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Optimality Conditions for (h, φ)-subdifferentiable Multiobjective Programming Problems with G-type I Functions

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Abstract In this paper, using generalized algebraic operations introduced by Ben-Tal [7], we introduce new classes of (h, φ)-subdifferentiable functions, called (h, φ)-G-type I functions and generalized (h, φ)-G-type I functions. Then, we consider a class of nonconvex (h, φ)-subdifferentiable multiobjective programming problems with locally Lipschitz functions in which the functions involved belong to aforesaid classes of (h, φ)-subdifferentiable nonconvex functions. For such (h, φ)-subdifferentiable vector optimization problems, we prove the sufficient optimality conditions for a feasible solution to be its (weak) Pareto solution. Further, we define a vector dual problem in the sense of Mond-Weir for the considered (h, φ)-subdifferentiable multiobjective programming problem and we prove several duality theorems for the aforesaid (h, φ)-subdifferentiable vector optimization problems also under (h, φ)-G-type I hypotheses.

Keywords (h, φ)-subdifferentiable multiobjective programming problem, optimality conditions, weak Pareto solution, (h, φ)-subdifferentiable vector Mond-Weir dual problem, (h, φ)-G-type I functions

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1. Introduction

The concept of convexity is of utmost significance in multi-objective programming since it allows proving optimality conditions and duality results for various classes of vector optimization problems in which the functions involved are convex. Nevertheless, there are many real-world problems in various areas of human activity which cannot be modeled by using convex vector optimization problems since the functions constituting them are not convex. Therefore, many concepts of generalized convexity have been derived to optimization theory to overcome limitations in using convexity notion to such vector optimization problems in which the functions involved are nonconvex (see, for example [3], [5], [7], [17], [18], [19], [24], [26], [29], [31], [32], [33], [34], and others).

In economics, we often seek to maximize efficiency in economic systems, which involves solving optimization problems with ratio objective functions. The benefit of second-order duality is preferred over first-order duality because it provides closer bounds. Here are some recent advancements and practical applications in the field of convexity generalizations, particularly focusing on higher-order and second-order duality concepts within nondifferentiable symmetric mathematical programming problems; [9, 10, 11, 13, 15, 27]

In 1977, Ben-Tal [7] introduced a set of generalized addition and multiplication operations. Avriel [5] made a remarkable contribution to the theory of convex functions by introducing a more general class of functions called

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(h, φ)-convex functions, using the concept of Ben-Tal's [7] generalized addition and multiplication operations. Moreover, he defined the concept of (h, φ)-differentiability and then the definition of (h, φ)-differentiable (h, φ)-convex functions.

Recently, Xu and Liu [29] have made a significant advancement in the field of generalized (h, φ)-convex functions by presenting a generalized necessary optimality condition based on the Kuhn-Tucker theorem. This contribution pertains to a class of mathematical programming problems and provides a fresh perspective on the theory of generalized (h, φ)-convex functions. This class of mathematical programming problems includes both single-objective and multi-objective optimization problems and assumes a general form of (h, φ)-differentiability. The research done by Xu and Liu [29] showcases the ongoing pursuit of understanding generalized (h, φ)-convex functions and their role in optimization theory. In [34], Zhang also made a valuable contribution to this area by exploring the sufficiency and duality of solutions in the context of a particular optimization problem which is a nonsmooth (h, φ)-semi-infinite optimization problem. The aforesaid works demonstrate the significance of continued further study in this field. Very recently, using generalized algebraic operations introduced by Ben-Tal [7], Antczak et al. [4] defined classes of (h, φ)-(b, F, ρ)-convex functions and established optimality conditions and duality results for (h, φ)-nondifferentiable multiobjective programming problems with aforesaid generalized convex functions.

In 1981, Hanson [18] introduced into optimization theory the notion of invexity for differentiable scalar optimization problems, which was considered a significant contribution to optimization theory as it presented an important extension of differentiable convex functions. Hanson utilized this concept to prove the sufficiency of Kuhn-Tucker optimality conditions and weak duality theorem in the Wolfe sense for such nonconvex extremum problems. Later, Hanson and Mond [19] generalized invexity notion derived for smooth optimization problems with inequality constraints to the more general concept which can be used for a larger class for nonconvex extremum problems. Namely, they proposed the notion of type I as well as type II functions, which is recognized as a significant breakthrough in this area of study.

In [6], Avriel et al. introduced the concept of G-convexity, in which G is a continuous and strictly increasing real-valued function. Based on the aforesaid Avriel's concept of generalized convexity, Antczak [1] generalized the aforesaid definition by introducing the notion of G-invexity for scalar optimization problems. Later, Antczak [2] extended the concept of G-invexity to the differentiable vectorial case. Then Antczak [2] used the aforesaid concept of generalized convexity in proving optimality conditions for a new class of differentiable vector optimization problems with G-invex functions (not necessarily with respect to the same G) and also he studied duality results [3] for such nonconvex multiobjective optimization problems. Later the concept of G-invexity has been defined also for nondifferentiable vector optimization problems with locally Lipschitz functions by Kang et al. [20]. Recently, in 2019, Dubey et al. [14] introduced a generalization of convexity termed (F, G_f)-convexity. This novel concept finds application in non-linear multiobjective programming problems, particularly when the functions involved in vector optimization problems are differentiable. The same year, Dubey et al. [12] introduced various definitions of (C, G_f)-invexity/(F, G_f)-invexity and constructed nontrivial numerical examples to illustrate the existence of such functions.

In this paper, we use the generalized algebraic operations introduced by Ben-Tal [7] to define the concept of (h, φ)-subdifferentiable G-type I functions and several concepts of (h, φ)-subdifferentiable generalized G-type I functions. However, the main aim of this paper is to establish optimality conditions and duality results for a new class of (h, φ)-nondifferentiable multiobjective programming problems with locally Lipschitz functions. Namely, we prove sufficient optimality conditions for a feasible solution to be a weak Pareto solution (a Pareto solution) in the considered (h, φ)-nondifferentiable vector optimization problems in which the functions constituting them are (h, φ)-subdifferentiable G-type I functions and/or (h, φ)-subdifferentiable generalized G-type I functions. The aforesaid sufficient optimality conditions are illustrated by an example of (h, φ)-nondifferentiable multiobjective programming problem with (h, φ)-subdifferentiable G-type I functions. Further, for the considered

(h, φ)-nondifferentiable multiobjective programming problem, we formulate its (h, φ)-nondifferentiable vector Mond-Weir dual problem and we establish several duality theorems also under assumptions that the functions involved are (h, φ)-nondifferentiable type I functions and/or (h, φ)-nondifferentiable generalized type I functions. The results established in the paper extend the concept of differentiable G-invexity introduced by Antczak [1, 2], nondifferentiable G-invexity defined by Kang et al. [20] and the concept of type I functions defined for (h, φ)-differentiable multiobjective programming problems by Yu and Liu [32]. Thus, the optimality and duality results established in the paper generalize similar results proved in the optimization literature by using generalized Ben-Tal's algebraic operations to a new class of nonconvex (h, φ)-subdifferentiable vector optimization problems.

2. Preliminaries

In this section, we provide some definitions and results that we shall use in the sequel. Throughout this paper, $\mathbb{R}$ denotes the set of all real numbers and $\mathbb{R}^n$ the set of n-dimensional real vectors. Then, for any vectors $x = (x_1, \ldots, x_n)^T, y = (y_1, \ldots, y_n)^T \in \mathbb{R}^n$, we define:

$$\begin{array}{llll} (i) x = y & \text{if and only if} & x_i = y_i & \text{for all} \quad i = 1, \ldots, n \\ (ii) x < y & \text{if and only if} & x_i < y_i & \text{for all} \quad i = 1, \ldots, n \\ (iii) x \leqq y & \text{if and only if} & x_i \leqq y_i & \text{for all} \quad i = 1, \ldots, n \\ (iv) x \leq y & \text{if and only if} & x \leqq y & \text{and} \quad x \neq y. \end{array}$$

Now, we recall the generalized notions of addition as well as multiplication, which were first introduced by Ben-Tal [7].

1) Assume that h is a continuous function that maps vectors in $\mathbb{R}^n$ to vector values which has an inverse function h^{-1}. Thus, for $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$, we define h-vector addition by

$$x \oplus y = h^{-1}(h(x) + h(y)), \tag{1}$$

where $\alpha \in \mathbb{R}$, $x \in \mathbb{R}^n$ and, moreover, h-scalar multiplication by

$$\alpha \otimes x = h^{-1}(\alpha h(x)). \tag{2}$$

2) Let φ be a continuous function with real values defined on $\mathbb{R}$ and let φ^{-1} denote its inverse function. Then the φ-scalar addition is defined by the relation:

$$\alpha [+] \beta = \varphi^{-1}(\varphi(\alpha) + \varphi(\beta)), \tag{3}$$

is satisfied for any $\alpha, \beta \in \mathbb{R}$. Further, the φ-scalar multiplication is defined by the relation which

$$\beta [\cdot] \alpha = \varphi^{-1}(\beta \varphi(\alpha)) \tag{4}$$

is satisfied for any $\alpha, \beta \in \mathbb{R}$.

3) For the given vectors $x \in \mathbb{R}^n$, $y \in \mathbb{R}^n$, the (h, φ)-inner product is defined

$$(x^T y)_{(h,\varphi)} = \varphi^{-1}(h(x)^T h(y)). \tag{5}$$

Putting $x^i \in \mathbb{R}^n$, where $i = 1, ..., n$, one writes

$$\oplus_{i=1}^n x^i = x^1 \oplus x^2 \oplus ... \oplus x^n, \tag{6}$$

and also, for $\alpha_i \in \mathbb{R}$, $i = 1, ..., n$, one writes

$$\left[\sum_{i=1}^{n} \alpha_i\right] = \alpha_1[+]\alpha_2[+]...[+]\alpha_n. \tag{7}$$

Moreover, for any $\alpha, \beta \in \mathbb{R}$ and $x, y \in \mathbb{R}^n$, the operations of subtraction are given by;

$$x\Theta y = x \oplus ((-1) \otimes y), \tag{8}$$

and

$$\alpha[-]\beta = \alpha[+]((-1)[\cdot]\beta), \tag{9}$$

Remark 2.1. *Note that the generalized algebraic operations defined above the following properties:*
a) It is worth noting $\psi[\cdot]\chi$ may not be equal to $\chi[\cdot]\psi$, where $\chi, \psi \in \mathbb{R}$.
b) 1 $\otimes x = x$ for any $x \in R^n$ and $1[\cdot]\omega = \omega$ for any $\omega \in \mathbb{R}$.
c) $\varphi(\chi[\cdot]\psi) = \chi\varphi(\psi)$ for any $\chi, \psi \in \mathbb{R}$.
d) $h(\omega \otimes x) = \omega h(x)$ for any $\omega \in \mathbb{R}$ and $x \in \mathbb{R}^n$.
e) $\chi[-]\psi = \varphi^{-1}(\varphi(\chi) - \varphi(\psi))$ for any $\chi, \psi \in \mathbb{R}$.

Now, we give the lemma which, for the generalized algebraic operations, presents their further properties introduced by Ben-Tal [7].

Lemma 2.1. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a (h, φ)-nondifferentiable function at the given point $\bar{x} \in \mathbb{R}^n$. Then, the following statements are true:*
a) Let $\omega_i \in \mathbb{R}$, $x^i \in \mathbb{R}^n$, where $i = 1, ..., n$. Then

$$\otimes_{i=1}^{n}(\omega_i \otimes x^i) = h^{-1}\left(\sum_{i=1}^{n} \omega_i h(x^i)\right),$$

$$\otimes_{i=1}^{n} x^i = h^{-1}\left(\sum_{i=1}^{n} h(x^i)\right).$$

b) Let $\chi_i, \psi_i \in \mathbb{R}$, i=1,...,n. Then

$$\left[\sum_{i=1}^{n}\right](\chi_i[\cdot]\psi_i) = \varphi^{-1}\left(\sum_{i=1}^{n} \chi_i\varphi(\psi_i)\right),$$

$$\left[\sum_{i=1}^{n}\right]\psi_i = \varphi^{-1}\left(\sum_{i=1}^{n} \varphi(\psi_i)\right).$$

The following properties of generalized algebraic operations were established by Ben-Tal [7].

Lemma 2.2. *The following statements are true:*

a) $\chi[\cdot](\psi[\cdot]\omega) = \psi[\cdot](\chi[\cdot]\omega) = (\chi\psi)[\cdot]\omega$, where $\chi, \psi, \omega \in \mathbb{R}$,

b) $\left[\sum_{i=1}^{m}\right](\chi_i[+]\psi_i) = \left[\sum_{i=1}^{m}\right](\chi_i)[+]\left[\sum_{i=1}^{m}\right](\psi_i)$, where $\chi_i, \psi_i \in \mathbb{R}, i = 1, ..., m$,

c) $\left[\sum_{i=1}^{m}\right](\chi_i[-]\psi_i) = \left[\sum_{i=1}^{m}\right](\chi_i)[-]\left[\sum_{i=1}^{m}\right](\psi_i)$, where $\chi_i, \psi_i \in \mathbb{R}$, $i = 1, ..., m$, *d)* $\psi[\cdot]\left[\sum_{i=1}^{m}\right](\chi_i) = \left[\sum_{i=1}^{m}\right](\psi[\cdot]\chi_i)$, *where* $\chi_i, \psi, \in \mathbb{R}$ *and* $i = 1, ..., m$.

e) $\omega[\cdot](\chi[-]\psi) = (\omega[\cdot]\chi)[-](\omega[\cdot]\psi)$, *where* $\chi, \psi, \omega \in \mathbb{R}$.

Lemma 2.3. *[7] Let* φ *be a strictly monotone function such that* $\varphi(0) = 0$. *Then, the following statements are true:*

a) Let $\chi, \psi, \omega \in \mathbb{R}, \omega \geqq 0$. *If* $\chi \leqq \psi$, *then* $\omega[\cdot]\chi \leqq \omega[\cdot]\psi$.

b) Let $\chi, \psi, \omega \in \mathbb{R}, \omega \geqq 0$. *If* $\chi < \psi$, *then* $\omega[\cdot]\chi < \omega[\cdot]\psi$.

c) Let $\chi, \psi, \omega \in \mathbb{R}, \omega > 0$. *If* $\chi < \psi$, *then* $\omega[\cdot]\chi < \omega[\cdot]\psi$.

d) Let $\chi, \psi, \omega \in \mathbb{R}, \omega < 0$. *If* $\chi \geqq \psi$, *then* $\omega[\cdot]\chi \leqq \omega[\cdot]\psi$.

e) Let $\chi_i, \psi_i \in \mathbb{R}$. *If* $\chi_i \leqq \psi_i$, *then* $\left[\sum_{i=1}^{m}\right]\chi_i \leqq \left[\sum_{i=1}^{m}\right]\psi_i$, *for all* $i = 1, 2, 3, ..., m$

f) Let $\chi_i, \psi_i \in \mathbb{R}, i = 1, ..., m$, *such that* $\chi_i \leqq \psi_i$ *and there exists at least one* $i^* \in \{1, ..., m\}$ *such that* $\chi_{i^*} < \psi_{i^*}$, *then* $\left[\sum_{i=1}^{m}\right]\chi_i < \left[\sum_{i=1}^{m}\right]\psi_i$.

If φ a strictly monotone function is extra assume to be an one-to-one continuous function, then next properties of generalized algebraic operations can be derived.

Lemma 2.4. *[7] Let* φ *be a strictly monotone function such that* $\varphi(0) = 0$. *Then, the following statements are true:*

a) $\chi < \psi \Longleftrightarrow \chi[-]\psi < 0$, *where* $\chi \in \mathbb{R}, \psi \in \mathbb{R}$.

b) $\chi \leqq \psi \Longleftrightarrow \chi[-]\psi \leqq 0$, *where* $\chi \in \mathbb{R}, \psi \in \mathbb{R}$.

c) $\chi[+]\psi < 0 \Longrightarrow \chi < (-1)[\cdot]\psi$, *where* $\chi \in \mathbb{R}, \psi \in \mathbb{R}$.

d) $\chi[+]\psi \leqq 0 \Longrightarrow \chi \leqq (-1)[\cdot]\psi$, *where* $\chi \in \mathbb{R}, \psi \in \mathbb{R}$.

In the following lemma, we will outline several crucial properties of generalized algebraic operations introduced by Ben-Tal, that will have significant importance in proving the main results in the paper. Namely, these properties given by Yu and Liu [32] will be fundamental tools in the construction of various mathematical formulations and solutions.

Lemma 2.5. *[32] The following statements are true:*
a) $\chi[\cdot](\psi[\cdot]\omega) = \psi[\cdot](\chi[\cdot]\omega) = \chi\psi[\cdot]\omega$, *where* $\chi, \psi, \omega \in \mathbb{R}$.

b) $\chi[\cdot]\left[\sum_{i=1}^{m}\right]\psi_i = \left[\sum_{i=1}^{m}\right]\chi[\cdot]\psi_i$, *where* $\psi_i, \chi \in \mathbb{R}$.

c) $\chi[\cdot](\psi[-]\omega) = \chi[\cdot]\psi[-]\chi[\cdot]\omega$, *where* $\chi, \psi, \omega \in \mathbb{R}$.

d) $\left[\sum_{i=1}^{m}\right](\psi_i[-]\omega_i) = \left[\sum_{i=1}^{m}\right]\psi_i[-]\left[\sum_{i=1}^{m}\right]\omega_i$, *where* $\psi_i, \omega_i \in \mathbb{R}, i = 1, ..., m$.

e) $\left((\oplus_{i=1}^m a_i)^T b \right)_{(h,\varphi)} = \left[\sum_{i=1}^m\right] (a_i^T b)_{(h,\varphi)}$, *where* $a_i, b \in \mathbb{R}^n, i = 1, ..., m$.

f) $\left((\omega \otimes a)^T b\right)_{(h,\varphi)} = \omega[\cdot] \left(a^T b\right)_{(h,\varphi)}$, *where* $a, b \in \mathbb{R}^n, \omega \in \mathbb{R}$.

g) $\left((\oplus_{i=1}^m \omega_i \otimes a_i)^T b \right)_{(h,\varphi)} = \left[\sum_{i=1}^m\right] \omega_i[\cdot] \left(a_i^T b\right)_{(h,\varphi)}$, *where* $a_i, b \in \mathbb{R}^n, \omega_i \in \mathbb{R}, i = 1, ..., m$.

Definition 2.1. *[16] We say that $G : \mathbb{R} \to \mathbb{R}$ is a strictly increasing if and ony if*

$$\forall \zeta, \psi \in \mathbb{R}, \zeta < \psi \Rightarrow G(\zeta) < G(\psi).$$

Definition 2.2. *A function $f : \mathbb{R}^n \supseteq X \to \mathbb{R}$ is said to be locally Lipschitz at $z \in X$ on X if there exists a constant $K_z > 0$ and $\delta > 0$ such that the inequality*

$$|f(x) - f(y)| \leq K_z \|x - y\|$$

holds for all $x, y \in N_\delta(z)$, where $N_\delta(z) := \{y \in \mathbb{R}^n : \|y - z\| < \delta\}$.

Now, we re-call the definition of a Clarke generalized directional (h,φ)-derivative and the definition of a (h,φ)-Clarke generalized gradient introduced by Yuan et al. [33].

Definition 2.3. *[33] The Clarke generalized directional (h,φ)-derivative of a locally Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}$ at $\bar{x} \in \mathbb{R}^n$ in the direction $d \in \mathbb{R}^n$ is defined by:*

$$f^*(\bar{x}; d) := \lim_{\substack{y \to \bar{x} \\ t \downarrow 0}} \frac{1}{t}[\cdot] \left(f(y \oplus (t \otimes d))[-]f(y)\right)$$

Definition 2.4. *[33] The Clarke generalized (h,φ)-subgradient of a locally Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}$ at $\bar{x} \in \mathbb{R}^n$ is defined by:*

$$\partial^* f(\bar{x}) := \left\{\xi^* \in \mathbb{R}^n : f^*(\bar{x}; d) \geqq ((\xi^*)^T d)_{(h,\varphi)}, \ \forall d \in \mathbb{R}^n\right\}$$

Now, we extend the aforesaid definitions of (h,φ)-Clarke generalized directional derivative and Clarke generalized (h,φ)-subdifferential for a locally Lipschitz vector-valued function $f = (f_1, \ldots, f_k) : \mathbb{R}^n \to \mathbb{R}^k$.

Definition 2.5. *The (h,φ)-Clarke generalized directional derivative of a locally Lipschitz vector-valued function $f = (f_1, ..., f_k) : \mathbb{R}^n \to \mathbb{R}^k$ at $\bar{x} \in \mathbb{R}^n$ in the direction $d \in \mathbb{R}^n$ is defined by:*

$$f^*(\bar{x}; d) := (f_1^*(\bar{x}; d), ..., f_k^*(\bar{x}; d))$$

Definition 2.6. *The Clarke generalized (h,φ)-subgradient of a locally Lipschitz vector-valued function $f : \mathbb{R}^n \to \mathbb{R}^k$ at $\bar{x} \in \mathbb{R}^n$ is defined by:*

$$\partial^* f(\bar{x}) := \partial^* f_1(\bar{x}) \otimes \ldots \otimes \partial^* f_k(\bar{x})$$

where $\partial^* f_i(\bar{x})$ for $i = 1, \ldots, k$ is the Clarke's generalized (h,φ)-subdifferential of f_i at $\bar{x}$.

Lemma 2.6. *Let $f : \mathbb{R}^n \to \mathbb{R}$, and $\omega \in \mathbb{R}$, $\bar{x} \in \mathbb{R}^n$ be such that $\omega[\cdot]f$ is a (h,φ)-nondifferentiable function at $\bar{x}$. Then, $\partial^*(\omega[\cdot]f(\bar{x})) = \omega \otimes \partial^* f(\bar{x})$.*

Now, by using generalized algebraic operations introduced by Ben-Tal [7] and the Clarke generalized directional (h,φ)-derivative of a locally Lipschitz vector-valued function defined in Definition 2.5, we extend the notion of a locally Lipschitz vector-valued G-invex function defined by Kang et al. [20].

Definition 2.7. *Let X be a nonempty open subset of $\mathbb{R}^n$ and $f : X \to \mathbb{R}^k$ be a locally Lipschitz function on X. If there exists $G_f = (G_{f_1}, \ldots, G_{f_k}) : \mathbb{R} \to \mathbb{R}^k$ is a differentiable vector-valued function with strictly increasing component functions $G_{f_i} : I_{f_i}(X) \to \mathbb{R}$, $i = 1, \ldots, k$, and $\eta : X \times X \to \mathbb{R}^n$ such that the inequalities*

$$G_{f_i}(f_i(x))[-]G_{f_i}(f_i(\bar{x})) \geqq G'_{f_i}(f_i(\bar{x}))[\cdot]A_i[\cdot]\eta(x,\bar{x}) \quad (>)$$

hold for all $x \in X$, $(x \neq \bar{x})$, and any $A_i \in \partial^* f_i(\bar{x})$, $i = 1, \ldots, k$, then f is said to be a nondifferentiable vector-valued (h,φ)-G_f-invex function at $\bar{x}$ on X with respect to η. If the inequalities above are fulfilled for all $\bar{x} \in X$, then f is said to be a nondifferentiable vector-valued (h,φ)-G_f-invex function on X with respect to η.

Based on the results established by Antczak [2] and using the concept of a generalized (h,φ)-convex function introduced by Ben-Tal [7], we prove the next result.

Proposition 2.1. *1. Let X be a nonempty open subset of $\mathbb{R}^n$ and $f : X \to \mathbb{R}^k$ be a locally Lipschitz function on X. Further, we assume that the function $f^*(u,\cdot) : \mathbb{R}^n \to \mathbb{R}^k$ is surjective at each point $u \in X$, and each function $G_{f_i} : I_{f_i}(X) \to \mathbb{R}$, $i = 1, \ldots, k$, is differentiable on $\mathbb{R}$ and strictly increasing and convex on $I_{f_i}(X)$. Then there exists a vector-valued function $\eta : X \times X \to \mathbb{R}^n$ such that f is a (h,φ)-G_f-invex function with respect to η on X.*

Proof
Assume that X is a nonempty open subset of $\mathbb{R}^n$, and $x, u \in X$. Let $w = f(x)$ and $v = f(u)$. Then, $w = (w_1, \ldots, w_k)$ and $v = (v_1, \ldots, v_k)$ are k-dimensional vectors. Since $f^*(u,\cdot)$ is surjective for every $u \in X$, it follows that $w - v \in \mathbb{R}^k$ can be expressed as a linear combination of the partial derivatives of $f^*(u,\cdot)$ evaluated at some point $\eta(x,u) \in \mathbb{R}^n$, where $\eta : X \times X \to \mathbb{R}^n$. Thus,

$$w[-]v = f^*(u,\eta(x,u)) \Rightarrow w_i[-]v_i = f_i^*(u,\eta(x,u)), \quad i = 1, \ldots, k. \tag{10}$$

Now, by convexity of $G_{f_i} : I_{f_i}(X) \to \mathbb{R}$, $i = 1, \ldots, k$, we have

$$G_{f_i}(f_i(x))[-]G_{f_i}(f_i(u)) = G_{f_i}(w_i)[-]G_{f_i}(v_i) \geqq G'_{f_i}(v_i)[\cdot](w_i[-]v_i), \quad i = 1, \ldots, k.$$

Using (10) in the above inequalities, we obtain

$$G_{f_i}(f_i(x))[-]G_{f_i}(f_i(u)) \geqq G'_{f_i}(f_i(u))[\cdot]f_i^*(u;\eta(x,u)), \quad i = 1, \ldots, k.$$

Thus, by the definition of the (h,φ)-Clarke's generalized subdifferential, we obtain that the inequalities

$$G_{f_i}(f_i(x))[-]G_{f_i}(f_i(u)) \geqq G'_{f_i}(f_i(u))[\cdot]A_i[\cdot]\eta(x,u), \quad i = 1, \ldots, k$$

are fulfilled for any $A_i \in \partial^* f_i(u)$, $i = 1, \ldots, k$. Then, by Definition 2.7, we conclude that f is (h,φ)-G_f-invex with respect to η on X. □

3. (h,φ)-nondifferentiable multiobjective programming and optimality

In the paper, consider an (h,φ)-nondifferentiable multiobjective programming problem defined by

$$\begin{cases} \min f(x) = (f_1(x), \ldots, f_k(x)), & (MP)_{(h,\varphi)} \\ \text{s.t.} \quad g_j(x) \leq 0, \quad j = 1, \ldots, m, \end{cases}$$

where each objective function $f_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, ..., k$, and each constraint function $g_j : \mathbb{R}^n \to \mathbb{R}$, $j = 1, \ldots, m$, are locally Lipschitz on $\mathbb{R}^n$. Further, we shall assume that each of the aforesaid functions constituting $(MP)_{(h,\phi)}$ is (h,ϕ)-subdifferentiable on $\mathbb{R}^n$. Let us define the set Ω of all feasible solutions in $(MP)_{(h,\phi)}$ by $\Omega = \{x \in \mathbb{R}^n : g_j(x) \leqq 0, j = 1, \ldots, m\}$ and the set of indices of inequality constraints that are active at the given feasible solution $\bar{x}$ by $J(\bar{x}) = \{j \in J : g_j(\bar{x}) = 0\}$.

Definition 3.1. *It is said that $\bar{x} \in \Omega$ is a weak Pareto solution of $(MP)_{(h,\varphi)}$ if there does not exist other $x \in \Omega$ such that $f(x) < f(\bar{x})$.*

Definition 3.2. *It is said that $\bar{x} \in \Omega$ is a Pareto solution of $(MP)_{(h,\varphi)}$ if there does not exist other $x \in \Omega$ such that $f(x) \leqq f(\bar{x})$ and there exists at least one $i^* \in \{1, .., k\}$ such that $f_{i^*}(x) < f_{i^*}(\bar{x})$.*

The concept of nondifferentiable (h,φ)-G-invexity (see Definition 2.6) can be generalized by introducing for a multiobjective programming problem with inequality constraints the concepts of nondifferentiable (h,φ)-G-type I, (h,φ)-G-pseudo-type I, (h,φ)-G-quasi-type I and (h,φ)-G-pseudo-quasi-type I functions. The aforesaid concepts of (h,φ)-generalized convexity allow for a more flexible way of characterizing optimality of optimization problems and can be useful in developing optimization algorithms that are tailored to specific types of constraints or problem structures.

In order to provide the definition of (h,φ)-G-type I functions and its various generalizations for the considered (h,φ)-subdifferentiable multiobjective programming problem $(MP)_{(h,\varphi)}$, we define a vector-valued function $\eta : X \times X \to \mathbb{R}^n$, where X is a nonempty subset of $\mathbb{R}^n$. Further, let $G_f = (G_{f_1}, \ldots, G_{f_k}) : \mathbb{R} \to \mathbb{R}^k$ be a differentiable vector-valued function with strictly increasing component functions $G_{f_i} : I_{f_i}(X) \to \mathbb{R}$, $i = 1, \ldots, k$, and $G_g = (G_{g_1}, \ldots, G_{g_m}) : \mathbb{R} \to \mathbb{R}^m$ be a differentiable vector-valued function with strictly increasing component functions $G_{g_j} : I_{g_j}(X) \to \mathbb{R}$, $j = 1, \ldots, m$, where $I_{f_i}(X)$, $i = 1, .., k$, and $I_{g_j}(X)$, $j = 1, ...m$, denote the range of f_i and the range of g_j, respectively, that is, the image of X under f_i, and the image of X under g_j, respectively. We also assume that each function $f_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, \ldots, k$, and each function $g_j : \mathbb{R}^n \to \mathbb{R}$, $j = 1, \ldots, m$, is locally Lipschitz (h,φ)-subdifferentiable on $\mathbb{R}^n$.

Definition 3.3. *(f, g) is said to be (h,φ)-G-type I at $\bar{x} \in \mathbb{R}^n$ on $\mathbb{R}^n$ with respect to η if the inequalities*

$$G_{f_i}(f_i(x))[-]G_{f_i}(f_i(\bar{x})) \geqq G'_{f_i}(f_i(\bar{x}))[\cdot](A_i^T \eta(x, \bar{x}))_{(h,\varphi)} \tag{11}$$

$$(-1)[\cdot]G_{g_j}(g_j(\bar{x})) \geqq G' g_j(g_j(\bar{x}))[\cdot](B_j^T \eta(x, \bar{x}))_{(h,\varphi)} \tag{12}$$

hold for all $x \in \mathbb{R}^n$ and any $A_i \in \partial^* f_i(\bar{x}), i = 1, \ldots, k, B_j \in \partial^* g_j(\bar{x}), j = 1, \ldots, m$, respectively. If inequalities (11) are strict for all $x \in \mathbb{R}^n, (x \neq \bar{x})$, and any $A_i \in \partial^* f_i(\bar{x}), i = 1, \ldots, k$, then (f, g) is said to be $(h,\varphi) - G$-type I at $\bar{x} \in \mathbb{R}^n$ on $\mathbb{R}^n$ with respect to η. If (11) and (12) are satisfied at any $\bar{x} \in \mathbb{R}^n$, then (f, g) is said to be (h,φ)-G-type I on $\mathbb{R}^n$ with respect to η. If (11) and (12) are satisfied for all $x \in X$, where X is a nonempty subset of $\mathbb{R}^n$, then (f, g) is said to be (h,φ)-G-type I at $\bar{x} \in X$ on X with respect to η.

Definition 3.4. *(f, g) is said to be (h,φ)-G-pseudo type I at $\bar{x} \in \mathbb{R}^n$ on $\mathbb{R}^n$ with respect to η if there exist $\bar{\lambda} = (\bar{\lambda}_1, \ldots, \bar{\lambda}_k) \geq 0$ and $\bar{\mu} = (\bar{\mu}_1, \ldots, \bar{\mu}_m) \geq 0$ such that the following relations hold:*

$$\begin{gathered} \left[\sum_{i=1}^{k}\right] \bar{\lambda}_i[\cdot]G'_{f_i}(f_i(\bar{x}))[\cdot](A_i^T \eta(x, \bar{x})_{(h,\varphi)} \geqq 0 \Rightarrow \\ \left[\sum_{i=1}^{k}\right] \bar{\lambda}_i[\cdot]G_{f_i}(f_i(x)) \geqq \left[\sum_{i=1}^{k}\right] \bar{\lambda}_i[\cdot]G_{f_i}(f_i(\bar{x})) \end{gathered} \tag{13}$$

$$\left[\sum_{j=1}^{m}\right] \bar{\mu}_j[\cdot]G'_{g_j}(g_j(\bar{x}))[\cdot](B_j^T\eta(x,\bar{x}))_{(h,\varphi)} \geqq 0 \Rightarrow \left[\sum_{j=1}^{m}\right] (-1)[\cdot]\bar{\mu}_j[\cdot]G_{g_j}(g_j(\bar{x})) \geqq 0 \tag{14}$$

hold for all $x \in \mathbb{R}^n$ and any $A_i \in \partial^* f_i(\bar{x}), i = 1, \ldots, k, B_j \in \partial^* g_j(\bar{x}), j = 1, \ldots, m$, respectively. If the second inequality in (13) are strict for all $x \in \mathbb{R}^n, (x \neq \bar{x})$, and any $A_i \in \partial^* f_i(\bar{x}), i = 1, \ldots, k$, then (f, g) is said to be $(h, \varphi) - G$-pseudo type I at $\bar{x} \in \mathbb{R}^n$ on $\mathbb{R}^n$ with respect to η. If (13) and (14) are satisfied at any $\bar{x} \in \mathbb{R}^n$, then (f, g) is said to be (h, φ)-G-pseudo type I on $\mathbb{R}^n$ with respect to η. If (13) and (14) are satisfied for all $x \in X$, where X is a nonempty subset of $\mathbb{R}^n$, then (f, g) is said to be (h, φ)-G-pseudo type I at $\bar{x} \in X$ on X with respect to η.

Definition 3.5. *(f, g) is said to be (h, φ)-G-quasi type I at $\bar{x} \in \mathbb{R}^n$ on $\mathbb{R}^n$ with respect to η if there exist $\bar{\lambda} = (\bar{\lambda}_1, \ldots, \bar{\lambda}_k) \geq 0$ and $\bar{\mu} = (\bar{\mu}_1, \ldots, \bar{\mu}_m) \geq 0$ such that the following relations hold:*

$$\left[\sum_{i=1}^{k}\right] \bar{\lambda}_i[\cdot]G_{f_i}(f_i(x)) \leqq \left[\sum_{i=1}^{k}\right] \bar{\lambda}_i[\cdot]G_{f_i}(f_i(\bar{x})) \Rightarrow \left[\sum_{i=1}^{k}\right] \bar{\lambda}_i[\cdot]G'_{f_i}(f_i(\bar{x}))[\cdot]\left(A_i^T\eta(x,\bar{x})\right)_{(h,\varphi)} \leqq 0, \tag{15}$$

$$\left[\sum_{j=1}^{m}\right] (-1)[\cdot]\bar{\mu}_j[\cdot]G_{g_j}(g_j(\bar{x})) \leqq 0 \Rightarrow \left[\sum_{j=1}^{m}\right] \bar{\mu}_j[\cdot]G'_{g_j}(g_j(\bar{x}))[\cdot](B_j^T\eta(x,\bar{x}))_{(h,\varphi)} \leqq 0 \tag{16}$$

hold for all $x \in \mathbb{R}^n$ and any $A_i \in \partial^* f_i(\bar{x}), i = 1, \ldots, k, B_j \in \partial^* g_j(\bar{x}), j = 1, \ldots, m$, respectively. If (15) and (16) are satisfied at any $\bar{x} \in \mathbb{R}^n$, then (f, g) is said to be (h, φ)-G-quasi type I on $\mathbb{R}^n$ with respect to η. If (15) and (16) are satisfied for all $x \in X$, where X is a nonempty subset of $\mathbb{R}^n$, then (f, g) is said to be (h, φ)-G-quasi type I at $\bar{x} \in X$ on X with respect to η.

Now, we prove the sufficient optimality conditions for a feasible solution of $(MP)_{(h,\varphi)}$ to be its weak Pareto solution.

Definition 3.6. *(f, g) is said to be (h, φ)-G-pseudo-quasi type I at $\bar{x} \in \mathbb{R}^n$ on $\mathbb{R}^n$ with respect to η if there exist $\bar{\lambda} = (\bar{\lambda}_1, \ldots, \bar{\lambda}_k) \geq 0$ and $\bar{\mu} = (\bar{\mu}_1, \ldots, \bar{\mu}_m) \geq 0$ such that the following relations hold:*

$$\left[\sum_{i=1}^{k}\right] \bar{\lambda}_i[\cdot]G'_{f_i}(f_i(\bar{x}))[\cdot]\left(A_i^T\eta(x,\bar{x})\right)_{(h,\varphi)} \geqq 0 \Rightarrow \left[\sum_{i=1}^{k}\right] \bar{\lambda}_i[\cdot]G_{f_i}(f_i(x)) \geqq \left[\sum_{i=1}^{k}\right] \bar{\lambda}_i[\cdot]G_{f_i}(f_i(\bar{x})) \tag{17}$$

$$\left[\sum_{j=1}^{m}\right](-1)[\cdot]\bar{\mu}_j[\cdot]G_{g_j}(g_j(\bar{x})) \leqq 0 \Rightarrow \left[\sum_{j=1}^{m}\right]\bar{\mu}_j[\cdot]G'_{g_j}(g_j(\bar{x}))[\cdot](B_j^T\eta(x,\bar{x}))_{(h,\varphi)} \leqq 0 \tag{18}$$

hold for all $x \in \mathbb{R}^n$ and any $A_i \in \partial^* f_i(\bar{x}), i = 1,\ldots,k, B_j \in \partial^* g_j(\bar{x}), j = 1,\ldots,m,$ respectively. If the second inequality in (17) are strict for all $x \in \mathbb{R}^n, (x \neq \bar{x})$, and any $A_i \in \partial^* f_i(\bar{x}), i = 1,\ldots,k$, then (f,g) is said to be (h,φ)-G-pseudo-quasi type I at $\bar{x} \in \mathbb{R}^n$ on $\mathbb{R}^n$ with respect to η. If (17) and (18) are satisfied at any $\bar{x} \in \mathbb{R}^n$, then (f,g) is said to be (h,φ)-G-pseudo-quasi type I on $\mathbb{R}^n$ with respect to η. If (17) and (18) are satisfied for all $x \in X$, where X is a nonempty subset of $\mathbb{R}^n$, then (f,g) is said to be (h,φ)-G-pseudo-quasi type I at $\bar{x} \in X$ on X with respect to η.

Theorem 3.1. *Let $\bar{x}$ be a feasible solution of the (h,φ)-subdifferentiable multiobjective programming problem $(MP)_{(h,\varphi)}G_f = (G_{f_1},\ldots,G_{f_k}) : \mathbb{R} \to \mathbb{R}^k$ be a differentiable vector-valued function with strictly increasing component functions $G_{f_i} : I_{f_i}(X) \to \mathbb{R}$, $i = 1,\ldots,k$, and $G_g = (G_{g_1},\ldots,G_{g_m}) : \mathbb{R} \to \mathbb{R}^m$ be a differentiable vector-valued function with strictly increasing component functions $G_{g_j} : I_{g_j}(X) \to \mathbb{R}$, $j = 1,\ldots,m$. Further, we assume that there exist $\bar{\lambda} \in \mathbb{R}^k$ and $\bar{\mu} \in \mathbb{R}^m$ such that*

$$0 \in \left(\oplus_{i=1}^k \bar{\lambda}_i[\cdot]G'_{f_i}(f_i(\bar{x})) \otimes \partial^* f_i(\bar{x})\right) \oplus \left(\oplus_{j\in J(\bar{x})}\bar{\mu}_j[\cdot]G'_{g_j}(g_j(\bar{x})) \otimes \partial^* g_j(\bar{x})\right) \tag{19}$$

$$\bar{\mu}_j[\cdot]G_{g_j}(g_j(\bar{x})) = 0, \quad j = 1,\ldots,m \tag{20}$$

$$\bar{\lambda} = (\bar{\lambda}_1,\ldots,\bar{\lambda}_k) \geq 0, \quad \bar{\mu} = (\bar{\mu}_1,\ldots,\bar{\mu}_m) \geqq 0 \tag{21}$$

If one of the following sets of hypotheses

a) (f,g) is (h,φ)-G-type I at $\bar{x} \in \Omega$ on Ω with respect to η,

b) (f,g) is (h,φ)-G-pseudo-quasi type I at $\bar{x} \in \Omega$ on Ω with respect to η is fulfilled, then $\bar{x}$ is a weak Pareto solution in $(MP)_{(h,\varphi)}$.

Proof
We assume that $\bar{x} \in \Omega$ and all hypotheses of this theorem are fulfilled. We proceed by contradiction. Suppose, contrary to the result, that $\bar{x}$ is not a weak Pareto solution of $(MP)_{(h,\varphi)}$. Then, by Definition 3.1, there exists another $\tilde{x} \in \Omega$ such that

$$f_i(\tilde{x}) < f_i(\bar{x}), \quad i = 1,\ldots,k. \tag{22}$$

Since $G_{f_i}, i = 1,\ldots,k$, are strictly increasing functions, by Definition 2.1, (22) yields

$$G_{f_i}(f_i(\tilde{x})) < G_{f_i}(f_i(\bar{x})), \quad i = 1,\ldots,k. \tag{23}$$

By Lemma 2.2 a), (23) gives

$$G_{f_i}(f_i(\tilde{x}))[-]G_{f_i}(f_i(\bar{x})) < 0, \quad i = 1,\ldots,k \tag{24}$$

Since (19) is satisfied, by Lemma 2.5 g), it follows that there exist $A_i \in \partial^* f_i(\bar{x}), i = 1,\ldots,k$, $B_j \in \partial^* g_j(\bar{x}), j = 1,\ldots,m$, such that

$$\left(\left(\oplus_{i=1}^{k}\bar{\lambda}_i[\cdot]G'_{f_i}(f_i(\bar{x}))\otimes A_i\right)^T\eta(\tilde{x},\bar{x})\right)_{(h,\varphi)}[+]$$
$$\left(\left(\oplus_{j\in J(\bar{x})}\bar{\mu}_j[\cdot]G'_{g_j}(g_j(\bar{x}))\otimes B_j\right)^T\eta(\tilde{x},\bar{x})\right)_{(h,\varphi)}=0 \tag{25}$$

Now, we prove this theorem under hypothesis a). Then, by Definition 3.1, the inequalities

$$G_{f_i}(f_i(\tilde{x}))[-]G_{f_i}(f_i(\bar{x}))\geqq\left(\left(G'_{f_i}(f_i(\bar{x}))[\cdot]A_i\right)^T\eta(\tilde{x},\bar{x})\right)_{(h,\varphi)}, \tag{26}$$

$$(-1)[\cdot]G_{g_j}(g_j(\bar{x}))\geqq\left(\left(G'_{g_j}(g_j(\bar{x}))[\cdot]B_j\right)^T\eta(\tilde{x},\bar{x})\right)_{(h,\varphi)} \tag{27}$$

hold for any $A_i\in\partial^* f_i(\bar{x}), i=1,\dots,k, B_j\in\partial^* g_j(\bar{x}), j=1,\dots,m$, respectively. Combining (24) and (26), we obtain that the inequalities

$$G'_{f_i}(f_i(\bar{x}))[\cdot](A_i^T\eta(\tilde{x},\bar{x}))_{(h,\varphi)}<0, i=1,\dots,k$$

hold for any $A_i\in\partial^* f_i(\bar{x}), i=1,\dots,k$. Then, by (21) and Lemma 2.3 b) and c), the above inequalities yield

$$\bar{\lambda}_i[\cdot]G'_{f_i}(f_i(\bar{x}))[\cdot](A_i^T\eta(\tilde{x},\bar{x}))_{(h,\varphi)}\leqq 0, i=1,\dots,k \tag{28}$$

$$\lambda_{i^*}[\cdot]G'_{f_{i^*}}(f_{i^*}(\bar{x}))[\cdot](A_{i^*}^T\eta(\tilde{x},\bar{x}))_{(h,\varphi)}<0 \text{ for at least one } i^*\in\{1,\dots,k\}. \tag{29}$$

Hence, by Lemma 2.3 f), (28) and (29) imply

$$\left[\sum_{i=1}^{k}\right]\bar{\lambda}_i[\cdot]G'_{f_i}(f_i(\bar{x}))[\cdot](A_i^T\eta(\tilde{x},\bar{x}))_{(h,\varphi)}<0.$$

Then, by Lemma 2.5 g), the inequality above gives that the inequality

$$\left(\oplus_{i=1}^{k}\left(\bar{\lambda}_i[\cdot]G'_{f_i}(f_i(\bar{x}))\otimes A_i\right)^T\eta(\tilde{x},\bar{x})\right)_{(\tilde{h},\varphi)}<0. \tag{30}$$

holds for any $A_i\in\partial^* f_i(\bar{x}), i=1,\dots,k$. By (21), (26) and Lemma 2.3 a), we get

$$(-1)[\cdot]\bar{\mu}_j[\cdot]G_{g_j}(g_j(\bar{x}))\geqq((\bar{\mu}_j[\cdot]G'_{g_j}(g_j(\bar{x}))[\cdot]B_j)^T\eta(\tilde{x},\bar{x}))_{(h,\varphi)}\quad j=1,\dots,m$$

Then, by (20), the above inequalities yield

$$\left(\left(\bar{\mu}_j[\cdot]G'_{g_j}(g_j(\bar{x}))[\cdot]B_j\right)^T\eta(\tilde{x},\bar{x})\right)_{(h,\varphi)}\leqq 0,\quad j=1,\dots,m \tag{31}$$

Hence, by Lemma 2.3 e), (31) implies that the inequality

$$\left[\sum_{j=1}^{m}\right]\bar{\mu}_j[\cdot]G'_{g_j}(g_j(\bar{x}))[\cdot](B_j^T\eta(\tilde{x},\bar{x}))_{(h,\varphi)}\leqq 0$$

holds for any $B_j\in\partial^* g_j(\bar{x}), j=1,\dots,m$. Then, by Lemma 2.5 g), the inequality above gives

$$\left(\left(\oplus_{j=1}^{m}(\bar{\mu}_j[\cdot]G'_{g_j}(g_j(\bar{x}))\otimes B_j\right)^T\eta(\tilde{x},\bar{x})\right)_{(h,\varphi)}\leqq 0. \tag{32}$$

Combining (30) and (32), we get that the inequality

$$\left(\oplus_{i=1}^{k}\left(\bar{\lambda}_i[\cdot]G'_{f_i}(f_i(\bar{x}))\otimes A_i\right)^T\eta(\tilde{x},\bar{x})\right)_{(\tilde{h},\varphi)}[+]$$
$$\left(\oplus_{j=1}^{m}\left(\bar{\mu}_j[\cdot]G'_{g_j}(g_j(\bar{x}))\otimes B_j\right)^T\eta(\tilde{x},\bar{x})\right)_{(h,\varphi)}<0$$

holds for any $A_i\in\partial^* f_i(\bar{x}), i=1,\ldots,k,\ B_j\in\partial^* g_j(\bar{x}), j=1,\ldots,m$, contradicting (25). Now, we prove this theorem under hypothesis b). By Lemma 2.3 a) and b), (23) gives

$$\bar{\lambda}_i[\cdot]G_{f_i}(f_i(\tilde{x}))\leqq\bar{\lambda}_i[\cdot]G_{f_i}(f_i(\bar{x})),\quad i=1,\ldots,k \tag{33}$$

$$\bar{\lambda}_{i^*}[\cdot]G_{f_{i^*}}(f_{i^*}(\tilde{x}))<\bar{\lambda}_{i^*}[\cdot]G_{f_{i^*}}(f_{i^*}(\bar{x}))\quad\text{for at least one}\quad i^*\in\{1,\ldots,k\}. \tag{34}$$

Hence, by Lemma 2.3 f), (33) and (34) yield

$$\left[\sum_{i=1}^{k}\right]\bar{\lambda}_i[\cdot]G_{f_i}(f_i(\tilde{x}))<\left[\sum_{i=1}^{k}\right]\bar{\lambda}_i[\cdot]G_{f_i}(f_i(\bar{x})). \tag{35}$$

By assumption, (f,g) is (h,φ)-G-pseudo-quasi type I at $\bar{x}\in\Omega$ on Ω with respect to η. Then, by Definition 3.6, (35) implies that the inequality.

$$\left[\sum_{i=1}^{k}\right]\bar{\lambda}_i[\cdot]G'_{f_i}(f_i(\bar{x}))[\cdot](A_i^T\eta(\tilde{x},\bar{x}))_{(h,\varphi)}<0$$

holds. Then, by Lemma 2.5 g), the inequality above implies that the inequality

$$\left(\oplus_{i=1}^{k}\left(\bar{\lambda}_i[\cdot]G'_{f_i}(f_i(\bar{x}))\otimes A_i\right)^T\eta(\tilde{x},\bar{x})\right)_{(h,\varphi)}<0 \tag{36}$$

holds for any $A_i\in\partial^* f_i(\bar{x})$, $i=1,\ldots,k$. Since (20) is satisfied, one has

$$(-1)[\cdot]\bar{\mu}_j[\cdot]G_{g_j}(g_j(\bar{x}))=0,\quad j=1,\ldots,m. \tag{37}$$

Hence, by Lemma 2.3 e), (37) gives

$$\left[\sum_{j=1}^{m}\right](-1)[\cdot]\bar{\mu}_j[\cdot]G_{g_j}(g_j(\bar{x}))=0.$$

By assumption, (f,g) is (h,φ)-G-pseudo-quasi type I at $\bar{x}\in\Omega$ on Ω with respect to η. Therefore, using Definition 3.6 again, we get that the inequality

$$\left[\sum_{j=1}^{m}\right]\bar{\mu}_j[\cdot]G'_{g_j}(g_j(\bar{x}))[\cdot](B_j^T\eta(\tilde{x},\bar{x}))_{(h,\varphi)}\leqq 0$$

holds for any $B_j\in\partial^* g_j(\bar{x}), j=1,\ldots,$ m. Then, by Lemma 2.5 g), the inequality above gives

$$\left(\oplus_{j=1}^{m}\left((\bar{\mu}_j[\cdot]G'_{g_j}(g_j(\bar{x}))\otimes B_j\right)^T\eta(\tilde{x},\bar{x})\right)_{(h,\varphi)}\leqq 0. \tag{38}$$

Combining (36) and (38), we obtain that the inequality

$$\left(\otimes_{i=1}^{k}\left(\bar{\lambda}_i[\cdot]G'_{f_i}(f_i(\bar{x}))\otimes A_i\right)^T\eta(\tilde{x},\bar{x})\right)_{(h,\varphi)}[+]$$
$$\left(\oplus_{j=1}^{m}\left((\bar{\mu}_j[\cdot]G'_{g_j}(g_j(\bar{x}))\otimes B_j\right)^T\eta(\tilde{x},\bar{x})\right)_{(h,\varphi)}<0$$

holds for any $A_i \in \partial^* f_i(\bar{x}), i = 1, \ldots, k, B_j \in \partial^* g_j(\bar{x}), j = 1, \ldots, m$, contradicting (25). This completes the proof of this theorem.

□

Now, we formulate the sufficient optimality conditions for a feasible solution of $(MP)_{(h,\varphi)}$ to be its Pareto solution.

Theorem 3.2. *Let $\bar{x}$ be a feasible solution of the (h,φ)-subdifferentiable multiobjective programming problem $(MP)_{(h,\varphi)}$. $G_f = (G_{f_1}, \ldots, G_{f_k}) : \mathbb{R} \to \mathbb{R}^k$ be a differentiable vector-valued function with strictly increasing component functions $G_{f_i} : I_{f_i}(X) \to \mathbb{R}$, $i = 1, \ldots, k$, and $G_g = (G_{g_1}, \ldots, G_{g_m}) : \mathbb{R} \to \mathbb{R}^m$ be a differentiable vector-valued function with strictly increasing component functions $G_{g_j} : I_{g_j}(X) \to \mathbb{R}$, $j = 1, \ldots, m$. Further, we assume that there exist $\bar{\lambda} \in \mathbb{R}^k$ and $\bar{\mu} \in \mathbb{R}^m$ such that*

$$0 \in \left(\oplus_{i=1}^{k}\bar{\lambda}_i[\cdot]G'_{f_i}(f_i(\bar{x}))\otimes\partial^* f_i(\bar{x})\right)\oplus\left(\oplus_{j\in J(\bar{x})}\bar{\mu}_j[\cdot]G'_{g_j}(g_j(\bar{x}))\otimes\partial^* g_j(\bar{x})\right), \tag{39}$$

$$\bar{\mu}_j[\cdot]G_{g_j}(g_j(\bar{x})) = 0, j = 1, \ldots, m, \tag{40}$$

$$\bar{\lambda} = (\bar{\lambda}_1, \ldots, \bar{\lambda}_k) \geq 0, \quad \bar{\mu} = (\bar{\mu}_1, \ldots, \bar{\mu}_m) \geqq 0. \tag{41}$$

If one of the following set of hypotheses

a) (f, g) is strictly (h,φ)-G-type I at $\bar{x} \in \Omega$ on Ω with respect to η

b) (f, g) is strictly (h,φ)-G-pseudo-quasi type I at $\bar{x} \in \Omega$ on Ω with respect to η, then $\bar{x}$ is a Pareto solution in $(MP)_{(h,\varphi)}$

4. (h,φ)-nondifferentiable Mond-Weir duality

In this section, for the considered (h,φ)-nondifferentiable multiobjective programming problem $(MP)_{(h,\varphi)}$, we define its (h,φ)-nondifferentiable vector dual problem in the Mond-Weir sense. Then, we prove several dual theorems between the two aforesaid (h,φ)-nondifferentiable vector optimization problems under appropriate (h,φ)-G-type I and/or generalized (h,φ)-G-type I hypotheses.

We now assume that $G_f = (G_{f_1}, \ldots, G_{f_k}) : \mathbb{R} \to \mathbb{R}^k$ is a differentiable vector-valued function with strictly increasing component functions $G_{f_i} : I_{f_i}(X) \to \mathbb{R}$, $i = 1, \ldots, k$, and $G_g = (G_{g_1}, \ldots, G_{g_m}) : \mathbb{R} \to \mathbb{R}^m$ is a differentiable vector-valued function with strictly increasing component functions $G_{g_j} : I_{g_j}(X) \to \mathbb{R}$, $j = 1, \ldots, m$.

Now, consider the following (h,φ)-nondifferentiable vector dual problem in the Mond-Weir sense defined by

$$\text{(VD)}_{(h,\varphi)} \quad f(y) := (f_1(y), \dots, f_k(y)) \to V\text{-min}$$
$$0 \in \left(\oplus_{i=1}^k \lambda_i [\cdot] G'_{f_i}(f_i(y)) \otimes \partial^* f_i(y)\right) \oplus \left(\oplus_{j\in J(\bar{x})} \mu_j [\cdot] G'_{g_j}(g_j(y)) \otimes \partial^* g_j(y)\right)$$
$$\left[\sum_{j=1}^m\right] \mu_j [\cdot] G_{g_j}(g_j(\bar{x})) \geqq 0,$$
$$x \in X, \lambda = (\lambda_1, \dots, \lambda_k) \geq 0, \left[\sum_{i=1}^k\right] \lambda_i = 1, \mu = (\mu_1, \dots, \mu_m) \geqq 0,$$

where all functions are defined in a similar way as for $(MP)_{(h,\varphi)}$.

Let us define the set Q of all feasible solutions in $(VD)_{(h,\varphi)}$, that is, the set of all solutions (y, λ, μ) satisfying all constraints of $(VD)_{(h,\varphi)}$. Further, by Y we define the following set $Y := \{y \in X : (y, \lambda, \mu) \in Q\}$.

Theorem 4.1. *(Weak duality) Let x and (y, λ, μ) be any feasible solutions in (MP)(h, ϕ) and (VD)$_{(h,\varphi)}$, respectively. Further, assume that at least one of the following hypotheses is fulfilled:*

1. *(f, g) is (h,φ)-G-type I at $y \in Y$ on $\Omega \cup Y$ with respect to η,*
2. *(f, g) is (h,φ)-G-pseudo-quasi type I at $y \in Y$ on $\Omega \cup Y$ with respect to η.*

Then, $f(x) \nless f(y)$.

Proof

Let x and (y, λ, μ) be any feasible solutions in (MP)$_{(h,\varphi)}$ and (VD)$_{(h,\varphi)}$, respectively. Suppose, contrary to the result, that $f(x) < f(y)$. Since G_{f_i}, $i = 1, \dots, k$, are strictly increasing functions, the inequality above yields:

$$G_{f_i}(f_i(x)) < G_{f_i}(f_i(y)), \quad i = 1, \dots, k. \tag{42}$$

By Lemma 2.4 a), (23) gives

$$G_{f_i}(f_i(x))[-]G_{f_i}(f_i(y)) < 0, \quad i = 1, \dots, k. \tag{43}$$

Now, we prove this theorem under hypothesis a). Then, by Definition 3.1, the inequalities

$$G_{f_i}(f_i(x))[-]G_{f_i}(f_i(y)) \geqq \left((G' f_i(f_i(y))[\cdot]A_i)^T \eta(x,y) \right)_{(h,\varphi)}, \tag{44}$$
$$(-1)[\cdot]G_{g_j}(g_j(y)) \geqq \left((G' g_j(g_j(y))[\cdot]B_j)^T \eta(x,y) \right)_{(h,\varphi)} \tag{45}$$

hold for any $A_i \in \partial^* f_i(y)$, $i = 1, \dots, k$, and $B_j \in \partial^* g_j(y)$, $j = 1, \dots, m$, respectively. Combining (43) and (44), we get that the inequalities

$$G'_{f_i}(f_i(y))[\cdot] \left(A_i^T \eta(x,y) \right)_{(h,\varphi)} < 0, i = 1, \dots, k$$

are satisfied for any $A_i \in \partial^* f_i(y)$, $i = 1, \dots, k$. Then, by $(y, \lambda, \mu) \in Q$ and Lemma 2.3 b) and c), the above inequalities give

$$\lambda_i [\cdot] G'_{f_i}(f_i(y))[\cdot] \left(A_i^T \eta(x,y) \right)_{(h,\varphi)} \leqq 0, i = 1, \dots, k \tag{46}$$

$$\lambda_{i^*} [\cdot] G'_{f_{i^*}}(f_{i^*}(y))[\cdot] \left(A_{i^*}^T \eta(x,y) \right)_{(h,\varphi)} < 0 \quad \text{for at least one} \quad i^* \in \{1, \dots, k\}. \tag{47}$$

Thus, by Lemma 2.3 f), (46) and (47) yield

$$\left[\sum_{i=1}^{k}\right] \lambda_i[\cdot]G'_{f_i}(f_i(y))[\cdot]\left(A_i^T\eta(x,y)\right)_{(h,\varphi)} < 0. \tag{48}$$

Hence, by Lemma 2.5 g), (48) implies that the inequality

$$(\oplus_{i=1}^{k}(\lambda_i[\cdot]G'_{f_i}(f_i(y)) \otimes A_i)^T\eta(x,y))_{(h,\varphi)} < 0 \tag{49}$$

is satisfied for any $A_i \in \partial^* f_i(y), i = 1, \ldots, k$. By $(y, \lambda, \mu) \in Q$ and Lemma 2.3 a), (45) yields

$$(-1)[\cdot]\mu_j[\cdot]G_{g_j}(g_j(y)) \geqq \left(\left(\mu_j[\cdot]G'_{g_j}(g_j(y))[\cdot]B_j\right)^T\eta(x,y)\right)_{(h,\varphi)} \qquad j = 1, ..., m. \tag{50}$$

Then, by the second constraint of $(VD)_{(h,\varphi)}$, (50) gives

$$\left(\left(\mu_j[\cdot]G'_{g_j}(g_j(y))[\cdot]B_j\right)^T\eta(x,y)\right)_{(h,\varphi)} \leqq 0, \quad j = 1, ..., m. \tag{51}$$

Hence, by Lemma 2.3 e), (51) implies that the inequality

$$\left[\sum_{j=1}^{m}\right] \mu_j[\cdot]G'_{g_j}(g_j(y))[\cdot](B_j^T\eta(x,y))_{(h,\varphi)} \leqq 0$$

is satisfied for any $B_j \in \partial^* g_j(y)$, $j = 1, \ldots, m$. Then, by Lemma 2.5 g), the inequality above gives

$$\left(\oplus_{j=1}^{m}\left(\mu_j[\cdot]G'_{g_j}(g_j(y)) \otimes B_j\right)^T\eta(x,y)\right)_{(h,\varphi)} \leqq 0. \tag{52}$$

Combining (49) and (52), we obtain that the inequality

$$\begin{gathered}(\oplus_{i=1}^{k}(\lambda_i[\cdot]G'_{f_i}(f_i(y)) \otimes A_i)^T\eta(x,y))_{(h,\varphi)}\,[+] \\ \left(\oplus_{j=1}^{m}\left(\mu_j[\cdot]G'_{g_j}(g_j(y)) \otimes B_j\right)^T\eta(x,y)\right)_{(h,\varphi)} < 0\end{gathered}$$

is satisfied for any $A_i \in \partial^* f_i(y)$, $i = 1, \ldots, k$, $B_j \in \partial^* g_j(y)$, $j = 1, \ldots, m$, contradicts the first constraint of $(VD)_{(h,\varphi)}$, that is, the feasibility of (y, λ, μ) in $(VD)_{(h,\varphi)}$. Now, we prove this theorem under hypothesis b). By Lemma 2.3 a) and b), (42) implies

$$\lambda_i[\cdot]G_{f_i}(f_i(x)) \leqq \lambda_i[\cdot]G_{f_i}(f_i(y)), \quad i = 1, \ldots, k. \tag{53}$$

$$\lambda_{i^*}[\cdot]G_{f_{i^*}}(f_{i^*}(x)) < \lambda_{i^*}[\cdot]G_{f_{i^*}}(f_{i^*}(y)), \quad \text{for at least one } i^* \in \{1, \ldots, k\}. \tag{54}$$

Hence, by Lemma 2.3 f), (53) and (54) yield

$$\left[\sum_{i=1}^{k}\right] \lambda_i[\cdot]G_{f_i}(f_i(x)) < \left[\sum_{i=1}^{k}\right] \lambda_i[\cdot]G_{f_i}(f_i(y)). \tag{55}$$

By assumption, (f, g) is (h, φ)-G-pseudo-quasi type I at $\bar{x} \in \Omega$ on Ω with respect to η. Then, by Definition 3.6, (55) implies that the inequality

$$\left[\sum_{i=1}^{k}\right]\lambda_i[\cdot]G'_{f_i}(f_i(y))[\cdot]\left(A_i^T\eta(x,y)\right)_{(h,\varphi)}<0 \tag{56}$$

is satisfied. Thus, by Lemma 2.5 g), (56) gives that the inequality

$$\left(\oplus_{i=1}^{k}\left(\lambda_i[\cdot]G'_{f_i}(f_i(y))\otimes A_i\right)^T\eta(x,y)\right)_{(h,\varphi)}<0 \tag{57}$$

is satisfied for any $A_i\in\partial^* f_i(\bar{x}), i=1,\dots,k$. By $(y,\lambda,\mu)\in Q$, we have

$$(-1)[\cdot]\mu_j[\cdot]G_{g_j}(g_j(y))=0,\quad j=1,\dots,m. \tag{58}$$

Hence, by Lemma 2.3 e), (58) gives

$$\left[\sum_{j=1}^{m}\right](-1)[\cdot]\mu_j[\cdot]G_{g_j}(g_j(y))=0.$$

By assumption, (f,g) is (h,φ)-G-pseudo-quasi type I at $y\in Y$ on $\Omega\cup Y$ with respect to η. Hence, by Definition 3.6, we have that the inequality

$$\left[\sum_{j=1}^{m}\right]\mu_j[\cdot]G'_{g_j}(g_j(y))[\cdot]\left(B_j^T\eta(x,y)\right)_{(h,\varphi)}\leqq 0$$

is satisfied for any $B_j\in\partial^* g_j(y)$, $j=1,\dots,m$, then, by Lemma 2.5 g), the inequality above implies that the inequality

$$\left(\oplus_{j=1}^{m}\left(\mu_j[\cdot]G'_{g_j}(g_j(y))\otimes B_j\right)^T\eta(x,y)\right)_{(h,\varphi)}\leqq 0 \tag{59}$$

is satisfied for any $B_j\in\partial^* g_j(y)$, $j=1,\dots,m$. Combining (57) and (59), we obtain that the inequality

$$\left(\oplus_{i=1}^{k}\left(\lambda_i[\cdot]G'_{f_i}(f_i(y))\otimes A_i\right)^T\eta(x,y)\right)_{(h,\varphi)}[+]$$
$$\left(\oplus_{j=1}^{m}\left(\mu_j[\cdot]G'_{g_j}(g_j(y))\otimes B_j\right)^T\eta(x,y)\right)_{(h,\varphi)}<0$$

is satisfied for any $A_i\in\partial^* f_i(y)$, $i=1,\dots,k$, $B_j\in\partial^* g_j(y)$, $j=1,\dots,m$, contradicts the first constraint of $(VD)_{(h,\varphi)}$, that is, the feasibility of (y,λ,μ) in $(VD)_{(h,\varphi)}$. This completes the proof of this theorem. □

If we assume slightly stronger hypotheses of generalized invexity, then we are in position to prove the following stronger result:

Theorem 4.2. *(Strong duality) Let x and (y,λ,μ) be any feasible solutions in $(MP)_{(h,\varphi)}$ and $(VD)_{(h,\varphi)}$, respectively. Further, assume that at least one of the following hypotheses is fulfilled:*

1. *(f,g) is strictly (h,φ)-G-type I at $y\in Y$ on $\Omega\cup Y$ with respect to η,*
2. *(f,g) is strictly (h,φ)-G-pseudo-quasi type I at $y\in Y$ on $\Omega\cup Y$ with respect to η.*

Then, $f(x)\not< f(y)$.

Theorem 4.3. *(Direct duality) Assume that $\bar{x} \in \Omega$ is a weak Pareto solution (a Pareto solution) in $(MP)_{(h,\phi)}$ and there exist $\bar{\lambda} \in \mathbb{R}^k$ and $\bar{\mu} \in \mathbb{R}^m$ such that the conditions*

$$0 \in \left(\oplus_{i=1}^{k} \bar{\lambda}_i [\cdot] G'_{f_i}(f_i(\bar{x})) \otimes \partial^* f_i(\bar{x})\right) \oplus \left(\oplus_{j \in J(\bar{x})} \bar{\mu}_j [\cdot] G'_{g_j}(g_j(\bar{x})) \otimes \partial^* g_j(\bar{x})\right), \tag{60}$$

$$\bar{\mu}_j [\cdot] G_{g_j}(g_j(\bar{x})) = 0, \quad j = 1, \ldots, m, \tag{61}$$

$$\bar{\lambda} = (\bar{\lambda}_1, \ldots, \bar{\lambda}_k) \geq 0, \quad \bar{\mu} = (\bar{\mu}_1, \ldots, \bar{\mu}_m) \geqq 0 \tag{62}$$

are fulfilled at $\bar{x}$ with these Lagrange multipliers. Then, $(\bar{x}, \bar{\lambda}, \bar{\mu})$ is feasible in $(VD)_{(h,\varphi)}$. If all hypotheses of Theorem 4.1 (resp. Theorem 4.2) are satisfied, then $(\bar{x}, \bar{\lambda}, \bar{\mu})$ is a weak efficient solution (an efficient solution) of a maximum type in problem $(VD)_{(h,\varphi)}$ and optimal values in both (h,φ)-nondifferentiable vector optimization problems are the same.

Proof
The feasibility of $(\bar{x}, \bar{\lambda}, \bar{\mu})$ in $(VD)_{(h,\varphi)}$ follows directly from (60)-(62). Then, the weak efficiency of a maximum type (efficiency of a maximum type) of $(\bar{x}, \bar{\lambda}, \bar{\mu})$ in $(VD)_{(h,\phi)}$ follows from weak duality (Theorem 4.1 or Theorem 4.2, respectively).

□

Theorem 4.4. *(Converse duality). Let $(\bar{y}, \bar{\lambda}, \bar{\mu})$ be a weakly efficient solution (an efficient solution) of a maximum type in the Mond-Weir vector dual problem $(VD)_{(h,\varphi)}$ with $\bar{y} \in \Omega$. Furthermore, assume that any one of the following hypotheses is fulfilled:*

1. *(f, g) is (strictly) (h,φ)-G-type I at $\bar{y} \in Y$ on $\Omega \cup Y$ with respect to η,*
2. *(f, g) is (strictly) (h,φ)-G-pseudo-quasi type I at $\bar{y} \in Y$ on $\Omega \cup Y$ with respect to η.*

Then $\bar{y}$ is a weak Pareto (a Pareto solution) in the considered (h,φ)-nondifferentiable multiobjective programming problem $(MP)_{(h,\varphi)}$.

Proof
Proof of this theorem is similar to the proof of Theorem 4.1 (resp. Theorem 4.2).

□

Theorem 4.5. *(Strict converse duality). Let $\bar{x}$ and $(\bar{y}, \bar{\lambda}, \bar{\mu})$ be feasible solutions in $(MP)_{(h,\varphi)}$ and $(VD)_{(h,\varphi)}$, respectively, such that*

$$f(\bar{x}) = f(\bar{y}). \tag{63}$$

Further, assume that (f, g) is strictly (h,φ)-G-type I at $\bar{y} \in Y$ on $\Omega \cup Y$ with respect to η. Then $\bar{x} = \bar{y}$ and, therefore, $\bar{x}$ is a Pareto optimal solution in $(MP)_{(h,\varphi)}$ whereas $(\bar{y}, \bar{\lambda}, \bar{\mu})$ is an efficient solution of a maximum type in $(VD)_{(h,\varphi)}$.

Proof
Let $\bar{x}$ and $(\bar{y}, \bar{\lambda}, \bar{\mu})$ be feasible solutions in $(MP)_{(h,\varphi)}$ and $(VD)_{(h,\varphi)}$, respectively, such that (63) is satisfied. Let us suppose $\bar{x} \neq \bar{y}$ as, if not, the result would be proved. By assumption, (f, g) is strictly (h,φ)-G-type I at $\bar{y} \in Y$ on $\Omega \cup Y$ with respect to η. Then, by Definition 3.1, the inequalities

$$G_{f_i}(f_i(\bar{x}))[-]G_{f_i}(f_i(\bar{y})) > \left(\left(G'_{f_i}(f_i(\bar{y}))[\cdot]A_i\right)^T \eta(\bar{x}, \bar{y})\right)_{(h,\varphi)}, \tag{64}$$

$$(-1)[\cdot]G_{g_j}(g_j(\bar{y})) \geqq \left(\left(G'_{g_j}(g_j(\bar{y}))[\cdot]B_j\right)^T \eta(\bar{x}, \bar{y})\right)_{(h,\varphi)} \tag{65}$$

hold for any $A_i \in \partial^* f_i(\bar{y}), \quad i = 1,\ldots,k, \quad B_j \in \partial^* g_j(\bar{y}), \quad j = 1,\ldots,m$, respectively. Combining (63) and (64), by Lemma 2.4 b), the inequalities

$$G'_{f_i}(f_i(\bar{y}))[\cdot](A_i^T \eta(\bar{x},\bar{y}))_{(h,\varphi)} < 0, \quad i = 1,\ldots,k$$

are satisfied for any $A_i \in \partial^* f_i(\bar{y}), \quad i = 1,\ldots,k$. Hence, by $(\bar{y},\bar{\lambda},\bar{\mu}) \in Q$ and Lemma 2.3 b) and c), we obtain that

$$\bar{\lambda}_i\,[\cdot]\,G'_{f_i}\,(f_i(\bar{y}))\,[\cdot]\,\left(A_i^T \eta(\bar{x},\bar{y})\right)_{(h,\varphi)} \leqq 0, \quad i = 1,\ldots,k. \tag{66}$$

$$\lambda_{i^*}\,[\cdot]\,G'_{f_{i^*}}\,(f_{i^*}(\bar{y}))\,[\cdot]\,\left(A_{i^*}^T \eta(\bar{x},\bar{y})\right)_{(h,\varphi)} < 0 \quad \text{for at least one } i^* \in \{1,\ldots,k\}. \tag{67}$$

Then, by Lemma 2.3 f), (66) and (67) imply

$$\left[\sum_{i=1}^{k}\right] \bar{\lambda}_i\,[\cdot]\,G'_{f_i}\,(f_i(\bar{x}))\,[\cdot]\,\left(A_i^T \eta(x,\bar{y})\right)_{(h,\varphi)} < 0. \tag{68}$$

Thus, by Lemma 2.5 g), (68) gives that the inequality

$$\left(\oplus_{i=1}^{k}\left(\bar{\lambda}_i\,[\cdot]\,G'_{f_i}\,(f_i(\bar{y})) \otimes A_i\right)^T \eta(\bar{x},\bar{y})\right)_{(h,\varphi)} < 0 \tag{69}$$

is satisfied for any $A_i \in \partial^* f_i(\bar{y}), i = 1,\ldots,k$. By $(\bar{y},\bar{\lambda},\bar{\mu}) \in Q$ and Lemma 2.3 a), (65) implies

$$(-1)[\cdot]\bar{\mu}_j[\cdot]G_{g_j}(g_j(\bar{y})) \geqq \left(\left(\bar{\mu}_j[\cdot]G'_{g_j}(g_j(\bar{y}))[\cdot]B_j\right)^T \eta(\bar{x},\bar{y})\right)_{(h,\varphi)} j = 1,\ldots,m. \tag{70}$$

Hence, by the second constraint of $(VD)_{(h,\varphi)}$, (70) gives

$$\left(\left(\bar{\mu}_j[\cdot]G'_{g_j}(g_j(\bar{y}))[\cdot]B_j\right)^T \eta(\bar{x},\bar{y})\right)_{(h,\varphi)} \leqq 0 \quad j = 1,\ldots,m. \tag{71}$$

Then, by Lemma 2.4 e), (71) yields that the inequality

$$\left[\sum_{j=1}^{m}\right] \bar{\mu}_j[\cdot]G'_{g_j}(g_j(\bar{y}))[\cdot]\left(B_j^T \eta(\bar{x},\bar{y})\right)_{(h,\varphi)} \leqq 0$$

is satisfied for any $B_j \in \partial^* g_j(\bar{y}), j = 1,\ldots,m$. Then, by Lemma 2.5 g), the inequality above gives

$$\left(\oplus_{j=1}^{m}\left(\bar{\mu}_j[\cdot]G'_{g_j}(g_j(\bar{y})) \otimes B_j\right)^T \eta(\bar{x},\bar{y})\right)_{(h,\varphi)} \leqq 0. \tag{72}$$

Combining (69) and (72), we obtain that the inequality

$$\left(\oplus_{i=1}^{k}\left(\bar{\lambda}_i\,[\cdot]\,G'_{f_i}\,(f_i(\bar{y})) \otimes A_i\right)^T \eta(\bar{x},\bar{y})\right)_{(h,\varphi)} [+]$$
$$\left(\oplus_{j=1}^{m}\left(\bar{\mu}_j[\cdot]G'_{g_j}(g_j(\bar{y})) \otimes B_j\right)^T \eta(\bar{x},\bar{y})\right)_{(h,\varphi)} < 0$$

is satisfied for any $A_i \in \partial^* f_i(\bar{y})$, $i = 1,\ldots,k$, $B_j \in \partial^* g_j(\bar{y})$, $j = 1,\ldots,m$, contradicting the first constraint of $(VD)_{(h,\varphi)}$, that is, the feasibility of $(\bar{y},\bar{\lambda},\bar{\mu})$ in $(VD)_{(h,\varphi)}$. The fact that $\bar{x}$ is a Pareto optimal solution in $(MP)_{(h,\varphi)}$ and $(\bar{y},\bar{\lambda},\bar{\mu})$ is an efficient solution of maximum type in $(VD)_{(h,\varphi)}$ follows from the weak duality. □

5. Conclusion

In the paper, a new class of (h,φ)-nondifferentiable multiobjective programming problems in which every component of involved functions is a Lipschitz function has been considered. The aforesaid class of (h,φ)-nondifferentiable multiobjective programming problems has been defined by using generalized algebraic operations introduced by Ben-Tal [7]. Namely, the class of (h,φ)-nondifferentiable multiobjective programming problems has been investigated in which the functions involved are (h,φ)-G-type I functions and/or generalized (h,φ)-G-type I functions. The introduced concepts of (h,φ)-nondifferentiable G-type I functions and/or generalized (h,φ)-G-type I functions turned out to be useful to development optimality conditions for a feasible solution to be a weak Pareto solution (resp. a Pareto solution) in the considered (h,φ)-nondifferentiable multiobjective programming problem. Further, for the considered (h,φ)-nondifferentiable multiobjective programming problem, its Mond-Weir vector dual problem has been defined also by using generalized algebraic operations introduced by Ben-Tal [7]. Then, several duality theorems have been established also under assumptions that the functions involved are (h,φ)-nondifferentiable G-type I functions and/or generalized (h,φ)-G-type I functions. Thus, the sufficiency of optimality conditions and several duality results in the sense of Mond-Weir have been proved for the aforesaid class of nonconvex (h,φ)-nondifferentiable multiobjective programming problems.

Thus, the results established earlier in the optimization literature have been extended to a new class of nonconvex (h,φ)-subdifferentiable multiobjective programming problems. It seems that the results established in the paper can be established in a similar way for other classes of (h,φ)-subdifferentiable vector optimization problems.

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Declarations

The authors declare that they have no conflict of interest.

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