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Multi-sensors search for lost moving targets using unrestricted effort

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Abstract This paper addresses the problem of searching for multiple targets using multiple sensors, where targets move randomly between a limited number of states at each time interval. Due to the potential value or danger of the targets, multiple sensors are employed to detect them as quickly as possible within a fixed number of search intervals. Each search interval has an available search effort and an exponential detection function is assumed. The goal is to develop an optimal search strategy that distributes the search effort across cells in each time interval and calculates the probability of not detecting the targets throughout the entire search period. The optimal search strategy that minimizes this probability is determined, the stability of the search is analyzed, and some special cases are considered. Additionally, we introduce the M-cells algorithm.

Keywords Search plan, Available resources, Lost targets, Multi-sensors, Probability of undetection.

AMS 2010 subject classifications 60K30,90B40

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1. Introduction

The investigation of search strategies for lost targets, whether stationary or randomly moving, is crucial and has found numerous applications recently. These include searching for missing persons on roads, cancer cells in the human body, the black box of a plane crash in the depths of the sea or ocean, underground gold mines, land and naval mines, and faulty units in extensive linear systems like electrical power lines, telephone lines, and mining systems (see [1–5]), among others. The process of searching for a randomly located target is also discussed (see [6, 7]). Often, the objective is to maximize the probability of locating the target while the search effort is constrained (see [8–10]). Additionally, a generalized search for a randomly moving target is presented (see [11]). Recently, studies have focused on a lost target acting as a random walker on one of two intersecting real lines, with the aim of detecting the target as quickly as possible (see [12]). A coordinated search algorithm for a lost target on a plane has been proposed (see [13]). Furthermore, a quasi-coordinated search technique for a lost target, assumed to move randomly on one of two disjoint lines according to a random walk motion, has been introduced (see [14]). In recent years, searches for a randomly moving coronavirus (COVID-19) among a finite set of different states and the detection of a randomly moving COVID-19 have been conducted (see [15, 16]). Recently, a method for searching for a randomly located target in a bounded area divided into a finite set of cells has been introduced (see [17]). Here, targets are assumed to be in one of several cells, which are not necessarily identical regions. Let the number of cells be m. Our objective is to develop a search strategy that minimizes the probability of not detecting the targets, where the total effort is unrestricted, and the sensors operate independently at the cell level.

The structure of this paper is as follows: Section 2 introduces the problem, its notations, and the solution. Section 3 examines the stability of the search. Section 4 discusses special cases. Section 5 presents the M-cells algorithm

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for detecting targets moving through a finite set of cells using sensors. Section 6 provides applications, and Section 7 lists the tables. Finally, the conclusion summarizes the work.

2. The search problem

The notations for the problem are summarized in Table 1.

K	Set of sensors, $K = \{1, 2, \cdots, L\}$.
J	Set of cells, $J = \{1, 2, \cdots, m\}$.
R	Set of targets, $R = \{1, 2, \cdots, w\}$.
I	Set of time intervals, $I = \{1, 2, \cdots, n\}$.
k	Sensor index.
j	Cell index.
r	Target index.
i	Time interval index.
P_{ij}^r	Probability that target r exist at cell j at time interval i.
$l_i(z)$	The effort during the i^{th} time interval.
z_{ij}^{kr}	The allocation of the search effort during interval i at cell j by sensor k to search about target r.
$b(i, j, k, r, z_{ij}^{kr})$	The conditional probability of detecting the target r at interval i with z_{ij}^{kr} effort given that the target is lcoated in state j by sensor k.

Table 1

For a search about set of targets R which are assumed to be hidden throughout set of cells J, set of sensors K must disperse their efforts throughout the set of cells during i time interval to decrease the probability of non-detection of the set of targets, not necessarily identical cells. The probability that the target r exist in state j at time interval i is represented by p_{ij}^r. Throughout the i^{th} time interval, the effort is made by $l_i(z)$. The sensors search activities are denoted by z_{ij}^{kr}, the effort put by the sensor k in the cell j during period i to detect target r. We call $Z = \left[\sum_{r=1}^{w}\sum_{k=1}^{L} Z_{ij}^{kr}\right]$ the search plan. The conditional probability of detecting the target r at time i with z_{ij}^{kr} effort is given by $b(i, j, k, r, z_{ij}^{kr})$, which is defined by the detection function. Here, it is assumed that the probability of detection target r in state j at time interval i by sensor k depends simply on the overall effort put out and not on the manner in which it is directed.

We assume that the sensors are independent at different time intervals and targets action is unrelated to the sensor activities. The probability of undetection of the targets over the duration, (see [7]), is given by

$$H(Z) = \prod_{i=1}^{n}\sum_{j=1}^{m}\prod_{r=1}^{w}\prod_{k=1}^{L} P_{ij}^{r}(1 - b(i, j, k, r, z_{ij}^{kr})) \tag{1}$$

and the effort is

$$l(Z) = \sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{r=1}^{w}\sum_{k=1}^{L} z_{ij}^{kr} \tag{2}$$

we want to minimize

$$(H(Z), l(Z)) \tag{3}$$

Subjected to constraints $Z_{ij}^{kr} \geq 0$ and $\sum_{j=1}^{m} P_{ij}^{r}$, $r = 1, 2, \cdots, w$, where the problem is called multi-objectives optimization problem. Let the detection function is exponential then the problem becomes

$$\left.\begin{array}{l}\min(H(Z), l(Z)), \text{ where} \\ H(Z) = \prod_{i=1}^{n} \sum_{j=1}^{m} \prod_{r=1}^{w} \prod_{k=1}^{L} P_{ij}^{r} e^{-z_{ij}^{kr}/T_{j}^{kr}}, \\ l(Z) = \sum_{i=1}^{n} \sum_{j=1}^{m} \sum_{r=1}^{w} \sum_{k=1}^{L} z_{ij}^{kr}, \\ l_i(Z) = \sum_{j=1}^{m} \sum_{r=1}^{w} \sum_{k=1}^{L} z_{ij}^{kr}, \\ Z_{ij}^{kr} \geq 0, k = 1, 2, \cdots, L, i = 1, 2, \cdots, n, r = 1, 2, \cdots, w \text{ and } j = 1, 2, \cdots, m, \sum_{j=1}^{m} P_{ij}^{r} = 1, \end{array}\right\} \tag{4}$$

where T_{j}^{kr} is the mean effort to detection target r by sensor k in state j. We use the so-called ε constraint approach to resolve this problem. The ideal resolution of the following problem may be used to describe the effective solutions to problem (4).

$$\left.\begin{array}{l}\min \prod_{i=1}^{n} \sum_{j=1}^{m} \prod_{r=1}^{w} \prod_{k=1}^{L} P_{ij}^{r} e^{-z_{ij}^{kr}/T_{j}^{kr}}, \text{ where} \\ l(Z) = \sum_{i=1}^{n} \sum_{j=1}^{m} \sum_{r=1}^{w} \sum_{k=1}^{L} z_{ij}^{kr} \leq \varepsilon, \\ l_i(Z) = \sum_{j=1}^{m} \sum_{r=1}^{w} \sum_{k=1}^{L} z_{ij}^{kr} \leq \varepsilon_i, Z_{ij}^{kr} \geq 0, \text{ and} \\ \sum_{j=1}^{m} P_{ij}^{r} = 1, \text{ and for all } k = 1, 2, \cdots, L, i = 1, 2, \cdots, n, r = 1, 2, \cdots, w \text{ and } j = 1, 2, \cdots, m. \end{array}\right\} \tag{5}$$

Since $Z_{ij}^{kr} \geq 0$ for all $k = 1, 2, \cdots, L$, $i = 1, 2, \cdots, n$, $r = 1, 2, \cdots, w$ and $j = 1, 2, \cdots, m$, $\sum_{j=1}^{m} \sum_{r=1}^{w} \sum_{k=1}^{L} z_{ij}^{kr} \leq \varepsilon_i$ provides the feasible domain ε_i, which must be $\varepsilon_i \geq 0$. Now, any unique solution to problem (5) is now an effective solution to the problem (4). The assurance of the uniqueness is result to convexity of $H(Z)$, over the following convex set.

$Z(\varepsilon_i) \{Z \in D^{\alpha\beta} / l_i(Z) \leq \varepsilon_i,\ Z_{ij}^{kr} \geq 0,$ for $\alpha = Lw,\ k = 1, 2, \cdots, L,\ r = 1, 2, \cdots, w$ and $\beta = nm,\ i = 1, 2, \cdots, n,\ j = 1, 2, \cdots, m\}$. We can prove that $H(Z)$ is convex function (see [18]) and the unique solution is guaranteed by the convexity of it. We are able to resolve (6), a nonlinear optimization problem with a set of linear constraints $Z(\varepsilon_i)$. $H(Z)$ is convex and hence using Kuhn-Tucker theorem,(see [19]) we obtain

$$\frac{\partial H(Z)}{\partial Z_{qc}^{sf}} + \sum_{q=1}^{h} U_q \frac{\partial g_q(Z)}{\partial Z_{qc}^{sf}},$$

$$g_q(Z) \leq 0,$$

$$U_q g_q(Z) = 0,$$

$$U_q \geq 0,$$

$g_i(Z) = l_i(Z) - \varepsilon_i \leq 0$, which implies that

$$\frac{-P^f_{q_c} e^{-Z^{sf}_{qc}/T^{sf}_e}}{T^{sf}_c} \prod_{\substack{i=1\\i\neq q}}^{n} \sum_{j=1}^{m} \prod_{\substack{r=1\\r\neq f}}^{w} \prod_{\substack{k=1\\k\neq s}}^{L} P^r_{ij} e^{-Z^{kr}_{ij}/T^{kr}_j} + U_q = 0,$$

$$l_q(Z) - \varepsilon_q \leq 0,$$

$$U_q(l_q(Z) - \varepsilon_q) = 0,$$

$$U_q \geq 0.$$

where U_q is the Lagrange variable, there are three cases:

1- $U_q = 0,\ i = 1, 2, \cdots, n$, that is, $l_q(Z) \leq \varepsilon_i$, and $\sum_{j=1}^{m} P^r_{ij} e^{-Z^{kr}_{ij}/T^{kr}_j} = 0$ but $P^r_{ij} \geq 0$ then $\sum_{j=1}^{m} P^r_{ij} e^{-Z^{kr}_{ij}/T^{kr}_j}$ $= 0$ is impossible.
2- $U_q = 0, i = 1, 2, \cdots, s, U_q > 0, i = s + 1, s + 2, \cdots, n$. We can conclude that this case is impossible.
3- $U_q > 0, q = 1, 2, \cdots, h$, we can get

$$Z^{kr}_{ij} = -T^{kr}_j \left\{ \ln\left(\left(\frac{T^{kr}_j}{P^r_{ij}}\right) \left(e^{-\frac{\varepsilon_i}{A}}\right) \cdot \prod_{j=1}^{m} \prod_{k=1}^{L} \prod_{r=1}^{w} \left(\left(\frac{P^r_{ij}}{T^{kr}_j}\right)^{\frac{T^{kr}_j}{A}} \right) \right) \right\} \tag{6}$$

then

$$H(Z) = \prod_{i=1}^{n} \sum_{j=1}^{m} \prod_{r=1}^{w} \prod_{k=1}^{L} (T^{kr}_j) \left(e^{-\frac{\varepsilon_i}{A}}\right) \cdot \prod_{j=1}^{m} \prod_{k=1}^{L} \prod_{r=1}^{w} \left(\left(\frac{P^r_{ij}}{T^{kr}_j}\right)^{\frac{T^{kr}_j}{A}} \right), \text{ where } A = \sum_{j=1}^{m} \sum_{k=1}^{L} \sum_{r=1}^{w} T^{kr}_j. \tag{7}$$

3. The stability of the search

The set of feasible parameters for problem (5) is described as $U = \{\varepsilon = Z(\varepsilon)/Z(\varepsilon) \neq \phi\}$.
Definition 1. Suppose that Z^* be an optimal solution for problem (5) corresponding to $\varepsilon_i \in Z(\varepsilon_i)$, then the stability set of the first kind is defined such that

$$\varepsilon_i \geq A \max_{j,k,r} \ln\left(\frac{T^{kr}_j}{P^r_{ij}}\right) + \sum_{j=1}^{m} \sum_{r=1}^{w} \sum_{k=1}^{L} T^{kr}_j \ln\left(\frac{P^r_{ij}}{T^{kr}_j}\right), \tag{8}$$

$$\varepsilon = \sum_{i=1}^{n} \varepsilon_i \geq A \sum_{i=1}^{n} \max_{j,k,r} \ln\left(\frac{T^{kr}_j}{P^r_{ij}}\right) + \sum_{i=1}^{n} \sum_{k=1}^{L} \sum_{j=1}^{m} \sum_{r=1}^{w} T^{kr}_j \ln(P^r_{ij}) - n \sum_{j=1}^{m} \sum_{k=1}^{L} \sum_{r=1}^{w} T^{kr}_j \ln(T^{kr}_j),$$

therefore

$$S(Z^*) = \Bigg\{ \varepsilon \in Z(\varepsilon) \Big/ \sum_{j=1}^{m} \sum_{r=1}^{w} \sum_{k=1}^{L} Z^{kr}_{ij} = \varepsilon_i,\ \ \varepsilon_i \geq A \max_{j,k,r} \ln\left(\frac{T^{kr}_j}{P^r_{ij}}\right) + \sum_{j=1}^{m} \sum_{r=1}^{w} \sum_{k=1}^{L} T^{kr}_j \ln\left(\frac{P^r_{ij}}{T^{kr}_j}\right),$$
$$\text{for all } i = 1, 2, \cdots, n \tag{9}$$
$$\varepsilon = \sum_{i=1}^{n} \varepsilon_i \geq A \sum_{i=1}^{n} \max_{k,j,r} \ln\left(\frac{T^{kr}_j}{P^r_{ij}}\right) + \sum_{i=1}^{n} \sum_{j=1}^{m} \sum_{k=1}^{L} \sum_{r=1}^{w} T^{kr}_j \ln(P^r_{ij}) - n \sum_{j=1}^{m} \sum_{k=1}^{L} \sum_{r=1}^{w} T^{kr}_j \ln(T^{kr}_j) \Bigg\}.$$

4. Special cases

First case (One sensor-multi moving targets):

We consider search for lost moving targets, by a sensor, and our purpose to obtain the search plan which made the probability of undetection is minimum, also find the stability of search. We can obtain the probability of undetection of the targets from equations (1, 2) at $k = 1$:

$$H_x(Z) = \prod_{i=1}^{n} \sum_{j=1}^{m} \prod_{r=1}^{w} P_{ij}^r e^{-z_{ij}^r / T_j^r}, \tag{10}$$

and the unrestricted effort is given by

$$l_x(Z) = \sum_{i=1}^{n} \sum_{j=1}^{m} \sum_{r=1}^{w} z_{ij}^r, \quad \sum_{j=1}^{m} P_{ij}^r = 1.$$

where Z_{ij}^r give the effort put in state j at time interval i to detect target r. Our aim is to find $Z = \left[Z_{ij}^r\right]$ to $\min(H_x(Z), I_x(Z))$. For this issue, we can get the problem formulation from equation (5) by put $k = 1$, then the corresponding single objective optimization problem becomes.

$$\left.\begin{array}{l} \min H_x(Z) = \prod\limits_{i=1}^{n} \sum\limits_{j=1}^{m} \prod\limits_{r=1}^{w} P_{ij}^r e^{-z_{ij}^r / T_j^r}, \text{ where } l_{x_i}(Z) = \sum\limits_{j=1}^{m} \sum\limits_{r=1}^{w} Z_{ij}^r \leq \varepsilon_i, \\ Z_{ij}^r \geq 0, \ i = 1, 2, \cdots, n, \ j = 1, 2, \cdots, m, \text{ and } r = 1, 2, \cdots, w, \ \sum\limits_{j=1}^{m} P_{ij}^r = 1. \end{array}\right\} \tag{11}$$

We can get the solution of this problem from equation (6-8) by put $k = 1$, we get:

$$Z_{ij}^r = -T_j^r \left\{ \ln \left(\frac{T_j^r}{P_{ij}^r} \right) \left(e^{-\frac{\varepsilon_i}{A}} \right) \cdot \prod_{j=1}^{m} \prod_{r=1}^{w} \left(\left(\frac{P_{ij}^r}{T_j^r} \right)^{\frac{T_j^r}{A_x}} \right) \right) \right\} \tag{12}$$

and then

$$H_x(Z) = \prod_{i=1}^{n} \sum_{j=1}^{m} \prod_{r=1}^{w} (T_j^r) \left(e^{-\frac{\varepsilon_i}{A_x}} \right) \left(\prod_{j=1}^{m} \prod_{r=1}^{w} \left(\frac{P_{ij}^r}{T_j^r} \right)^{\frac{T_j^r}{A_x}} \right), \qquad \text{where } A_x = \sum_{j=1}^{m} \sum_{r=1}^{w} T_j^r. \tag{13}$$

The stability set becomes

$$\begin{array}{l} S_x(Z^*) = \left\{ \varepsilon_i \in Z(\varepsilon_i) \middle/ \sum\limits_{j=1}^{m} \sum\limits_{r=1}^{w} z_{ij}^r = \varepsilon_i, \right. \\ \left. \varepsilon_i \geq A_x \max\limits_{j,r} \ln \left(\frac{T_j^r}{P_{ij}^r} \right) + \sum\limits_{j=1}^{m} \sum\limits_{r=1}^{w} T_j^r \ln \left(\frac{P_{ij}^r}{T_j^r} \right) \text{ for all } i = 1, 2, \cdots, n \right\} \end{array} \tag{14}$$

Second case (Multi Sensors-One Moving Target):

We consider search for a lost moving target, by set of sensors, and our purpose to obtain the search plan which made the probability of undetection is minimum, also find the stability of search. We can obtain the probability of undetection of the target from equations (1, 2) at $r = 1$:

$$H_y(Z) = \prod_{i=1}^{n} \sum_{j=1}^{m} \prod_{k=1}^{L} P_{ij} e^{-z_{ij}^k / T_j^k} \tag{15}$$

and the unrestricted effort is given by $l_y(Z) = \sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{k=1}^{L} z_{ij}^k, \sum_{j=1}^{m} P_{ij} = 1$,
where Z_{ij}^k give the effort put in state j at time interval i to detect the target by searcher k. Our aim is to find $z = [z_{ij}^k]$ to $\min(H_y(Z), l_y(Z))$. For this issue, we can get the problem formulation from equation (5) by put $r = 1$, then the corresponding single objective optimization problem becomes.

$$\left.\begin{array}{l} \min H_y(Z) = \prod_{i=1}^{n}\sum_{j=1}^{m}\prod_{k=1}^{L} P_{ij} e^{-z_{ij}^k/T_j^k}, \text{ where } l_{y_i}(Z) = \sum_{j=1}^{m}\sum_{k=1}^{L} Z_{ij}^k \leq \varepsilon_i, \\ Z_{ij}^k \geq 0,\ i = 1,2,\cdots,n,\ j = 1,2,\cdots,m, \text{ and } k = 1,2,\cdots,L,\ \sum_{j=1}^{m} P_{ij} = 1. \end{array}\right\} \tag{16}$$

We can get the solution of this problem from equations (6-8) by put $r = 1$, we get:

$$Z_{ij}^k = -T_j^k \cdot \ln\left(\left(\frac{T_j^k}{P_{ij}}\right)\left(e^{-\frac{\varepsilon_i}{A_y}}\right)\left(\prod_{k=1}^{L}\prod_{j=1}^{m}\left(\frac{P_{ij}}{T_j^k}\right)^{\frac{T_j^k}{A_y}}\right)\right) \tag{17}$$

and then

$$H(Z) = \prod_{i=1}^{n}\sum_{j=1}^{m}\prod_{k=1}^{L}(T_j^k)\left(e^{-\frac{\varepsilon_i}{A_y}}\right)\left(\prod_{k=1}^{L}\prod_{j=1}^{m}\left(\frac{P_{ij}}{T_j^k}\right)^{\frac{T_j^k}{A_y}}\right) \text{ where } A_y = \sum_{k=1}^{L}\sum_{j=1}^{m} T_j^k. \tag{18}$$

The stability set becomes

$$S_y(Z^*) = \left\{\varepsilon \in Z^*(\varepsilon) \middle/ \sum_{j=1}^{m}\sum_{k=1}^{L} Z_{ij}^k = \varepsilon_i, \right. \\ \left. \varepsilon_i \geq A_y \max_{j,k} \ln\left(\frac{T_j^k}{P_{ij}}\right) + \sum_{j=1}^{m}\sum_{k=1}^{L} T_j^k \ln\left(\frac{P_{ij}}{T_j^k}\right), \text{ for all } i = 1,2,\cdots,n\right\}. \tag{19}$$

Third case (One sensor-one moving target):

We consider search for a lost moving target, by a sensor and our purpose to obtain the search plan which made the probability of undetection is minimum, also find the stability of search, this case is studied (see [1]). We can obtain the probability of undetection of the target from equations (1, 2) at $r = 1$ and $k = 1$:

$$H_b(Z) = \prod_{i=1}^{n}\sum_{j=1}^{m} P_{ij} e^{-Z_{ij}/T_j} \tag{20}$$

and the unrestricted effort is given by $l_b(Z) = \sum_{i=1}^{n}\sum_{j=1}^{m} z_{ij}, \sum_{j=1}^{m} P_{ij} = 1$,
where Z_{ij} give the effort put in state j at time interval i. Our aim is to find $Z = [Z_{ij}]$ to $\min(H_b(Z), l_b(Z))$. For this issue, we can get the problem formulation from equation (5) by put $r = 1$ and $k = 1$, then the corresponding single objective optimization problem becomes

$$\left.\begin{array}{l} \min H_b(Z) = \prod_{i=1}^{n}\sum_{j=1}^{m} P_{ij} e^{-z_{ij}/T_j}, \text{ where } l_{b_i}(Z) = \sum_{j=1}^{m} Z_{ij} \leq \varepsilon_i, \\ Z_{ij} \geq 0,\ i = 1,2,\cdots,n, \text{ and } j = 1,2,\cdots,m,\ \sum_{j=1}^{m} P_{ij} = 1. \end{array}\right\} \tag{21}$$

We can get the solution of this problem from equations (6-8) by put $r=1$ and $k=1$, we get:

$$Z_{ij} = -T_j \left\{ \ln\left(\frac{T_j}{P_{ij}}\right)\left(e^{-\frac{\varepsilon_i}{A_b}}\right) \prod_{j=1}^{m} \left(\frac{P_{ij}}{T_j}\right)^{\frac{T_j}{A_b}} \right) \right\} \tag{22}$$

and then

$$H_b(Z) = \prod_{i=1}^{n}\sum_{j=1}^{m}(T_j)\left(e^{-\frac{\varepsilon_i}{A_b}}\right)\left(\prod_{j=1}^{m}\left(\frac{P_{ij}}{T_j}\right)^{\frac{T_j}{A_b}}\right), \quad \text{where } \; A_b = \sum_{j=1}^{m} T_j, \tag{23}$$

$$\begin{aligned} S_b(Z^*) = & \left\{ \varepsilon_i \in Z(\varepsilon_i) \Big/ \sum_{j=1}^{m} Z_{ij} \le \varepsilon_i, \right. \\ & \left. \varepsilon_i \ge A_b \max_j \ln\left(\frac{T_j}{P_{ij}}\right) + \sum_{j=1}^{m} T_j \ln\left(\frac{P_{ij}}{T_j}\right), \quad \text{for all } \; i = 1,2,\cdots,n \right\}. \end{aligned} \tag{24}$$

Forth case (Multi sensors-multi located targets):

We consider search for lost targets, which are located throughout a limited number of states and the object is to detect the targets as quickly as possible, by multi-sensors, through one interval. Let the targets are located in one of m states with probabilitym P_j, $j = 1,2,\cdots,m.$. We can obtain the probability of undetection of the targets from equations (1, 2) at $i=1$:

$$H_o(Z) = \sum_{j=1}^{m}\prod_{k=1}^{L}\prod_{r=1}^{w} P_j^r e^{-Z_j^{kr}/T_j^{kr}} \tag{25}$$

and the unrestricted effort is given by $l_o(Z) = \sum\limits_{j=1}^{m}\sum\limits_{r=1}^{w}\sum\limits_{k=1}^{L} Z_j^{kr}$, $\sum\limits_{j=1}^{m} P_j = 1$, where Z_j^{kr} give the effort put in state j, by sensor k to detect target r. Our aim is to find $Z = [Z_j^{kr}]$ to $\min(H_o(Z), l_o(Z))$. For this issue, we can get the problem formulation from equation (5) by put $i=1$, then the corresponding single objective optimization problem becomes.

$$\left. \begin{aligned} & \min H_o(Z) = \sum_{j=1}^{m}\prod_{k=1}^{L}\prod_{r=1}^{w} P_j^r e^{-Z_j^{kr}/T_j^{kr}}, \text{ where } l_o(Z) = \sum_{j=1}^{m}\sum_{r=1}^{w}\sum_{k=1}^{L} z_j^{kr} \le \varepsilon_0, \\ & Z_j^{kr} \ge 0, \; k = 1,2,\cdots,L, \; r = 1,2,\cdots,w \text{ and } j = 1,2,\cdots,m, \; \sum_{j=1}^{m} P_j^r = 1. \end{aligned} \right\} \tag{26}$$

We can get the solution of this problem from equations (6-8) by put $i=1$, we get:

$$Z_j^{kr} = -T_j^{kr} \left\{ \ln\left(\frac{T_j^{kr}}{P_j^r}\right)\left(e^{-\frac{\varepsilon_0}{A_o}}\right) \cdot \prod_{j=1}^{m}\prod_{k=1}^{L}\prod_{r=1}^{w}\left(\frac{P_j^r}{T_j^{kr}}\right)^{\frac{T_j^{kr}}{A_o}} \right\} \tag{27}$$

and then

$$H_o(Z) = \sum_{j=1}^{m}\prod_{r=1}^{w}\prod_{k=1}^{L}(T_j^{kr})\left(e^{-\frac{\varepsilon_0}{A_o}}\right) \cdot \prod_{j=1}^{m}\prod_{k=1}^{L}\prod_{r=1}^{w}\left(\left(\frac{P_j^r}{T_j^{kr}}\right)^{\frac{T_j^{kr}}{A_o}}\right), \quad \text{where } \; A_o = \sum_{j=1}^{m}\sum_{k=1}^{L}\sum_{r=1}^{w} T_j^{kr}. \tag{28}$$

The stability set becomes

$$S_o(Z^*) = \Bigg\{ \varepsilon \in Z(\varepsilon) \Big/ \sum_{j=1}^{m} \sum_{r=1}^{w} \sum_{k=1}^{L} Z_j^{kr} = \varepsilon_0, \\ \varepsilon_0 \geq A_0 \max_{j,k,r} \ln \left(\frac{T_j^{kr}}{P_j^r} \right) + \sum_{j=1}^{m} \sum_{r=1}^{w} \sum_{k=1}^{L} T_j^{kr} \ln \left(\frac{P_j^r}{T_j^{kr}} \right), \\ \text{for all } j = 1, 2, \cdots, m, \ k = 1, 2, \cdots, L \text{ and } r = 1, 2, \cdots, w. \Bigg\} \tag{29}$$

Fifth case (One sensor-one located target):

We consider search for a lost target, which is located throughout limited number of states, and the object is to detect the target as quickly as possible, by a sensor, through one interval. Let the target is located in one of m states with probability P_j, $j = 1, 2, \cdots, m$. We can obtain the probability of undetection of the target from equations (1, 2) at $i = 1$, $r = 1$ and $k = 1$:

$$H_\theta(Z) = \sum_{j=1}^{m} P_j e^{-Z_j/T_j} \tag{30}$$

and the unrestricted effort is given by $l_\theta(Z) = \sum_{j=1}^{m} z_j$, $\sum_{j=1}^{m} P_j = 1$, where Z_j give the effort put in state j. Our aim is to find $Z = [Z_j]$ to $\min(H_\theta(Z), l_\theta(Z))$. For this issue, we can get the problem formulation from equation (5) by put $i = 1$, $r = 1$ and $k = 1$, then the corresponding single objective optimization problem becomes.

$$\left. \begin{aligned} &\min H_\theta(Z) = \sum_{j=1}^{m} P_j e^{-Z_j/T_j} \text{ where } l_o(Z) = \sum_{j=1}^{m} z_j \leq \varepsilon_0, \\ &Z_j \geq 0, \ j = 1, 2, \cdots, m, \text{ and } \sum_{j=1}^{m} P_j = 1. \end{aligned} \right\} \tag{31}$$

We can get the solution of this problem from equations (6-8) by put $i = 1$, $r = 1$ and $k = 1$, we get:

$$Z_j = -T_j \left\{ \ln \left(\frac{T_j}{P_j} \right) \left(e^{-\frac{\varepsilon_0}{A_\theta}} \right) \cdot \prod_{j=1}^{m} \left(\frac{P_j}{T_j} \right)^{\frac{T_j}{A_\theta}} \right\} \tag{32}$$

and then

$$H_\theta(Z) = \sum_{j=1}^{m} T_j \left(e^{-\frac{\varepsilon_0}{A_\theta}} \right) \cdot \prod_{j=1}^{m} \left(\frac{P_j}{T_j} \right)^{\frac{T_j}{A_\theta}}, \text{ where } A_\theta = \sum_{j=1}^{m} T_j. \tag{33}$$

The stability set becomes

$$S_\theta(Z^*) = \left\{ \varepsilon \in Z(\varepsilon) \Big/ \sum_{j=1}^{m} Z_j = \varepsilon_0, \varepsilon_0 \geq A_\theta \max_j \ln \left(\frac{T_j}{P_j} \right) + \sum_{j=1}^{m} T_j \ln \left(\frac{P_j}{T_j} \right), \text{ for all } j = 1, 2, \cdots, m \right\}. \tag{34}$$

5. Algorithm

We update the previously described algorithm (see [20]). The goal of this method is to determine the optimal allocation of the search effort and the probability of undetection for M-cells.

The algorithm's steps may be summed up as follows:

Step 1: Input the values of:

$n \equiv$ the number of time intervals;
$m \equiv$ the number of states (cells);
$L \equiv$ the number of sensors;
$w \equiv$ the number of targets;
$P_{0r} \equiv$ the probability distribution of the initial state, for target r;
$P_r \equiv$ one-step transition probability matrix, for target r;
$T_j^{kr} \equiv$ the factor due to the search about the target r in cell j using sensor k;

Step 2: Compute $P_{ij}^r,\ i=1,2,\cdots,n,\ j=1,2,\cdots,m,\ r=1,2,\cdots,w$, which equivalent to the probability distribution of the moving target r, among the cells during the interval i.

Step 3: Input the arbitrary values of ε_i and test the stability condition

$$\varepsilon_i \geq A \max_{j,k,r} \ln\left(\frac{T_j^{kr}}{P_{ij}^r}\right) + \sum_{j=1}^{m}\sum_{r=1}^{w}\sum_{k=1}^{L} T_j^{kr} \ln\left(\frac{P_{ij}^r}{T_j^{kr}}\right),$$

if this condition is satisfied go to step 4 elsewhere go to step 3.
Step 4: Compute Z_{ij}^{kr} and $H(Z)$ from equations (6) and (7), respectively and then go to step 5.

Step 5: End (Stop).

6. Applications

6.1. Multi sensors-multi moving targets with unrestricted effort:

Suppose the targets move randomly and effort unrestricted, in accordance with three states Markov chain with transition matrices M_1, M_2, M_3 for them,

$$M_1 = \begin{vmatrix} 1/4 & 0 & 3/4 \\ 1/2 & 1/4 & 1/4 \\ 1/3 & 1/3 & 1/3 \end{vmatrix}, \qquad M_2 = \begin{vmatrix} 2/9 & 3/9 & 4/9 \\ 3/12 & 4/12 & 5/12 \\ 4/15 & 5/15 & 6/15 \end{vmatrix}, \qquad \text{and} \quad M_3 = \begin{vmatrix} 1/2 & 1/4 & 1/4 \\ 0 & 1/2 & 1/2 \\ 1/3 & 0 & 2/3 \end{vmatrix},$$

the initial probabilities are given by

$$M_1^0 = \begin{vmatrix} 1/3 & 1/3 & 1/3 \end{vmatrix}, \qquad M_2^0 = \begin{vmatrix} 1/2 & 1/4 & 1/4 \end{vmatrix}, \qquad \text{and} \quad M_3^0 = \begin{vmatrix} 1/2 & 1/3 & 1/6 \end{vmatrix},$$

where $i,j=1,2,3$, the distribution of the states of the targets P_{ij}^r, is given by $M_1^0M_1$, $M_2^0M_2$ and $M_3^0M_3$, see [21]. The values P_{ij}^r of are

$$\begin{array}{llllll}
P_{11}^1=0.361, & P_{12}^1=0.195, & P_{13}^1=0.444, & P_{21}^1=0.336, & P_{22}^1=0.197, & P_{23}^1=0.467, \\
P_{31}^1=0.338, & P_{32}^1=0.205, & P_{33}^1=0.457, & P_{11}^2=0.240, & P_{12}^2=0.333, & P_{13}^2=0.427, \\
P_{21}^2=0.251, & P_{22}^2=0.333, & P_{23}^2=0.416, & P_{31}^2=0.250, & P_{32}^2=0.333, & P_{33}^2=0.417, \\
P_{11}^3=0.305, & P_{12}^3=0.292, & P_{13}^3=0.403, & P_{21}^3=0.287, & P_{22}^3=0.222, & P_{23}^3=0.491, \\
P_{31}^3=0.307, & P_{32}^3=0.183, & P_{33}^3=0.510, & & &
\end{array}$$

Letting $T_{11}^1=0.01,\ T_{21}^1=0.02,\ T_{31}^1=0.03,\ T_{11}^2=0.04,\ T_{21}^2=0.05,\ T_{31}^2=0.06,\ T_{11}^3=0.07,\ T_{21}^3=0.08,$ $T_{31}^3=0.09, T_{12}^1=0.1, T_{22}^1=0.2, T_{32}^1=0.3, T_{12}^2=0.4, T_{22}^2=0.5, T_{32}^2=0.6, T_{12}^3=0.7, T_{22}^3=0.8, T_{32}^3=0.9$, and existence k sensors $k = 1, 2$. For arbitrary values of ε_i, which must satisfy that $l_i(Z) = \sum_{j=1}^{m} \sum_{r=1}^{w} \sum_{k=1}^{L} z_{ij}^{kr} \leq \varepsilon_i$ and equation of the stability (8), the values of The allocation about the search effort and The probability of undetection $H(Z)$ are given in Table 2.

And we will observe from the Table 2, that by increasing the arbitrary values of ε_i, the probability of undedection decreasing.

6.2. *One sensor-multi moving targets with unrestricted effort:*

We consider the data in case 6.1 and suppose there exists one sensor. Letting $T_1^1 = 1, T_2^1 = 1.2, T_3^1 = 1.4, T_1^2 = 1.6, T_2^2 = 1.8$, $T_3^2 = 2$, $T_1^3 = 2.2$, $T_2^3 = 2.4$, $T_3^3 = 2.6$ and existence one sensor. For arbitrary values of ε_i, which must satisfy that $l_{x_i}(Z) = \sum_{j=1}^{m} \sum_{r=1}^{w} z_{ij}^{r} \leq \varepsilon_i$ and equation of the stability (14), the values of the allocation about the search effort and the probability of undetection $H_x(Z)$ are given in Table 3.

And we will observe from Table 3, that by increasing the arbitrary values of ε_i, The probability of undetection decreasing.

6.3. *Multi sensors-multi located targets with unrestricted effort:*

Assume that the targets are located in one of five states with the following probabilities $P_1^1 = 0.10$, $P_2^1 = 0.15$, $P_3^1 = 0.20$, $P_4^1 = 0.25$ and $P_5^1 = 0.30$, $P_1^2 = 0.15$, $P_2^2 = 0.20$, $P_3^2 = 0.25$, $P_4^2 = 0.10$ and $P_5^2 = 0.30$ for all $r = 1, 2$, $j = 1, 2, 3, 4, 5$ and $k = 1, 2$. Letting $T_1^{11} = 0.01$, $T_2^{11} = 0.02$, $T_3^{11} = 0.03$, $T_4^{11} = 0.04$, $T_5^{11} = 0.05$, $T_1^{12} = 0.06$, $T_2^{12} = 0.07$, $T_3^{12} = 0.08$, $T_4^{12} = 0.09$, $T_5^{12} = 0.1$, $T_1^{21} = 0.2$, $T_2^{21} = 0.3$, $T_3^{21} = 0.4$, $T_4^{21} = 0.5$, $T_5^{21} = 0.6$, $T_1^{22} = 0.7$, $T_2^{22} = 0.8$, $T_3^{22} = 0.9$, $T_4^{22} = 0.95$, $T_5^{22} = 0.99$. The values of Z_j^{kr} and $H_o(Z)$, for arbitrary values of ε_0, which must satisfying condition of stability set $S_o(Z^*)$ are given in Table 4.

6.4. *Multi sensors-multi moving targets with restricted effort:*

We can calculate the previous case 6.1., where the effort is restricted. In state restricted effort, while in this state we put $l(Z)$ as a fixed value V, for illustration, we used $l(Z) = \sum_{i=1}^{n} l_i(Z) = V$, as an example, consider the case in example 6.1. but restricted effort, given by $L_i(Z) = 15$ for $i = 1, 2, 3$, $l(Z) = \sum_{i=1}^{3} l_i(Z) = 90$ units of effort, the values of the allocation about the restricted search effort and The probability of undetection $H(Z)$ are given in Table 5.

6.5. *One sensor-multi moving targets with restricted effort:*

We can calculate the previous case 6.2., where the effort is restricted. In state restricted effort, while in this state we put $l(Z)$ as a fixed value V, for illustration, we used $l(Z) = \sum_{i=1}^{n} l_i(Z) = V$, as an example, consider the case in example 6.2 but restricted effort, given by $l_i(Z) = 30$, for $i = 1, 2, 3$, $l(Z) = \sum_{i=1}^{3} l_i(Z) = 90$ units of effort, the values of The allocation about the restricted search effort and The probability of undetection $H(Z)$ are given in Table 6.

7. Tables of the search

Table 2

			Cell 1	**Cell 2**	**Cell 3**	**Cell 1**	**Cell 2**	**Cell 3**	**Cell 1**	**Cell 2**	**Cell 3**
ε_1			6.3			8.4			10		
1^{st} time interval	Sensor 1	Target 1	0.0485	0.0706	0.1181	0.0528	0.0791	0.1308	0.0560	0.0856	0.1405
		Target 2	0.1218	0.1560	0.1886	0.1387	0.1772	0.2141	0.1517	0.1934	0.2335
		Target 3	0.1893	0.1980	0.2351	0.2190	0.2319	0.2733	0.2416	0.2578	0.3024
	Sensor 2	Target 1	0.2524	0.2406	0.4753	0.2948	0.3255	0.6026	0.3271	0.3901	0.6995
		Target 2	0.3027	0.4372	0.5496	0.4724	0.6493	0.8041	0.6017	0.8110	0.9980
		Target 3	0.3737	0.4077	0.7299	0.6706	0.7471	1.1117	0.8969	1.0057	1.4026
ε_2			6.8			9.1			12		
1^{st} time interval	Sensor 1	Target 1	0.0488	0.0729	0.1226	0.0534	0.0822	0.1366	0.0593	0.0939	0.1541
		Target 2	0.1274	0.1610	0.1932	0.1460	0.1843	0.2211	0.1695	0.2135	0.2562
		Target 3	0.1920	0.2065	0.2621	0.2245	0.2436	0.3039	0.2655	0.2905	0.3566
	Sensor 2	Target 1	0.2553	0.2635	0.5202	0.3018	0.3564	0.6596	0.3603	0.4736	0.8353
		Target 2	0.3587	0.4864	0.5940	0.5446	0.7188	0.8728	0.7789	1.0117	1.2243
		Target 3	0.4058	0.3152	1.0131	0.7311	0.6870	1.4312	1.1412	1.1556	1.9585
ε_3			6.3			8.4			10		
1^{st} time interval	Sensor 1	Target 1	0.0499	0.0757	0.1249	0.0551	0.0862	0.1407	0.0624	0.1008	0.1625
		Target 2	0.1314	0.1661	0.1993	0.1524	0.1923	0.2309	0.1815	0.2287	0.2745
		Target 3	0.2037	0.1774	0.2748	0.2405	0.2194	0.3220	0.2914	0.2776	0.3875
	Sensor 2	Target 1	0.2661	0.2916	0.5433	0.3187	0.3966	0.7009	0.3914	0.5421	0.9191
		Target 2	0.3985	0.5360	0.6550	0.6086	0.7986	0.9702	0.8995	1.1623	1.4065
		Target 3	0.5165	0.2581	1.1527	0.8842	0.6783	1.6254	1.3933	1.2601	2.2800
The probability of undetection $H(Z)$			$5.2466e^{-21}$			$1.0838e^{-24}$			$5.9016e^{-29}$		

		Cell 1	Cell 2	Cell 3	Cell 1	Cell 2	Cell 3	Cell 1	Cell 2	Cell 3
ε_1		30			45			60		
1st time interval	Target 1	0.8961	0.2753	1.4017	1.8221	1.3864	2.6980	2.7480	2.4975	3.9943
	Target 2	0.3298	1.3278	2.3373	1.8112	2.9945	4.1891	3.2927	4.6612	6.0410
	Target 3	0.8714	2.1528	4.4952	2.9085	4.3750	6.9026	4.9455	6.5972	9.3100
ε_2		35			50			65		
2nd time interval	Target 1	1.1361	0.6647	1.9039	2.0620	1.7758	3.2002	2.9879	2.8869	4.4965
	Target 2	0.8902	1.8826	2.9024	2.3717	3.5492	4.7543	3.8532	5.2159	6.6061
	Target 3	1.4321	3.0096	5.8507	3.4691	5.2319	8.2581	5.5061	7.4541	10.6655
ε_3		40			55			70		
3rd time interval	Target 1	1.4519	1.0804	2.3012	2.3779	2.1915	3.5975	3.3038	3.3026	4.8938
	Target 2	1.3810	2.4321	3.5191	2.8624	4.0987	5.3709	4.3439	5.7654	7.2228
	Target 3	2.2391	2.6691	6.7845	4.2761	4.8913	9.1919	6.3131	7.1136	11.5993
The probability of undetection $H_x(Z)$		6.2484e^{-8}			1.5019e^{-11}			3.6102e^{-15}		

Table 3

		Cell 1	Cell 2	Cell 3	Cell 4	Cell 5
ε_0		10				
Sensor 1	Target 1	0.0375	0.0691	0.0999	0.1302	0.1601
	Target 2	0.1414	0.1731	0.2033	0.1346	0.2462
Sensor 2	Target 1	0.1550	0.2398	0.3325	0.4355	0.5510
	Target 2	0.0534	0.3240	0.6268	0.0710	1.3218
$H_o(Z)$		0.0002				
ε_0		12				
Sensor 1	Target 1	0.0404	0.0749	0.1086	0.1418	0.1746
	Target 2	0.1588	0.1935	0.2265	0.1607	0.2753
Sensor 2	Target 1	0.2131	0.3269	0.4486	0.5806	0.7252
	Target 2	0.2566	0.5562	0.8881	0.3468	1.6091
$H_o(Z)$		0.0001				

Table 4

			Cell 1	Cell 2	Cell 3
ε_1			15		
1st time interval	Sensor 1	Target 1	0.0661	0.1921	0.3123
		Target 2	0.1058	0.2439	0.3386
		Target 3	0.1708	0.2941	0.3933
	Sensor 2	Target 1	0.4281	1.0057	1.6040
		Target 2	0.5921	1.3160	1.8138
		Target 3	1.0026	1.6041	2.3117
ε_2			15		
2nd time interval	Sensor 1	Target 1	0.0654	0.1937	0.3080
		Target 2	0.1060	0.2439	0.3390
		Target 3	0.1723	0.2926	0.4112
	Sensor 2	Target 1	0.4209	1.0213	1.5654
		Target 2	0.5948	1.3147	1.6405
		Target 3	1.0171	1.5879	2.5040
ε_3			15		
3rd time interval	Sensor 1	Target 1	0.0654	0.1936	0.3126
		Target 2	0.1068	0.2438	0.3018
		Target 3	0.1716	0.2927	0.4148
	Sensor 2	Target 1	0.4217	1.0207	1.6054
		Target 2	0.6027	1.3138	1.5025
		Target 3	1.0100	1.5884	2.5527
The probability of undetection $H(Z)$			5.8885e^{-34}		

Table 5

		Cell 1	Cell 2	Cell 3
ε_1		30		
1st time interval	Target 1	0.8961	0.2753	1.4017
	Target 2	0.3298	1.3278	2.3373
	Target 3	0.8714	2.1528	4.4952
ε_2		30		
2nd time interval	Target 1	0.8274	0.2943	1.4719
	Target 2	0.3964	1.3270	2.2851
	Target 3	0.7530	2.2689	5.0482
ε_3		30		
3rd time interval	Target 1	0.8347	0.3397	1.4370
	Target 2	0.3933	1.3209	2.2845
	Target 3	0.8810	1.1876	5.1795
The probability of undetection $H(Z)$		1.0049e^{-06}		

Table 6

8. Conclusion

In this paper, we extend previous results concerning the search for randomly moving targets among a finite set of states, where a fixed number is assigned to the search effort needed to find the targets and the states are equivalent, with the intention of reducing the probability of detection. We use finite number of sensors, each of them use any amount of effort to find the targets. Since the targets are valuable or dangerous, the search effort may be unconstrained, our aim is to minimize the probability that the targets won't be detect and the search effort also. We find a search strategy that meets our objectives and observe its algorithm. Finally, we propose for the problem applications.

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