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Dominant Mixed Metric Dimension of Graph

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Abstract For $k-$ordered set $W = \{s_1, s_2, \ldots, s_k\}$ of vertex set G, the representation of a vertex or edge a of G with respect to W is $r(a|W) = (d(a, s_1), d(a, s_2), \ldots, d(a, s_k))$ where a is vertex so that $d(a, s_i)$ is a distance between the vertex a and the vertices in W and $a = uv$ is edge so that $d(a, s_i) = min\{d(u, s_i), d(v, s_i)\}$. The set W is a mixed resolving set of G if $r(a|W) \neq r(b|W)$ for every pair a, b of distinct vertices or edge of G. The minimum mixed resolving set W is a mixed basis of G. If G has a mixed basis, then its cardinality is called a mixed metric dimension, denoted by $dim_m(G)$. A set W of vertices in G is a dominating set for G if every vertex of G that is not in W is adjacent to some vertex of W. The minimum cardinality of the dominant set is the domination number, denoted by $\gamma(G)$. A vertex set of some vertices in G that is both mixed resolving and dominating set is a mixed resolving dominating set. The minimum cardinality of the dominant set with mixed resolving is called the dominant mixed metric dimension, denoted by $\gamma_{mr}(G)$. In our paper, we investigate the establishment of sharp bounds of the dominant mixed metric dimension of G and determine the exact value of some family graphs.

Keywords Mixed resolving set, dominating set, mixed resolving dominating set, dominant mixed metric dimension.

AMS 2010 subject classifications 05C12

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1. Introduction

In this paper, all graphs are nontrivial and connected graph, for detail definition of graph see [1, 4, 5]. The concept of metric dimension was independently introduced by Slater [6], Harrary and Melter [7]. Slater considered the minimum resolving set of a graph as the location of the placement of a minimum number of sonar/loran detecting devices in a network. So, the position of every vertex in the network can be uniquely described in terms of its distances to the devices in the set. Applications of metric dimension problem can also be found in network and verification, robot navigation, combinatorial optimization, pharmaceutical chemistry, and strategies for the mastermind game.

We have the vertex set and edge set, respectively are $V(G)$ and $E(G)$. The distance of u and v and denoted by $d(u, v)$ is the length of a shortest path of the vertices u to v. For the set $W = \{s_1, s_2, \ldots, s_k\} \subset V(G)$. The vertex representations of the vertex x to the set W is an ordered k-tuple, $r(x|W) = (d(x, s_1), d(x, s_2), \ldots, d(x, s_k))$. The set W is called the resolving set of G if every vertices of G has different vertex representations. The resolving set having minimum cardinality is called basis and its cardinality is called metric dimension of G and denoted by $dim(G)$ [9].

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The concept of metric dimension can be extended to distinguishing pairs of edges, if the distance of a vertex s and an edge uv is defined by $d(uv, s) = min\{d(u, s), d(v, s)\}$. Then, a vertex $s \in V(G)$ distinguishes a pair of edges e, $f \in E(G)$ if $d(s, e) \neq d(s, f)$. There are graphs in which none of the smallest metric basis distinguishes all pairs of edges, so they were motivated to introduce a notion of an edge metric basis as any set $S \subset V(G)$ which distinguishes all pairs of edges, the edge metric dimension, denoted by $dim_e(G)$ as the size of a smallest edge metric basis [11].

Kelenc, et al [12] introduced the new variant of metric dimension and edge metric dimension of graphs which is called mixed metric dimension. A vertex $w \in V(G)$ distinguishes two elements (vertices or edges) $x, y \in E(G) \cup V(G)$ if $d_G(w, x) \neq d_G(w, y)$. A set S of vertices in a connected graph G is a mixed metric generator for G if every two distinct elements (vertices or edges) of G are distinguished by some vertex of S. The smallest cardinality of a mixed metric generator for G is called the mixed metric dimension and is denoted by $dim_m(G)$.

A vertex v in a graph G is said to dominate itself as well as its neighbors. A set W of vertices in G is a dominating set for G if every vertex of G is dominated by some vertex of W. The minimum cardinality of a dominating set is domination number, denoted by $\gamma(G)$. In recent years, there exists additional properties for dominating set, for example independent dominating set require a dominating to be independent, connected dominating set require a dominating set to induce a connected graphs and total dominating sets are not defined for graphs having an isolated vertex. For more detail about other conditional domination number in [8]. Brigham, et al. [10] combined the concept of metric dimension and dominating set by term resolving domination number, denoted by $\gamma_r(G)$ and got a result that $max\{dim(G), \gamma(G)\} \leq \gamma_r(G) \leq dim(G) + \gamma(G)$. A vertex set of some vertices in G that is both mixed resolving and dominating set is a mixed resolving dominating set. The minimum cardinality of mixed resolving dominating set is called dominant mixed metric dimension, denoted by $\gamma_{mr}(G)$. There are some previous results about dominant metric dimension, edge metric dimension, mixed metric dimension, dominant edge metric dimension as follows.

Theorem 1
Let G be a connected graph, then

$$max\{\gamma(G), dim(G)\} \leq \gamma_r(G) \leq min\{\gamma(G) + dim(G), n - 1\}.$$

Theorem 2
Let G be a connected graph, then

$$max\{\gamma(G), dim_e(G)\} \leq \gamma_{er}(G) \leq n - 1.$$

Theorem 3
Let G be a connected graph, then

$$max\{dim(G), dim_e(G)\} \leq dim_m(G).$$

2. Results and Discussion

In this section, the property of the dominant mixed resolving set is needed to facilitate the proof of the main result.

Lemma 1
Let G be a connected graph. If there is no dominant mixed resolving set of G with cardinality k, then any set $W \subset V(G)$ with $|W| < k$, is not a dominant mixed resolving set.

Proof
Let G be a connected graph. Suppose that there is no dominant mixed resolving set of G with cardinality k and there exists a dominant mixed resolving set $T \subset V(G)$ with $|T| < k$ so that for every $u, v \in V(G)$ we have $r(u|T) \neq r(v|T)$ and T is a dominating set of G. Moreover, there exists a subset $U \subset V(G)$ such that $|T \cup U| = k$. Since T is a mixed resolving set and a dominating set of G, one can easily see that $T \cup U$ is a mixed resolving set and a dominating set of G. So that, $T \cup U$ is a dominant mixed resolving set of G which is a contradiction. Thus the result follows and the proof is completed. □

We present basic result for the dominant mixed metric dimension of graphs. We also give the bounds for $\gamma_{mr}(G)$ as follows.

Lemma 2
Let G be a connected graph of order n, then

$$max\{\gamma(G), dim_m(G)\} \leq \gamma_{mr}(G) \leq min\{\gamma(G) + dim_m(G), n\}.$$

Proof
Let G be a connected graph of order n. Since the dominant mixed metric dimension of a graph G is greater than its domination number and its mixed metric dimension, then $max\{\gamma(G), dim_m(G)\} \leq \gamma_{mr}(G)$. Furthermore, since the mixed resolving set and the dominating set of a graph are possible to not intersect, and since a subset of $V(G)$ which consists of n vertices in graph G always becomes mixed resolving set and dominating set of graph G, then $\gamma_{mr}(G) \leq min\{\gamma(G) + dim_m(G), n\}$. □

The following lemma show the property of mixed resolving set of G.

Lemma 3
Let G be a connected graph and W be a mixed resolving set. If $W \subset V(G)$, then for every $u_i, u_j \in W$ with $i \neq j$, $r(u_i|W) \neq r(u_j|W)$.

Proof
Let G be a connected graph and $W = \{u_1, u_2, u_3, \dots, u_k\} \subset V(G)$. Since for every $u_i, u_j \in W$ with $i \neq j$ causes $d(u_i, u_i) = 0$ and $d(u_i, u_j) \neq 0$, then there exists 0 on i-th element of $r(u_i|W)$ for every $u_i \in W$. It is consequently that $r(u_i|W) \neq r(u_j|W)$. □

Furthermore, the dominant mixed metric dimension of path, cycle, star, wheel, friendship, and complete graph are presented in the Theorems 4, 5, 6, 7, 8, and 9, respectively.

Lemma 4
[12] Let the path graph P_n with $n \geq 2$. Then the cardinally of mixed resolving set of P_n, is 2.

Theorem 4
Let P_n be path graph for $n \geq 5$, then $\gamma_{mr}(P_n) = \lceil \frac{n-4}{3} \rceil + 2$.

Proof
Choose $W = \{u_1, u_n, u_i; i \equiv 1(mod3)\} \subset V(P_n)$ for $n \geq 5$. Based on Lemma 4, $\{u_1, u_n\} \subset W$ so that W is mixed resolving set. We know that $u_2 \sim u_1$, $u_{n-1} \sim u_n$, $u_{i-1} \sim u_i$, and $u_i \sim u_{i+1}$, it implies W is a dominant mixed resolving set. Now, assume that $\gamma_{mr}(P_n) < \lceil \frac{n-4}{3} \rceil + 2$. Take $|S| = \lceil \frac{n-4}{3} \rceil + 1$. There are three conditions for the resolver or dominator vertex as follows.

1. Choose $S = W - \{u_1\}$, then we have the same representation for vertex and edge, $r(u_1u_2|S) = r(u_2|S)$. It is contradiction.
2. Choose $S = W - \{u_n\}$, then we have the same representation for vertex and edge, $r(u_nu_{n-1}|S) = r(u_{n-1}|S)$. It is contradiction.
3. Choose $S = W - \{u_{3k+1}\}$ where u_{3k+1} is a vertex between u_i with $i \equiv 1(mod3)$ and $1 \leq k \leq \frac{n-2}{3}$, then we have condition that there is a vertex not adjacent to vertex in S namely $u_{3k} \nsim u_{3k+1}$. It is contradiction.

It is clear that S is not dominant mixed resolving set. Based on Lemma 1 that $|W| = \lceil \frac{n-4}{3} \rceil + 2$. Thus, $\gamma_{mr}(P_n) = \lceil \frac{n-4}{3} \rceil + 2$. □

Lemma 5
[12] Let the cycle graph C_n with $n \geq 7$. Then the cardinally of mixed resolving set of C_n is 3.

Theorem 5
Let C_n be cycle graph for $n \geq 7$, then $\gamma_{mr}(C_n) = \gamma(C_n)$.

Proof
Let $W = \{u_i; i \equiv 1(mod 3)\}$ be a subset of $V(C_n)$ for $n \geq 7$. By Lemma 5, we have $\{u_1, u_5, u_6\} \subset W$ so that W is mixed resolving set. We know that $u_{i-1} \sim u_i$, and $u_i \sim u_{i+1}$ such that W is a dominant mixed resolving set. Furthermore, assume that $\gamma_{mr}(C_n) < \lceil \frac{n}{3} \rceil$. Take $|S| = \lceil \frac{n}{3} \rceil - 1$. Choose $S = W - \{u_{3k+1}\}$ where u_{3k+1} is a vertex between u_i with $i \equiv 1(mod3)$ and $1 \leq k \leq \lceil \frac{n}{3} \rceil$, then we have condition that there is a vertex not adjacent to vertex in S. It is clear that S is not dominant mixed resolving set. Based on Lemma 1, $|W| = \gamma(C_n)$. Thus, $\gamma_{mr}(C_n) = \gamma(C_n)$. □

Star graph, denoted by S_n, has vertex set $V(S_n) = \{v, v_i; 1 \leq i \leq n\}$ and edge set $E(S_n) = \{vv_i; 1 \leq i \leq n\}$ [2].

Theorem 6
Let S_n be star graph for $n \geq 7$, then $\gamma_{mr}(S_n) = n$.

Proof
We choose $W = \{v_i; 1 \leq i \leq n\} \subset V(S_n)$. Every vertices in W has distinct representation as follows:

$$r(x|W) = \begin{cases} (\underbrace{1,1,1,\ldots,1}_{n}), & \text{if } x = v \\ (\underbrace{2,2,2,\ldots,2}_{i-1}, 0, \underbrace{2,2,2,\ldots,2}_{n-i}), & \text{if } x = v_i \end{cases}$$
$$r(vv_i|W) = (\underbrace{1,1,1,\ldots,1}_{i-1}, 0, \underbrace{1,1,1,\ldots,1}_{n-i})$$

Based on the vertex or edge representation above, W is mixed resolving set. We know that $v \sim v_i$ such that W is a dominant mixed resolving set. Assume that $\gamma_{mr}(S_n) < n$. Take $|S| = n - 1$. Choose $S = W - \{v_k\}$ where $1 \leq k \leq n$, then we have condition that there is a vertex not adjacent to vertex in S namely $v_k \nsim y$ with $y \in S$. It is clear that S does not dominant mixed resolving set. Based on Lemma 1, we get $|W| = n$. Therefore, $\gamma_{mr}(S_n) = n$. □

Wheel graph, denoted by W_n, has vertex set $V(W_n) = \{v, v_i; 1 \leq i \leq n\}$ and edge set $E(W_n) = \{vv_i; 1 \leq i \leq n\} \cup \{v_1v_n, v_iv_{i+1}; 1 \leq i \leq n+1\}$.

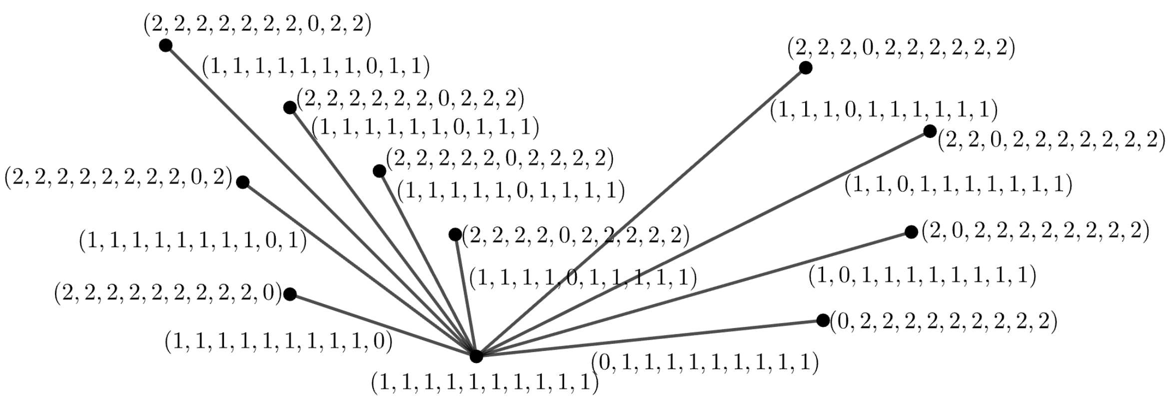

Figure 1. Dominant Mixed Metric Dimension of S_{10} is 10

Theorem 7
Let W_n be wheel graph for $n \geq 4$, then $\gamma_{mr}(W_n) = n$.

Proof
Take $W = \{v_i; 1 \leq i \leq n\} \subset V(W_n)$ for $n \geq 4$. Every vertices in W has distinct representation as follows:

$$r(x|W) = \begin{cases} (\underbrace{1,1,1,\dots,1}_{n}), & \text{if } x = v \\ (\underbrace{2,2,2,\dots,2}_{i-2},1,0,1,\underbrace{2,2,2,\dots,2}_{n-i-1}), & \text{if } x = v_i, 2 \leq i \leq n-1 \\ (0,1,\underbrace{2,2,2,\dots,2}_{n-3},1), & \text{if } x = v_1 \\ (1,\underbrace{2,2,2,\dots,2}_{n-3},1,0), & \text{if } x = v_n \end{cases}$$

$$r(e|W) = \begin{cases} \underbrace{1,1,1,\dots,1}_{i-},0,\underbrace{1,1,1,\dots,1}_{n-i}), & \text{if } e = vv_i \\ (\underbrace{2,2,2,\dots,2}_{i-2},1,0,0,1,\underbrace{2,2,2,\dots,2}_{n-i-2}), & \text{if } e = v_iv_{i+1}, 2 \leq i \leq n-2 \\ (0,0,1,\underbrace{2,2,2,\dots,2}_{n-4},1), & \text{if } e = v_1v_2 \\ (0,1,\underbrace{2,2,2,\dots,2}_{n-4},1,0), & \text{if } e = v_1v_n \\ (1,\underbrace{2,2,2,\dots,2}_{n-4},1,0,0), & \text{if } e = v_{n-1}v_n \end{cases}$$

It is clear that, W is mixed resolving set.
We know that $v \sim v_i$ such that W is a dominant mixed resolving set. Next, by Lemma 2, we obtained

$$\gamma_{mr}(W_n) \geq max\{dim_m(W_n), \gamma(W_n)\} = max\{n, 1\} = n.$$

Thus, $\gamma_{mr}(W_n) = n$. □

We give results of dominant mixed metric dimension of graph with $n-1$ and n for n is the number of vertices in G which attain the upper bounds. Their graph attained the upper bound namely friendship graphs and complete graph.

Friendship graph has vertex set $V(Fr_n) = \{u, u_i, v_i; 1 \leq i \leq n\}$ and edge set $E(Fr_n) = \{uu_i, uv_i, u_iv_i; 1 \leq i \leq n\}$ [3]. While, for complete graph K_n with vertex set $V(K_n) = \{u_i; 1 \leq i \leq n\}$ and $E(K_n) = \{u_iu_{i+k}; 1 \leq i \leq n, 1 \leq k \leq n-i\}$ [2]. Theorems 8 and 9 show the dominant mixed metric dimension of friendship graphs and complete graph, respectively.

Theorem 8
Let Fr_n be a friendship graph with $n \geq 2$, then $\gamma_{mr}(Fr_n) = 2n$.

Proof
Let $W = \{u_i, v_i; 1 \leq i \leq n\}$ be a subset of Fr_n with $n \geq 2$. Every vertices in W has distinct representation as follows

$$r(x|W) = \begin{cases} (\underbrace{1,1,1,\dots,1}_{2n-1}), & \text{if } x = u \\ (\underbrace{2,2,2,\dots,2}_{i-1},0,\underbrace{2,2,2,\dots,2}_{n-i},\underbrace{2,2,2,\dots,2}_{i-1},1,\underbrace{2,2,2,\dots,2}_{n-i}), & \text{if } x = u_i, 1 \le i \le n \\ (\underbrace{2,2,2,\dots,2}_{i-1},1,\underbrace{2,2,2,\dots,2}_{n-i},\underbrace{2,2,2,\dots,2}_{i-1},0,\underbrace{2,2,2,\dots,2}_{n-i}), & \text{if } x = v_i, 1 \le i \le n \end{cases}$$

$$r(e|W) = \begin{cases} (\underbrace{2,2,\dots,2}_{i-1},0,\underbrace{2,2,\dots,2}_{n-i},\underbrace{2,2,\dots,2}_{i-1},0,\underbrace{2,2,\dots,2}_{n-i}), & \text{if } e = u_iv_i, 1 \le i \le n \\ (\underbrace{1,1,\dots,1}_{i-1},0,\underbrace{1,1,\dots,1}_{n-i},\underbrace{1,1,\dots,1}_{i-1},1,\underbrace{1,1,\dots,1}_{n-i}), & \text{if } e = uu_i, 1 \le i \le n \\ (\underbrace{1,1,\dots,1}_{i-1},1,\underbrace{1,1,\dots,1}_{n-i},\underbrace{1,1,\dots,1}_{i-1},0,\underbrace{1,1,\dots,1}_{n-i}), & \text{if } e = uv_i, 1 \le i \le n \end{cases}$$

The vertex representations above show that W is mixed resolving set. We show that W is also dominating set since for $u \notin W$ adjacent to $u_i, v_i \in W$ such that W is dominant mixed resolving set with $|W| = 2n$ and $\gamma_{mr}(Fr_n) \le 2n$.
Next, assume $\gamma_{mr}(Fr_n) < 2n$. Taking $|S| = 2n-1$, then there are two cases: $v_k \notin S$ or $v_l \notin S$.

1. for $v_k \notin S$, then we get the same representation for vertex and edge, $r(u|S) = r(uv_k|S) = (\underbrace{1,1,1,\dots,1}_{2n-1})$. It is contradiction.
2. for $v_l \notin S$, then we get the same representation for vertex and edge, $r(u|S) = r(uv_l|S) = (\underbrace{1,1,1,\dots,1}_{2n-1})$. It is contradiction.

Since S is not dominant mixed resolving set of Fr_n. Thus, $\gamma_{mr}(Fr_n) = 2n$. □

Theorem 9
Let K_n be a complete graph with $n \ge 3$, then $\gamma_{mr}(K_n) = n$.

Proof
Take $W = \{u_i; 1 \le i \le n\}$ as a subset of K_n with $n \ge 3$. Every vertices in W has distinct representation as follows

$$r(u_i|W) = (\underbrace{1,1,1,\dots,1}_{i-1},0,\underbrace{1,1,1,\dots,1}_{n-i}),$$
$$r(u_iu_{i+k}|W) = (\underbrace{1,1,1,\dots,1}_{i-1},0,\underbrace{1,1,1,\dots,1}_{k-1},0,\underbrace{1,1,1,\dots,1}_{n-i-k}).$$

It is shows that W is mixed resolving set. Clearly the set W is obvious dominant mixed resolving set with $|W| = n$ and $\gamma_{mr}(K_n) \le n$.
Now, suppose that $\gamma_{mr}(K_n) < n$ and $|S| = n-1$, for $u_k \notin S$. Since the representation for vertex and edge are $r(u_k|S) = r(u_ku_l|S) = (\underbrace{1,1,1,\dots,1}_{l-1},0,\underbrace{1,1,1,\dots,1}_{n-l-1})$, then S is not dominant mixed resolving set of K_n. Thus, $\gamma_{mr}(K_n) = n$. □

Next, the dominant mixed metric dimension of graphs resulting on corona product is provided.

Definition 1
[13] Let G and H be two connected graphs. The corona product between G and H, denoted by $G \odot H$, is defined as the graph created by taking one duplicate of G and $|V(G)|$ copies of H and connecting the j-th vertex of G to every vertices in the j-th duplicate of H.

Proposition 1
[14] Let $G \odot H$ be a corona product of G and H, then

$$dim_m(G \odot H) = |V(G)||V(H)|.$$

The following theorem shows the dominant mixed metric dimension of corona product.

Theorem 10
Let $G \odot H$ be a corona product of G and H, Then

$$\gamma_{mr}(G \odot H) = |V(G)||V(H).$$

Proof
We choose $W = \{u_{i,j}; 1 \le i \le n, 1 \le j \le m\}$ as a subset of $V(G \odot H)$. Every vertices in W can be present as the following distinct representations:

1. For every vertices $u_{i,j} \in W$. Based on Lemma 3, $r(u_{k,j}|W) \neq r(u_{l,j}|W)$.
2. For $d_{G \odot H}(u_k, u_{k,j}) = 1$ and
$$d_{G \odot H}(u_k, u_{l,j}) = d_G(u_k, u_l) + d_{G \odot H}(u_l, u_{l,j}) = d_G(u_k, u_l) + 1.$$
We have $r(u_k|W) \neq r(u_l|W)$.
3. For $d_{G \odot H}(u_k u_{k+1}, u_{k,j}) = 1$ and
$$\begin{aligned} d_{G \odot H}(u_k u_{k+1}, u_{l,j}) &= min\{d(u_k, u_l), d(u_{k+1}, u_l)\} + d_{G \odot H}(u_l, u_{l,j}) \\ &= min\{d(u_k, u_l), d(u_{k+1}, u_l)\} + 1. \end{aligned}$$
We have $r(u_k u_{k+1}|W) \neq r(u_l u_{k+1}|W)$.
4. For $d_{G \odot H}(u_k u_{k,s}, u_{k,j}) = 1$ with $s \neq j$, $d_{G \odot H}(u_k u_{k,s}, u_{k,s}) = 0$ and
$$d_{G \odot H}(u_k u_{k,j}, u_{l,j}) = d_{G \odot H}(u_k, u_l) + d_{G \odot H}(u_L, u_{l,j}) = d_{G \odot H}(u_k, u_l) + 1,$$
then there exists 0 on $i + j$-th element of $r(u_k u_{k,j}|W)$ for every $u_{k,j} \in W$. It is consequently that $r(u_k u_{k,j}|W) \neq r(u_k u_{k,s}|W)$.
5. For $d_H(u_k u_{k+1}, u_{k,j}) = 1$ and
$$\begin{aligned} d_{G \odot H}(u_k u_{k+1}, u_{l,j}) &= min\{d(u_k, u_l), d(u_{k+1}, u_l)\} + d_{G \odot H}(u_l, u_{l,j}) \\ &= min\{d(u_k, u_l), d(u_{k+1}, u_l)\} + 1. \end{aligned}$$
We get $r(u_k u_{k+1}|W) \neq r(u_l u_{k+1}|W)$.
6. For
$$d_H(u_{k,s} u_{k,s+1}, u_{k,j}) = \begin{cases} 2, & \text{if } u_{k,j} \nsim u_{k,s+1}, u_{k,j} \nsim u_{k,s} \\ 1, & \text{if } u_{k,j} \sim u_{k,s+1} \text{ or } u_{k,j} \nsim u_{k,s} \end{cases}$$
and
$$\begin{aligned} d_{G \odot H}(u_{k,s} u_{k,s+1}, u_{l,j}) &= d_{G \odot H}(u_{k,s}, u_k) + d_{G \odot H}(u_k, u_{l,j}) \\ &= d_{G \odot H}(u_k, u_{l,j}) + 1, \end{aligned}$$
then there exists 0 on $i + j$-th element of $r(u_{k,s} u_{k,j}|W)$ for every $u_{k,j} \in W$. Such that, we obtained $r(u_{k,s} u_{k,j}|W) \neq r(u_{k,r} u_{k,j}|W)$.

The vertex or edge representations above show that W is mixed resolving set.
We know that $v \sim v_i$ such that W is a dominant mixed resolving set.
Furthermore, Based on Lemma 2, we have

$$\begin{aligned} \gamma_{mr}(G \odot H) \ge max\{dim_m(G \odot H), \gamma(G \odot H)\} &= max\{|V(G)||V(H)|, |V(G)|\} \\ &= |V(G)||V(H)|. \end{aligned}$$

Thus, $\gamma_{mr}(G \odot H) = |V(G)||V(H)|$. □

3. Conclusion

We have characterized the bounds of dominant mixed metric of graphs which depend on domination number or mixed metric dimension of graph. There are some open problem for this topic as follows.

Open Problem 1
Determine all graph with the characterization $\gamma_{mr}(G) = 2$?

Open Problem 2
Determine all graph with the characterization $\gamma_{mr}(G) = dim_m(G)$ or $\gamma_{mr}(G) = \gamma(G)$?

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