observed failures has not yet reached m, we do not remove any items from the experiment by setting $K_{j+1} = K_{j+2} = \cdots = K_{m-1} = 0$ and $K^* = n - m - \sum_{i=1}^{j} K_i$. This setting can be seen as a design that guarantees m observed failure times while keeping the total test time not too far away from the ideal test time T (depicted in Figure 1(b)). Note that if we set $T = 0$, we will have a traditional type-II censoring method. However, if $T \to \infty$, the Adaptive-IIPH process becomes a progressive type-II censoring technique.

Adaptive-IIPH significantly impacts real-life applications, as evidenced by its widespread use in literature. Most recently, Alslman and Helu (2023) developed new methods for estimating the stress strength of the Inverse Weibull distribution using the Adaptive-IIPH censoring scheme. Asadi et al. (2022) employed Adaptive-IIPH censoring to conduct accelerated life tests on virus-containing microdroplets, monitoring Virus-MD persistence during coughs at different time points. Alotaibi et al. (2022) utilized Adaptive-IIPH censoring for testing sodium sulfur battery lifetimes in a chemical application employing the XLindley distribution. Furthermore, Helu and Samawi (2021) applied Adaptive-IIPH censoring to radar-evaluated rainfall data from 52 cumulus clouds in South Florida, highlighting its versatile utility in various fields.

Estimating the proportion of machines or electronic devices with similar failure time ranges is crucial in reliability analysis, especially when dealing with different sources or stress levels. Various overlap coefficients (OVL), such as Matusia's measure ρ, Morisita's measure λ, and Weitzman's measure Δ, are utilized to achieve this. These coefficients represent the common area between two probability density functions. The depiction of

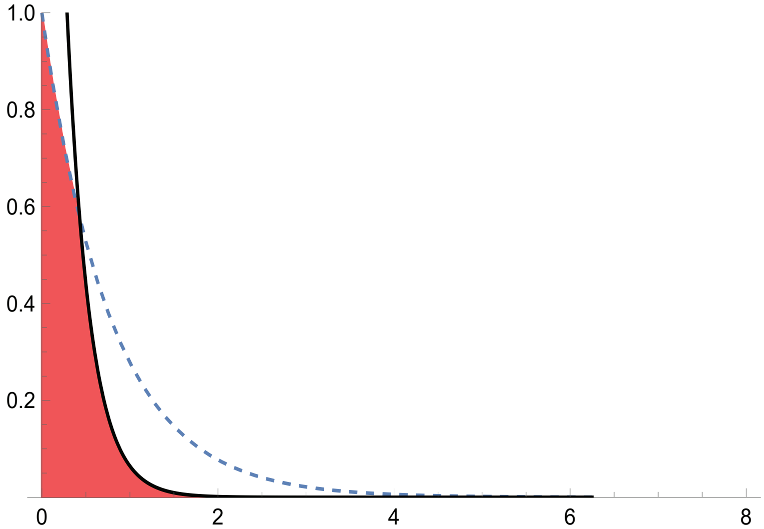

Figure 2. Overlap of two densities.

OVL for two distributions in Figure 2 displays the natural interpretations of OVL as a fraction of probability mass under either density, represented by the shaded area in Figure 2.

OVL has found widespread use in various practical applications as well. It has been utilized in quantitative ecology, as demonstrated by Gastwirth (1975). Furthermore, OVL has been applied to electromyographic assessment of muscular asymmetry by Ferrario et al. (2000) and in treatment assessment during clinical trials, as discussed by Mizuno et al. (2005).

For a deeper exploration of the various applications of overlap coefficients, interested readers can refer to the works of Wang and Tian (2017) and Martinez-Camblor (2022).

The mathematical form of the OVL measures are as follows: Suppose two samples of observations are drawn from two continuous distributions $f_1(x)$ and $f_2(x)$. Then the overlap measures are defined as follows:

$$\begin{aligned}
&\text{Matusita's Measure (1955):} && \rho = \int \sqrt{f_1(x)f_2(x)}dx, \\
&\text{Morisita's Measure (1959):} && \lambda = \frac{2\int f_1(x)f_2(x)dx}{\int [f_1(x)]^2dx + \int [f_2(x)]^2dx}, \\
&\text{Weitzman's Measure (1970):} && \Delta = \int \min(f_1(x), f_2(x))dx.
\end{aligned}$$

It is possible to adapt these measures for discrete distributions by using summations. They can also be extended to multivariate distributions. They are quantified on a scale from 0 to 1, with values near 0 indicating significant inequality (or disagreement) and 1 suggesting exact equality (perfect agreement) between density functions.

The mathematical structure of these measures is intricate, and there are no results available on the exact sampling distributions of their estimators. Prior work includes Smith (1982) on discrete Weitzman's measure, Mishra et al. (1986) on sampling properties under homogeneity assumptions, Mulekar and Mishra's (1994) simulations on normal densities, and Lu et al.'s (1989) study of sampling variability. Additionally, Dixon (1993) applied bootstrapping and jackknife techniques, while Mulekar and Mishra (2000) addressed inference problems.

The sampling behavior of a nonparametric estimator of OVL was analyzed by Helu and Samawi (2011). Samawi et al. (2017) conducted a study investigating the similarities and distinctions between the maximum of the Youden index (J) and overlap coefficient (OVL), highlighting the advantages of OVL over J.

In this article, our primary focus lies in making inferences regarding the measure of overlap (OVL) while utilizing Adaptive-IIPH censoring data from two independent Lomax distributions.

In Section 2, we introduce the Lomax distribution and derive the measures, as well as introduce the estimators. Moving on to Section 3, we discuss the approximate biases, establish confidence intervals via the delta method and bootstrap techniques for the OVL measures. In Section 4, we present the outcomes of our simulations and engage in a comprehensive discussion. Finally, in Section 5, we showcase a practical example using real-life data.

2. The model

The Lomax distribution, also known as Pareto Type-II distribution, belongs to the category of decreasing failure rate distributions. This has been noted in the research conducted by Chahkandi and Ganjali (2009). Initially introduced by Lomax (1954) as a model for business failure data, it has now found widespread applications in the field of lifetime distribution and its various extensions. These extensions are comprehensively discussed in the research of Marshall and Olkin (2007). According to Bryson (1974), when we assume that the population distribution exhibits heavy-tailed characteristics, The Lomax distribution can be an excellent alternative to more conventional lifetime distributions such as the exponential, Weibull, or gamma distributions. For a more detailed exploration of the Lomax distribution and its various applications, Arnold (2001) provides extensive information. A random variable U is said to have a Lomax distribution with probability density function (pdf)

$$f(u) = \frac{\beta}{\theta}(1+\beta u)^{-\left(1+\frac{1}{\theta}\right)},\ u>0 \tag{1}$$

where, $\theta > 0$ and $\beta > 0$ are the shape and scale parameters, respectively. The cumulative distribution function (cdf) corresponding to (1) for $u > 0$, is

$$F(u) = 1-(1+\beta u)^{-\frac{1}{\theta}}. \tag{2}$$

For known β, the distribution of $X = \log(1+\beta U)$ is a one-parameter exponential distribution $(Exp(\theta))$, with pdf and cdf as follows:

$$g(x) = \frac{1}{\theta}e^{-x/\theta}, \quad \text{for } x>0, \theta>0 \tag{3}$$

and

$$G(x) = 1-e^{-\frac{x}{\theta}}, \quad \text{for } x>0, \theta>0 \tag{4}$$

Because the transformation $X = \log(1+\beta U)$ is a one-to-one function that strictly increases, both the original data set U and the transformed data set X have an equivalent impact on overlap measures. Furthermore, transforming the data to X simplifies the computation of essential properties.

Let $R = \frac{\theta_1}{\theta_2}$, as in Helu and Samawi (2011), the continuous version of the three proposed overlap measures can be expressed as a function of R as follows:

$$\rho = \frac{2\sqrt{R}}{1+R}, \tag{5}$$

$$\lambda = \frac{4R}{(1+R)^2}, \tag{6}$$

and

$$\Delta = 1 - R^{\frac{1}{1-R}}\left|1-\frac{1}{R}\right|, R \neq 1. \tag{7}$$

According to Mulekar and Mishra (2000), ρ, λ, and Δ are not monotone for all $R > 0$. However, they exhibit certain properties, such as symmetry in R, meaning that $OVL(R) = OVL(\frac{1}{R})$. They also remain invariance under linear transformations, $Y = aX + b$, $a \neq 0$ and attain the maximum value of 1 at $R = 1$.

2.1. Maximum likelihood estimates

The joint pdf of the Adaptive-IIPH censored sample coming from an absolutely continuous distribution with pdf $f(.)$ and cdf $F(.)$ (see Balakrishnan and Cramer, 2014) can be written as

$$L(\theta_1|\mathbf{U}) = C[1-F(u_m)]^{K^*} \prod_{i=1}^{m} f(u_i) \prod_{i=1}^{J} [1-F(u_i)]^{K_i}, \tag{8}$$

where,

$$C = n(n-K_1-1)(n-K_1-K_2-2)...(n-K_1-K_2-...-K_{m-1}-m+1),$$

is the normalizing constants. Let $\mathbf{U} = U_{1:m_1:n_1} < U_{2:m_1:n_1} < ... < U_{m_1:m_1:n_1}$ be an Adaptive-IIPH censoring sample for a random sample of size m_1 from Lomax(θ_1, β) distribution under the censoring scheme $\{n_1, m_1, K_1, ..., K_{J_1}, 0, ..., 0, K^* = n_1 - m_1 - \sum_{i=1}^{J_1} K_i\}$ such that $U_{J_1:m_1:n_1} < T_1 < U_{J_1+1:m_1:n_1}$. For simplicity, let $U_i = U_{i:m_1:n_1}$. Since the lifetime of product U follows Lomax(θ_1, β) with known (given) scale parameter β, then by substituting Eqs. (1) and (2) into Eq. 8, the log-likelihood function reads

$$\begin{aligned} l_x \propto & -\frac{K^*}{\theta_1}\log(1+\beta u_{m_1}) + m_1 \log\left(\frac{\beta}{\theta_1}\right) - \left(1+\frac{1}{\theta_1}\right)\sum_{i=1}^{m_1}\log(1+\beta u_i) \\ & -\sum_{i=1}^{J_1}\frac{K_i}{\theta_1}\log(1+\beta u_i). \end{aligned} \tag{9}$$

Using the transformation $X = \log(1+\beta U)$, the order statistic $X_1 < X_2 < ... < X_{m_1}$ will be the corresponding Adaptive-IIPH from the one parameter exponential distribution with mean θ_1. Hence, the associated log-likelihood function of the observed transformed data becomes

$$l_x \propto -m_1 \log\theta_1 - \frac{\left(K^* x_{m_1} + (\theta_1+1)\sum_{i=1}^{m_1} x_i + \sum_{i=1}^{J_1} K_i x_i\right)}{\theta_1}. \tag{10}$$

Similarly for the second data set, $\mathbf{V} = \{V_1 < V_2 < ... < V_{m_2}\}$ is an Adaptive-IIPH censoring sample under the scheme $\{n_2, m_2, L_1, ..., L_{J_2}, 0, ..., 0, L^* = n_2 - m_2 - \sum_{i=1}^{J_2} L_i\}$ such that $V_{J_2} < T_2 < V_{J_2+1}$ from Lomax(θ_2, β) .. Using the transformation $Y = \log(1+\beta V)$, the order statistic $Y_1 < Y_2 < ... < Y_{m_2}$ will be the corresponding Adaptive-IIPH from the one parameter exponential distribution with mean θ_2. Hence, the joint associated log-likelihood function of the observed transformed data becomes

$$l_y \propto -m_2 \log\theta_2 - \frac{\left(L^* y_{m_2} + (\theta_2+1)\sum_{i=1}^{m_2} y_i + \sum_{i=1}^{J_2} L_i y_i\right)}{\theta_2}. \tag{11}$$

The MLEs of the parameters θ_1 and θ_2 can be obtained by taking the first derivative of Eqs. 10 and 11 with respect to θ_1 and θ_2, respectively, and equating the normal equations to 0 to get

$$\hat{\theta}_1 = \frac{K^* x_{m_1} + \sum_{i=1}^{m_1} x_i + \sum_{i=1}^{J_1} K_i x_i}{m_1}, \tag{12}$$

$$\hat{\theta}_2 = \frac{\left(L^* y_{m_2} + \sum_{i=1}^{m_2} y_i + \sum_{i=1}^{J_2} L_i y_i\right)}{m_2}. \tag{13}$$

Viveros and Balakrishnan (1994; page 88) showed that when the underlying distribution is an exponential with unknown mean θ, and when data $W_{1:m:n} < W_{2:m:n} < \cdots < W_{m:m:n}$ are based on progressively type-II censored sample with censoring scheme $\mathbf{K} = (K_1, K_2, \ldots, K_m)$, $\hat{\theta} = \frac{\sum_{i=1}^{m}(K_i+1)w_i}{m}$ is the MLE of θ, and $\hat{\theta} \sim Gamma(m, \frac{\theta}{m})$ in which $Gamma(.,.)$ denote the Gamma distribution. Cramer and Iliopolous (2010; Theorems

5 and 7) showed that the MLE when data are based on Adaptive-IIPH coincide with the MLE in deterministic progressive type-II censoring schemes. Thus, the distribution of this particular random variable is invariant with respect to random (fixed) progressive type-II censoring procedure. Thus, we obtain $\hat{\theta}_i \sim G(m_i, \frac{\theta_i}{m_i}); i = 1, 2.$ Consequently, the means and variances of the MLEs in (12) and (13) are

$$E(\hat{\theta}_1) = \theta_1, \qquad E(\hat{\theta}_2) = \theta_2, \tag{14}$$

$$Var(\hat{\theta}_1) = \frac{\theta_1^2}{m_1}, \qquad Var(\hat{\theta}_2) = \frac{\theta_2^2}{m_2}, \tag{15}$$

by the invariant property, the MLE of R is $\hat{R} = \frac{\hat{\theta}_1}{\hat{\theta}_2}$. Hence, $\frac{\theta_2}{\theta_1}\hat{R}$ has F-distribution with $2m_1$ and $2m_2$ degrees of freedom ($F_{2m_1,2m_2}$). Thus, the variance of $\hat{R}$ is given by:

$$Var(\hat{R}) = \frac{m_2^2(m_1 + m_2 - 1)}{m_1(m_2 - 1)^2(m_2 - 2)} R^2. \tag{16}$$

Clearly, an unbiased estimator of R is given by $\hat{R}^* = \frac{(m_2-1)}{m_2}\hat{R}$ with variance $Var(\hat{R}^*) = \frac{(m_1+m_2-1)}{m_1(m_2-2)}R^2$ and hence $Var(\hat{R}^*) < Var(\hat{R})$. Since the OVL measures are functions of R, therefore, based on the MLE estimate of R, the OVL measures can be estimated by

$$\hat{\rho} = \frac{2\sqrt{\hat{R}^*}}{1 + \hat{R}^*}, \tag{17}$$

$$\hat{\lambda} = \frac{4\hat{R}^*}{\left(1 + \hat{R}^*\right)^2}, \tag{18}$$

and,

$$\hat{\Delta} = 1 - \hat{R}^{*\frac{1}{1-\hat{R}^*}} \left|1 - \frac{1}{\hat{R}^*}\right|, \hat{R}^* \neq 1. \tag{19}$$

3. Asymptotic properties of OVL

Using the delta method, the asymptotic variance and bias for OVL measures are as follows: Let $OVL = g(\hat{R}^*)$, then the asymptotic variance are given by

$$Var(\hat{\rho}) = \sigma_{\hat{\rho}}^2 \cong \frac{(m_1 + m_2 - 1)}{m_1(m_2 - 2)} \frac{R(1 - R)^2}{(1 + R)^4}, \tag{20}$$

$$Var(\hat{\lambda}) = \sigma_{\hat{\lambda}}^2 \cong \frac{16(m_1 + m_2 - 1)}{m_1(m_2 - 2)} \frac{R^2(1 - R)^2}{(1 + R)^6}, \tag{21}$$

$$Var(\hat{\Delta}) = \sigma_{\hat{\Delta}}^2 \cong \frac{(m_1 + m_2 - 1)}{m_1(m_2 - 2)} \frac{R^{\frac{2}{1-R}}(\log R)^2}{(1 - R)^2}. \tag{22}$$

with the asymptotic bias

$$Bias(\hat{\rho}) \cong \frac{(m_1 + m_2 - 1)}{m_1(m_2 - 2)} \times \frac{\sqrt{R}(3R^2 - 6R - 1)}{2(1 + R)^3}, \tag{23}$$

$$Bias(\hat{\lambda}) \cong \frac{(m_1 + m_2 - 1)}{m_1(m_2 - 2)} \times \frac{4R^2(R - 2)}{(1 + R)^4}, \tag{24}$$

and,

$$Bias(\hat{\Delta}) \cong \left\{ \begin{array}{ll} H(R)\frac{(m_1+m_2-1)}{m_1(m_2-2)}, & R>1 \\ -H(R)\frac{(m_1+m_2-1)}{m_1(m_2-2)}, & R<1 \end{array} \right\}, \tag{25}$$

where, $H(R) = R^2 \left[\frac{R^{\frac{2R-1}{1-R}} R\{2R - \log R - 2\} \log R - (R-1)^2}{(R-1)^3} \right]$.

Consistent estimators for the above variances and biases can be obtained by substituting R by $\hat{R}^*$ in the above formulas.

3.1. Interval estimation

Two types of interval estimation for the OVL measure are considered, namely the asymptotic confidence interval and the bootstrap confidence interval that were introduced by Efron (1992). For a large sample, normal approximation to the sampling distribution using the delta-method, works fairly well. Therefore, the asymptotic $100(1-\alpha)\%$ confidence interval for the OVL measures is given by:
$\left\{ \widehat{OVL} \mp \hat{\sigma}_{\widehat{OVL}} Z_{\alpha/2} \right\}$, where $Z_{\alpha/2}$ is the $\frac{\alpha}{2}$ upper quantile of the standard normal distribution.

There is an obvious bias involved in all OVL measure estimates, however, for large samples, they work fairly well. Thus, the bias corrected interval can be computed as follows:

$$\left(\widehat{OVL} - Bias(\widehat{OVL}) \right) \pm \hat{\sigma}_{\widehat{OVL}} Z_{\alpha/2}. \tag{26}$$

However, uniform bootstrap resampling approach for estimating bootstrap confidence intervals as described by Efron (1992), is designed for one sample case. For a two-sample case, the uniform resampling rules will apply to each sample separately and independently (see Helu and Samawi, 2011).

Let $\mathbf{X} = (X_1, X_2, \cdots, X_{m_1})$ and $\mathbf{Y} = (Y_1, Y_2, \cdots, Y_{m_2})$ be two independent Adaptive-IIPH samples. Assume that the parameter of interest is the OVL coefficient. Let S be an estimate of OVL based on the mentioned two random samples. For B uniform re-samples, say $(X_{i1}^*, X_{i2}^*, ..., X_{im_1}^*)$ and $(Y_{i1}^*, Y_{i2}^*, ..., Y_{im_2}^*), i = 1, 2, ..., B$, let $S_1^*, S_2^*, ..., S_B^*$ be the re-sampling realization of S. Then, the uniform re-sampling approximation to the $100(1-\alpha)\%$ bootstrap confidence limits can be obtained as follows: Let $S_{(1)}^*, S_{(2)}^*, ..., S_{(B)}^*$ be the order statistics of $S_1^*, S_2^*, ..., S_B^*$. Define $\omega_1 = integer(B(\alpha))$ and $\omega_2 = integer(B(1-\alpha))$. Then the uniform re-sampling approximation of the $100(1-\alpha)\%$ confidence interval is $\left(\frac{S_{(\omega_1)}^* + S_{(\omega_1+1)}^*}{2}, \frac{S_{(\omega_2)}^* + S_{(\omega_2+1)}^*}{2} \right)$.

4. Simulation Study

This simulation study aims to rigorously compare the performance of maximum likelihood estimators for the measures of overlap. These estimators are derived from diverse sets of Adaptive-IIHP censoring samples, as described by Ng et al. (2009), generated from two independent Lomax distributions. The algorithm proceeds as follows:

1. Generate two independent progressive type-II censored samples, denoted as $U_1, U_2, ..., U_{m_1}$ and $V_1, V_2, ..., V_{m_2}$ from Lomax (θ_1, β) and Lomax (θ_2, β), respectively. Use censoring schemes $\mathbf{K} = (K_1, K_2, \ldots, K_{m_1})$ and $\mathbf{L} = (L_1, L_2, \ldots, L_{m_2})$ as proposed by Balakrishnan and Cramer (2014).
2. Determine the values of J_1 and J_2, such that $U_{J_1} < T_1 < U_{J_1+1}$ and $V_{J_2} < T_2 < V_{J_2+1}$. Then, remove $U_{J_1+2}, \ldots, U_{m_1}$ and $V_{J_2+2}, \ldots, V_{m_2}$.
3. Generate the first $m_1 - j_1 - 1$ order statistics from the truncated distribution $\frac{f_1(u)}{1-F_1(u_{J_1+1})}$ as $U_{J_1+2}, \ldots, U_{m_1}$, and adjust the censoring scheme to $\mathbf{K} = (K_1, \ldots, K_{J_1}, 0, \ldots, 0, K^* = n_1 - m_1 - \sum_{i=1}^{J_1} K_i)$. Similarly, generate the first $m_2 - j_2 - 1$ order statistics from the truncated distribution $\frac{f_2(v)}{1-F_2(v_{J_2+1})}$ as $V_{J_2+2}, \ldots, V_{m_2}$,

and update the censoring scheme to $\mathbf{L} = (L_1, \ldots, L_{J_2}, 0, \ldots, 0, L^* = n_2 - m_2 - \sum_{i=1}^{J_2} L_i)$. Use the transformation $X = \log(1 + \beta U)$ and $Y = \log(1 + \beta V)$ as in Section 2.

4. Calculate $\hat{\theta}_1$ and $\hat{\theta}_2$, and subsequently obtain the estimates of the measures of overlap $\hat{\rho}$, $\hat{\lambda}$, and $\hat{\Delta}$.

In this study, we executed a total of 10 000 simulations, each corresponding to one of four distinct values of R. Specifically:

1. When $R = 0.005$, the resulting parameter values are as follows: $\rho = 0.1407, \lambda = 0.0198$, and $\Delta = 0.0311$.
2. For $R = 0.05$, we observe $\rho = 0.4259, \lambda = 0.1814$, and $\Delta = 0.1886$.
3. When $R = 0.2$, the associated parameter values are $\rho = 0.70, \lambda = 0.50$, and $\Delta = 0.42$.
4. Lastly, $R = 0.8$ yielded parameter values of $\rho = 0.994, \lambda = 0.988$, and $\Delta = 0.918$.

These simulations are conducted based on four distinct sets of population parameters: $\beta = 1$, $(\theta_1, \theta_2) = (0.005, 1), (0.1, 2), (0.1, 0.5)$, and $(0.8, 1)$. This comprehensive range of parameter combinations allow us to explore varying degrees of similarity between the two Lomax distributions. Additionally, three primary stopping times are considered: $T_1 = X_{\lfloor \frac{m}{4} \rfloor}$, $T_2 = X_{\lfloor \frac{4m}{5} \rfloor}$, and $T_3 = X_m + 2$.

We then compute the associated approximate 95% confidence intervals, $|Bias|$, mean squared error (MSE), length of the confidence intervals (L) and coverage probability (Cov) using Taylor and bootstrap approximation techniques. The bootstrap approximation is based on $B = 1000$ resamples. For illustrative purposes we generate the censoring samples using $n = n_1 = n_2 = 20, 30, m = m_1 = m_2 = 5, 10, 20$, and set $\mathbf{K} = \mathbf{L}$, employing three censoring schemes:

- Scheme-I: $(n - m, 0^{*(m-1)})$, known as scheme-I, where $n - m$ units are removed just after the first failure.
- Scheme-II: $\left(0^{*(m-1)}, n - m\right)$, known as scheme-II, where $n - m$ units are removed after the last failure.
- Scheme-III: $\left(\frac{n-m}{2}, 0^{*(m-2)}, \frac{n-m}{2}\right)$, known as scheme-III, where $\frac{n-m}{2}$ units are removed after the first and last failures. For brevity, we use the notation 0^{*p} to denote p successive zeros. Thus, the scheme $(9, 0, 0, 0, 0, 0)$ is denoted by $(9, 0^{*5})$.

4.1. Data analysis and comparison study

Our study aims to examine how overlap estimators perform when applied to samples drawn from two Lomax distributions of varying degrees of similarity based on Adaptive-IIPH censored data. Our research has revealed an essential relationship between the similarity of two distributions and the accuracy of the estimators. This helps to shed light on the behavior of the estimators and their effectiveness in real-world scenarios.

Most favorable estimators tend to have minimal $|Bias|$, the smallest MSE, and the shortest confidence intervals (L). These desirable properties manifest prominently when a substantial disagreement exists between the two Lomax density distributions i.e., when $\rho = 0.14, \lambda = 0.019$ and $\Delta = 0.03$, as depicted in Tables 6 and 10 where we can notice that $|Bias|$ and MSE are almost zero with very small L and a coverage probability (Cov) that is quite close to the nominal level.

Conversely, as the similarity between the source distributions increases, i.e., $\widehat{OVL}$ approaches 1, we consistently observe an escalation in $|Bias|$, MSE, and L and a decrease in Cov across all OVL estimators. Based on this pattern, these estimators appear less accurate and precise as the source distributions become more alike You can refer to Tables 6-9 for more information.

Interestingly, a notable inverse relationship surfaces concerning coverage probability. As the source distributions become more alike, the coverage probability, Cov, decreases for the estimators $\hat{\rho}$ and $\hat{\lambda}$. However, this trend diverges for $\hat{\Delta}$, where the Cov improves with increasing similarity between the two densities (See Tables 8 & 9).

Furthermore, as the values of $\hat{\rho}$ and $\hat{\lambda}$ approach 1, signifying strong agreement between the source distributions, we observe a similar behavior pattern: intriguingly, the $\hat{\Delta}$ estimator deviates from this pattern. As $\hat{\Delta}$ approaches 1, indicating maximum similarity, only the $|Bias|$ of the $\hat{\Delta}$ estimator declines, while MSE, L and Cov increase (compare Tables 8 & 9). This suggests that the $\hat{\Delta}$ possesses unique characteristics, performing optimally when the source distributions completely agree or disagree.

Moreover, it's crucial to underscore the consistent behavior of the $\hat{\Delta}$ regarding coverage. Specifically, as $\hat{\Delta}$ approaches the extremes of 0 or 1, there is a consistent increase in Cov values. This highlights the remarkable stability of the $\hat{\Delta}$ estimator in scenarios where the source distributions either fully align or diverge.

It is noteworthy that when there exists a substantial disagreement between the two Lomax densities, there are minimal differences between the three stopping times. Additionally, when the ratio of m/n is large $(\geq 2/3)$, $|Bias|$, MSE, L, and coverage probability show noticeable improvement.

Shifting our focus to the bootstrap method, results presented in Tables 10-13 align with the observations made in Tables 6-9. Furthermore, the bootstrap results indicate no significant impact from varying censoring schemes or OVL values, except for the consistent coverage values, which remain stable regardless of the source distributions aligning or diverging.

5. Real life data

In this section, we present a real life data to demonstrate the effectiveness of our proposed method in practical situations. We used data sets that contain information on aircraft windshields' failure times. These data sets include both low- and high-quality variants. Aircraft windshields are designed with multiple layers of materials to withstand extreme conditions during flight and play a critical role in ensuring aircraft safety and performance. Therefore, data on their performance is routinely collected and analyzed, measured in 100,000-hour increments. The data used in this research is obtained from Helu and Samawi (2017). Table 1 displays the failure times for low (Data 1) and high (Data 2) quality windshields (see Table 1).

The legitimacy of the Lomax model For Data 1 & 2 is assessed using Kolmogorov-Smirnov (K-S), Anderson-Darling (A-D) and chi-square tests with $\beta = 1$, $\theta_1 = 0.0907$ and $\theta_2 = 0.1340$, respectively. Table 2 shows that the Lomax model fits both data sets well with a significance level of 0.05.

Three different artificial Adaptive-IIHP censored data are created for both sets using the same censoring schemes as those in Section 4. The associated stopping time for each scheme and the generated censored samples are given in Tables 3 and 4.

The estimates of the OVLs are calculated based on $m_1 = 32$, $m_2 = 32$. The corresponding MLEs, $|Bias|$, asymptotic variance and 95% confidence intervals for OVLs, using Taylor approximation and bootstrap methods, are reported in Table 5. The results illustrate the estimates' proximity to unity, indicating a high level of agreement between the two data sets. The $|Bias|$, asymptotic variances, and confidence interval lengths are notably minimal. The asymptotic variances approach zero. Notably, estimates based on Scheme I demonstrate the closest resemblance to the complete case.

Table 1. Aircraft windshields' failure times

Data 1	0.0075	0.0085	0.0138	0.0165	0.0205	0.0258	0.0290	0.0298	0.0343
	0.0388	0.0422	0.0425	0.0552	0.0578	0.0642	0.0642	0.0685	0.0702
	0.0705	0.0723	0.0725	0.0778	0.0853	0.0887	0.0898	0.0905	0.1015
	0.1035	0.1040	0.1057	0.1085	0.1163	0.1182	0.1213	0.1380	0.1458
	0.1458	0.1553	0.1783	0.1965	0.3113	0.3245			
Data 2	0.0085	0.0138	0.0165	0.0775	0.0258	0.0290	0.0298	0.0334	0.0398
	0.0425	0.0642	0.0643	0.0702	0.0705	0.0725	0.0778	0.0887	0.0898
	0.0905	0.0948	0.0968	0.1015	0.1035	0.1070	0.1127	0.1182	0.1213
	0.1425	0.1428	0.1457	0.1458	0.1508	0.1553	0.1737	0.1783	0.2057
	0.2598	0.3007	0.3458	0.4235	0.4975	0.4988			

Table 2. Test statistic and p-value for Data 1 & Data 2

Data	K-S(p-value)	A-D(p-value)	chi-squared(p-value)
Data 1	0.1927 (0.0768)	2.1384 (0.2250)	8.3344 (0.0801)
Data 2	0.1778 (0.1240)	1.3404 (0.1233)	3.8000 (0.4338)

Table 3. Artificial Adaptive-IIHP censored samples for Data 1

scheme	T	censored data for Data 1								
I	T_1	0.0075	0.0085	0.0165	0.0205	0.0258	0.0290	0.0425	0.0552	0.0578
		0.0642	0.0642	0.1057	0.1085	0.1163	0.1182	0.1213	0.1380	0.1458
		0.1553	0.1965	0.3245	0.0685	0.0702	0.0723	0.0725	0.0778	0.0887
		0.0898	0.0898	0.1015	0.1035	0.1040				
II	T_2	0.0075	0.0085	0.0165	0.0205	0.0258	0.0290	0.0298	0.0343	0.0388
		0.0422	0.0552	0.0642	0.0642	0.0685	0.0702	0.0705	0.0723	0.0725
		0.0887	0.0898	0.0905	0.1015	0.1035	0.1057	0.1163	0.1182	0.1213
		0.1380	0.1458	0.1553	0.1965	0.3245				
III	T_3	0.0138	0.0165	0.0205	0.0258	0.0290	0.0343	0.0388	0.0422	0.0425
		0.0552	0.0702	0.0705	0.0723	0.0725	0.0853	0.0887	0.0898	0.0905
		0.1015	0.1040	0.1057	0.1085	0.1163	0.1182	0.1213	0.1380	0.1458
		0.1458	0.1553	0.1783	0.3113	0.3245				

Table 4. Artificial Adaptive-IIHP censored samples for Data II

scheme	T	censored data for Data 2								
I	T_1	0.0085	0.0138	0.0165	0.0258	0.0298	0.0334	0.0398	0.0425	0.0642
		0.0643	0.0702	0.0705	0.0887	0.0898	0.0905	0.0948	0.0968	0.1015
		0.1035	0.1070	0.1127	0.1182	0.1213	0.1425	0.1458	0.1508	0.1783
		0.2057	0.3007	0.4235	0.4975	0.4988				
II	T_2	0.0085	0.0165	0.0258	0.0290	0.0298	0.0334	0.0398	0.0425	0.0642
		0.0643	0.0705	0.0725	0.0775	0.0778	0.0887	0.0887	0.0898	0.0905
		0.0948	0.1015	0.1182	0.1213	0.1425	0.1428	0.1457	0.1458	0.1508
		0.1553	0.3007	0.4235	0.4975	0.4988				
III	T_3	0.0398	0.0905	0.0085	0.1428	0.0705	0.0258	0.4235	0.1457	0.4988
		0.2057	0.0725	0.0425	0.0165	0.1508	0.0948	0.1553	0.0290	0.0702
		0.0642	0.0898	0.1015	0.0887	0.1425	0.0298	0.1737	0.0138	0.1035
		0.1182	0.1127	0.1070	0.1783	0.3007				

Table 5. Results based on failure times of aircraft windshields

				Asymptotic Inference 95% confidence		Bootstrap Inference 95% confidence	
Scheme	**Coeff**	**MLEs ($\|Bias\|$)**	**Asymptotic variance**	**Lower**	**Upper**	**Lower**	**Upper**
Complete	ρ	0.9941(0.0043)	0.000084	0.9804	1	0.9839	0.9991
	λ	0.9882(0.0084)	0.000332	0.9609	1	0.9680	0.9982
	Δ	0.9199(0.0050)	0.00385	0.8032	1	0.8677	0.9691
1	ρ	0.9925(0.0077)	0.00019	0.9732	1	0.9104	0.9999
	λ	0.9850(0.0152)	0.00075	0.9466	1	0.8288	0.9999
	Δ	0.9097(0.0088)	0.00684	0.7564	1	0.6863	0.9942
2	ρ	0.9826(0.0082)	0.0004	0.9504	1	0.9150	0.9999
	λ	0.9656(0.0157)	0.0016	0.9018	1	0.8371	0.9998
	Δ	0.8627(0.0084)	0.0067	0.7106	1	0.6945	0.9905
3	ρ	0.9872(0.0080)	0.0003	0.9603	1	0.9241	0.9998
	λ	0.9746(0.0155)	0.0012	0.9212	1	0.8539	0.9997
	Δ	0.8823(0.0086)	0.0068	0.7296	1	0.7115	0.9866

6. Concluding Remarks

Our investigation of overlap estimators using Adaptive-IIPH censored data from two Lomax distributions revealed important insights. Favorable estimators exhibited minimal bias and high accuracy when source distributions disagreed, while their accuracy decreased with increasing distribution similarity. Notably, the $\hat{\Delta}$ estimator performed optimally in cases of complete agreement or disagreement between source distributions. We recommend using the $\hat{\Delta}$ estimator for applications that require high accuracy and precision. Additionally, a higher m/n ratio resulted in improved estimator performance. This study offers valuable guidance for selecting the most suitable overlap estimator based on the similarity of distributions.

Table 6. Taylor approximation: absolute value of bias ($|Bias|$), length (L), mean squared error (MSE) & coverage probability (Cov), when $R = 0.005, \rho = 0.1407, \lambda = 0.0198$ and $\Delta = 0.0311$

3* (n,m)	Scheme	Estimate	$T_1 = X_{\left[\frac{m}{4}\right]}$ $\lvert Bias\rvert$	MSE	L	Cov	$T_2 = X_{\left[\frac{4\times m}{5}\right]}$ $\lvert Bias\rvert$	MSE	L	Cov	$T_3 = X_{[m]+2}$ $\lvert Bias\rvert$	MSE	L	Cov
(20,6)	I	ρ	0.0078542	0.0022360	0.1758549	0.9707	0.0078052	0.0022176	0.1747471	0.9647	0.0078746	0.0022641	0.1760899	0.9618
		λ	0.0001180	0.0002286	0.0509102	0.9329	0.0001187	0.0002295	0.0504984	0.9270	0.0001275	0.0002458	0.0515873	0.9217
		Δ	0.0006033	0.0003425	0.0657005	0.9435	0.0005986	0.0003410	0.0651585	0.9372	0.0006128	0.0003567	0.0661448	0.9364
	II	ρ	0.0080241	0.0023867	0.1787519	0.9412	0.0082482	0.0026189	0.1819413	0.8741	0.0078733	0.0022634	0.1760616	0.9621
		λ	0.0001563	0.0002986	0.0544833	0.8936	0.0002437	0.0004514	0.0601193	0.8366	0.0001275	0.0002457	0.0515730	0.9205
		Δ	0.0006515	0.0004045	0.0685246	0.9093	0.0007308	0.0005202	0.0725072	0.8518	0.0006126	0.0003566	0.0661289	0.9367
	III	ρ	0.0080106	0.0023719	0.1785624	0.9437	0.0081101	0.0024947	0.1797937	0.9014	0.0078735	0.0022643	0.1760595	0.9615
		λ	0.0001511	0.0002894	0.0541273	0.8991	0.0001967	0.0003704	0.0571060	0.8564	0.0001279	0.0002464	0.0515933	0.9211
		Δ	0.0006465	0.0003973	0.0682749	0.9141	0.0006886	0.0004602	0.0702694	0.8728	0.0006129	0.0003571	0.0661397	0.9350
(20,12)	I	ρ	0.0033405	0.0009200	0.1159606	0.9637	0.0033676	0.0009354	0.1168271	0.9609	0.0033644	0.0009347	0.1167056	0.9584
		λ	0.0000421	0.0000822	0.0329093	0.9406	0.0000438	0.0000855	0.0334671	0.9396	0.0000440	0.0000859	0.0334443	0.9360
		Δ	0.0002507	0.0001336	0.0430840	0.9499	0.0002553	0.0001376	0.0436474	0.9476	0.0002551	0.0001378	0.0435999	0.9432
	II	ρ	0.0033700	0.0009479	0.1167397	0.9223	0.0034459	0.0010032	0.1189505	0.8753	0.0033655	0.0009355	0.1167393	0.9557
		λ	0.0000479	0.0000932	0.0339441	0.8998	0.0000585	0.0001131	0.0359894	0.8663	0.0000442	0.0000862	0.0334763	0.9348
		Δ	0.0002597	0.0001446	0.0439014	0.9103	0.0002770	0.0001644	0.0456821	0.8723	0.0002554	0.0001382	0.0436276	0.9424
	III	ρ	0.0033566	0.0009356	0.1163822	0.9405	0.0034052	0.0009675	0.1178605	0.9221	0.0033662	0.0009360	0.1167591	0.9556
		λ	0.0000453	0.0000883	0.0334863	0.9152	0.0000503	0.0000978	0.0346566	0.9018	0.0000443	0.0000864	0.0334947	0.9347
		Δ	0.0002557	0.0001397	0.0435374	0.9256	0.0002654	0.0001499	0.0446191	0.9111	0.0002556	0.0001383	0.0436438	0.9421
(30,10)	I	ρ	0.0041341	0.0011492	0.1288188	0.9630	0.0041476	0.0011576	0.1291974	0.9600	0.0041582	0.0011641	0.1294887	0.9574
		λ	0.0000548	0.0001068	0.0368360	0.9355	0.0000560	0.0001091	0.0371085	0.9315	0.0000571	0.0001111	0.0373226	0.9321
		Δ	0.0003129	0.0001698	0.0480068	0.9437	0.0003154	0.0001723	0.0482685	0.9431	0.0003174	0.0001744	0.0484701	0.9432
	II	ρ	0.0042162	0.0012195	0.1308356	0.9030	0.0043120	0.0013160	0.1328536	0.8133	0.0041587	0.0011645	0.1295033	0.9562
		λ	0.0000697	0.0001348	0.0391727	0.8798	0.0000999	0.0001895	0.0424486	0.8193	0.0000572	0.0001112	0.0373354	0.9321
		Δ	0.0003353	0.0001971	0.0499097	0.8903	0.0003678	0.0002426	0.0522910	0.8220	0.0003175	0.0001745	0.0484816	0.9421
	III	ρ	0.0042003	0.0012062	0.1304463	0.9162	0.0042699	0.0012744	0.1319757	0.8475	0.0041596	0.0011652	0.1295246	0.9554
		λ	0.0000667	0.0001293	0.0387267	0.8903	0.0000859	0.0001644	0.0410303	0.8389	0.0000573	0.0001115	0.0373585	0.9315
		Δ	0.0003310	0.0001919	0.0495466	0.9008	0.0003536	0.0002225	0.0512662	0.8441	0.0003178	0.0001748	0.0485009	0.9419
(30,20)	I	ρ	0.0019180	0.0005261	0.0886037	0.9626	0.0019127	0.0005238	0.0883645	0.9549	0.0019163	0.0005260	0.0885168	0.9501
		λ	0.0000230	0.0000450	0.0251553	0.9483	0.0000229	0.0000449	0.0250449	0.9420	0.0000232	0.0000454	0.0251548	0.9408
		Δ	0.0001439	0.0000753	0.0330218	0.9539	0.0001432	0.0000749	0.0328913	0.9474	0.0001439	0.0000756	0.0329975	0.9439
	II	ρ	0.0019321	0.0005394	0.0891016	0.9011	0.0019428	0.0005503	0.0894549	0.8449	0.0019166	0.0005263	0.0885256	0.9494
		λ	0.0000257	0.0000502	0.0258171	0.8952	0.0000282	0.0000550	0.0263598	0.8492	0.0000232	0.0000455	0.0251675	0.9389
		Δ	0.0001482	0.0000806	0.0335448	0.8989	0.0001517	0.0000852	0.0339554	0.8479	0.0001440	0.0000757	0.0330073	0.9428
	III	ρ	0.0019254	0.0005332	0.0888633	0.9304	0.0019269	0.0005363	0.0888798	0.9030	0.0019167	0.0005264	0.0885313	0.9494
		λ	0.0000244	0.0000478	0.0255071	0.9192	0.0000253	0.0000495	0.0256661	0.8929	0.0000233	0.0000456	0.0251735	0.9386
		Δ	0.0001462	0.0000781	0.0332990	0.9259	0.0001472	0.0000797	0.0333960	0.8961	0.0001440	0.0000757	0.0330124	0.9421

Table 7. Taylor approximation: absolute value of bias ($Bias$), length (L), mean squared error (MSE) & coverage probability (Cov), when $R = 0.05, \rho = 0.4259, \lambda = 0.1814$ and $\Delta = 0.1886$

3* (n,m)	Scheme	Estimate	$T_1 = X_{\left[\frac{m}{4}\right]}$				$T_2 = X_{\left[\frac{4\times m}{5}\right]}$				$T_3 = X_{[m]+2}$			
			$\|Bias\|$	MSE	L	Cov	$\|Bias\|$	MSE	L	Cov	$\|Bias\|$	MSE	L	Cov
(20,6)	I	ρ	0.0273201	0.0163161	0.4789260	0.9619	0.0271328	0.0161493	0.4759521	0.9560	0.0273937	0.0162840	0.4773923	0.9507
		λ	0.0084135	0.0129086	0.4053003	0.9365	0.0083651	0.0127724	0.4012606	0.9288	0.0087626	0.0131856	0.4059027	0.9259
		Δ	0.0058290	0.0081485	0.3339108	0.9524	0.0057760	0.0080665	0.3311823	0.9439	0.0058787	0.0082620	0.3341260	0.9430
	II	ρ	0.0279587	0.0164845	0.4775929	0.9197	0.0286451	0.0163809	0.4699172	0.8367	0.0273888	0.0162804	0.4773311	0.9508
		λ	0.0099660	0.0143434	0.4151969	0.8923	0.0124432	0.0160490	0.4216586	0.8172	0.0087589	0.0131808	0.4058061	0.9258
		Δ	0.0061334	0.0087775	0.3395973	0.9175	0.0065319	0.0096422	0.3445424	0.8471	0.0058771	0.0082594	0.3340595	0.9429
	III	ρ	0.0279142	0.0164916	0.4781303	0.9251	0.0282322	0.0163689	0.4722302	0.8712	0.0273895	0.0162778	0.4772490	0.9503
		λ	0.0098012	0.0142197	0.4147523	0.8995	0.0112280	0.0152451	0.4171807	0.8477	0.0087702	0.0131886	0.4057888	0.9241
		Δ	0.0061068	0.0087205	0.3393136	0.9227	0.0063219	0.0091918	0.3407829	0.8748	0.0058784	0.0082626	0.3340419	0.9442
(20,12)	I	ρ	0.0116199	0.0068820	0.3187838	0.9582	0.0117264	0.0069527	0.3202740	0.9540	0.0117142	0.0069386	0.3198609	0.9512
		λ	0.0032471	0.0052171	0.2694835	0.9444	0.0033571	0.0053646	0.2728329	0.9424	0.0033634	0.0053619	0.2723700	0.9366
		Δ	0.0024657	0.0033972	0.2218541	0.9540	0.0025038	0.0034673	0.2238907	0.9520	0.0025010	0.0034638	0.2235717	0.9466
	II	ρ	0.0117456	0.0069133	0.3181839	0.9106	0.0120437	0.0070365	0.3196569	0.8539	0.0117188	0.0069406	0.3198806	0.9504
		λ	0.0035561	0.0055478	0.2728770	0.8984	0.0040841	0.0061022	0.2811110	0.8528	0.0033719	0.0053712	0.2725007	0.9352
		Δ	0.0025255	0.0035261	0.2234054	0.9112	0.0026490	0.0037706	0.2281668	0.8642	0.0025029	0.0034677	0.2236442	0.9456
	III	ρ	0.0116895	0.0068992	0.3184015	0.9324	0.0118811	0.0069995	0.3201207	0.9095	0.0117216	0.0069418	0.3198935	0.9509
		λ	0.0034208	0.0054063	0.2713832	0.9169	0.0036973	0.0057253	0.2770016	0.8989	0.0033767	0.0053766	0.2725793	0.9350
		Δ	0.0024991	0.0034698	0.2226910	0.9283	0.0025735	0.0036129	0.2260068	0.9106	0.0025040	0.0034700	0.2236873	0.9451
(30,10)	I	ρ	0.0143865	0.0085236	0.3530566	0.9563	0.0144394	0.0085549	0.3535985	0.9524	0.0144792	0.0085762	0.3539921	0.9493
		λ	0.0041484	0.0065787	0.2992959	0.9390	0.0042147	0.0066583	0.3007278	0.9362	0.0042677	0.0067144	0.3017433	0.9343
		Δ	0.0030625	0.0042403	0.2461547	0.9502	0.0030824	0.0042775	0.2470228	0.9477	0.0030974	0.0043053	0.2476755	0.9453
	II	ρ	0.0147184	0.0086244	0.3527090	0.8832	0.0150592	0.0085933	0.3488100	0.7834	0.0144814	0.0085774	0.3540051	0.9489
		λ	0.0048862	0.0073364	0.3072310	0.8737	0.0059910	0.0081839	0.3128539	0.7872	0.0042712	0.0067185	0.3018035	0.9345
		Δ	0.0032135	0.0045543	0.2502874	0.8902	0.0033980	0.0049437	0.2536116	0.8028	0.0030983	0.0043071	0.2477085	0.9462
	III	ρ	0.0146558	0.0086061	0.3527600	0.8971	0.0149177	0.0086128	0.3504874	0.8239	0.0144849	0.0085787	0.3540076	0.9477
		λ	0.0047470	0.0071992	0.3057655	0.8861	0.0055228	0.0078528	0.3107367	0.8178	0.0042783	0.0067261	0.3018908	0.9342
		Δ	0.0031849	0.0044956	0.2494919	0.9013	0.0033200	0.0047815	0.2521915	0.8323	0.0030998	0.0043102	0.2477537	0.9449
(30,20)	I	ρ	0.0066768	0.0039628	0.2439366	0.9577	0.0066565	0.0039460	0.2433648	0.9536	0.0066712	0.0039548	0.2435842	0.9470
		λ	0.0018216	0.0029779	0.2076300	0.9493	0.0018121	0.0029597	0.2067274	0.9423	0.0018301	0.0029832	0.2073364	0.9392
		Δ	0.0014223	0.0019569	0.1703983	0.9567	0.0014159	0.0019463	0.1698096	0.9491	0.0014214	0.0019568	0.1701624	0.9449
	II	ρ	0.0067381	0.0039771	0.2435491	0.8911	0.0067836	0.0039813	0.2429978	0.8299	0.0066723	0.0039550	0.2435737	0.9455
		λ	0.0019715	0.0031429	0.2098567	0.8897	0.0020998	0.0032727	0.2113613	0.8356	0.0018331	0.0029865	0.2073773	0.9377
		Δ	0.0014513	0.0020197	0.1713835	0.8956	0.0014740	0.0020695	0.1720283	0.8391	0.0014220	0.0019580	0.1701791	0.9451
	III	ρ	0.0067093	0.0039704	0.2437122	0.9212	0.0067173	0.0039642	0.2431984	0.8929	0.0066730	0.0039553	0.2435749	0.9451
		λ	0.0019017	0.0030673	0.2088217	0.9182	0.0019479	0.0031125	0.2089981	0.8878	0.0018343	0.0029880	0.2074016	0.9374
		Δ	0.0014377	0.0019905	0.1709112	0.9260	0.0014436	0.0020055	0.1708676	0.8935	0.0014222	0.0019586	0.1701912	0.9441

Table 8. Taylor approximation: absolute value of bias ($|Bias|$), length (L), mean squared error (MSE) & coverage probability (Cov), when $R = 0.2, \rho = 0.7, \lambda = 0.5$ and $\Delta = 0.42$

3* (n,m)	Scheme		$T_1 = X_{[\frac{m}{4}]}$				$T_2 = X_{[\frac{4\times m}{5}]}$				$T_3 = X_{[m]}+2$			
		Estimate	$\lvert Bias\rvert$	MSE	L	Cov	$\lvert Bias\rvert$	MSE	L	Cov	$\lvert Bias\rvert$	MSE	L	Cov
(20,6)	I	ρ	0.0562546	0.0281264	0.6126518	0.9034	0.0557921	0.0279431	0.6106681	0.9054	0.0559284	0.0277509	0.6071673	0.8943
		λ	0.0575088	0.0537097	0.8535895	0.8986	0.0566880	0.0529309	0.8459899	0.8938	0.0574538	0.0528118	0.8426389	0.8841
		Δ	0.0176547	0.0302777	0.6626746	0.9607	0.0174599	0.0299434	0.6580928	0.9522	0.0175964	0.0302156	0.6604309	0.9474
	II	ρ	0.0557064	0.0268148	0.5923238	0.8524	0.0538973	0.0248834	0.5626493	0.7633	0.0559201	0.027748	0.6071495	0.8940
		λ	0.0588714	0.0513039	0.8202386	0.8335	0.0585295	0.0472085	0.7665091	0.7345	0.0574381	0.0527998	0.8425271	0.8840
		Δ	0.0176569	0.0306192	0.6609926	0.9205	0.0166425	0.030583	0.6516884	0.8491	0.0175924	0.0302082	0.6603335	0.9472
	III	ρ	0.055832	0.0269591	0.5945166	0.8578	0.0545661	0.0256274	0.5744046	0.7984	0.0559050	0.0277328	0.6069438	0.8945
		λ	0.0588837	0.0516343	0.8243695	0.8387	0.0585119	0.0486076	0.7858314	0.7714	0.0574315	0.0527647	0.8420961	0.8835
		Δ	0.0177059	0.0306268	0.6616996	0.9255	0.0171406	0.0304478	0.6536668	0.8751	0.0175740	0.0302032	0.6602110	0.9469
(20,12)	I	ρ	0.0245060	0.0116603	0.4109311	0.9290	0.0246464	0.0116073	0.4096799	0.9204	0.0246004	0.0115860	0.4092171	0.9211
		λ	0.0250462	0.0236956	0.5894077	0.9257	0.0255019	0.0237796	0.5899660	0.9198	0.0254305	0.0236847	0.5885137	0.9169
		Δ	0.0076707	0.0131185	0.4428679	0.9562	0.0077515	0.0132614	0.4450848	0.9524	0.0077348	0.0132328	0.4444544	0.9483
	II	ρ	0.0243908	0.0112985	0.4031750	0.8737	0.0244312	0.0108853	0.3934397	0.8035	0.0246007	0.0115786	0.4090475	0.9182
		λ	0.0254828	0.0229066	0.5760317	0.8614	0.0264199	0.0221931	0.5626014	0.7920	0.0254459	0.0236706	0.5882654	0.9153
		Δ	0.0077289	0.0131720	0.4416812	0.9087	0.0078591	0.0134279	0.4438710	0.8540	0.0077376	0.0132366	0.4444810	0.9488
	III	ρ	0.0244383	0.0114477	0.4064483	0.8971	0.0245632	0.0112544	0.4019322	0.8626	0.0246011	0.0115743	0.4089504	0.9173
		λ	0.0253018	0.0232319	0.5816347	0.8873	0.0260001	0.0230153	0.5770663	0.8545	0.0254559	0.0236631	0.5881299	0.9148
		Δ	0.0077022	0.0131463	0.442106	0.9297	0.0078165	0.0133497	0.4446501	0.9083	0.0077392	0.0132390	0.4444979	0.9483
(30,10)	I	ρ	0.0301362	0.0143943	0.4530857	0.9191	0.0301874	0.0143580	0.4522663	0.9157	0.0302199	0.0143381	0.4517276	0.9124
		λ	0.0309343	0.0288916	0.6461867	0.9122	0.0311370	0.0288954	0.6457965	0.9097	0.0312608	0.0288975	0.6455462	0.9072
		Δ	0.0094673	0.0161706	0.4900649	0.9529	0.0095007	0.0162344	0.4908796	0.9489	0.0095151	0.0162800	0.4915098	0.9472
	II	ρ	0.0299532	0.0136703	0.4383383	0.8335	0.0292229	0.0127344	0.4179451	0.7324	0.0302211	0.0143349	0.4516607	0.9116
		λ	0.0319459	0.0274364	0.6222027	0.8210	0.0320912	0.0251123	0.5839731	0.7088	0.0312701	0.0288946	0.6454683	0.9067
		Δ	0.0095902	0.0163688	0.4892762	0.8824	0.0093565	0.0163694	0.4840696	0.7848	0.0095166	0.0162823	0.4915269	0.9472
	III	ρ	0.0299872	0.0137968	0.4409804	0.8501	0.0295406	0.0130943	0.4259188	0.7675	0.0302203	0.0143277	0.4515170	0.9124
		λ	0.0317714	0.0276851	0.6264288	0.8365	0.0321122	0.0260237	0.5989469	0.7538	0.0312827	0.0288826	0.6452509	0.9056
		Δ	0.0095710	0.0163303	0.4893699	0.8953	0.0094781	0.0163720	0.4861507	0.8243	0.0095185	0.0162848	0.4915263	0.9463
(30,20)	I	ρ	0.0141938	0.0066069	0.3133374	0.9382	0.0141488	0.0065994	0.3131467	0.9351	0.0141637	0.0065852	0.3127598	0.9305
		λ	0.0146660	0.0139679	0.4575892	0.9378	0.0145704	0.0138924	0.4562539	0.9295	0.0146346	0.0138839	0.4560091	0.9254
		Δ	0.0044456	0.0076289	0.3396541	0.9568	0.0044277	0.0075940	0.3387812	0.9499	0.0044382	0.0076116	0.3390946	0.9445
	II	ρ	0.0141373	0.0064204	0.3082446	0.8625	0.0140679	0.0062692	0.3039185	0.8005	0.0141623	0.0065813	0.3126556	0.9287
		λ	0.0148938	0.0135253	0.4485924	0.8553	0.0150231	0.0131398	0.4405622	0.7931	0.0146389	0.0138742	0.4558187	0.9227
		Δ	0.0044732	0.0076520	0.3388453	0.8912	0.0044842	0.0076584	0.3378817	0.8309	0.0044386	0.0076119	0.3390736	0.9439
	III	ρ	0.0141629	0.0065038	0.3105594	0.8957	0.0141133	0.0064347	0.3086631	0.8676	0.0141622	0.0065798	0.3126168	0.9286
		λ	0.0147926	0.0137238	0.4526672	0.8921	0.0148066	0.0135222	0.4486665	0.8588	0.0146414	0.0138711	0.4557565	0.9229
		Δ	0.0044600	0.0076406	0.3391822	0.9213	0.0044578	0.0076257	0.3383269	0.8915	0.0044390	0.0076123	0.3390736	0.9435

Table 9. Taylor approximation: absolute value of bias ($|Bias|$), length (L), mean squared error (MSE) & coverage probability (Cov), when $R = 0.8, \rho = 0.994, \lambda = 0.988$ and $\Delta = 0.918$

3* (n,m)	Scheme	Estimate	$T_1 = X_{[\frac{m}{4}]}$ $|Bias|$	MSE	L	Cov	$T_2 = X_{[\frac{4\times m}{5}]}$ $|Bias|$	MSE	L	Cov	$T_3 = X_{[m]}+2$ $|Bias|$	MSE	L	Cov
(20,6)	I	ρ	0.0629316	0.0130830	0.3136123	0.7513	0.0628740	0.0134689	0.3215902	0.7624	0.0620400	0.0135154	0.3232825	0.7575
		λ	0.1102586	0.0414021	0.5723596	0.7490	0.1093838	0.0421911	0.5843029	0.7587	0.1076603	0.0422024	0.5862373	0.7556
		Δ	0.0150567	0.0570747	0.9227707	0.9787	0.0146734	0.0567018	0.9195845	0.9758	0.0137353	0.0565873	0.9184364	0.9727
	II	ρ	0.0590975	0.0141734	0.3397860	0.7740	0.0547394	0.0158385	0.3770439	0.7897	0.0620410	0.0135230	0.3233951	0.7584
		λ	0.1004830	0.0430643	0.6075083	0.7646	0.0868790	0.0450793	0.6505005	0.7620	0.1076447	0.0422167	0.5863889	0.7551
		Δ	0.0107958	0.0556012	0.9091969	0.9563	0.0071651	0.0529574	0.8838795	0.8982	0.0137556	0.0565805	0.9183799	0.9730
	III	ρ	0.0594711	0.0140324	0.3368168	0.7737	0.0510893	0.0153106	0.3648882	0.7880	0.0620064	0.0135420	0.3237355	0.7591
		λ	0.1014769	0.0428360	0.6037576	0.7656	0.0923564	0.0446436	0.6383941	0.7672	0.1075356	0.0422480	0.5868227	0.7559
		Δ	0.0111250	0.0557894	0.9109468	0.9596	0.0083894	0.0539651	0.8938710	0.9242	0.0137865	0.0565591	0.9181924	0.9725
(20,12)	I	ρ	0.0275208	0.0032187	0.1628338	0.7742	0.0272212	0.0031885	0.1615351	0.7713	0.0272009	0.0032354	0.1632825	0.7730
		λ	0.0507949	0.0111295	0.3086751	0.7726	0.0502370	0.0110056	0.3060242	0.7693	0.0501144	0.0111321	0.3089275	0.7718
		Δ	0.0062111	0.0246413	0.6116880	0.9782	0.0076198	0.0246492	0.6117445	0.9769	0.0074989	0.0246109	0.6112407	0.9748
	II	ρ	0.0265605	0.0036448	0.1776207	0.7894	0.0252343	0.0039950	0.1905921	0.7984	0.0271785	0.0032424	0.1635548	0.7720
		λ	0.0480718	0.0121881	0.3319599	0.7842	0.0448373	0.0129953	0.3518632	0.7883	0.0500579	0.0111493	0.3093672	0.7710
		Δ	0.0064388	0.0242585	0.6065633	0.9529	0.0048296	0.0239060	0.6017070	0.9236	0.0074038	0.0246045	0.6111543	0.9750
	III	ρ	0.0269620	0.0034738	0.1717513	0.7827	0.0262530	0.0035898	0.1767399	0.7932	0.0271649	0.0032464	0.1636988	0.7731
		λ	0.0491913	0.0117718	0.3228442	0.7797	0.0475789	0.0120399	0.3308045	0.7886	0.0500232	0.0111591	0.3095925	0.7714
		Δ	0.0071758	0.0244160	0.6086845	0.9647	0.0060913	0.0242971	0.6070468	0.9537	0.0074158	0.0246007	0.6111032	0.9747
(30,10)	I	ρ	0.0336360	0.0045955	0.1932728	0.7726	0.0334671	0.0045954	0.1930931	0.7715	0.0333632	0.0045893	0.1931137	0.7729
		λ	0.0613047	0.0155428	0.3631435	0.7718	0.0609635	0.0155067	0.3624600	0.7701	0.0607800	0.0154833	0.3624353	0.7721
		Δ	0.0092964	0.0302925	0.6771687	0.9768	0.0090529	0.0302793	0.6769795	0.9760	0.0089251	0.0302767	0.6769169	0.9743
	II	ρ	0.0315784	0.0053300	0.2160390	0.7868	0.0293956	0.0063966	0.2484373	0.8095	0.0333520	0.0045913	0.1931229	0.7724
		λ	0.0558305	0.0171604	0.3970567	0.7800	0.0491309	0.0193890	0.4433215	0.7865	0.0607525	0.0154848	0.3623991	0.7711
		Δ	0.0064249	0.0295431	0.6679282	0.9385	0.0040293	0.0283724	0.6531454	0.8677	0.0089111	0.0302739	0.6768811	0.9742
	III	ρ	0.0319478	0.0052097	0.2123686	0.7868	0.0300684	0.0059971	0.2362642	0.8081	0.0333304	0.0045981	0.1932623	0.7717
		λ	0.0567872	0.0169152	0.3918320	0.7808	0.0516861	0.0185810	0.4263932	0.7901	0.0606956	0.0154975	0.3625554	0.7706
		Δ	0.0068682	0.0296759	0.6695963	0.9456	0.0046989	0.0288378	0.6591284	0.8959	0.0088116	0.0302666	0.6767912	0.9744
(30,20)	I	ρ	0.0157053	0.0012227	0.1023877	0.7731	0.0157279	0.0012593	0.1043478	0.7764	0.0156734	0.0012672	0.1044796	0.7738
		λ	0.0297394	0.0044292	0.1976157	0.7732	0.0297213	0.0045462	0.2011134	0.7761	0.0295968	0.0045672	0.2012317	0.7735
		Δ	0.0042661	0.0141598	0.4649246	0.9802	0.0043669	0.0141353	0.4645130	0.9777	0.0041457	0.0141286	0.4643954	0.9754
	II	ρ	0.0152284	0.0014716	0.1147454	0.7888	0.0148511	0.0016746	0.1247492	0.7995	0.0156638	0.0012727	0.1047359	0.7750
		λ	0.0283352	0.0051574	0.2186248	0.7842	0.0272243	0.0057232	0.2352612	0.7909	0.0295677	0.0045838	0.2016696	0.7754
		Δ	0.0040723	0.0139703	0.4616445	0.9459	0.0034117	0.0138066	0.4587685	0.9089	0.0040910	0.0141246	0.4643273	0.9738
	III	ρ	0.0154432	0.0013625	0.1093403	0.7831	0.0152934	0.0014704	0.1148985	0.7913	0.0156599	0.0012745	0.1048255	0.7756
		λ	0.0289600	0.0048435	0.2095048	0.7822	0.0284677	0.0051597	0.2190266	0.7883	0.0295566	0.0045893	0.2018241	0.7761
		Δ	0.0044444	0.0140551	0.4631190	0.9618	0.0041435	0.0139732	0.4617028	0.9464	0.0040641	0.0141233	0.4643049	0.9741

Table 10. Bootstrap results: absolute value of bias ($|Bias|$), length (L), mean squared error (MSE) & coverage probability (Cov), when $R = 0.005, \rho = 0.1407, \lambda = 0.0198$ and $\Delta = 0.03114$

3*	(n,m)	Scheme		$T_1 = X_{[\frac{m}{4}]}$				$T_2 = X_{\left[\frac{4\times m}{5}\right]}$				$T_3 = X_{[m]}+2$			
			Estimate	$\|Bias\|$	MSE	L	Cov	$\|Bias\|$	MSE	L	Cov	$\|Bias\|$	MSE	L	Cov
	(20,6)	I	ρ	0.0057719	0.0021693	0.1718689	0.937	0.0060681	0.0023285	0.1767447	0.938	0.0063525	0.0025073	0.1797093	0.950
			λ	0.0039358	0.0003375	0.0633661	0.937	0.0041810	0.0003834	0.0661163	0.938	0.0043293	0.0005461	0.0673430	0.950
			Δ	0.0037974	0.0004283	0.0731760	0.937	0.0040114	0.0004724	0.0758094	0.938	0.0041714	0.0005809	0.0770663	0.950
		II	ρ	0.0082319	0.0031671	0.2095153	0.945	0.0131046	0.0060223	0.2746659	0.947	0.0063667	0.0025124	0.1799279	0.949
			λ	0.0059441	0.0006063	0.0843087	0.945	0.0105502	0.0017741	0.1310053	0.947	0.0043435	0.0005450	0.0674629	0.949
			Δ	0.0055539	0.0006954	0.0931233	0.945	0.0093248	0.0016828	0.1334303	0.947	0.0041827	0.0005814	0.0771835	0.949
		III	ρ	0.0079313	0.0030251	0.2053284	0.944	0.0111045	0.0048143	0.2485872	0.939	0.0064118	0.0025226	0.1808699	0.948
			λ	0.0056887	0.0005595	0.0818932	0.944	0.0085835	0.0012233	0.1112053	0.939	0.0043854	0.0005398	0.0679887	0.948
			Δ	0.0053357	0.0006527	0.0908633	0.944	0.007732	0.0012374	0.1166626	0.939	0.0042177	0.0005804	0.0776895	0.948
	(20,12)	I	ρ	0.0027247	0.0008590	0.1149772	0.957	0.0029247	0.0009988	0.1166813	0.945	0.0030117	0.0010065	0.1200596	0.942
			λ	0.0017041	0.0000969	0.0369412	0.957	0.0018071	0.0001251	0.0378388	0.945	0.0018786	0.0001246	0.0392068	0.942
			Δ	0.0017315	0.0001432	0.0457371	0.957	0.0018366	0.0001757	0.0466124	0.945	0.0019038	0.0001766	0.0481555	0.942
		II	ρ	0.0039793	0.0013400	0.1397905	0.948	0.0054716	0.0020278	0.1650620	0.948	0.0030345	0.0010107	0.1205079	0.941
			λ	0.0025686	0.0001772	0.0474050	0.948	0.0036613	0.0003309	0.0595455	0.948	0.0018938	0.0001248	0.0393854	0.941
			Δ	0.0025604	0.0002427	0.0570842	0.948	0.0035662	0.0004092	0.0694439	0.948	0.0019192	0.0001771	0.0483513	0.941
		III	ρ	0.0034443	0.0011387	0.1296862	0.947	0.0041116	0.0015030	0.1412575	0.940	0.0030488	0.0010151	0.1207990	0.943
			λ	0.0021937	0.0001415	0.0430086	0.947	0.0026542	0.0002145	0.0483154	0.940	0.0019035	0.0001255	0.0395058	0.943
			Δ	0.0022046	0.0001997	0.0523862	0.947	0.0026382	0.0002833	0.0578976	0.940	0.0019287	0.0001779	0.0484832	0.943
	(30,10)	I	ρ	0.0034625	0.0011171	0.1290878	0.952	0.0035330	0.0011183	0.1302099	0.960	0.0036491	0.0011874	0.1328062	0.955
			λ	0.0021874	0.0001373	0.0427940	0.952	0.0022231	0.0001455	0.0431090	0.960	0.0023068	0.0001494	0.0443288	0.955
			Δ	0.0022027	0.0001949	0.0521607	0.952	0.0022414	0.0002013	0.0525420	0.960	0.0023200	0.0002098	0.0538257	0.955
		II	ρ	0.0057561	0.0020389	0.1698244	0.946	0.0091697	0.0035282	0.2225386	0.956	0.0036538	0.0011857	0.1328795	0.955
			λ	0.0038776	0.0003315	0.0619040	0.946	0.0067314	0.0007644	0.0920474	0.956	0.0023099	0.0001491	0.0443387	0.955
			Δ	0.0037657	0.0004109	0.0718598	0.946	0.0062372	0.0008297	0.1001930	0.956	0.0023234	0.0002094	0.0538427	0.955
		III	ρ	0.0053110	0.0018380	0.1623589	0.948	0.0076540	0.0028450	0.2005434	0.954	0.0036705	0.0011861	0.1331724	0.955
			λ	0.0035336	0.0002845	0.0580703	0.948	0.0054279	0.0005503	0.0785065	0.954	0.0023216	0.0001491	0.0444427	0.955
			Δ	0.0034563	0.0003610	0.0680675	0.948	0.0051285	0.0006295	0.0878331	0.954	0.0023349	0.0002094	0.0539614	0.955
	(30,20)	I	ρ	0.0017891	0.0005424	0.0877089	0.945	0.0017255	0.0004811	0.0887829	0.955	0.0017623	0.0005162	0.0898892	0.949
			λ	0.0010244	0.0000549	0.0267887	0.945	0.0010205	0.0000478	0.0272002	0.955	0.0010433	0.0000508	0.0275441	0.949
			Δ	0.0010708	0.0000854	0.0340056	0.945	0.0010590	0.0000751	0.0345083	0.955	0.0010829	0.0000801	0.0349269	0.949
		II	ρ	0.0027930	0.0008605	0.1127807	0.951	0.0035817	0.0012324	0.1318379	0.945	0.0017852	0.0005201	0.0904284	0.952
			λ	0.0016880	0.0000970	0.0360656	0.951	0.0022949	0.0001614	0.0442643	0.945	0.0010574	0.0000515	0.0277290	0.952
			Δ	0.0017317	0.0001432	0.0447414	0.951	0.0022980	0.0002225	0.0536219	0.945	0.0010977	0.0000809	0.0351495	0.952
		III	ρ	0.0023316	0.0007117	0.1019545	0.948	0.0025873	0.0008291	0.1109394	0.951	0.0017928	0.0005225	0.0906322	0.951
			λ	0.0013793	0.0000762	0.0319410	0.948	0.0016046	0.0000945	0.0355510	0.951	0.0010623	0.0000518	0.0278004	0.951
			Δ	0.0014268	0.0001153	0.0400356	0.948	0.0016357	0.0001391	0.0440948	0.951	0.0011027	0.0000813	0.0352344	0.951

Table 11. Bootstrap results: absolute value of bias ($|Bias|$), length (L), mean squared error (MSE) & coverage probability (Cov), when $R = 0.05, \rho = 0.4259, \lambda = 0.1814$ and $\Delta = 0.1886$

3* (n,m)	Scheme	Estimate	$T_1 = X_{[\frac{m}{4}]}$ $\vert Bias\vert$	MSE	L	Cov	$T_2 = X_{[\frac{4\times m}{5}]}$ $\vert Bias\vert$	MSE	L	Cov	$T_3 = X_{[m]+2}$ $\vert Bias\vert$	MSE	L	Cov
(20,6)	I	ρ	0.0094978	0.0129907	0.4235773	0.937	0.0097378	0.0136084	0.4323874	0.938	0.0104861	0.0132736	0.4396057	0.950
		λ	0.0194719	0.0138432	0.4140023	0.937	0.0202544	0.0146998	0.4247872	0.938	0.0213622	0.0147013	0.4287571	0.950
		Δ	0.0129532	0.0083320	0.3305778	0.937	0.0135303	0.0088960	0.3395280	0.938	0.0142336	0.0090488	0.3437481	0.950
	II	ρ	0.0126486	0.0171257	0.4917633	0.945	0.0191223	0.0247619	0.5792681	0.947	0.0104968	0.0132940	0.4400079	0.949
		λ	0.0259875	0.0194883	0.4928500	0.945	0.0378904	0.0321148	0.6001988	0.947	0.0214023	0.0147710	0.4293260	0.949
		Δ	0.0178690	0.0119328	0.3985085	0.945	0.0270900	0.0203964	0.5010937	0.947	0.0142657	0.0090909	0.3441934	0.949
	III	ρ	0.0122549	0.0166390	0.4843928	0.944	0.0165280	0.0218074	0.5471864	0.939	0.0105486	0.0133754	0.4417017	0.948
		λ	0.0252023	0.0187734	0.4843507	0.944	0.0333555	0.0275673	0.5609553	0.939	0.0215318	0.0149129	0.4315642	0.948
		Δ	0.0172891	0.0114536	0.3910387	0.944	0.0230782	0.0172850	0.4621257	0.939	0.0143736	0.0091768	0.3459848	0.948
(20,12)	I	ρ	0.0050470	0.0058923	0.3024659	0.957	0.0054515	0.0065995	0.3056943	0.945	0.0056654	0.0066636	0.3136868	0.942
		λ	0.0101427	0.0053418	0.2787546	0.957	0.0106353	0.0062976	0.2824768	0.945	0.0110002	0.0063983	0.2914379	0.942
		Δ	0.0063694	0.0032874	0.2224686	0.957	0.0067738	0.0038402	0.2256195	0.945	0.0069784	0.0038800	0.2323326	0.942
	II	ρ	0.0069741	0.0086450	0.3589377	0.948	0.0091878	0.0119409	0.4112578	0.948	0.0057017	0.0066987	0.3147307	0.941
		λ	0.0142612	0.0085010	0.3387726	0.948	0.0187339	0.0128143	0.3966464	0.948	0.0110855	0.0064175	0.2924645	0.941
		Δ	0.0091621	0.0051477	0.2698208	0.948	0.0123369	0.0077661	0.3171392	0.948	0.0070336	0.0038944	0.2331630	0.941
	III	ρ	0.0061811	0.0075306	0.3364148	0.947	0.0072120	0.0093945	0.3612012	0.940	0.0057249	0.0067243	0.3154023	0.943
		λ	0.0125412	0.0071878	0.3144940	0.947	0.0145194	0.0095797	0.3412721	0.940	0.0111350	0.0064462	0.2931757	0.943
		Δ	0.0079828	0.0043711	0.2505796	0.947	0.0094188	0.0057851	0.2723580	0.940	0.0070667	0.0039113	0.2337210	0.943
(30,10)	I	ρ	0.0062764	0.0074177	0.3350111	0.952	0.0064567	0.0072630	0.3380036	0.960	0.0066154	0.0078107	0.3435340	0.955
		λ	0.0125689	0.0070565	0.3136631	0.952	0.0127941	0.0071344	0.3153762	0.960	0.0131511	0.0075272	0.3222788	0.955
		Δ	0.0080038	0.0042902	0.2496368	0.952	0.0081689	0.0043136	0.2515303	0.960	0.0084151	0.0045634	0.2566152	0.955
	II	ρ	0.0095358	0.0120137	0.4209876	0.946	0.0142513	0.0177257	0.5140453	0.956	0.0066236	0.0078020	0.3437599	0.955
		λ	0.0196853	0.0128972	0.4086553	0.946	0.0288333	0.0212418	0.5156304	0.956	0.0131709	0.0075149	0.3223657	0.955
		Δ	0.0129611	0.0078073	0.3263921	0.946	0.0197366	0.0131431	0.4201960	0.956	0.0084293	0.0045565	0.2567261	0.955
	III	ρ	0.0089342	0.0110818	0.4063776	0.948	0.0121448	0.0152767	0.4782279	0.954	0.0066512	0.0078042	0.3444821	0.955
		λ	0.0183643	0.0116627	0.3919223	0.948	0.0249387	0.0176339	0.4730264	0.954	0.0132333	0.0075152	0.3229916	0.955
		Δ	0.0120308	0.0070491	0.3125371	0.948	0.0168537	0.0107832	0.3819409	0.954	0.0084704	0.0045572	0.2572593	0.955
(30,20)	I	ρ	0.0036759	0.0038644	0.2354817	0.945	0.0035030	0.0034464	0.2382119	0.955	0.0035674	0.0037094	0.2411036	0.949
		λ	0.0065141	0.0032788	0.2108528	0.945	0.0064497	0.0029083	0.2142501	0.955	0.0065897	0.0030976	0.2166533	0.949
		Δ	0.0041307	0.0020634	0.1696722	0.945	0.0040288	0.0018320	0.1720391	0.955	0.0041215	0.0019586	0.1740503	0.949
	II	ρ	0.0053210	0.0059082	0.2972158	0.951	0.0063791	0.0079782	0.3402727	0.945	0.0036095	0.0037318	0.2424851	0.952
		λ	0.0101865	0.0053265	0.2729093	0.951	0.0129548	0.0078616	0.3205403	0.945	0.0066767	0.0031270	0.2179933	0.952
		Δ	0.0064683	0.0032858	0.2180943	0.951	0.0082746	0.0047494	0.2550044	0.945	0.0041767	0.0019747	0.1751001	0.952
	III	ρ	0.0045583	0.0049761	0.2709806	0.948	0.0048714	0.0056654	0.2922562	0.951	0.0036235	0.0037476	0.2430010	0.951
		λ	0.0085267	0.0043581	0.2461344	0.948	0.0096118	0.0051763	0.2692787	0.951	0.0067057	0.00314300	0.2184961	0.951
		Δ	0.0054068	0.0027121	0.1972199	0.948	0.0060411	0.0031786	0.2148612	0.951	0.0041951	0.0019842	0.1754943	0.951

Table 12. Bootstrap results: absolute value of bias ($|Bias|$), length (L), mean squared error (MSE) & coverage probability (Cov), when $R = 0.2, \rho = 0.7, \lambda = 0.5$ and $\Delta = 0.42$

3* (n,m)	Scheme		$T_1 = X_{[\frac{m}{4}]}$				$T_2 = X_{[\frac{4\times m}{5}]}$				$T_3 = X_{[m]}+2$			
			$\lvert Bias\rvert$	MSE	L	Cov	$\lvert Bias\rvert$	MSE	L	Cov	$\lvert Bias\rvert$	MSE	L	Cov
(20,6)	I	ρ	0.0117477	0.015306	0.4520037	0.938	0.0124744	0.0159014	0.4582487	0.938	0.0123077	0.0151538	0.4711041	0.953
		λ	0.0205008	0.0305288	0.6375286	0.938	0.0214902	0.0309625	0.6459255	0.938	0.0216617	0.0292849	0.6600009	0.953
		Δ	0.0164230	0.0221759	0.5580971	0.938	0.0170966	0.0225292	0.5678643	0.938	0.017472	0.0210157	0.5780715	0.953
	II	ρ	0.0173613	0.0189994	0.5056227	0.957	0.0283296	0.0226172	0.5720952	0.978	0.0123457	0.0151013	0.4713209	0.952
		λ	0.0304653	0.0351571	0.6961822	0.957	0.0492816	0.0381734	0.7637851	0.978	0.0216993	0.0292323	0.6604776	0.952
		Δ	0.0238102	0.0257647	0.6240167	0.957	0.0393319	0.0282494	0.6979926	0.978	0.0174671	0.0210003	0.5786043	0.952
	III	ρ	0.0167805	0.0186198	0.4996590	0.951	0.0236159	0.0205869	0.5471290	0.974	0.0124580	0.0151167	0.4721985	0.951
		λ	0.0294620	0.0347652	0.6898956	0.951	0.0416742	0.0364771	0.7410927	0.974	0.0218503	0.0292777	0.6617918	0.951
		Δ	0.0229306	0.0254761	0.6168704	0.951	0.0330614	0.0271488	0.6735641	0.974	0.0175051	0.0210673	0.5800995	0.951
(20,12)	I	ρ	0.0051924	0.0081654	0.3522493	0.957	0.0056191	0.0086741	0.3547387	0.945	0.0057734	0.0086649	0.3613832	0.942
		λ	0.0150004	0.0174741	0.5136336	0.957	0.0195769	0.0186152	0.5160287	0.945	0.0121476	0.0187232	0.5263887	0.942
		Δ	0.0075834	0.0113431	0.4190711	0.957	0.0083219	0.0124018	0.4220805	0.945	0.0087366	0.0126256	0.4329965	0.942
	II	ρ	0.0077081	0.0111095	0.4034235	0.948	0.0110582	0.0140089	0.4477545	0.948	0.0058300	0.0087364	0.3624388	0.941
		λ	0.0155373	0.0230542	0.5792616	0.948	0.0195532	0.0280009	0.6321406	0.948	0.0122220	0.0188477	0.5275483	0.941
		Δ	0.0109682	0.0159236	0.4893388	0.948	0.0153288	0.0200989	0.5478277	0.948	0.0087820	0.0126957	0.4342903	0.941
	III	ρ	0.0066184	0.0099493	0.3835531	0.947	0.0080926	0.0115844	0.4054907	0.940	0.0058606	0.0087640	0.3630609	0.943
		λ	0.013589	0.0209350	0.5541380	0.947	0.0146447	0.0240177	0.5805475	0.940	0.0122687	0.0189043	0.5283669	0.943
		Δ	0.0094657	0.0141743	0.462099	0.947	0.0116148	0.0168391	0.4907971	0.940	0.0088164	0.0127407	0.4351819	0.943
(30,10)	I	ρ	0.0063220	0.0098149	0.3813168	0.952	0.0065258	0.0092576	0.3864415	0.960	0.0067700	0.0102060	0.3896721	0.955
		λ	0.0131349	0.020749	0.5524642	0.952	0.0137229	0.0198060	0.5582560	0.960	0.0121018	0.0214703	0.5625002	0.955
		Δ	0.0095183	0.0140212	0.4607232	0.952	0.0099746	0.0135339	0.4643317	0.960	0.0101267	0.0146796	0.4707832	0.955
	II	ρ	0.0111383	0.0141172	0.4539873	0.946	0.0193504	0.0183464	0.5289440	0.975	0.0067787	0.0102024	0.3901201	0.955
		λ	0.0197449	0.0280094	0.6410094	0.946	0.0337041	0.0333715	0.7223986	0.975	0.0121252	0.0214542	0.5629440	0.955
		Δ	0.0157761	0.0201779	0.5586898	0.946	0.0271128	0.0245698	0.6482635	0.975	0.0101429	0.0146647	0.4710832	0.955
	III	ρ	0.0101599	0.0133057	0.4425067	0.948	0.0159464	0.0165252	0.5018451	0.962	0.0068184	0.0102117	0.3909943	0.955
		λ	0.0187450	0.0266800	0.6278937	0.948	0.0278609	0.0312857	0.6940647	0.962	0.0121964	0.0214546	0.5639524	0.955
		Δ	0.0145597	0.0190826	0.5437723	0.948	0.0225508	0.0229354	0.6162352	0.962	0.0101831	0.0146654	0.4719769	0.955
(30,20)	I	ρ	0.0043043	0.0056780	0.2852449	0.945	0.0036912	0.0050746	0.2872892	0.955	0.0036233	0.0055372	0.2909235	0.949
		λ	0.0069500	0.0123430	0.4193755	0.945	0.0066839	0.0111278	0.4246590	0.955	0.0069573	0.0119694	0.4288250	0.949
		Δ	0.0054071	0.0075049	0.3276875	0.945	0.0051606	0.0067049	0.3318073	0.955	0.0052665	0.0072067	0.3357459	0.949
	II	ρ	0.0053870	0.0082616	0.3477700	0.951	0.0067238	0.0101398	0.3842908	0.945	0.0037242	0.0055508	0.2924221	0.952
		λ	0.0190466	0.0175064	0.5065918	0.951	0.0137173	0.0215014	0.5568929	0.945	0.0069851	0.0120183	0.4310227	0.952
		Δ	0.0079416	0.0113462	0.4116783	0.951	0.0100717	0.0148207	0.4668933	0.945	0.0053283	0.0072512	0.3376696	0.952
	III	ρ	0.0046563	0.0071328	0.3219558	0.948	0.0049234	0.0077694	0.3408670	0.951	0.0037335	0.0055702	0.2929728	0.951
		λ	0.0096027	0.0152661	0.4711108	0.948	0.0099987	0.0167407	0.4991944	0.951	0.0069020	0.0120601	0.4318050	0.951
		Δ	0.0067645	0.0096235	0.3764679	0.948	0.0072995	0.0108885	0.4052772	0.951	0.0053489	0.0072820	0.3383823	0.951

Table 13. Bootstrap results: absolute value of bias ($|Bias|$), length (L), mean squared error (MSE) & coverage probability (Cov), when $R = 0.8, \rho = 0.994, \lambda = 0.988$ and $\Delta = 0.918$

3* (n,m)	Scheme		$T_1 = X_{[\frac{m}{4}]}$				$T_2 = X_{[\frac{4\times m}{5}]}$				$T_3 = X_{[m]}+2$			
			$\|Bias\|$	MSE	L	Cov	$\|Bias\|$	MSE	L	Cov	$\|Bias\|$	MSE	L	Cov
(20,6)	I	ρ	0.0308060	0.0072922	0.2657358	0.958	0.0320181	0.0083925	0.2712829	0.953	0.0334323	0.0084632	0.2797605	0.961
		λ	0.0530997	0.0225076	0.4527855	0.958	0.0554572	0.0251130	0.4595161	0.953	0.0577409	0.0253653	0.4722637	0.961
		Δ	0.0592748	0.0314768	0.5230580	0.958	0.0632343	0.0337075	0.5272644	0.953	0.0645894	0.0341479	0.5372285	0.961
	II	ρ	0.0404262	0.0123259	0.3363409	0.954	0.0553779	0.0223033	0.4382527	0.953	0.0334900	0.0084394	0.2797889	0.960
		λ	0.0680466	0.0350072	0.5489102	0.954	0.0901309	0.0298864	0.6716731	0.953	0.0578427	0.0253067	0.4723510	0.960
		Δ	0.0715550	0.0456126	0.5919078	0.954	0.0870255	0.0690497	0.6798790	0.953	0.0646732	0.0340842	0.5374120	0.960
	III	ρ	0.0393436	0.0116744	0.3284257	0.953	0.0495402	0.0176619	0.3964957	0.957	0.0336891	0.0084849	0.2806474	0.958
		λ	0.0663976	0.0342370	0.5386186	0.953	0.0820931	0.0289835	0.6238621	0.957	0.0581917	0.0254326	0.4735348	0.958
		Δ	0.0700636	0.0439369	0.5844043	0.953	0.0815505	0.0585026	0.6455887	0.957	0.0650003	0.0342020	0.5382088	0.958
(20,12)	I	ρ	0.0173379	0.0020832	0.1513348	0.973	0.0175658	0.0024003	0.1556503	0.971	0.0182105	0.0023980	0.1595477	0.971
		λ	0.0316969	0.0071062	0.2761606	0.973	0.0319776	0.0080826	0.2829716	0.971	0.0331436	0.0080897	0.2895726	0.971
		Δ	0.0425090	0.0126440	0.3902280	0.973	0.0427863	0.0139027	0.3950117	0.971	0.0439004	0.0139004	0.4009329	0.971
	II	ρ	0.0235105	0.0039179	0.2017465	0.971	0.0297500	0.0065997	0.2532006	0.965	0.0183343	0.0024280	0.1610704	0.970
		λ	0.0418336	0.0128028	0.3572402	0.971	0.0517037	0.0205691	0.4345758	0.965	0.0333583	0.0081828	0.2921088	0.970
		Δ	0.0509934	0.0201239	0.4538805	0.971	0.0595242	0.0292889	0.5102579	0.965	0.0438742	0.0140371	0.4030005	0.970
	III	ρ	0.0209405	0.0031072	0.1814503	0.970	0.0236037	0.0043345	0.2070162	0.967	0.0184068	0.0024423	0.1617829	0.969
		λ	0.0376597	0.0103269	0.3252193	0.970	0.0418290	0.0140166	0.3650784	0.967	0.0334830	0.0082276	0.2933013	0.969
		Δ	0.0472376	0.0169756	0.4291940	0.970	0.0513068	0.0215928	0.4585986	0.967	0.0439778	0.0141028	0.4040131	0.969
(30,10)	I	ρ	0.0206886	0.0030118	0.1777876	0.971	0.0211355	0.0028474	0.1815704	0.981	0.0215274	0.0032947	0.1864027	0.975
		λ	0.0373451	0.0099889	0.3192320	0.971	0.0381079	0.0095286	0.3259365	0.981	0.0386286	0.0108660	0.3330923	0.975
		Δ	0.0479382	0.0164325	0.4243346	0.971	0.0471415	0.0160237	0.4306349	0.981	0.0485550	0.0176450	0.4353935	0.975
	II	ρ	0.0313163	0.0069072	0.2591363	0.965	0.0448860	0.0134974	0.3585237	0.965	0.0215706	0.003295	0.1868645	0.974
		λ	0.0545513	0.0213105	0.4430167	0.965	0.0750240	0.0288637	0.5778535	0.965	0.0386985	0.010871	0.3338478	0.974
		Δ	0.0622430	0.0298801	0.5166619	0.965	0.0782035	0.0487759	0.6136568	0.965	0.0485744	0.017670	0.4359764	0.974
	III	ρ	0.0294368	0.0060963	0.2441454	0.965	0.0394689	0.0103934	0.3189418	0.971	0.0216841	0.0033105	0.1876902	0.978
		λ	0.0516392	0.0190040	0.4211409	0.965	0.0669488	0.0309312	0.5265924	0.971	0.0388869	0.0109200	0.3351916	0.978
		Δ	0.0598637	0.0272054	0.4211409	0.965	0.0725808	0.0406911	0.5773566	0.971	0.0486067	0.0177410	0.4371220	0.978
(30,20)	I	ρ	0.0108596	0.0010398	0.1034992	0.973	0.0112099	0.0008851	0.1001572	0.978	0.0113659	0.0010517	0.1043244	0.976
		λ	0.0202895	0.0036895	0.1940241	0.973	0.0210902	0.0031436	0.1883089	0.978	0.0212949	0.0036870	0.1955236	0.976
		Δ	0.0311329	0.0076617	0.3178154	0.973	0.0343604	0.0069596	0.3145213	0.978	0.0332594	0.0073536	0.3206024	0.976
	II	ρ	0.0166170	0.0021866	0.1484602	0.974	0.0213490	0.0032993	0.1817661	0.969	0.0114955	0.0010519	0.1049895	0.977
		λ	0.0303940	0.0073928	0.2710037	0.974	0.0385094	0.0108234	0.3253150	0.969	0.0215325	0.0036904	0.1967327	0.977
		Δ	0.0413493	0.0128827	0.3854701	0.974	0.0484320	0.0173575	0.4287055	0.969	0.0334982	0.0073816	0.3218615	0.977
	III	ρ	0.0140299	0.0016392	0.1283727	0.973	0.0162253	0.0019463	0.1415079	0.971	0.0115421	0.0010583	0.1052925	0.977
		λ	0.0259133	0.0056444	0.2371304	0.973	0.0298870	0.0066069	0.2593496	0.971	0.0216161	0.0037126	0.1972620	0.977
		Δ	0.0371079	0.0104482	0.3568513	0.973	0.0419273	0.0116938	0.3758565	0.971	0.0336998	0.0074208	0.3223643	0.977

REFERENCES

Alslman, M., & Helu, A. (2023). Reliability Estimation for the Inverse Weibull Distribution Under Adaptive Type-II Progressive Hybrid Censoring: Comparative Study. Statistics, Optimization & Information Computing, 11(2), 216-242.

Alotaibi, R., Nassar, M., & Elshahhat, A. (2022). Computational Analysis of XLindley Parameters Using Adaptive Type-II Progressive Hybrid Censoring with Applications in Chemical Engineering. Mathematics, 10(18), 3355.

Arnold, B. C. (2001). Pareto processes. Handbook of Statistics, 19, 1-33.

Asadi, S., Panahi, H., Swarup, C., & Lone, S. A. (2022). Inference on adaptive progressive hybrid censored accelerated life test for Gompertz distribution and its evaluation for virus-containing micro droplets data. Alexandria Engineering Journal, 61(12), 10071-10084.

Balakrishnan, N., & Cramer, E. (2014). The art of progressive censoring. Statistics for industry and technology.

Bryson, M. C. (1974). Heavy-tailed distributions: properties and tests. Technometrics, 16(1), 61-68.

Chahkandi, M., & Ganjali, M. (2009). On some lifetime distributions with decreasing failure rate. Computational Statistics & Data Analysis, 53(12), 4433-4440.

Cramer, E., & Iliopoulos, G. (2010). Adaptive progressive Type-II censoring. Test, 19, 342-358.

Dixon, P. M. (1993). The bootstrap and the jackknife: Describing the precision of ecological indices. In Scheiner, S. M., & Gurevitch, J. (Eds.), Design and analysis of ecological experiments. London: Chapman & Hall.

Efron, B. (1992). Bootstrap methods: another look at the jackknife. In Breakthroughs in statistics: Methodology and distribution (pp. 569-593). New York, NY: Springer New York.

Gastwirth, J. L. (1975). Statistical measures of earnings differentials. The American Statistician, 29(1), 32-35.

Ferrario, V. F., Sforza, C., Colombo, A., & Ciusa, V. (2000). An electromyographic investigation of masticatory muscles symmetry in normo-occlusion subjects. Journal of oral rehabilitation, 27(1), 33-40.

Helu, A., & Samawi, H. (2011). On inference of overlapping coefficients in two lomax populations using different sampling methods. Journal of Statistical Theory and Practice, 5(4), 683-696.

Helu, A., & Samawi, H. (2017). On Marginal Distributions under Progressive Type II Censoring: Similarity/Dissimilarity Properties. Open Journal of Statistics, 7(4), 633-644.

Helu, A., & Samawi, H. (2021). Statistical analysis based on adaptive progressive hybrid censored data from Lomax distribution. Statistics, Optimization & Information Computing, 9(4), 789.

Kundu, D., & Joarder, A. (2006). Analysis of Type-II progressively hybrid censored data. Computational Statistics & Data Analysis, 50(10), 2509-2528.

Lomax, K. S. (1954). Business failures: Another example of the analysis of failure data. Journal of the American statistical association, 49(268), 847-852.

Lu, R. P., Smith, E. P., & Good, I. J. (1989). Multivariate measures of similarity and niche overlap. Theoretical Population Biology, 35(1), 1-21.

Marshall, A. W., & Olkin, I. (2007). Life distributions (Vol. 13). Springer, New York.

Martinez-Camblor, P. (2022). About the use of the overlap coefficient in the binary classification context. Communications in Statistics-Theory and Methods, 1-11.

Matusita, K. (1955). Decision rules, based on the distance, for problems of fit, two samples, and estimation. The Annals of Mathematical Statistics, 631-640.

Mishra, S. N., Shah, A. K., & Lefante, J. J. (1986). Overlapping coefficient: the generalized t approach. Communications in Statistics-Theory and Methods, 15(1), 123-128.

Mizuno, S., Yamaguchi, T., Fukushima, A., Matsuyama, Y., & Ohashi, Y.(2005). Overlap coefficient for assessing the similarity of pharmacokinetic data between ethnically different populations. Clinical Trials, 2(2), 174-181.

Morisita, M. (1959). Measuring of interspecific association and similarity between communities. Memoirs of the Faculty of Science of Kyushu University Series E Bilogy, 3:65-80.

Mulekar, M. S., & Mishra, S. N. (1994). Overlap coefficients of two normal densities: equal means case. Journal of the Japan Statistical Society, Japanese Issue, 24(2), 169-180.

Mulekar, M. S., & Mishra, S. N. (2000). Confidence interval estimation of overlap: equal means case. Computational statistics & data analysis, 34(2), 121-137.

Ng, H. K. T., Kundu, D., & Chan, P. S. (2009). Statistical analysis of exponential lifetimes under an adaptive type-II progressive censoring scheme. Naval Research Logistics (NRL), 56(8), 687-698.

Samawi, H. M., Yin, J., Rochani, H., & Panchal, V. (2017). Notes on the overlap measure as an alternative to the Youden index: how are they related?. Statistics in medicine, 36(26), 4230-4240.

Smith, E. P. (1982). Niche breadth, resource availability, and inference. Ecology, 63(6), 1675-1681.

Viveros, R., & Balakrishnan, N. (1994). Interval estimation of parameters of life from progressively censored data. Technometrics, 36(1), 84-91.

Wang, D., & Tian, L. (2017). Parametric methods for confidence interval estimation of overlap coefficients. Computational Statistics & Data Analysis, 106, 12-26.

Weitzman, M. S. (1970). Measures of overlap of income distributions of white and Negro families in the United States (Vol. 22). US Bureau of the Census.