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Portfolio selection problem: main knowledge and models (A systematic review)

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Abstract The challenge in portfolio optimization lies in creating a collection of assets that attains a target expected return while mitigating risk. This problem is often framed as an optimization task, specifically Mean Variance Optimization (MVO). MVO involves formulating an objective function, typically quadratic, that is contingent on the composition of the portfolio, and linear constraints that represents the portfolio's asset allocation restriction. Several improvements have been proposed, such as adding constraints to MVO or using alternative risk measures. As a result, even though MVO model remains the most widely studied type of portfolio optimization, different types of portfolio optimization models, risk/return measurements and constraints have been suggested and used since its invention. In this work, we delve into the various risk and return measures, constraints, and mathematical models commonly used in portfolio optimization. We discuss the key risk measures employed in portfolio optimization, including the Sharpe ratio, beta, maximum drawdown and others, We explore the constraints commonly applied in portfolio optimization. Furthermore, we delve into the mathematical models utilized in portfolio optimization. Then, we emphasize the interplay between risk and return measures, constraints, and mathematical models in portfolio optimization. By providing a comprehensive overview of risk and return measures, constraints, and mathematical models, this work aims to enhance the understanding of portfolio optimization techniques and facilitate informed decision-making in the field of investment management. To illustrate different knowledge and models, several experiments were conducted on well-known real data portfolios.

Keywords Portfolio optimization problem, Risk and Return measurements, Mean Variance Optimization (MVO), portfolio's asset allocation, decision-making.

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1. Introduction

Portfolio optimization, often referred to as the portfolio selection problem, involves crafting a collection of assets or securities with the goal of attaining a targeted expected return while mitigating risk. The foundational principles of modern portfolio theory were pioneered by Markowitz in 1952 with the introduction of the mean-variance model. Within this model, the portfolio's variance serves as a metric for risk assessment [1]. The formulation of the portfolio selection problem involves an optimization task with real-valued variables.The objective function adopts a quadratic structure, contingent on the portfolio's asset allocation, and linear constraints depict the limitations on this allocation. Markowitz's seminal work from (1959) provides an introductory presentation of this model [2].

Despite its foundational significance, portfolio optimization remains a intricate and formidable optimization challenge, given its high-dimensional nature and the nonlinear interconnections between asset returns. In response to these complexities, previous scholars, like Rolland (1997) [3] and Chang et al. (2000) [4], have suggested local search techniques, exemplifying ways to address the intricacies of the portfolio selection problem.

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This work aims to delve into various risk and return measures, constraints, and mathematical models commonly employed in portfolio optimization. Firstly, we examine key risk measures used in portfolio optimization, including beta, maximum drawdown, Treynor ratio, and Jensen's alpha. These measures quantitatively assess different aspects of risk and return, providing valuable insights for informed portfolio construction. Next, we explore the constraints typically applied in portfolio optimization, encompassing factors such as asset allocation limits, quantity constraints, and cardinality constraints. The impact of these constraints on portfolio construction and their implications for risk and return trade-offs are discussed. We emphasize the significance of aligning constraints with specific investment objectives and risk preferences. Furthermore, we delve into the mathematical models utilized in portfolio optimization. In addition to the traditional mean-variance optimization (MVO), we discuss advanced optimization methods, such as the Capital Asset Pricing Model (CAPM)[5], which are the inspiration for different advanced methods including Value-at-Risk (Var) optimization [6], Conditional Value-at-Risk (CvaR) optimization [7], and multi-period optimization. These models offer diversified approaches for addressing the complexities of the portfolio selection problem. Finally, the interplay between risk and return measures, constraints, and mathematical models in portfolio optimization is emphasized, highlighting the significance of a holistic approach. And it is illustrated by several experiments conducted on the assets from the well-known S&P500 index. These real-life examples provide practical insights into the implementation and implications of different optimization techniques.[8, 9]

Additionally, when attempting to implement models for financial optimization, especially in portfolio optimization, one encounters the challenge of choosing suitable metrics for risk and return, or imposing constraints based on the model's objectives. Another hurdle is the varied formulations found in different papers, each introducing distinct risk and return measurements along with different constraints. This paper addresses these challenges by utilizing two prominent models in the financial optimization context. It aims to categorize and elucidate the diverse risk and return metrics available and explores key constraints that can be integrated into new models and formulations. In summary, the paper offers a comprehensive overview of risk and return measures, constraints, and mathematical models used in portfolio optimization. By examining various techniques and improvements beyond the traditional MVO, the paper aims to enhance the understanding of portfolio optimization techniques and equip investors with the knowledge to make better-informed investment decisions.

2. Main Knowledge

Portfolio Theory (MPT), introduced by Harry Markowitz in 1952 is one of the cornerstone theories in portfolio optimization is the Modern[1]. MPT serves as the foundation for numerous advancements in finance and portfolio optimization. It provides a framework for constructing portfolios that aim to maximize expected returns while managing risk efficiently. MPT aligns with the Efficient Market Hypothesis (EMH), which posits that markets are efficient, and asset prices reflect all available information. Additionally, MPT emphasizes that the risk of a portfolio is not solely determined by the risks of individual assets, but rather by the interactions among the assets. Diversification is thus recognized as a crucial risk management strategy, as the correlations between assets significantly impact portfolio performance. In the context of MVO (Mean Variance Optimization), the process involves estimating expected returns and standard deviation as measures of return and risk, respectively. These measures are utilized to construct an efficient frontier, representing portfolios that offer the highest expected return for a given level of risk. Subsequently, investors can select portfolios from the efficient frontier that align with their risk tolerance and investment objectives. The combination of MPT and MVO provides a comprehensive approach to portfolio optimization, enabling investors to make informed decisions and achieve a balance between risk and return. This concept leads us to one of the fundamental principles in finance known as the risk-return compromise or trade-off. According to this principle, to potentially attain higher returns, investors must be willing to accept a higher level of risk. As a result, risk and return have been central aspects explored during the evolution and advancement of the field of investing. In response, more sophisticated measures have emerged to account for various factors [10].

2.1. Return

In the context of investment, the concept of return is defined by the Oxford dictionary as the profit gained from an investment during a specific time, represented as a ratio of the initial investment. In the following section, we explore the diverse types of returns that have been introduced and employed in the context of portfolio management. These returns are broadly categorized into two main types: Absolute returns and relative returns.

That said, let.

- *Pc: Current Price of the asset,*
- *Pi: Initial Price of the asset,*
- R_p *: Portfolio return.*
- R_f *: Risk free rate.*

2.1.1. Absolute returns

1. Simple returns which are the percentage change in the price of an asset over a period. Used to calculate the returns of individual assets in a portfolio. Then the simple returns can be used to calculate the expected return of a portfolio, which is the anticipated amount of returns that a portfolio may generate, by taking the weighted average of the simple returns of the individual assets.

$$\textit{Simple Return} = \frac{P_c - P_i}{P_i} \tag{1}$$

   Equation (1) shows the mathematical formulation for calculating simple return.

2. In contrast to simple returns which are easy to calculate and are the most used in practice for centuries since their introduction in the early days of finance. Log returns are additive (That is, if we have a series of log returns for a period of time, we can add them up to get the total log return for that period.), more tractable statistically, and better for modeling stock prices thanks to their normal distribution. For that, they are the most used in academic research since their introduction [12]. To calculate the log return of a portfolio, you can use the weighted average of the log returns of the individual assets in the portfolio.

$$r_t = \ln\left(\frac{P_t}{P_{t-1}}\right) \tag{2}$$

   Equation (2) shows the mathematical formulation for calculating logarithmic return.Where P_t is the price of the asset at time t and P_{t-1} is the price of the asset at time t-1.

3. We can also mention total returns which measure the overall return of an investment, considering all forms of return, including price appreciation, dividends, and interest. Which will be referring to as C_R.

$$\textit{Total Return} = \frac{P_c + C_R - P_i}{P_i} \tag{3}$$

Equation (3) shows the mathematical formulation for calculating total return.

2.1.2. ***Relative returns*** :

Risk-adjusted return Is a crucial concept in finance that aims to evaluate the profit or potential profit of investment, relative to the level of risk accepted to achieve it. It enables investors to compare and assess different investment opportunities based on their risk-return trade-offs.

In that, various risk-adjusted return measures have been introduced:

4. Excess Return (ER): as one of the first forms of risk adjusted return ER was introduced to represent the difference between the portfolio return and the risk-free rate of return.

$$ER = R_p - R_f \tag{4}$$

   Equation (4) shows the excess return.

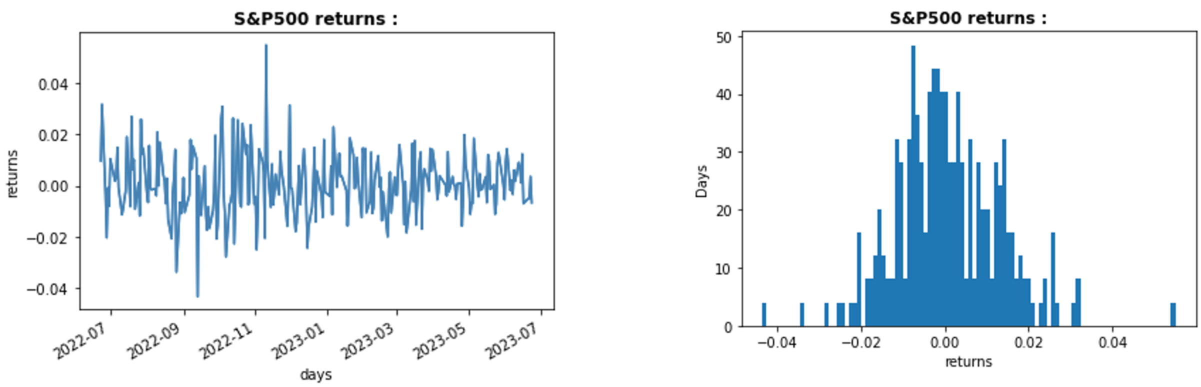

Figure 1. An example of the S&P500 returns over a 1 year period (Line graph and Histogram) .

5. Sharpe Ratio: Firstly, proposed by William F. Sharpe in 1966 [13]. The Sharpe Ratio is defined as the excess return of an asset (or portfolio) over the risk-free rate, divided by the asset's volatility (standard deviation). Which paved the way for more sophisticated risk adjusted return formulas .
6. Jensen's alpha: measures the excess return of an investment compared to its expected return, adjusting for its beta "β". [14]

$$R_p = \alpha + R_f + \beta_p * (R_m - R_f) \tag{5}$$

   Equation (5) shows the Jensen's aplha. Where $R_p - R_f$ is the portfolio excess return and $\beta_p * (R_m - R_f)$ is the expected excess return.

7. Treynor Ratio: introduced by Jack L. Treynor in 1965, which also incorporates the risk-free rate but employs systematic risk (β) as a measure of risk, rather than total volatility. The Treynor Ratio is especially relevant for evaluating the performance of diversified portfolios, where unsystematic risk is diversified away [15].

$$\textit{Treynor ratio} = \frac{R_p - R_f}{\beta_p} \tag{6}$$

   Equation (6) shows the Treynor's ratio.

8. Sortino ratio: proposed by Frank A. Sortino and Robert van der Meer in 1991[16], it focuses on downside risk. It uses the downside deviation (or semi-deviation) in the denominator instead of total volatility. The Sortino Ratio is particularly valuable when assessing investments where minimizing downside risk is a primary concern .

$$\textit{Sortino ratio} = \frac{R_p - R_f}{D_R} \tag{7}$$

   Equation (7) shows the Treynor's ratio. Where D_R is the Downside standard deviation which is the average of squared negative returns

2.2. Risk

Risk manifests in various forms, but its fundamental definition revolves around the potential for an investment's outcome or return to deviate from the expected result, often resulting in losses or unfavorable outcomes. This comprehensive concept encompasses multiple dimensions, such as market volatility, economic uncertainties, and interest rate fluctuations. Two primary categories of risk exist within investment contexts.

Systematic risk, also known as market risk or undiversifiable risk, represents factors that affect the entire market or a particular segment of it. These elements are beyond an investor's control and can impact a wide range of assets. Examples of systematic risk include economic recessions, geopolitical events, and changes in interest rates. Diversification across various assets may help mitigate, but not eliminate, the impact of systematic risk.

Conversely, **Unsystematic risk**, also referred to as specific risk or diversifiable risk, is unique to individual investments or companies. It arises from factors that are specific to a particular asset or its industry, such as company mismanagement, labor strikes, or product recalls. Unsystematic risk can be minimized through diversification, as holding a diversified portfolio of assets reduces the impact of adverse events affecting any single investment.

2.2.1. ***Unsystematic Risk*** :
Total risk measures, also known as unsystematic risk measures, capture the overall variability or dispersion of returns for a particular investment, considering all sources of risk that are specific to that investment.

One commonly employed metric for overall risk assessment is the standard deviation, denoted as σ , measuring the spread of returns relative to the mean return [20]. The higher the standard deviation, the greater the total risk. There are numerous other unsystematic risk measurements available,the most prominent ones are:

1. Variance: Also known as mean Absolute Deviation (MAD) which measures the average deviation of individual asset returns from the portfolio's expected return. It provides a simple and intuitive measure of total risk that considers both positive and negative deviations from the mean return [21].

$$\sigma_p = \frac{1}{N}\sqrt{\sum_{i=1}^{N} \omega_i \ * \sigma_i \ * \ \mathrm{Cor}_{i,p}} \tag{8}$$

Equation (8) shows the mean absolute deviation. Where;
ω_i : is the weight for asset i.
σ_i : is the standard deviation of asset i.
$\mathrm{Cor}_{i,p}$: is the correlation between asset I and the portfolio.

1. Semi-variance: focuses on negative deviations from the expected return, penalizing downside volatility more than upside volatility. It is particularly useful when investors are more concerned about minimizing losses than maximizing gains. [22]

$$\sigma_v = \frac{1}{N}\sum_{i=1}^{n} min((R_i \ - \overline{R}),\ 0)^2 \tag{9}$$

Equation (9) shows the semi variance. Where;
R_i: is the return of asst i.
$\overline{R}$: is the mean return of the observations below the threshold.

2. Value at Risk (Var): Var quantifies the maximum potential loss an investment portfolio may experience at a given confidence level over a specified time horizon t. VaR is defined as the α-th quantile of the portfolio's profit and loss distribution P&L(t). It helps investors understand the downside risk and aids in setting risk tolerances. [6]

$$VaR_i(t) = inf\{x \mid P[P\&L(t) \le x] \ge i\} \tag{10}$$

Equation (10) shows the mathematical formulation to calculate value at risk. Where x represents a quantile of the distribution of returns.

3. Conditional Value at Risk (CvaR): CVaR, also known as expected shortfall, calculates the average of all returns below the VaR level. It provides a more comprehensive view of potential losses beyond the VaR threshold. It is the expected value of the tail of the distribution beyond VaR. [7]

$$CVaR_i(t) = \frac{1}{(1-i)}\int_0^i \mathrm{VaR}_{\,\beta}(t)\ d\beta \tag{11}$$

Equation (11) shows the mathematical formulation to calculate the conditional value at risk. **Where x represents a quantile of the distribution of returns.

2.2.2. ***Systematic risk*** :

Systematic risk measures, also known as market risk measures, focus on the risk that cannot be diversified away and is inherent to the overall market or a particular asset class. The market return being (*Rm*).

4. Beta (β): The most prominent systematic risk measure is beta (β), which quantifies the sensitivity of an investment's returns to fluctuations in the market benchmark[5]. A β >1 indicates the asset is more volatile than the market, while a β< 1 suggests it is less volatile . ù

$$\beta = \frac{Cov(Ri,\ Rm)}{Var(Rm)} \tag{12}$$

   Equation (12) shows the mathematical formulation of beta .

5. Alpha (α): represents the excess return earned by an asset or portfolio above what would be expected based on its beta and the risk-free rate. It measures an investment's ability to outperform the market[25].

$$\alpha = R_i - [R_f + \beta * (R_m - R_f)] \tag{13}$$

   Equation (13) shows the mathematical formulation of alpha.

6. R-Squared (R²): Definition: R-Squared quantifies the proportion of an asset's variability that can be explained by market movements (captured by beta). It provides insights into how well an asset's performance aligns with overall market performance [26].

$$R^2 = (\mathrm{Cor}_{R_i,R_m})^2 \tag{14}$$

   Equation (14) shows the mathematical formulation of R^2.

7. Tracking Error measures, the deviation of an investment portfolio's returns (*Ri*), from its benchmark index (Rbi) [27]. It indicates how closely the portfolio's performance follows the benchmark.

$$TE = \sqrt{\sum_{i=0}^{n} (\omega_i \ * \ (R_i \ - \ R_{bi}))^2} \tag{15}$$

   Equation (15) shows the mathematical formulation of the tracking error measure.

8. Maximum Drawdown: The maximum drawdown represents the maximum loss experienced by an investment during a specific period. It assesses the largest loss an investor would have incurred had they invested at the peak and sold at the trough [28].

$$MD = \frac{P_m - P_k}{P_m} \tag{16}$$

   Equation (16) shows the simple return. Where;
   P_m : Peak value
   P_k: Trough value

2.2.3. ***Portfolio comparison*** :

We can compare portfolios using the efficient frontier where no other portfolio can offer more expected return for that same level of risk. The question remains how we can compare two portfolios generated by the MVO and which coexist on the efficient frontier.

The answer is performing one of the following measures on the portfolios that reside on the efficient frontier:

9. Relative Return RR: Definition: Relative Return measures the performance of an investment or portfolio relative to a benchmark or market index.

$$RR = R_p - R_b \tag{17}$$

Equation (17) shows the mathematical formulation for calculating relative return. Where;
R_b: is the return of a Benchmark

10. Information Ratio (IR): Definition: Information Ratio evaluates the risk-adjusted return of a portfolio relative to a benchmark, considering the tracking error or standard deviation of the active return [30].

$$IR = \frac{RR}{TE} \tag{18}$$

Equation (18) shows the mathematical formulation of the Information ratio. Where;
RR: is the relative return
TE: is the tracking error

11. For this reason, in this paper we're going to be using the Sharpe Ratio [31]. which is a key metric used to assess the risk-adjusted return of a portfolio. It is calculated as the difference between the portfolio return and the risk-free rate, divided by the portfolio risk:

$$SR = \frac{R_p - R_f}{\sigma_p} \tag{19}$$

Equation (19) shows the mathematical formulation of the Sharpe ratio.

3. Models

Accordingly, the portfolio returns $E[R]$ which is an aggregate measure of the returns of the assets included in the portfolio.Is calculated as follows:

$$E[R] = \sum_{i=1}^{N} \omega i * Ri \tag{20}$$

The portfolio risk (σp) on the other hand isn't just a weighted average of portfolio returns, but instead is the result of the interaction of the assets within the portfolio:

$$\sigma_p = \frac{1}{N}\sqrt{\sum_{i=1}^{N} \omega i \ * \sigma i \ * \ \mathrm{Cor}_{i,p}} \tag{21}$$

Thus, the problem can be considered as an optimization problem with the objective function of maximizing the return.

$$Maximize(E[R]) \tag{22}$$

Subject to $\sigma p \leq \ \sigma p_c$

Where;
α_{pc}: is the target risk which the maximal level of risk accepted for the portfolio.
$\mathrm{Cor}_{i,p}$: The correlation between Asset i return and Portfolio return.
α_i: Standard deviation of the returns of asset i.
ω_i: The weight of asset i in the portfolio.
R_i: Expected return of asset i.

3.1. MVO

Mean-Variance Optimization (MVO), introduced by Harry Markowitz in 1952[1], is a fundamental concept in modern portfolio theory that aims to construct an efficient investment portfolio . The main objective of MVO is to find the optimal allocation of assets in a portfolio to achieve a desired level of return while minimizing the portfolio's overall risk.

In the context of MVO, the expected return of a portfolio is denoted as E(Rp), and the risk of the portfolio is typically measured by its variance or standard deviation, denoted as Var (Rp) or α (Rp), respectively.

The mathematical formulation of the MVO formulates the portfolio optimization problem as an optimization with a quadratic objective function and linear constraints. Aiming to answer the question:

“ Given a target return and weights constraints, how to allocate the portfolio so risk is minimized?” [2]

Thus, the problem can be considered as an optimization problem with the objective function of maximizing the return.

$$Minimize\left(\sum_{i=1}^{N}\omega i\ *\sigma i\ *\ \mathrm{Cor}_{i,p}\ \right) \tag{23}$$

Subject to;

$$\textstyle\sum_{i=1}^{N} Ri*\omega_i = R_{pc}$$

$$\sum_{i=1}^{N}\omega i\ = 1$$

$$\mathit{0} \leq \omega i\ \leq \mathit{1}$$

The objective function (23) seeks to minimize the variance or standard deviation of the portfolio's returns, which represents its overall risk. The constraints ensure that the portfolio's weights add up to 1, meaning that the entire investment is fully allocated, and the expected return of the portfolio is a weighted average of the expected returns of individual assets.

MVO aims to find the set of portfolio weights that lie on the (efficient frontier/ Markowitz bullet) which represents the optimal trade-off between risk and return. The efficient frontier is a curve that connects the portfolios with the lowest possible risk for each level of expected return. The investor can choose any point on this curve based on their risk tolerance and return objectives.

3.2. CAPM

The Capital Asset Pricing Model (CAPM), introduced by William F. Sharpe in 1964 [5], is a foundational theory in modern finance that establishes the relationship between an asset's expected return and its systematic risk within a well-diversified portfolio. The CAPM provides a systematic framework for pricing risky assets and estimating their required rate of return. It assumes that investors are rational and risk-averse, aiming to maximize their utility by making efficient portfolio choices [5].

The central equation of the CAPM is as follows:

$$\textit{E[Ri]} = R_f + \beta_i * (E[R_m] - R_f) \tag{24}$$

Where;
$E[R_m]$: Expected return of the market portfolio.
$E[Ri]$: Expected return of asset i.
R_f : Risk-free rate of return.

β_i : Beta of asset i.

The CAPM suggests that an asset's expected return *E[Ri]* is equal to the risk-free rate Rf plus a premium proportional to its beta β_i times the excess return of the market portfolio $E[R_m] - R_f$.
β measures the sensitivity of the asset's returns to movements in the overall market. An asset with a higher beta is expected to experience larger price fluctuations in response to market movements, and thus, it should provide a higher expected return to compensate investors for the additional risk [5] .
The CAPM has been extensively studied and applied in empirical finance, and its practical implications have been explored in various real-world scenarios. Researchers have investigated the validity of the CAPM assumptions, the estimation of market risk premiums, and the implications of the model's predictions for portfolio construction and asset pricing [38].

While the CAPM has played a crucial role in financial theory and practice, critics have raised concerns about its underlying assumptions, such as the efficient markets hypothesis and homogenous expectations. Subsequent research has explored alternative asset pricing models, like the Fama-French Three-Factor Model and the Arbitrage Pricing Theory, which aim to capture additional factors that influence asset returns beyond just market risk.

In summary, the Capital Asset Pricing Model has been an influential theory in finance, providing insights into how investors price and evaluate risky assets. Despite its limitations, the CAPM has paved the way for further developments in asset pricing and portfolio management, and it remains a fundamental reference point in modern financial analysis.

3.3. Constraints

Portfolio optimization involves a process that achieves specific objectives while adhering to various constraints. These constraints are essential to ensure that the resulting portfolio aligns with the investor's preferences, risk tolerance, and investment objectives [40].
Some of the most used constraints are:

1. Budget constraint requires that the total allocation of assets in the portfolio should not exceed the total available investment budget.
$$\sum_{i=1}^{N} \omega i = 1 \tag{25}$$
Equation (25) shows a mathematical formulation of the budget constraint. Where ω_i is the weight associated with asset i, and N is the number of the available assets.

2. Quantity constraints: constraints specify the minimum and maximum weights that can be assigned to each individual asset in the portfolio. These constraints help to control the level of diversification and concentration in the portfolio, ensuring that the investor does not over or under allocate to specific assets.
$$\omega_{min} \leq \omega_i \leq \omega_{max} \tag{26}$$
Equation (26) shows a mathematical formulation of the quantity constraint. Such us $i \epsilon \{1, n\}$. Where;
ω_{max}: the maximum weight allowed for an asset i,
and ω_{min}: the minimum weight allowed for an asset i.

3. Cardinality constraints: It limits the number of assets that can be included in the final portfolio [40]. The cardinality constraint is particularly useful when an investor wants to control the level of diversification or prefers to work with a more concentrated portfolio. TWhich ensures that the number of assets included in the portfolio does not exceed the specified cardinality K .
$$\sum_{i=0}^{n} \mathbb{1}(\omega i) \leq K \tag{27}$$

Equation (27) shows a mathematical formulation of the cardinality constraint. Where;

$$\mathbb{1}(x) = \begin{cases} 1, & if x \neq 0 \\ 0, & else \end{cases}$$

4. Risk constraints limit the total risk exposure of the portfolio, typically measured by the portfolio's standard deviation or variance. By setting an upper limit on risk, investors can manage their risk tolerance and avoid excessive exposure to market fluctuations.
5. Return Objective: While not a constraint per se, the return objective defines the target return that the investor aims to achieve with the portfolio. The optimization process aims to maximize the portfolio's return while adhering to the specified constraints.
6. Sector or industry constraints restrict the exposure of the portfolio to specific sectors or industries. Investors may want to limit the allocation to certain sectors to manage sector-specific risks or comply with regulatory requirements.
7. Liquidity constraints ensure that the portfolio remains sufficiently liquid, allowing the investor to buy or sell assets without significantly impacting their prices. These constraints are crucial, especially for large institutional investors who need to manage cash flows effectively.

4. Experimentation

4.1. Case study (S&P500)

The Standard & Poor's 500 (S&P 500) stands as a benchmark stock market index, meticulously designed to monitor the performance of 500 prominent companies listed on United States stock exchanges. Revered as the most renowned portfolio, it is widely acknowledged as the standard for evaluating the overall vitality of the U.S. stock market. The index employs a market capitalization-weighted approach, wherein companies with higher market values wield a more substantial influence on the index's overall performance.
Consequently, investors frequently turn to it as a barometer for gauging the U.S. economy's well-being, leveraging its insights to inform their investment decisions.

4.2. Data description

Since the inception of portfolio optimization, the landscape of created portfolios has expanded exponentially through continuous refinement and optimization. Among the vast array of portfolios, certain standout ones have emerged, known as benchmark portfolios, with examples including S&P 500, NASDAQ 100, Russel 1000, and Russel 2000, which are also referred to as market indices. For the purposes of this study, we primarily focused on the S&P 500, a prominent index encompassing more than 500 stocks.

Throughout time, the constituent stocks of these portfolios may undergo changes in name or undergo stock splits, wherein a company divides its shares to decrease the price and increase the total number of shares available. To ensure data consistency, we retained only those assets with available data, accounting for stock splits that have occurred over the past 10 years. By curating the dataset in this manner, we aimed to maintain data integrity and relevance, facilitating accurate and meaningful analyses in our study.

Namely, our data consists of 437 out of the S&P500 stocks. Those are the stocks with a 10 year span data is available. Since there's only 252 trading days in one year on the New York stock exchange (NYSE) and according to the USA national holiday calendar. Several days are missing from the data we had. Which can also be caused by technical glitches and company circumstances.

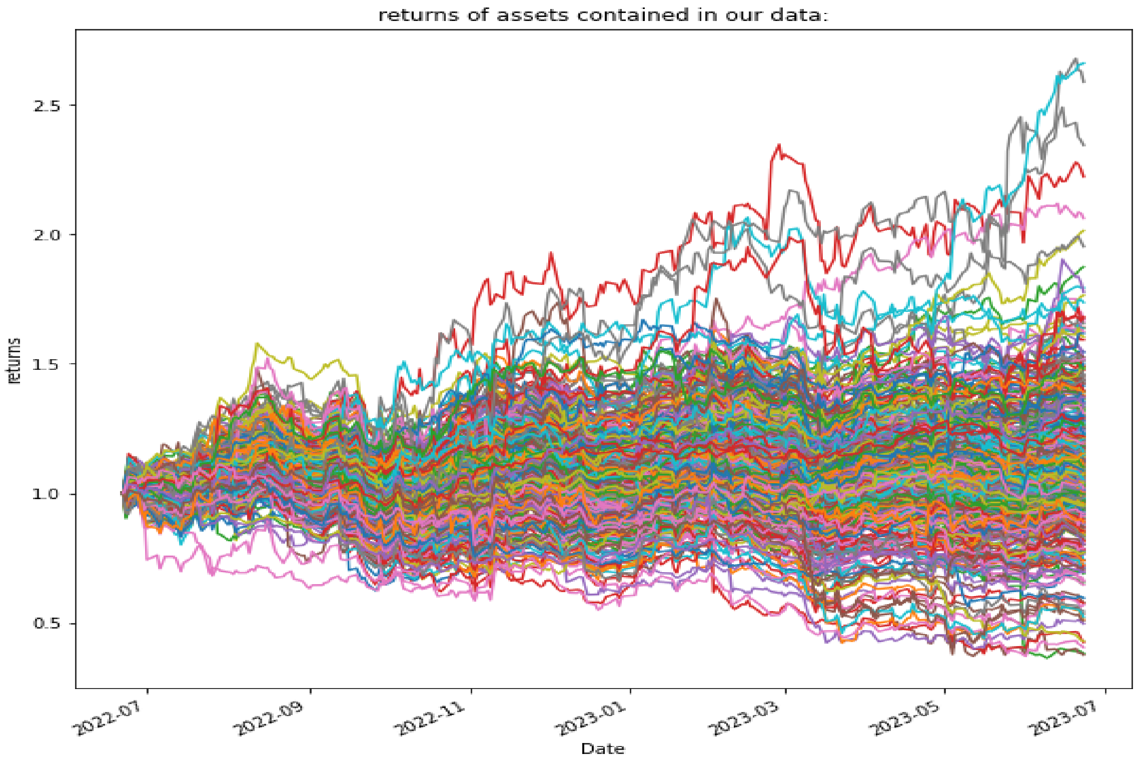

Figure 2. An example of the numerous assets contained in our data over a one-year period.

Hence, the obligation of data cleaning. We first tried linear interpolation, which hadn't much use since it relies a lot on future data. For that reason, we used the forward into backward filling, which depends on the data already available and filling the data according to it.

Which resulted in a time series data of 437 stocks from the S&P500 index with all data available over the last 10 years. A daily data consisting of those 437 stocks Opening price, Highest and lowest prices of the day, the daily closing price which already accounts for stock splits and finally the adjusted price which accounts for dividends payments which are regular profit-sharing payments made between a company and its investors.

4.3. Statistical testing

One of the most mentioned limitations of the MVO are its assumptions on the normality of the return's distribution. For that, we performed statistical testing on our data to incorporate that, here's the example on the International Business Machines Corporation (IBM) stock.

As in figure 3, as though returns show to have a bell curve shape. It's obvious that it had heavy tails. For that reason, we tried the t-distribution which didn't give us as good results as for the mixture gaussian model, also known as Gaussian Mixture Model (GMM)[44], is a probabilistic model used in statistics and machine learning for modeling complex data distributions. It is a combination of multiple Gaussian (normal) distributions, each representing a component of the overall data distribution. GMM is a generative model, which means it can be used to model the underlying distribution of a dataset and generate new data points that follow a similar distribution.

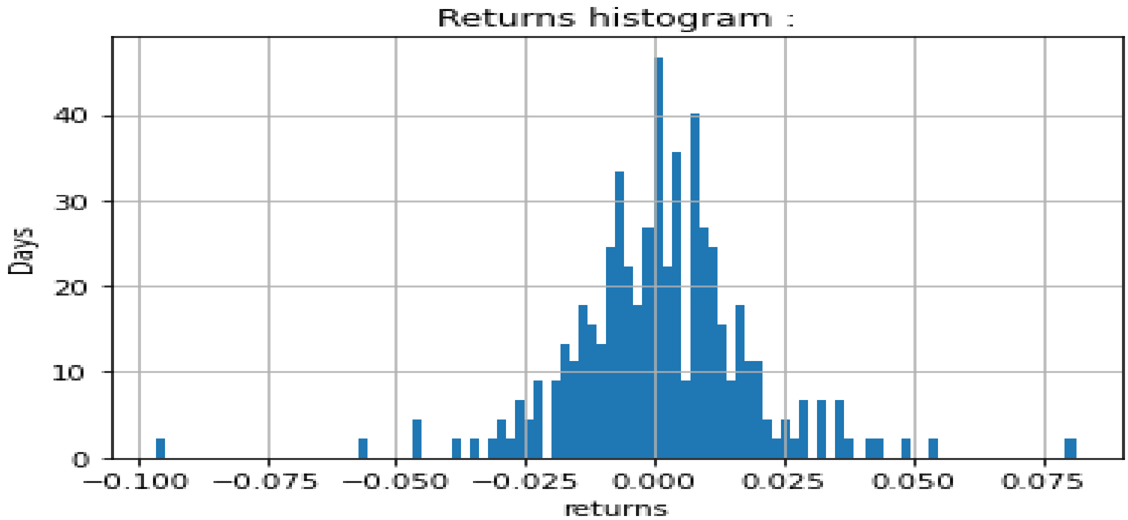

Figure 3. IBM returns histogram .

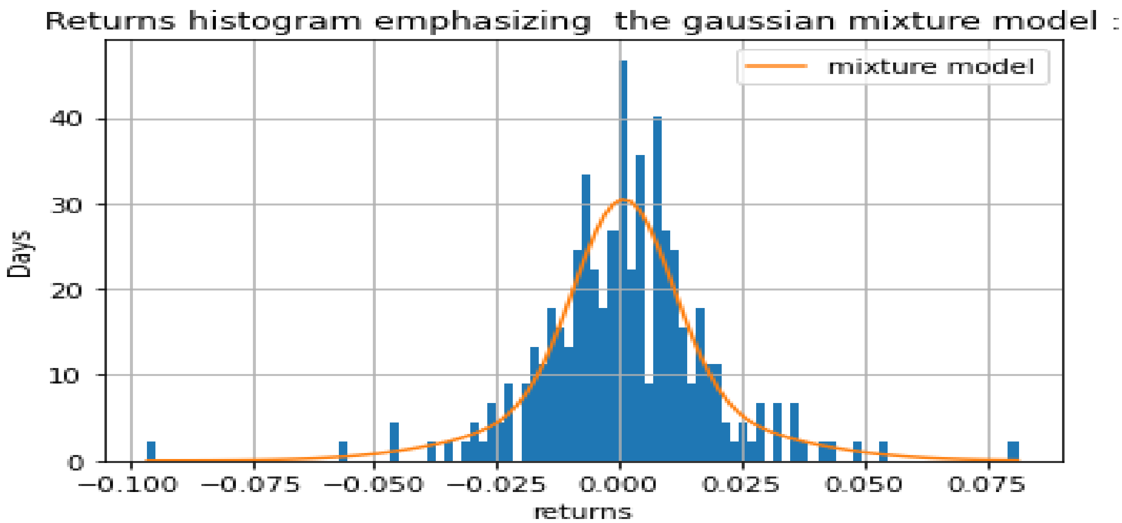

Figure 4. IBM returns histogram emphasizing the gaussian mixture model.

5. Results

5.1. Example 1

To illustrate the above we conducted the MVO on a technology focused subset of stocks from the dataset presented above such as Google, Nvidia and Netflix. The following figure represents the set of portfolios constructed from the selected stocks in a risk/return plan. And the efficient frontier can be found on the upper part of the figure:

That said, as Tobin stated in his Separation Theorem " Every optimal portfolio is a combination of a risk-free asset and the market portfolio ". [45] In other words, the one with the highest Sharpe ratio. Accordingly, here's the figure above when the portfolios are combined with a risk-free asset and colored according to their Sharpe ratio:

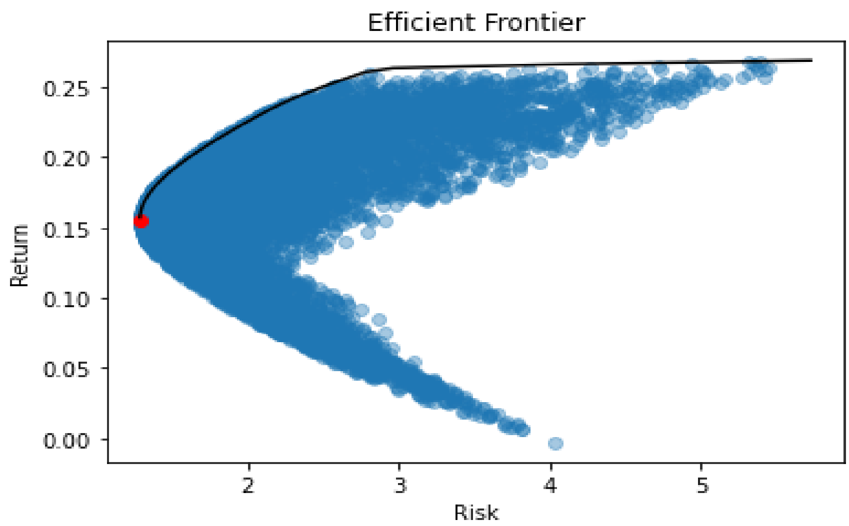

Figure 5. The efficient frontier on the different portfolios generated.

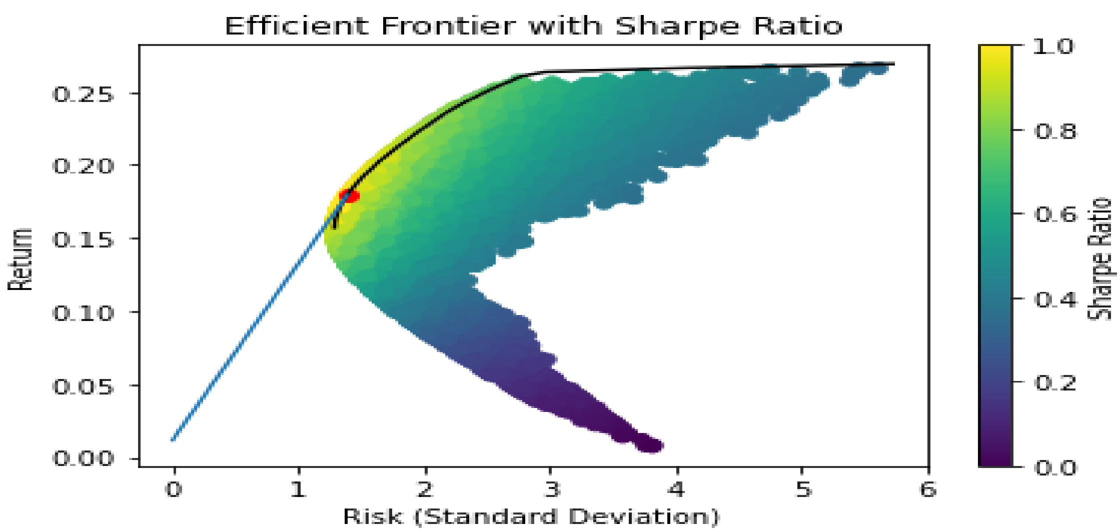

Figure 6. Efficient frontier emphasizing the Sharpe ratio.

We then selected the 5 best portfolios according to the MVO/Sharpe ratio approach and then performed the different risk return measurements introduced above.

The following tables give a summary of all the measurements mentioned above.

Table 1. Here are the absolute return measurements applied to the 5 selected portfolios.

	Mean return	Log return	Excess Return
Portfolio 1	0.182309	0.089820	0.112504
Portfolio 2	0.178610	0.103450	0.108805

	Mean return	Log return	Excess Return
Portfolio 3	0.179774	0.120429	0.109969
Portfolio 4	0.180591	0.113222	0.110786
Portfolio 5	0.174595	0.090866	0.104790

Table 2. Here are the relative return measurements applied to the 5 selected portfolios.

	Sharpe Ratio	Sortino Ratio	Treynor Ratio
Portfolio 1	0.79368	0.68080	6.526327
Portfolio 2	0.79350	0.67624	7.173840
Portfolio 3	0.79681	0.68041	8.303109
Portfolio 4	0.79669	0.68128	7.474107
Portfolio 5	0.79582	0.67316	8.222190

Table 3. Here are the unsystematic risk measurements applied to the 5 selected portfolios.

	Variance	Semi Variance	VaR	CvaR
Portfolio 1	1.409230	0.518878	2.010951	2.611962
Portfolio 2	1.380832	0.497567	2.066951	2.542309
Portfolio 3	1.386280	0.518559	2.005802	2.585181
Portfolio 4	1.392714	0.517393	1.995379	2.594624
Portfolio 5	1.347372	0.489455	1.918185	2.507187

Table 4. Here are the systematic risk measurements applied to the 5 selected portfolios.

	Max Drawdown	R^2	Beta	Alpha
Portfolio 1	-2.929000	-0.010931	0.023338	0.151380
Portfolio 2	-2.627593	-0.010956	0.020716	0.147786
Portfolio 3	-3.021091	-0.011186	0.018038	0.149056
Portfolio 4	-2.990874	-0.011103	0.020148	0.149789
Portfolio 5	-2.716175	-0.011028	0.017586	0.143895

5.2. Example 2

To embrace the fundamental principle of diversification, a cornerstone of Modern Portfolio Theory, we present a second illustrative example. In this scenario, we veer away from a concentration in technology-laden equities. Instead, we seek to enhance portfolio diversification by incorporating stocks from distinct categories. Notable additions include shares from Kohl's Corporation, Starbucks, and Boeing, representing commodity-centric equities, along with a foray into other sectors. Notably, Newmont Corporation, a gold mining enterprise, is also introduced. The resultant outcomes are as follows.

The following figure represents the set of portfolios constructed from the selected stocks in a risk/return plan. And the efficient frontier can be found on the upper part of the figure:

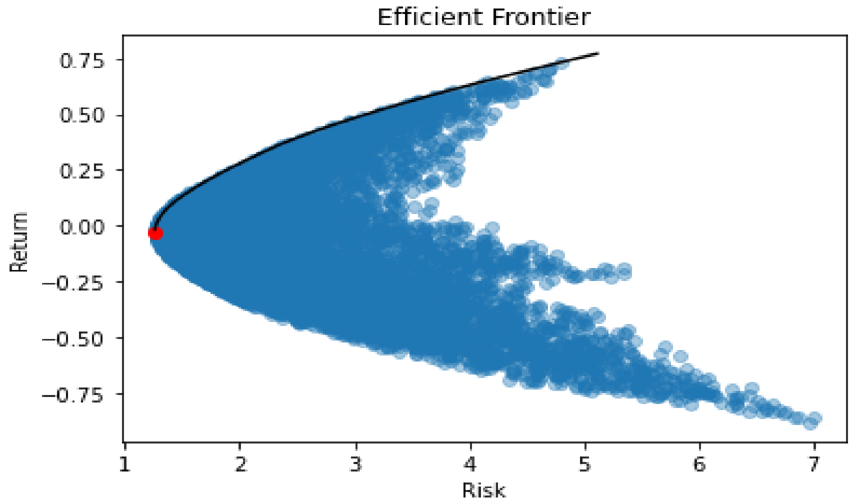

Figure 7. The efficient frontier on the different portfolios generated.

Accordingly, here's the figure above when the portfolios are combined with a risk-free asset and colored according to their Sharpe ratio: Same as the example above, we then selected the best portfolios according to

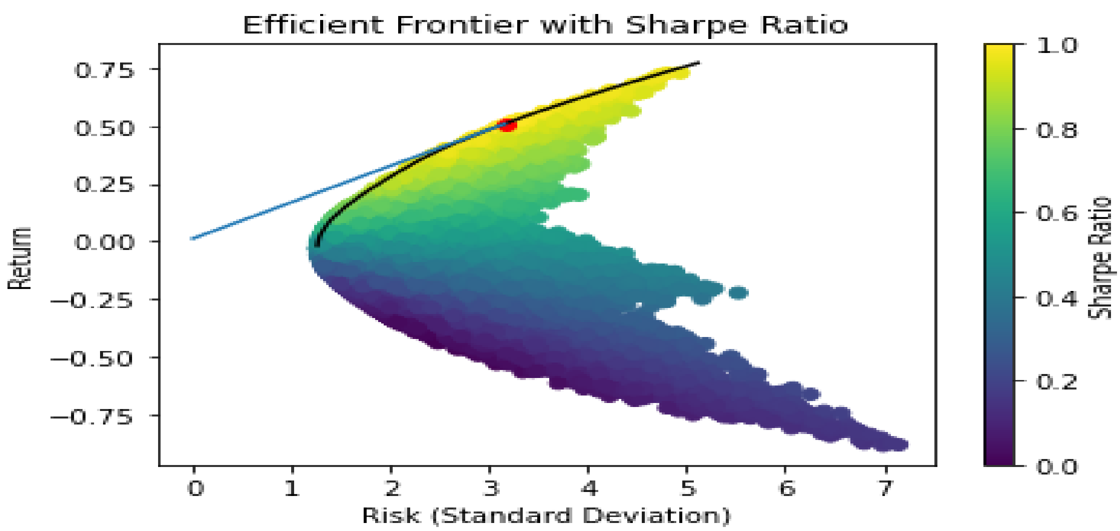

Figure 8. Efficient frontier emphasizing the Sharpe ratio

the MVO/Sharpe ratio approach and then performed the different risk return measurements introduced above. Which can be seen in the following tables:
The following tables give a summary of all the measurements mentioned above. **Table 5.** Here are the absolute

return measurements applied to the 5 selected portfolios.

	Mean return	Log return	Excess Return
Portfolio 1	0.438509	0.438533	0.308314
Portfolio 2	*0.392153*	0.228770	0.261958
Portfolio 3	*0.464360*	0.353327	0.434165
Portfolio 4	*0.464367*	0.350493	0.434172
Portfolio 5	*0.421884*	0.445889	0.391689

Table 6. Here are the relative return measurements applied to the 5 selected portfolios.

	Sharpe Ratio	Sortino Ratio	Treynor Ratio
Portfolio 1	0.87490	-0.194908	20.282043
Portfolio 2	*0.85047*	-0.193651	16.117867
Portfolio 3	*0.84683*	-0.191912	19.942289
Portfolio 4	*0.84397*	-0.191885	16.248180
Portfolio 5	*0.84314*	-0.191362	16.857955

Table 7. Here are the unsystematic risk measurements applied to the 5 selected portfolios.

	Variance	Semi Variance	Value at Risk (VaR)	Conditional Value at Risk (CvaR)
Portfolio 1	2.470604	2.608934	3.943834	5.841626
Portfolio 2	*2.240278*	2.178763	3.709194	5.413386
Portfolio 3	*2.658297*	2.975683	4.416247	6.178680
Portfolio 4	*2.662696*	3.032056	4.350503	6.304190
Portfolio 5	*2.420248*	2.481323	3.933731	5.683849

Table 8. Here are the systematic risk measurements applied to the 5 selected portfolios.

	R^2	Max Drawdown	Beta	Alpha
Portfolio 1	-0.035908	-56.427506	0.023100	0.469428
Portfolio 2	*-0.035381*	-6.562240	0.026192	0.423196
Portfolio 3	*-0.03453*	-2.361047	0.024790	0.495347
Portfolio 4	*-0.034351*	-37.666507	0.030426	0.495578
Portfolio 5	*-0.034753*	-2.410615	0.026805	0.452951

Conclusion

While acknowledging the diversity in formulations across various studies, it's essential to note that a comprehensive and exhaustive comparison of all the papers is challenging. However, we tried our best throughout this paper, to emphasize the interplay between risk and return measures, constraints, and mathematical models in portfolio optimization. We illustrate 2 of the main models in the context of portfolio optimization (MVO and CAPM). The different risk and return measurements and the importance of considering practical considerations, such as transaction costs and liquidity, in portfolio optimization.

Through our efforts in this paper, we frequently encountered the application of local search algorithms. As a result, we are inclined to believe that local search algorithms hold significant promise for improving the efficiency and effectiveness of portfolio optimization.

Subsequently, our intention is to extend these algorithms to various iterations of the portfolio selection problem, encompassing both discrete and continuous versions. Additionally, we aim to explore hybrid approaches, combining local search with other search paradigms.

By providing a comprehensive overview of risk and return measures, constraints, and mathematical models, this paper aims to enhance the understanding of portfolio optimization techniques and facilitate informed decision-making in the field of investment management.

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