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Generalization of Power Lindley Distribution: Properties and Applications

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Abstract This article introduces the generalized Kumaraswamy power Lindley (GKPL) distribution, a novel probabilistic model derived by combining the generalized Kumaraswamy (GK-G) family with the power Lindley (PL) distribution. The GKPL distribution encompasses a wide range of distributions, including Kumaraswamy power Lindley, Kumaraswamy Lindley, generalized power Lindley, generalized Lindley, power Lindley, and the well-known Lindley, as special cases. Fundamental properties are derived, such as the hazard rate function, survival function, quantile function, reverse hazard function, moments, mean residual life function, entropy, and order statistics. To determine the parameters of the GKPL distribution, four estimation methods, including maximum likelihood, least squares, Cramer-von Mises, and Anderson-Darling methods, are used to estimate the parameters of the GKPL distribution. The effectiveness of the estimation techniques is assessed by employing Monte Carlo simulations. The adaptability and validity of the proposed GKPL distribution are compared with alternative models, including the Kumaraswamy power Lindley (KPL), Extended Kumaraswamy power Lindley (EKPL), type II generalized Topp Leone-power Lindley (TIIGTLPL), exponentiated generalized power Lindley (EGPL), generalized Kumaraswamy Weibull (GKW), generalized Kumaraswamy log-logistic (GKLLo), and generalized Kumaraswamy generalized power Gompertz (GKGPGo) distributions, through analyses of three real datasets.

Keywords Generalized Kumaraswamy family, Kumaraswamy power Lindley, Power Lindley, Maximum likelihood estimation, Moments

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Introduction

The Lindley (L) distribution, suggested by D. Lindley [28], is a useful continuous probability distribution commonly utilized in the analysis of lifetime data. It is a mixture of exponential E(ν) and gamma G(2,ν) distributions. The cumulative distribution function (CDF) and probability density function (PDF) of the L distribution are defined as follows

$$G_L(x) = 1 - \left[1 + \frac{\nu x}{\nu + 1}\right] e^{-\nu x}, \tag{1}$$

and

$$g_L(x) = \frac{\nu^2}{\nu + 1} (1 + x) e^{-\nu x}, \tag{2}$$

where the scale parameter $\nu > 0$ and $x > 0$.

Several families of distributions have been suggested and studied, for example beta generalized (B-G) [22], Kumaraswamy generalized (K-G) [12], alpha power transformed (APT) [30], a new alpha power transformed

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(NAPT) [18], marshall-Olkin generalized (MO-G) [49], exponentiated generalized (E-G) [25], transmuted generalized (Trm-G) [45], and truncated generalized (Trc-G) [8].

To improve the flexibility of the L distribution, the power Lindley (PL) distribution was proposed by Ghitany et al. [23], using the power transformation $X = Y^{1/\xi}$ when $Y \sim L(\nu)$. The CDF and PDF of the PL distribution are defined as follows

$$G_{PL}(x) = 1 - \left[1 + \frac{\nu x^{\xi}}{\nu + 1}\right] e^{-\nu x^{\xi}}, \tag{3}$$

and

$$g_{PL}(x) = \frac{\xi \nu^2}{\nu + 1} x^{\xi - 1} \left(1 + x^{\xi}\right) e^{-\nu x^{\xi}}, \tag{4}$$

where $x > 0$, the shape parameter is $\xi > 0$, and the scale parameter is $\nu > 0$.

Researchers have introduced numerous extensions of the L and PL distributions, such as beta Lindley [32, 34], kurmaraswamy Lindley (KL) [11, 43], beta generalized power Lindley (BGPL) [19], exponentiated power Lindley (EPL) [7], exponentiated kumaraswamy Lindley (EKL) [39], kumaraswamy power Lindley (KPL) [38], exponentiated kumaraswamy power Lindley (EKPL) [20], exponentiated generalized Lindley (EGL) [35], odd log-logistic Lindley (OLL-L) [40], odd log-logistic Lindley Poisson (OLL-LP) [40], odd log-logistic Marshall-Olkin Lindley (OLLMO-L) [6], odd log-logistic Marshall-Olkin power Lindley (OLLMO-PL) [5], transmuted kumaraswamy Lindley (TKL) [21], generalized power Lindley (GPL) [46], alpha power transformed Lindley (APTL) [14], exponentiated generalized power Lindley (EGPL) [33], alpha power transformed power Lindley (APTPL) [27], odd Lindley kumaraswamy (OLK) [44], odd log-logistic Lindley-Exponential (OLLLi-Ex) [3], new alpha power transformed power Lindley (NAPTPL) [2], alpha power transformed extended power Lindley (APTEPL) [15], type II generalized Topp Leone-power Lindley (TIIGTL-PL) [26], odd log-logistic generalised Lindley (OLLG-L) [41], and extended odd Weibull Lindley (EOWL) [16].

Recently, the generalized Kumaraswamy generalized (GK-G) family was presented by Nofal et al. [37]. The GK-G family is a more flexible generator family and the new models generated by this family have wider applications in marketing, reliability, medicine, engineering, and other areas. This method has been used by many authors to introduce very important models such as the generalized Kumaraswamy Weibull (GKW), generalized Kurmaraswamy log-logistic (GKLLo), generalized Kumaraswamy gamma (GKGa) distributions [37], and the generalized Kumaraswamy generalized power Gompertz (GKPGo) distribution [31].

The main aim of this study is to introduce a novel lifetime distribution by combining the GK-G family with the PL distribution; called the generalized Kumaraswamy power Lindley (GKPL) distribution.

The considerations outlined below provide significant justifications for investigating the proposed distribution.

1. The GKPL distribution demonstrates enhanced flexibility as its PDF can exhibit various shapes, such as being uni-modal, symmetric, decreasing, left-skewed, and right-skewed.
2. The GKPL distribution has a wider application in marketing, reliability, medicine, engineering, and other areas.
3. The GKPL distribution consistently outperforms other generated models.
4. The closed quantile function of the GKPL distribution streamlines computations, making tasks such as generating random numbers more straightforward.

This study is presented in the following sections: In Section 1 the new extension of the power Lindley distribution is proposed, the reliability and hazard rate functions are also discussed and the linear representation of the density function is presented in this section. In Section 2, Statistical properties of the proposed distribution, such as moments, the quantile function for the median, the mean residual life function, Reny Entropy, and order Statistics are derived. Four estimation methods that are maximum likelihood, least square, Anderson-Darling, and Cramer-von Mises estimations are discussed in Section 3. Also, simulation studies for GKPL distribution are used to demonstrate the efficiency of the estimation methods in Section 3. The performance of the GKPL distribution is illustrated using three real-life datasets in Section 4. Ultimately, the findings and conclusions are presented in Section 5.

1. The generalized Kumaraswamy power Lindley (GKPL) distribution

The CDF and PDF of GK-G family are defined, respectively, as follows

$$F_{GK}(x) = \frac{1 - \left[1 - \eta \left[G(x)\right]^{\beta}\right]^{\gamma}}{1 - (1-\eta)^{\gamma}}, \tag{5}$$

and

$$f_{GK}(x) = \frac{\beta\gamma\eta g(x)}{1-(1-\eta)^{\gamma}} \left[G(x)\right]^{\beta-1} \left[1 - \eta \left[G(x)\right]^{\beta}\right]^{\gamma-1}, \tag{6}$$

where $0 < \eta \leq 1$, $\beta > 0$ and $\gamma > 0$ are shape parameters.

Thus by substituting (3) in (5), the CDF of the GKPL distribution is obtained by

$$F_{GKPL}(x) = \frac{1 - \left[1 - \eta \left[1 - \left[1 + \frac{\nu x^{\xi}}{\nu+1}\right] e^{-\nu x^{\xi}}\right]^{\beta}\right]^{\gamma}}{1 - (1-\eta)^{\gamma}}, \tag{7}$$

also by substituting (3) and (4) into (6) the PDF of the GKPL distribution is given by

$$\begin{aligned} f_{GKPL}(x) = {} & \frac{\beta\gamma\eta\xi\nu^{2}}{(\nu+1)(1-(1-\eta)^{\gamma})} x^{\xi-1} \left(1 + x^{\xi}\right) e^{-\nu x^{\xi}} \\ & \times \left[1 - \left[1 + \frac{\nu x^{\xi}}{\nu+1}\right] e^{-\nu x^{\xi}}\right]^{\beta-1} \\ & \times \left[1 - \eta \left[1 - \left[1 + \frac{\nu x^{\xi}}{\nu+1}\right] e^{-\nu x^{\xi}}\right]^{\beta}\right]^{\gamma-1}, \end{aligned} \tag{8}$$

where $0 < \eta <= 1$, $\beta > 0$, $\gamma > 0$, and $\xi > 0$ are shape parameters and $\nu > 0$ is scale parameter.

Definition 1. *The R. V. X is said to follow the GKPL distribution with the parameters β, γ, η, ξ, and ν, written as $X \sim GKPL(\beta, \gamma, \eta, \xi, \nu)$, if its CDF or PDF are given by (7) or (8), respectively.*

Figure 1 represented the PDF plots of the GKPL distribution with the selected values of its parameters β, γ, η, ξ, and ν. It is possible to evaluate how flexible the new proposed generalization is. Using the Figures 1a, 1b, 1c, and 1d uni-modal behavior can be seen in the density curves, and the curves of density are right-skewed, symmetric, platykurtic, and leptokurtic regarding to the different values of its parameters.

1.1. Survival function (SF) and Hazard rate function (HRF) of GKPL distribution

The SF and HRF of the GKPL distribution are obtained as follows

$$S_{GKPL}(x) = 1 - \left[\frac{1 - \left[1 - \eta \left[1 - \left[1 + \frac{\nu x^{\xi}}{\nu+1}\right] e^{-\nu x^{\xi}}\right]^{\beta}\right]^{\gamma}}{1 - (1-\eta)^{\gamma}}\right], \tag{9}$$

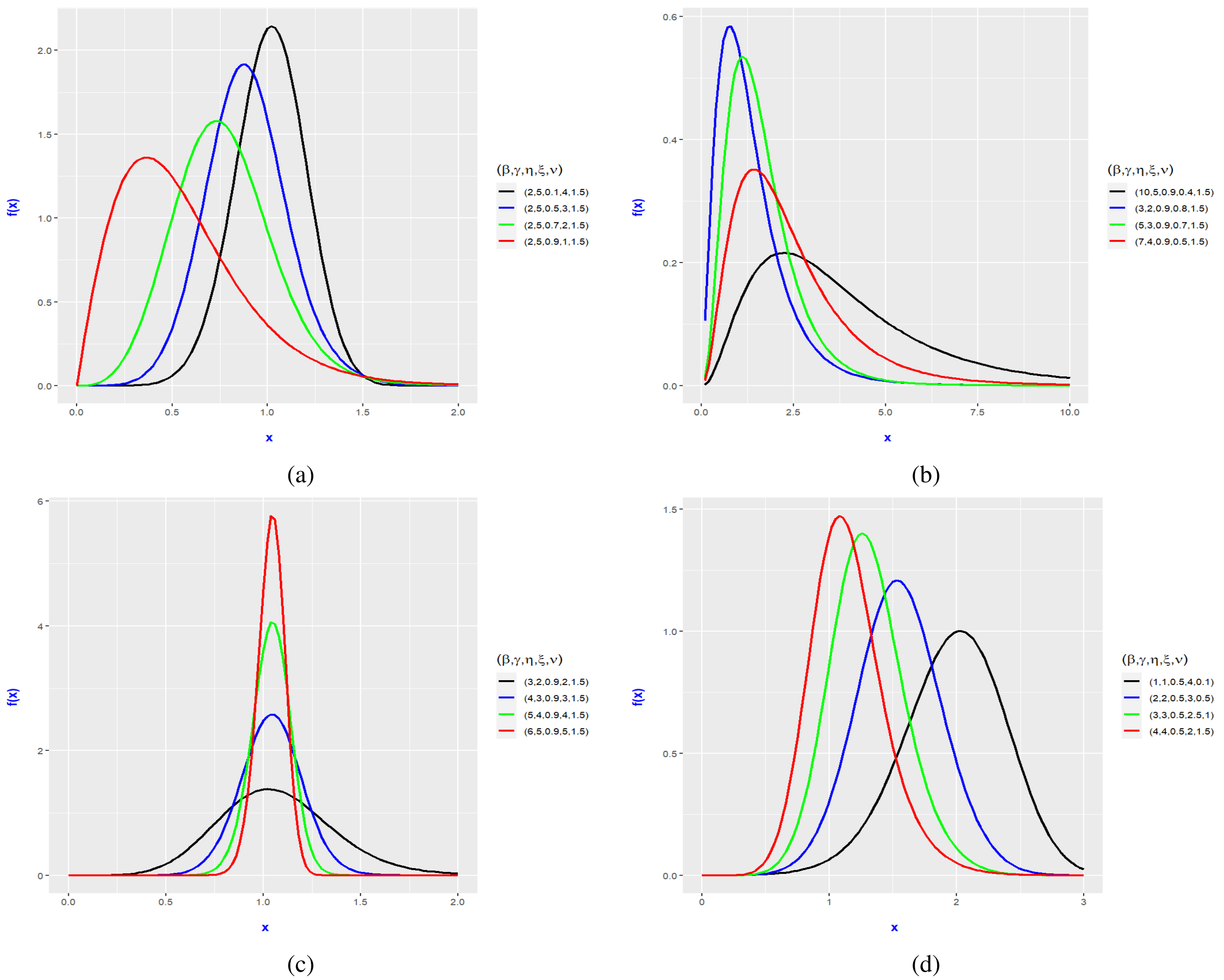


Figure 1. Plots of the PDF of the GKPL distribution for various parameter values.

and

$$h_{GKPL}(x) = \frac{\beta\gamma\eta\xi\nu^2 x^{\xi-1}(1+x^\xi)e^{-\nu x^\xi}\left[1-\left[1+\frac{\nu x^\xi}{\nu+1}\right]e^{-\nu x^\xi}\right]^{\beta-1}\left[1-\eta\left[1-\left[1+\frac{\nu x^\xi}{\nu+1}\right]e^{-\nu x^\xi}\right]^\beta\right]^{\gamma-1}}{(\nu+1)\left[\left[1-\eta\left[1-\left[1+\frac{\nu x^\xi}{\nu+1}\right]e^{-\nu x^\xi}\right]^\beta\right]^\gamma-(1-\eta)^\gamma\right]}. \tag{10}$$

In addition, the reverse hazard rate function (RHRF) and cumulative hazard rate function (CHRF), respectively, are given by

$$\pi_{GKPL}(x) = \frac{\beta\gamma\eta\xi\nu^2 x^{\xi-1}(1+x^\xi)e^{-\nu x^\xi}\left[1-\left[1+\frac{\nu x^\xi}{\nu+1}\right]e^{-\nu x^\xi}\right]^{\beta-1}\left[1-\eta\left[1-\left[1+\frac{\nu x^\xi}{\nu+1}\right]e^{-\nu x^\xi}\right]^\beta\right]^{\gamma-1}}{(\nu+1)\left[1-\left[1-\eta\left[1-\left[1+\frac{\nu x^\xi}{\nu+1}\right]e^{-\nu x^\xi}\right]^\beta\right]^\gamma\right]}, \tag{11}$$

and

$$
\begin{aligned}
H_{GKPL}(x) &= \int_0^x h_{GKPL}(t)dt = -\log(1 - F_{GKPL}(x)) \\
&= -\log\left[1 - \left[\frac{1 - \left[1 - \eta\left[1 - \left[1 + \frac{\nu x^\xi}{\nu+1}\right] e^{-\nu x^\xi}\right]^\beta\right]^\gamma}{1 - (1-\eta)^\gamma}\right]\right]. \qquad (12)
\end{aligned}
$$

Figures 2a, 2b and 2c depict the HRF plots of the GKPL distribution for various parameter values. The HRF curves exhibit decreasing, increasing, and upside-down decreasing patterns.

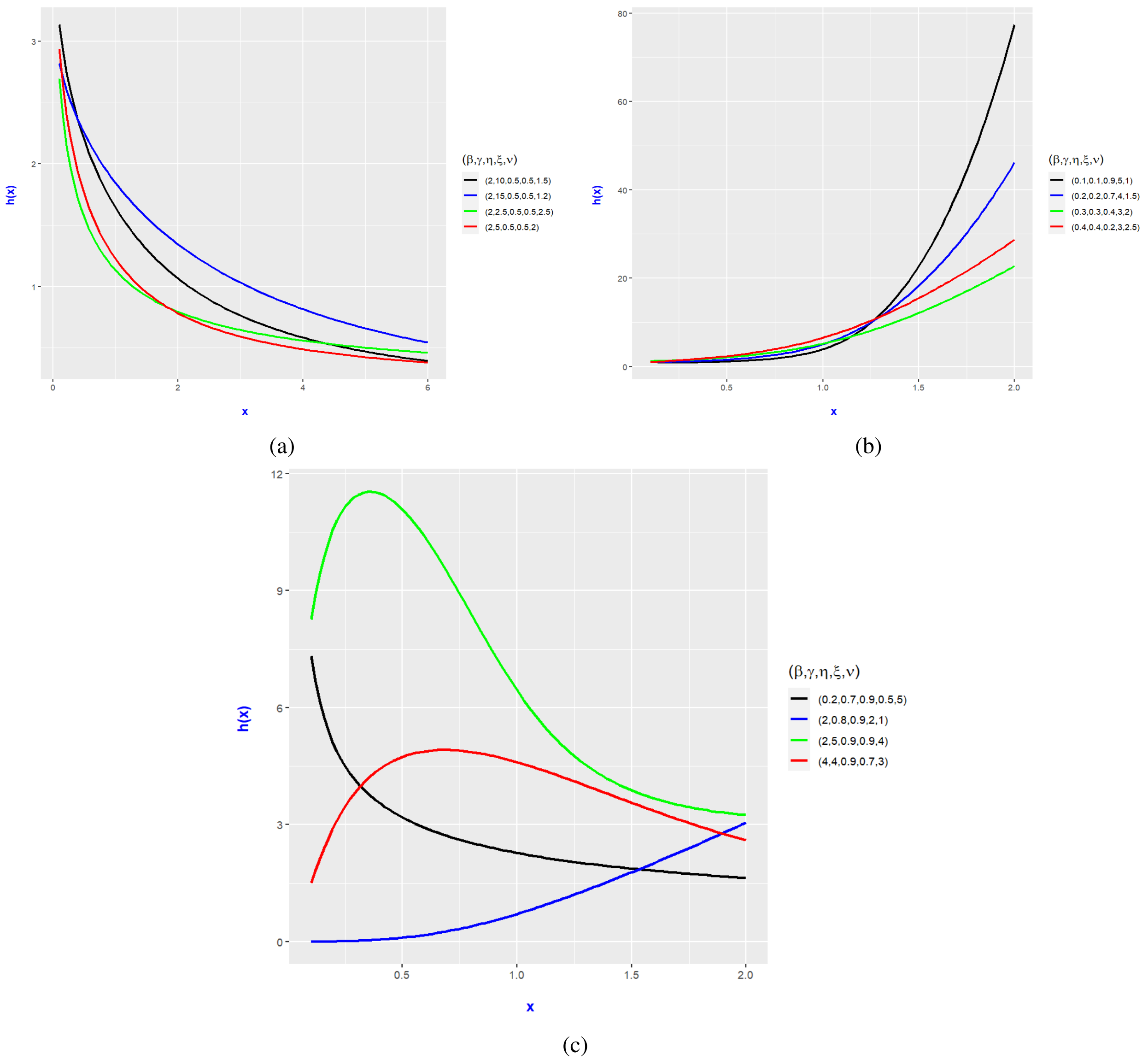


Figure 2. The HRF plots of the GKPL distribution for various parameter values.

1.2. Special cases of GKPL distribution

The significance of the GKPL distribution is accentuated by the fact that it includes various specialized sub-models, as outlined in Table 1.

Table 1. Some spacial cases of the GKPL distribution

Parameters					Distribution	Ref.
β	γ	η	ξ	ν		
			1		GKL	New
		1			KPL	[38]
		1	1		KL	[11]
	1				GPL	[47]
	1		1		GL	[36]
1	1				PL	[23]
1	1		1		L	[28]

1.3. The generalized Kumaraswamy Lindley (GKL) distribution

When $\xi = 1$, the GKPL distribution is reduced to the generalized kumaraswamy Lindley (GKL) distribution with parameters β, γ, η, and ν. The random variable X of GKL distribution is written as $X \sim GKL(\beta, b, \eta, \nu)$, and its PDF is defined by

$$f_{GKL}(x) = \frac{\beta\gamma\eta\nu^2}{(\nu+1)(1-(1-\eta)^\gamma)}\,(1+x)\,e^{-\nu x} \times \left[1-\left[1+\frac{\nu x}{\nu+1}\right]e^{-\nu x}\right]^{\beta-1} \times \left[1-\eta\left[1-\left[1+\frac{\nu x}{\nu+1}\right]e^{-\nu x}\right]^{\beta}\right]^{\gamma-1}, \quad (13)$$

also, the corresponding CDF of the GKL distribution is defined as follows

$$F_{GKL}(x) = \frac{1-\left[1-\eta\left[1-\left[1+\frac{\nu x}{\nu+1}\right]e^{-\nu x}\right]^{\beta}\right]^{\gamma}}{1-(1-\eta)^\gamma}, \quad (14)$$

furthermore, the PDF of the GKL distribution in (13) can be obtained by substituting (1) and (2) in (6), also the CDF of the GKL distribution in (14) can be obtained by substituting (1) in (5). Additionally, its SF, HRF, and special cases can be obtained by assigning specific values to the relevant parameters.

Proposition 1. *Let a random variable $Z \sim GKPL(1, 1, \eta, 1, \nu)$. Then the random variable $X = -\frac{1}{\delta}\log(1-e^{-Z})$ has a Lindley exponential (LE) distribution [9] with the parameters ν and δ. Its CDF is defined as*

$$F_X(x) = \frac{\left(1+\nu-\nu\log\left(1-e^{-\delta x}\right)\right)\left(1-e^{-\delta x}\right)^\nu}{\nu+1}. \quad (15)$$

Proof. Since the $Z \sim GKPL(1, 1, \eta, 1, \nu)$. Therefore, the CDF of Z is defined as

$$F_Z(z) = 1-\left(\frac{1+\nu+\nu z}{1+\nu}\right)e^{-\nu z}. \quad (16)$$

Now,

$$\begin{aligned}
F_X(x) &= P\left(X \leq x\right) \\
&= P\left(-\frac{1}{\beta}\log(1-e^{-Z}) \leq x\right) \\
&= P\left(\log(1-e^{-Z}) \geq -\delta x\right) \\
&= P\left(1-e^{-Z} \geq e^{-\delta x}\right) \\
&= P\left(e^{-Z} \leq 1-e^{-\delta x}\right) \\
&= P\left(-Z \leq \log(1-e^{-\delta x})\right) \\
&= P\left(Z \geq -\log(1-e^{-\delta x})\right) \\
&= 1 - P\left(Z \leq -\log(1-e^{-\delta x})\right) \\
&= 1 - F_Z\left(-\log(1-e^{-\delta x})\right).
\end{aligned}$$

From (16) we get that

$$F_X(x) = \frac{\left(1+\nu-\nu\log\left(1-e^{-\delta x}\right)\right)\left(1-e^{-\delta x}\right)^{\nu}}{\nu+1}.$$

So, the random variable $X \sim LE(\nu)$.

Proposition 2. *Let a random variable $Z \sim GKPL(1,1,\eta,1,\theta)$. Then the random variable $X = \frac{1}{\lambda}Z^{\frac{1}{\delta}}$ has a Lindley weibull (LW) distribution [10] with the parameters λ, δ, and ν. Its CDF is defined as*

$$F_X(x) = 1 - \frac{1+\nu+\nu(\lambda x)^{\delta}}{1+\nu}e^{-\nu(\lambda x)^{\delta}}. \tag{17}$$

Proof. It is similar to proof of Proposition 1.

1.4. Linear representations

By using the binomial series expansion

$$(1-z)^{w} = \sum_{i=0}^{\infty}(-1)^{i}\binom{w}{i}z^{i}. \tag{18}$$

The PDF of the GKPL distribution in (8) is given by

$$\begin{aligned}
f_{GKPL}(x) &= \frac{\beta\gamma\xi\nu^{2}}{(\nu+1)(1-(1-\eta)^{\gamma})}x^{\xi-1}\left(1+x^{\xi}\right)e^{-\nu x^{\xi}} \\
&\quad\times\sum_{i=0}^{\infty}(-1)^{i}\eta^{i+1}\binom{\gamma-1}{i}\left[1-\left[1+\frac{\nu x^{\xi}}{\nu+1}\right]e^{-\nu x^{\xi}}\right]^{\beta i+\beta-1},
\end{aligned}$$

again using binomial expression we obtained

$$f_{GKPL}(x) = \frac{\beta\gamma\xi}{d}\sum_{i,j,k=0}^{\infty}\mathfrak{D}_{(i,j,k)}x^{\xi k+\xi-1}\left(1+x^{\xi}\right)e^{-\nu(j+1)x^{\xi}}, \tag{19}$$

where $\mathfrak{D}_{i,j,k} = (-1)^{i+j}\frac{\eta^{i+1}\nu^{k+2}}{(\nu+1)^{k+1}}\binom{\gamma-1}{i}\binom{\beta i+\beta-1}{j}\binom{j}{k}$ and $d = 1-(1-\eta)^{\gamma}$.

2. Statistical properties of the GKPL distribution

2.1. Moments

Here is the clarification of the GKPL distribution's row moments, central moments, and conditional moments.

The r^{th} moment $E(X^r)$ of X, when $X \sim GKPL(\beta,\gamma,\eta,\xi,\nu)$, is defined as follows

$$\mu_r^{'} = \frac{\beta\gamma}{d}\sum_{i,j,k=0}^{\infty}\mathfrak{D}_{(i,j,k)}\left[\frac{\Gamma(k+\frac{r}{\xi}+1)}{[\nu(j+1)]^{(k+\frac{r}{\xi}+1)}}+\frac{\Gamma(k+\frac{r}{\xi}+2)}{[\nu(j+1)]^{(k+\frac{r}{\xi}+2)}}\right], \tag{20}$$

the q^{th} moment about the mean μ of X is obtained by

$$\mu_q = \sum_{r=0}^{q}(-1)^r\binom{q}{r}\mu_{q-r}^{'}\mu^r, \tag{21}$$

where

$$\mu_{q-r}^{'} = \frac{\beta\gamma}{d}\sum_{i,j,k=0}^{\infty}\mathfrak{D}_{(i,j,k)}\left[\frac{\Gamma(k+\frac{q-r}{\xi}+1)}{[\nu(j+1)]^{(k+\frac{q-r}{\xi}+1)}}+\frac{\Gamma(k+\frac{q-r}{\xi}+2)}{[\nu(j+1)]^{(k+\frac{q-r}{\xi}+2)}}\right],$$

and the conditional moments of X is defined as follows

$$E(X^r/X>t) = \frac{\beta\gamma}{\mathfrak{A}}\sum_{i,j,k=0}^{\infty}\mathfrak{D}_{(i,j,k)}\left[\frac{\Gamma(k+\frac{1}{\xi}+1,\nu(j+1)t^{\xi})}{[\nu(j+1)]^{k+\frac{1}{\xi}+1}}+\frac{\Gamma(k+\frac{1}{\xi}+2,\nu(j+1)t^{\xi})}{[\nu(j+1)]^{k+\frac{1}{\xi}+2}}\right] \tag{22}$$

where

$$\mathfrak{A} = \left[\left[1-\eta\left[1-\left[1+\frac{\nu x^{\xi}}{\nu+1}\right]e^{-\nu x^{\xi}}\right]^{\beta}\right]^{\gamma}-(1-\eta)^{\gamma}\right].$$

2.2. Moment generating function (MGF)

The MGF obtained from the known formula

$$\begin{aligned}\mathfrak{M}_x(t) &= E(e^{tx})\\ &= \int_0^{\infty}e^{tx}f(x)dx.\end{aligned}$$

Using the series representation $e^{tx} = \sum_{r=0}^{\infty}\frac{t^r}{r!}x^r$ [24].
Then

$$\begin{aligned}\mathfrak{M}_x(t) &= \sum_{r=0}^{\infty}\frac{t^r}{r!}\int_0^{\infty}x^r f(x)dx\\ &= \sum_{r=0}^{\infty}\frac{t^r}{r!}\ \mu_r^{'}(x),\end{aligned} \tag{23}$$

By substituting (20) in (23). Then the MGF of the GKPL distribution is obtained by

$$\mathfrak{M}_x(t) = \frac{\beta\gamma}{d}\sum_{i,j,k,r=0}^{\infty}\mathfrak{D}_{(i,j,k)}\frac{t^r}{r!}\left[\frac{\Gamma(k+\frac{r}{\xi}+1)}{[\nu(j+1)]^{(k+\frac{r}{\xi}+1)}}+\frac{\Gamma(k+\frac{r}{\xi}+2)}{[\nu(j+1)]^{(k+\frac{r}{\xi}+2)}}\right]. \tag{24}$$

2.3. Characteristic function (CF)

The CF of the GKPL is defined as follows

$$\mathfrak{Q}_x(t) = \frac{\beta\gamma}{d} \sum_{i,j,k,r=0}^{\infty} \mathfrak{D}_{(i,j,k)} \frac{(it)^r}{r!} \left[\frac{\Gamma(k+\frac{r}{\xi}+1)}{[\nu(j+1)]^{(k+\frac{r}{\xi}+1)}} + \frac{\Gamma(k+\frac{r}{\xi}+2)}{[\nu(j+1)]^{(k+\frac{r}{\xi}+2)}} \right], \tag{25}$$

where $i = \sqrt{-1}$.

2.4. Quantile function (QF)

Since the QF is given by $Q(u) = F^{-1}(u) = x$. Therefore from (7) we get

$$F(Q(u)) = \frac{1 - \left[1 - \eta\left[1 - \left[1 + \frac{\nu Q^{\xi}(u)}{1+\nu}\right] e^{-\nu Q^{\xi}(u)}\right]^{\beta}\right]^{\gamma}}{1-(1-\eta)^{\gamma}} = u.$$

Assuming $d = 1-(1-\eta)^{\gamma}$ and solving the above equation for $Q(u)$ we obtain that

$$Q(u) = \left[-\frac{1+\nu}{\nu} - \frac{1}{\nu} W_{(-1)} \left[(1+\nu)e^{-(1+\nu)} \left[\left(\frac{1}{\eta} \left[1-(1-ud)^{\frac{1}{\gamma}} \right] \right)^{\frac{1}{\beta}} - 1 \right] \right] \right]^{\frac{1}{\xi}}. \tag{26}$$

where $W_{(-1)}(.)$ is the negative branch of the Lambert function.

2.5. Generate GKPL random variables

To generate random variables X_i ; $i = 0, 1, 2, ..., n$ from the GKPL distribution with the parameters $\beta, \gamma, \eta, \xi,$ and ν, we follow

- Generate $U_i \sim Uniform(0, 1)$; i=0,1,2,...,n
- From (26), set

$$X_i = \left(-\frac{\nu+1}{\nu} - \frac{1}{\nu} W_{(-1)} \left[-(\nu+1)e^{-(\nu+1)} \left(1 - \left[\frac{1}{\eta} \left(1 - [1-U_i d]^{\frac{1}{\gamma}} \right) \right]^{\frac{1}{\beta}} \right) \right] \right)^{\frac{1}{\xi}}, \tag{27}$$

where $i = 0, 1, 2, ..., n$ and $W_{(-1)}(.)$ is the negative branch of the Lambert function.

2.6. Mean residual life (MRL) function

The MRL function is given as

$$m(x) = E[X-t|X>t] = \frac{\int_x^{\infty} t f(t) dt}{\bar{F}(x)} - x. \tag{28}$$

Considering the individual integration as follows

$$\int_x^{\infty} t f(t) dt = \frac{\beta\gamma\xi}{d} \sum_{i,j,k=0}^{\infty} \mathfrak{D}_{(i,j,k)} \int_x^{\infty} t^{\xi(k+1)} \left(1+t^{\xi}\right) e^{-\nu(j+1)t^{\xi}} dt. \tag{29}$$

Solving the above integration by using the incomplete gamma function $\Gamma(a,t) = \int_t^{\infty} x^{a-1} e^{-x} dx$, we get

$$\int_x^{\infty} t f(t) dt = \frac{\beta\gamma}{d} \sum_{i,j,k=0}^{\infty} \mathfrak{D}_{(i,j,k)} \left[\frac{\Gamma(k+\frac{1}{\xi}+1, \nu(j+1)x^{\xi})}{[\nu(j+1)]^{k+\frac{1}{\xi}+1}} + \frac{\Gamma(k+\frac{1}{\xi}+2, \nu(j+1)x^{\xi})}{[\nu(j+1)]^{k+\frac{1}{\xi}+2}} \right]. \tag{30}$$

Substituting (30) and (9) into (28), therefore the $m(x)$ can be obtained as follows

$$m(x)=\frac{\beta\gamma\sum_{i,j,k=0}^{\infty}\mathfrak{D}_{(i,j,k)}\left[\frac{\Gamma(k+\frac{1}{\xi}+1,\nu(j+1)x^{\xi})}{[\nu(j+1)]^{k+\frac{1}{\xi}+1}}+\frac{\Gamma(k+\frac{1}{\xi}+2,\nu(j+1)x^{\xi})}{[\nu(j+1)]^{k+\frac{1}{\xi}+2}}\right]}{d-\left[1-\left[1-\eta\left[1-\left[1+\frac{\nu x^{\xi}}{\nu+1}\right]e^{-\nu x^{\xi}}\right]^{\beta}\right]^{\gamma}\right]}-x. \quad (31)$$

2.7. Entropy

The Renyi Entropy is defined as follows

$$H_{\rho}(f(x))=\frac{1}{1-\rho}\log\int_{0}^{\infty}f^{\rho}(x)dx, \quad (32)$$

where $\rho>0$, and $\rho\neq 1$ [42].

To obtain the $H_{\rho}(f(x))$, we have to find $f^{\rho}(x)$. Through the PDF of the GKPL distribution in (8) and by applying the binomial expansion (18), we can get that

$$f^{\rho}(x)=\left(\frac{\beta\gamma\xi}{d}\right)^{\rho}\sum_{i,j,k,l=0}^{\infty}\mathfrak{D}_{(i,j,k,l)}\;x^{(\xi l+\xi\rho-\rho)}e^{-\nu(\rho+j)x^{\xi}},$$

where $\mathfrak{D}_{(i,j,k,l)}=(-1)^{i+j}\frac{\eta^{\rho+i}\nu^{2\rho+k}}{(\nu+1)^{\rho+j}}\binom{\rho(\gamma-1)}{i}\binom{\beta\rho+\beta i-\rho}{j}\binom{j}{k}\binom{\rho+k}{l}$.

Now,

$$\int_{0}^{\infty}f^{\rho}(x)\;dx=\left(\frac{\beta\gamma\xi}{d}\right)^{\rho}\sum_{i,j,k,l=0}^{\infty}\mathfrak{D}_{(i,j,k,l)}\;\int_{0}^{\infty}x^{(\xi l+\xi\rho-\rho)}e^{-\nu(\rho+j)x^{\xi}}\;dx.$$

By using Gamma function $\Gamma(a)=\int_{0}^{\infty}t^{a-1}e^{-t}dt$, the above integration will be solved as follows

$$\int_{0}^{\infty}f^{\rho}(x)\;dx=\left(\frac{\beta\gamma}{d}\right)^{\rho}\xi^{\rho-1}\sum_{i,j,k,l=0}^{\infty}\mathfrak{D}_{(i,j,k,l)}\frac{\Gamma(l+\rho-\frac{\rho}{\xi}+\frac{1}{\xi})}{[\nu(j+\rho)]^{l+\rho-\frac{\rho}{\xi}+\frac{1}{\xi}}}. \quad (33)$$

Substituting (33) in (32) the Renyi entropy of the GKPL distribution is given by

$$H_{\rho}(f_{GKPL}(x))=\frac{1}{1-\rho}\log\left[\left(\frac{\beta\gamma}{d}\right)^{\rho}\xi^{\rho-1}\sum_{i,j,k,l=0}^{\infty}\mathfrak{D}_{(i,j,k,l)}\frac{\Gamma(l+\rho-\frac{\rho}{\xi}+\frac{1}{\xi})}{[\nu(j+\rho)]^{l+\rho-\frac{\rho}{\xi}+\frac{1}{\xi}}}\right]. \quad (34)$$

2.8. Order Statistics

Let $X_i\sim GKPL(\beta,\gamma,\eta,\xi,\nu), i=1,2,...,m$ be Identically Independently random variables, and Let $X_{i:n}$ be the order statistics. Then the i^{th} PDF of order statistics [13] is defined as follows

$$f_{X_{(i)}}(x)=\frac{m!}{(i-1)!(m-i)!}f_X(x)\left[F_X(x)\right]^{i-1}\left[1-F_X(x)\right]^{m-i}. \quad (35)$$

Following the binomial expansion in (18)

$$[1-F_X(x)]^{m-i}=\sum_{l=0}^{\infty}(-1)^l\binom{m-i}{l}[F_X(x)]^l .$$

Therefore

$$f_{X_{(i)}}(x)=\frac{m!}{(i-1)!(m-i)!}\sum_{l=0}^{\infty}(-1)^l\binom{m-i}{l}f_X(x)\,[F_X(x)]^{i+l-1} . \tag{36}$$

By substituting (5) and (6) into (36), the PDF of i^{th} order statistics of GKPL distribution is defined as follows

$$\begin{aligned}f_{X_{(i)}}(x)=&\frac{m!}{(i-1)!(m-i)!}\frac{\beta\gamma\eta}{d}\sum_{l=0}^{\infty}(-1)^l\binom{m-i}{l}g(x)\,[G(x)]^{\beta-1}\\ &\times\left[1-\eta\,[G(x)]^{\beta}\right]^{\gamma-1}\left[\frac{1-\left[1-\eta\,[G(x)]^{\beta}\right]^{\gamma}}{d}\right]^{i+l-1},\end{aligned}$$

where the $G(x)$ and $g(x)$ are the CDF and PDF of the PL distribution, which are defined in (3) and (4), respectively.

Using the binomial expansion (18), we get

$$f_{X_{(i)}}(x)=\frac{\beta\gamma\;m!}{(i-1)!(m-i)!}\sum_{l,j,k=0}^{\infty}\mathfrak{M}_{(l,j,k)}g(x)\,[G(x)]^{\beta k+\beta-1} , \tag{37}$$

where $\mathfrak{M}_{(l,j,k)}=(-1)^{l+j+k}\frac{\eta^{k+1}}{d^{i+l}}\binom{m-i}{l}\binom{i+l-1}{j}\binom{\gamma j+\gamma-1}{k}$.

3. Estimation Methods and Simulation Study

Let $X_i\sim GKPL(\beta,\gamma,\eta,\xi,\nu); i=1,2,...,m$ and the $X_{1:m},X_{2:m},...,X_{m:m}$ are the order observations of a random sample of size m of the GKPL distribution.

The maximum likelihood estimates (MLEs) of $\beta,\gamma,\eta,\xi,$ and ν, say $\hat\beta_{MLE},\hat\gamma_{MLE},\hat\eta_{MLE},\hat\xi_{MLE},$ and $\hat\nu_{MLE}$ can be obtained by minimizing the log likelihood function ℓ

$$\begin{aligned}\ell=&\,m\log\beta+m\log\gamma+m\log\eta+m\log\xi+2m\log\nu-m\log(\nu+1)-m\log\left[1-(1-\eta)^{\gamma}\right]\\ &+(\xi-1)\sum_{i=1}^{m}\log x_i+\sum_{i=1}^{m}\log\left(1+x_i^{\xi}\right)-\nu\sum_{i=1}^{m}x_i^{\xi}+(\beta-1)\sum_{i=1}^{m}\log\left[1-\left[1+\frac{\nu x^{\xi}}{\nu+1}\right]e^{-\nu x^{\xi}}\right]\\ &+(\gamma-1)\sum_{i=1}^{m}\log\left[1-\eta\left[1-\left[1+\frac{\nu x^{\xi}}{\nu+1}\right]e^{-\nu x^{\xi}}\right]^{\beta}\right],\end{aligned} \tag{38}$$

or by simultaneously solving the equations $\frac{\partial\ell}{\partial\beta}$, $\frac{\partial\ell}{\partial\gamma}$, $\frac{\partial\ell}{\partial\eta}$, $\frac{\partial\ell}{\partial\xi}$, and $\frac{\partial\ell}{\partial\nu}$ when equating them to zero.

The least square estimates (LSEs) $\hat\beta_{LSE},\hat\gamma_{LSE},\hat\eta_{LSE},\hat\xi_{LSE},$ and $\hat\nu_{LSE}$ of $\beta,\gamma,\eta,\xi,$ and ν, respectively, can be obtained by minimizing the function

$$LSE=\sum_{i=1}^{m}\left(F_{GKPL}(x_{(i)})-\frac{i}{m+1}\right)^2 . \tag{39}$$

The Anderson-Darling estimates (ADE) $\hat{\beta}_{ADE}, \hat{\gamma}_{ADE}, \hat{\eta}_{ADE}, \hat{\xi}_{ADE}$, and $\hat{\nu}_{ADE}$ of β, γ, η, ξ, and ν, respectively, can be obtained by minimizing

$$AD = -m - \sum_{i=1}^{m} \frac{2i-1}{m} \left(\log F_{GKPL}(x_{(i)}) + \log \left[1 - F_{GKPL}(x_{(m+1-i)})\right] \right). \quad (40)$$

The Cramer-Von Mises estimates (CVME) $\hat{\beta}_{CVM}, \hat{\gamma}_{CVM}, \hat{\eta}_{CVM}, \hat{\xi}_{CVM}$, and $\hat{\nu}_{CVM}$ of β, γ, η, ξ, and ν, respectively, can be obtained by minimizing

$$CVM = \frac{1}{12m} + \sum_{i=1}^{m} \left(F_{GKPL}(x_{(i)}) - \frac{2i-1}{2m} \right). \quad (41)$$

Some simulation studies have been discussed to investigate and compare the performance of the MLE, LSE, ADE, and CVME for estimating the parameters of the GKPL for many sample size n=50, 100, 150, 200, 250 and two sets of parameters; set 1 ($\beta = 2, \gamma = 3, \eta = 0.5, \xi = 3, \nu = 1.5$) and set 2 ($\beta = 0.25, \gamma = 0.75, \eta = 0.9, \xi = 1.5, \nu = 2$). This investigation is performed by using the mean values of estimates (Mean), mean squared errors (MSE), root mean squared errors (RMSE), and average biases (Bias). The sequential procedures of the simulation analysis are outlined as follows:

(1) Choose the initial values of the parameters from the above two cases.
(2) Generate samples of sizes n = 50, 100, 150, and 250 from the GKPL distribution using (27).
(3) Obtain the ML, LS, AD, and CVM estimates by solving (38), (39), (40), and (41) respectively.
(4) Calculate the MSEs, RMSEs and ABs of the different parameters for the GKPL distribution using the expressions $MSE = \frac{1}{n}\sum_{i=0}^{n}(X_i - \hat{X}_i)^2$, $RMSE = \sqrt{\frac{1}{n}\sum_{i=0}^{n}(X_i - \hat{X}_i)^2}$ and $Bais = \frac{1}{n}\sum_{i=0}^{n}(X_i - \hat{X}_i)$.
(5) Replicate the aforementioned steps for each sample size 10,000 iterations.

The results of simulation studies of the MLE, LSE, ADE, and CVME are shown in Tables 2 and 3. The simulation results confirm that the estimators for the proposed GKPL distribution show consistent behavior. As the sample size increases, the estimators converge to true values, and all metrics such as MSE, RMSE, and Bias decrease.

4. Applications

The usefulness and adaptability of the GKPL distribution are demonstrated through three applications of lifetime datasets. We will evaluate how well the GKPL model fits the data in comparison to other competing models such as the Kumaraswamy power Lindley (KPL), Extended Kumaraswamy power Lindley (EKPL), type II generalized Topp Leone-power Lindley (TIIGTLPL), exponentiated generalized power Lindley (EGPL), generalized Kumaraswamy Weibull (GKW), generalized Kurmaraswamy log-logistic (GKLLo), generalized Kumaraswamy generalized power Gompertz (GKGPGo), odd-Burr generalized Frechet (OB-F), the generalized odd log-logistic reciprocal Weibull (GOLLRW), the generalized odd log-logistic reciprocal Rayleigh (GOLLRR), and the generalized Odd Log-Logistic Frechet (GOLLF) distributions. We utilize a range of goodness-of-fit (GOF) metrics to assess and compare the newly articulated distribution with existing distributions such as the The negative Log-Likelihood ($-\ell$), Akaike information criterion (AIC), Bayesian information criterion (BIC), Corrected Akaike information criterion (AICc), Anderson-Darling (A^*), Cramer-von Mises (W^*), Kolmogorov-Smirnov (K-S) with its p-value. In general, the models that have lower values of such GOF measures and higher p-values are often better at fitting the data. The parameters of models have been estimated using the MLE method.

Table 2. Results of simulation study for case 1.

		(2, 3, 0.5 , 3, 1.5)															
		MLE				LSE				ADE				CVME			
n	Par	Mean	MSE	RMSE	Bias	Mean	MSE	RMSE	Bias	Mean	MSE	RMSE	Bias	Mean	MSE	RMSE	Bias
30	β	4.1410	21.3642	4.6221	2.1410	3.5326	12.4889	5.5340	2.5326	3.5699	7.5786	2.6649	1.5699	3.9543	8.7181	2.9282	1.9543
	γ	4.2360	10.6960	3.2705	4.2360	7.3157	58.9939	7.6808	4.3157	4.5020	13.5325	3.6787	1.5020	4.2639	11.3896	3.5458	1.2639
	η	0.1040	0.7525	0.8905	0.4960	0.1992	0.4343	0.5841	0.3398	0.2946	0.6611	0.4286	0.3541	0.2149	0.4151	0.6443	0.2851
	ξ	3.2618	2.6455	1.6265	0.2618	2.8263	1.0333	1.4165	0.5737	2.8353	1.2079	1.0990	0.1647	2.8963	0.9837	0.9918	0.6715
	ν	2.0848	1.7419	1.3198	0.5848	1.8593	0.9982	1.7058	0.4593	1.9966	0.8798	0.9380	0.4966	1.8921	0.5665	0.8827	0.3921
50	β	3.4758	17.6673	4.2032	1.4758	3.5138	12.1890	4.9182	1.7514	2.9489	6.7057	2.5895	1.4892	3.1451	8.5078	2.9168	1.1451
	γ	4.1717	7.6638	2.6408	3.1717	5.3432	39.8235	6.3106	2.3432	4.3410	11.4785	3.3880	1.3410	4.2060	11.1128	3.3336	1.2060
	η	0.1109	0.6115	0.7820	0.3891	0.1982	0.3378	0.5812	0.3008	0.3283	0.4447	0.4186	0.3171	0.2449	0.3185	0.5644	0.2551
	ξ	3.1736	2.0167	1.4201	0.1736	3.0258	1.0148	1.3224	0.5258	2.8858	0.8930	0.9450	0.1542	3.0870	0.8332	0.8540	0.5702
	ν	1.8647	1.1175	1.0571	0.3647	1.8439	0.8851	0.9408	0.4439	1.8149	0.5224	0.7228	0.3149	1.8579	0.4884	0.8819	0.3579
100	β	2.9611	5.3867	2.3209	0.9611	3.0799	4.2928	2.0719	1.5799	3.2319	5.4994	2.3451	1.2319	3.1057	5.6253	2.3718	1.1057
	γ	4.4988	6.1949	2.4889	1.4988	3.3080	5.3695	2.3172	2.3080	4.1343	5.4761	2.3401	1.1343	3.5129	7.4822	3.2423	2.5129
	η	0.2441	0.1251	0.4537	0.2559	0.3776	0.1438	0.3791	0.2224	0.3281	0.3035	0.4096	0.1719	0.3929	0.1553	0.3749	0.1971
	ξ	2.8733	0.7744	0.8800	0.1267	2.5986	0.5853	0.7651	0.4014	2.5483	0.6261	0.7912	0.1517	2.6881	0.5884	0.7671	0.3119
	ν	1.8154	0.5649	0.7516	0.3154	1.9140	0.5124	0.7158	0.4140	1.8123	0.4296	0.6554	0.3123	1.8408	0.3867	0.6218	0.3408
150	β	2.6483	4.3412	2.0836	0.6483	2.3411	3.9398	0.9694	1.3411	2.5060	1.6316	1.2774	0.5060	2.7062	3.0061	1.8031	0.8622
	γ	4.4682	4.9098	2.3485	1.4682	3.2866	4.5548	2.2249	1.6287	3.5326	3.7568	1.9383	0.5326	3.5079	6.0462	1.4304	0.9079
	η	0.2715	0.1161	0.4196	0.2285	0.3504	0.1429	0.3339	0.1496	0.4404	0.1503	0.3876	0.0596	0.3824	0.1523	0.3672	0.1176
	ξ	2.9510	0.4568	0.6758	0.0490	2.8270	0.2960	0.5440	0.3730	2.7862	0.3840	0.6197	0.1138	2.8676	0.5440	0.5865	0.2324
	ν	1.7644	0.5118	0.7154	0.2644	1.6068	0.2013	0.4486	0.4068	1.6747	0.3897	0.5383	0.2747	1.7370	0.3517	0.5817	0.2370
200	β	2.5221	3.0256	1.7394	0.5221	2.6074	2.5188	0.5871	0.6074	2.5985	1.5827	1.2571	0.5048	2.6374	2.2537	1.5012	0.6374
	γ	4.2636	4.1260	1.7916	1.2636	3.6809	3.9224	2.2187	0.6809	3.6709	3.6925	1.3222	0.5314	3.8909	5.6051	2.3675	0.8909
	η	0.2787	0.1141	0.3796	0.2213	0.3519	0.1102	0.2585	0.1481	0.3911	0.1169	0.3420	0.1089	0.3749	0.1340	0.3661	0.1109
	ξ	2.9866	0.4876	0.6583	0.0134	2.8151	0.2344	0.4591	0.2849	2.8362	0.3463	0.5885	0.1138	2.8911	0.4870	0.5221	0.1789
	ν	1.6627	0.4509	0.6715	0.1627	1.5726	0.2002	0.4442	0.3264	1.6983	0.3205	0.5261	0.1983	1.7305	0.3402	0.5479	0.2305
250	β	2.5068	1.7207	1.3118	0.5068	2.5561	1.7203	0.3116	0.5561	2.4970	1.5697	1.2529	0.4970	2.5710	2.2051	1.4850	0.5710
	γ	4.0209	3.2328	1.3515	1.2209	3.4474	3.6412	1.9082	0.4474	3.5116	3.5350	1.2802	0.5116	3.7732	4.6528	2.7664	0.7732
	η	0.2977	0.1124	0.3790	0.2050	0.3760	0.1016	0.1972	0.1240	0.3906	0.1103	0.3066	0.1038	0.3898	0.1242	0.3174	0.1002
	ξ	2.8620	0.3567	0.2973	0.0133	2.8463	0.2217	0.4494	0.2537	2.7620	0.3389	0.5822	0.1124	2.8041	0.4199	0.4480	0.1559
	ν	1.7908	0.4462	0.6044	0.1291	1.5116	0.1025	0.3500	0.3116	1.6782	0.3128	0.5106	0.1782	1.7845	0.3345	0.5383	0.2845

4.1. Data set 1: Growth Hormone data.

This dataset, studied in Ref. [4], includes the expected time it took for children to reach a certain age after receiving growth hormone treatment.

The ML estimates of parameters with their standard errors (SE) (in brackets) are provided in Table 4 , while Table 5 includes the measures of GOF. Table 5 provides clear evidence that the GKPL distribution is a better fit than the other competing distributions for fitting growth hormone data. Figures 3a and 3b display the fitted PDF plots of the GKPL distribution and other competing distributions while Figures 3c and 3d display the fitted CDF plots of the GKPL distribution and other competing distributions. Additionally, the conclusions inferred from the findings presented in Table 5 are further reinforced by Figures 3a to 3d.

4.2. Data set 2: The annual mortality rate due to Alzheimer's disease.

This data set represents the annual mortality rate due to Alzheimer's disease in India in the period 1990−2019. It has been obtained from https://ourworldindata.org.

The ML estimates of parameters with their SE (in brackets) are provided in Table 6, while Table 7 includes the GOF measures. From Table 7, it is clear that the GKPL distribution provides a better fit than the other competing distributions. Figures 4a to 4d display the fitted PDF and CDF plots of the GKPL distribution and other competing distributions for annual mortality rate due to Alzheimer's disease. Additionally, the conclusions inferred from the findings presented in Table 7 are further reinforced by Figures 4a to 4d

4.3. Data set 3: Wingo data

This data set, known as the Wingo data, provides a complete sample obtained from a clinical trial, listing the duration of relief (in hours) for 50 patients suffering from arthritis. (see [1, 17, 29, 48]).

Table 3. Results of simulation study for case 2.

		(0.25, 0.75, 0.9 , 1.5, 2)															
n	Par	MLE				LSE				ADE				CVME			
		Mean	MSE	RMSE	Bias	Mean	MSE	RMSE	Bias	Mean	MSE	RMSE	Bias	Mean	MSE	RMSE	Bias
30	β	0.1890	0.0194	0.1392	0.0610	0.3051	0.2437	0.4090	0.2551	0.2534	0.0492	0.2710	0.1034	0.3514	0.2226	0.4502	0.2614
	γ	1.0248	3.2986	1.8162	1.2748	0.4520	1.9811	1.4075	0.2980	0.7211	2.3944	1.5474	0.2289	1.8981	11.0065	3.3176	1.1481
	η	0.8238	0.5142	0.1191	0.4762	0.7507	0.3953	0.6087	0.1493	0.6116	0.1337	0.3657	0.2884	0.4819	0.5328	0.5644	0.4181
	ξ	2.6669	3.3378	1.8270	1.1669	1.3738	0.4824	0.6945	0.3262	1.9756	1.0856	1.0419	0.4756	1.9888	1.4984	1.2241	0.4888
	ν	1.7348	1.7292	1.3150	0.3652	2.7463	2.5506	1.5971	0.7463	1.7428	2.3948	1.6283	0.9572	1.8591	0.7800	1.0093	0.5409
50	β	0.2364	0.0144	0.1200	0.0336	0.4372	0.1633	0.4042	0.1872	0.3437	0.0400	0.2000	0.0937	0.3437	0.2013	0.4486	0.2544
	γ	1.6539	3.2219	1.8108	0.9039	0.9934	0.8982	0.9478	0.2934	0.6374	1.9294	1.1640	0.2226	1.4037	3.7954	1.9482	0.6537
	η	0.5247	0.4555	0.6749	0.3753	0.7262	0.3286	0.5732	0.1438	0.8124	0.0513	0.2299	0.0876	0.5745	0.2652	0.5150	0.3255
	ξ	2.1353	2.1543	1.4677	0.6353	1.1913	0.4327	0.4824	0.3087	1.2706	0.4715	0.6867	0.2294	1.1616	0.4962	0.7044	0.3384
	ν	1.7156	0.9745	0.9872	0.2844	2.2831	2.4229	0.6503	0.2831	2.4832	1.5227	1.2340	0.4832	2.5171	0.6182	1.0091	0.5171
100	β	0.2275	0.0091	0.0955	0.0225	0.2674	0.0179	0.1338	0.0174	0.2507	0.0076	0.0872	0.0037	0.2450	0.0115	0.1074	0.0050
	γ	1.2584	2.7107	1.6464	0.5084	0.4639	0.8835	0.9400	0.2861	0.6179	1.5867	1.0894	0.1321	0.8358	1.7322	0.8557	0.5858
	η	0.8201	0.0437	0.2091	0.1799	0.8172	0.0394	0.1985	0.0828	0.8225	0.0328	0.2299	0.0775	0.7637	0.1439	0.3793	0.1363
	ξ	1.7974	0.4859	0.6971	0.2974	1.3810	0.3420	0.3848	0.1190	1.5152	0.3346	0.5784	0.1152	1.7018	0.4917	0.7012	0.3382
	ν	1.9165	0.7506	0.8664	0.1835	2.1847	1.5183	0.6199	0.1847	2.1485	1.3146	1.0557	0.1485	2.0326	0.6116	0.8995	0.4326
150	β	0.2323	0.0078	0.0882	0.0177	0.2351	0.0061	0.0779	0.0149	0.2527	0.0056	0.0745	0.0027	0.2455	0.0080	0.0892	0.0045
	γ	0.3799	1.1665	1.0801	0.3701	0.2797	0.7178	0.8472	0.2703	0.4159	1.5007	1.0250	0.1311	0.7904	0.9490	0.8442	0.4596
	η	0.7564	0.0398	0.1995	0.1436	0.8794	0.0216	0.1469	0.0206	0.8896	0.0189	0.1374	0.0104	0.8774	0.0214	0.1463	0.0226
	ξ	1.6794	0.4794	0.6612	0.1794	1.5225	0.3404	0.3749	0.0225	1.4059	0.3108	0.5147	0.0941	1.4889	0.2926	0.5409	0.3291
	ν	2.3408	0.5341	0.7109	0.1408	2.7800	1.2074	0.6067	0.1800	2.8685	1.2469	1.0499	0.1385	2.0551	0.6116	0.8975	0.3551
200	β	0.2557	0.0061	0.0781	0.0157	0.2824	0.0127	0.0777	0.0124	0.2972	0.0056	0.0737	0.0022	0.3142	0.0280	0.1674	0.0642
	γ	0.2638	0.3830	0.6188	0.2862	0.3436	0.4353	0.6598	0.4064	0.2095	0.4078	0.6386	0.1305	0.0938	0.8893	0.9430	0.6562
	η	0.7735	0.0258	0.2362	0.1265	0.9039	0.0130	0.1141	0.0039	0.8706	0.0180	0.1374	0.0294	0.9209	0.0065	0.0805	0.0209
	ξ	1.4361	0.3006	0.3172	0.0639	1.3259	0.3346	0.2784	0.0108	1.2214	0.2859	0.5147	0.2786	1.1714	0.2907	0.7062	0.3286
	ν	2.4635	0.4693	0.6851	0.1335	2.8863	1.1844	0.4780	0.1763	2.9237	1.0898	1.0243	0.1237	2.0850	0.4496	0.8942	0.3450
250	β	0.2623	0.0052	0.0647	0.0123	0.2423	0.0060	0.0774	0.0077	0.2674	0.0054	0.0733	0.0017	0.2389	0.0075	0.0863	0.0111
	γ	0.5732	0.3420	0.5848	0.1768	0.7527	0.3177	0.5636	0.0027	0.5711	0.4020	0.6340	0.1289	0.8503	0.3677	0.6063	0.1003
	η	0.8393	0.0253	0.1591	0.0607	0.8761	0.0096	0.0978	0.0039	0.8941	0.0099	0.0995	0.0059	0.9311	0.0013	0.0648	0.0888
	ξ	1.5469	0.2364	0.2862	0.0469	1.7013	0.2952	0.2286	0.0013	1.4584	0.2464	0.4964	0.0416	1.8462	0.2495	0.7059	0.3262
	ν	2.2120	0.3670	0.5311	0.1120	2.0036	0.7628	0.8734	0.0036	2.3408	0.9000	0.9487	0.0408	1.9829	0.3861	0.8866	0.0171

Table 4. The MLE and SE (in brackets) of the parameters for growth hormone data.

Model	β	γ	η	ξ	ν	δ
GKPL	37.7442	6.1649	0.1053	0.5448	2.0799	
	70.9842	23.2155	0.4144	0.2180	1.5652	
GKW	56.7565	6.2794	0.0923	0.5363	1.8879	
	83.0970	119.7632	1.8369	0.1835	1.1928	
GKLLo	15.8313	6.3937	0.4978	0.5819		2.4518
	22.5604	8.3574	4.5898	0.4180		0.3307
GKGPGo	8.4968	18.1197	0.7078	0.8956	2.0019	0.1625
	24.0527	48.7510	4.5413	0.7029	1.3673	0.1841
KPL	37.6626	1.1719		0.5241	2.1475	
	57.6503	1.1126		0.1057	1.2277	
EKPL	1.7038	31.9083	18.3725	0.3981	0.3711	
	2.2629	194.3861	20.3895	0.5402	0.1866	
TIIGTPL	4.8164	0.1303		1.3380	1.1579	
	1.9610	0.0229		0.0036	0.0037	
EGPL	2.3891	52.9725		0.4916	1.2262	
	6.1316	84.6959		0.1860	2.2911	

Table 5. The GOF measures for growth hormone data.

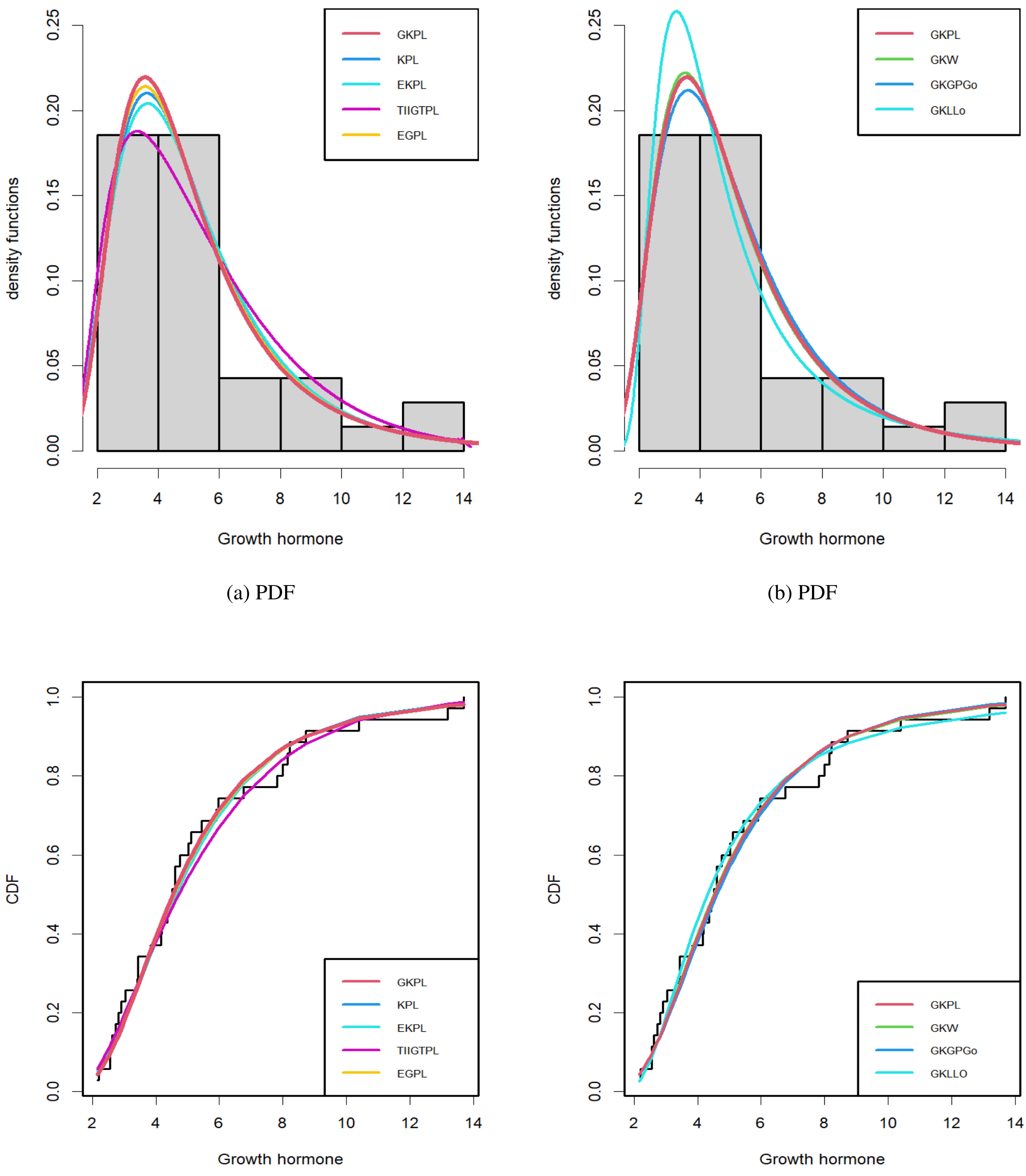
Model	$-\ell$	AIC	BIC	AICc	A*	W*	K-S	p-value
GKPL	77.9431	165.8861	173.6629	167.9551	0.3060	0.0409	0.0851	0.9617
GKW	77.8481	165.6963	173.4730	167.7652	0.2852	0.0380	0.0886	0.9464
GKLLo	77.7564	165.5128	173.2895	167.5817	0.2794	0.0433	0.0983	0.8877
GKGPGo	77.9962	167.9923	177.3244	170.9923	0.3433	0.0485	0.0863	0.9570
KPL	78.1489	164.2979	170.5193	165.6312	0.3407	0.0462	0.0891	0.9440
EKPL	78.2876	166.5751	174.3519	168.6441	0.3505	0.0489	0.0893	0.9427
TIIGTPL	77.9499	163.8997	170.1211	165.2330	0.3717	0.0578	0.1012	0.8663
EGPL	77.9572	163.9145	170.1359	165.2478	0.3118	0.0428	0.0867	0.9549

Table 6. The MLEs and SE (in brackets) of the parameters for annual mortality rate.

Model	β	γ	η	ξ	ν	δ
GKPL	20.086	27.704	0.034	0.577	1.627	
	21.436	49.016	0.059	0.169	0.847	
GKW	7.438	3.554	0.241	0.879	0.539	
	8.683	9.456	0.644	0.362	0.616	
GKLLo	3.219	4.939	0.051	2.621		2.624
	5.275	7.245	0.061	0.517		2.284
GKGPGo	7.068	9.013	0.130	0.351	0.367	1.101
	5.318	21.306	0.302	0.365	0.300	1.980
KPL	3.236	0.095		1.496	0.860	
	0.023	0.017		0.003	0.003	
EKPL	2.500	0.091	0.789	0.447	1.755	
	0.008	0.024	0.186	0.004	0.004	
TIIGTPL	10.481	0.117		1.272	1.404	
	0.164	0.021		0.018	0.009	
EGPL	0.246	66.295		0.483	9.112	
	0.028	157.754		0.213	1.652	

Table 7. The GOF measures for annual mortality rate.

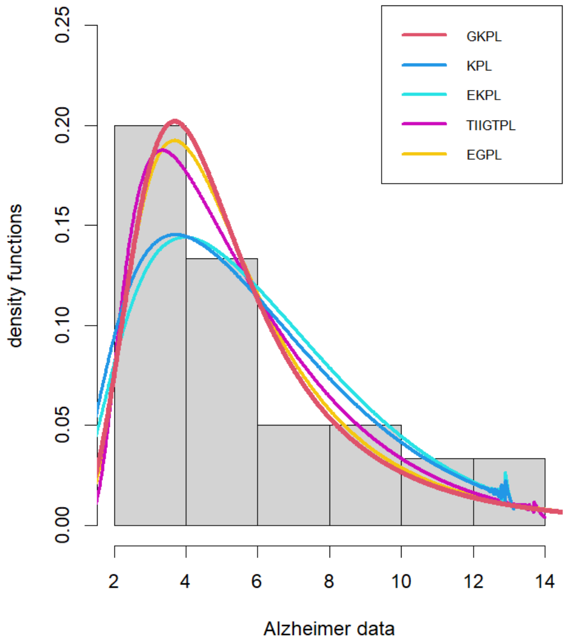
model	$-\ell$	AIC	BIC	AICc	AD	CVM	K-S	p-value
GKPL	70.509	151.018	158.024	153.518	0.451	0.054	0.091	0.948
GKW	70.908	151.815	158.821	154.315	0.549	0.073	0.091	0.944
GKLLo	70.551	151.102	158.108	153.602	0.454	0.056	0.091	0.940
GKGPGo	71.649	155.297	163.705	158.950	0.621	0.083	0.105	0.860
KPL	70.770	149.540	155.145	151.140	0.640	0.105	0.114	0.790
EKPL	70.655	151.311	158.317	153.811	0.867	0.160	0.138	0.571
TIIGTPL	70.208	151.415	158.020	158.015	0.458	0.061	0.094	0.931
EGPL	70.272	148.544	154.149	150.144	0.479	0.066	0.092	0.940



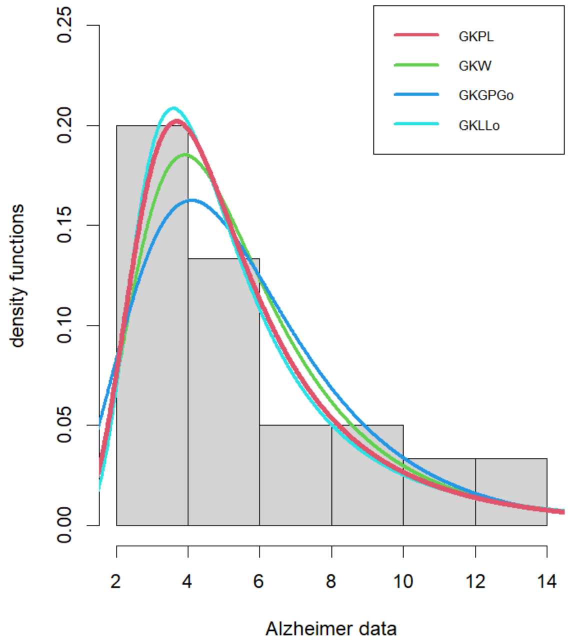
(a) PDF (b) PDF

(c) CDF (d) CDF

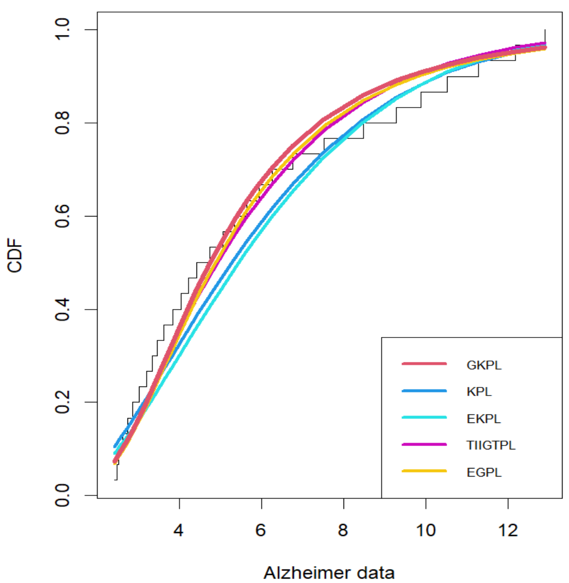
Figure 3. PDF and CDF fit plots of the GKPL, GKW, GKLLo, GKGPGo, KPL, EKPL, TIIGTPL, and EGPL distributions for growth hormone data.



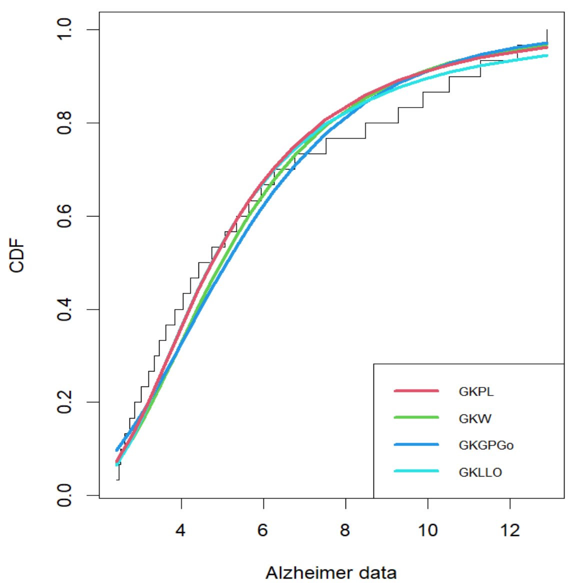
(a) PDF



(b) PDF



(c) CDF



(d) CDF

Figure 4. PDF and CDF fit plots of the GKPL, GKW, GKLLo, GKGPGo, KPL, EKPL, TIIGTPL, and EGPL distributions for annual mortality rate.

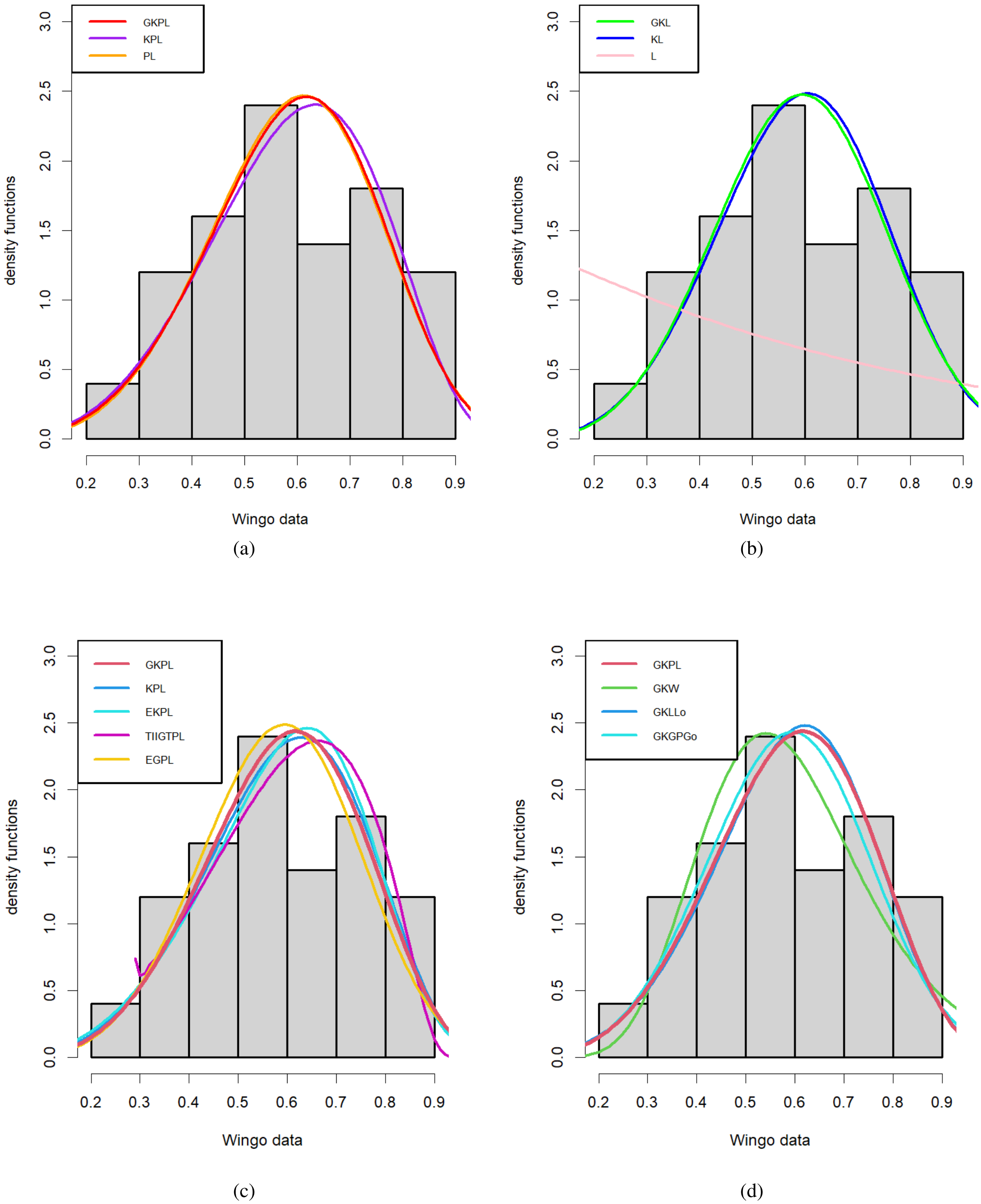
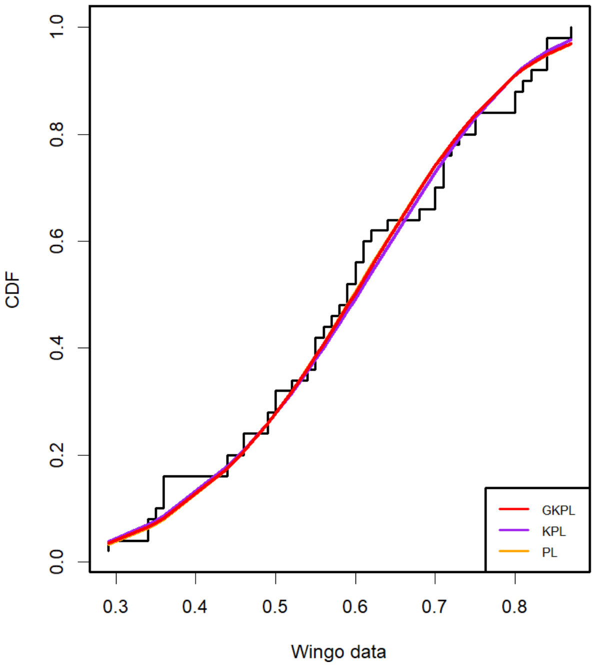
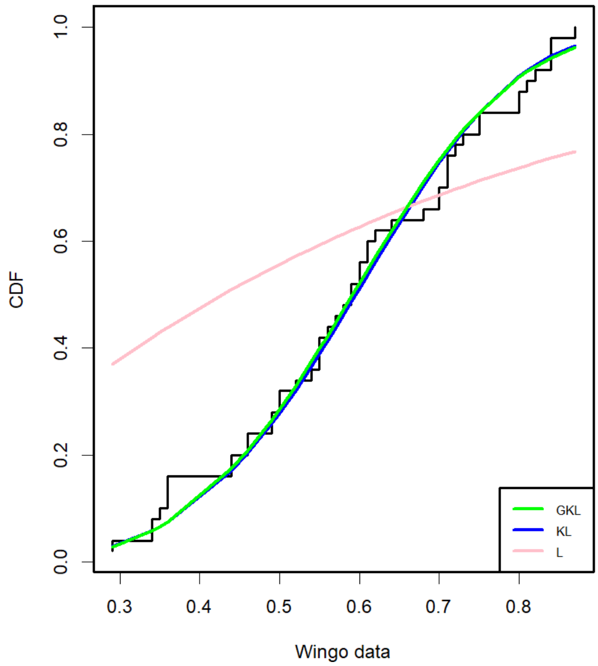


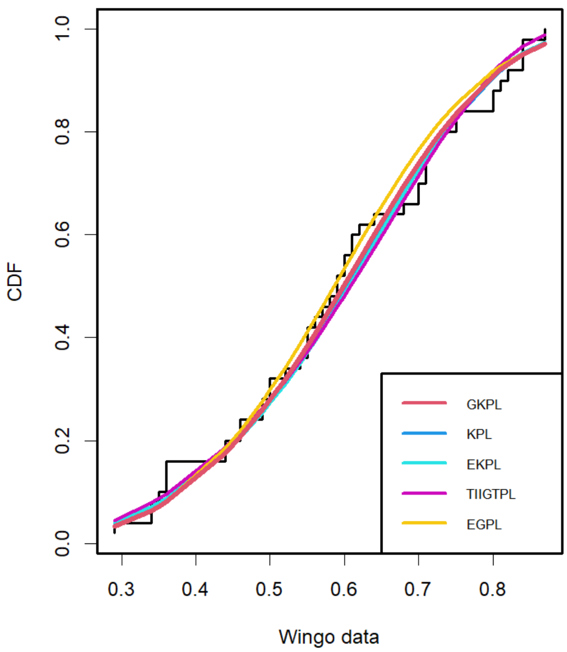
Figure 5. PDF fit plots of the GKPL distribution, its sub-models, and other competitive distributions for Wingo data.



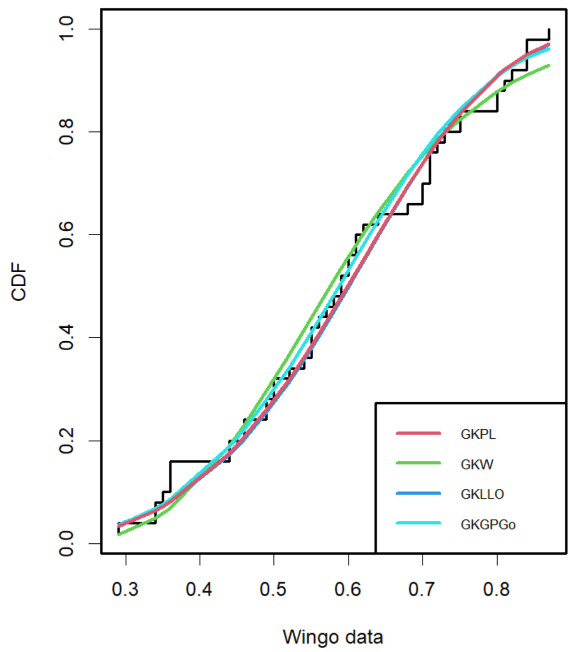
(a)



(b)



(c)



(d)

Figure 6. CDF fit plots of the GKPL distribution, its sub-models, and other competitive distributions for Wingo data.

The ML estimates of parameters with their SE (in brackets) are provided in Table 8, while Table 9 includes the measures of GOF for the fitted distributions. According to the findings in Table 9, it is clear that the GKPL distribution outperforms the other competing distributions for fitting the Wingo dataset. Figures 5a and 5b display the fitted PDF plots of the GKPL distribution and some special distributions while Figures 5c and 5d show the fitted PDF plots of the GKPL distribution and other competitive distributions. Figures 6a to 6d display the fitted CDF plots of the GKPL distribution and other distributions.

Table 8. The MLEs and SE (in brackets) of the parameters for Wingo data.

Model	β	γ	η	ξ	ν	δ
GKPL	1.7933 3.0139	12.1037 77.1592	0.2676 1.4345	2.1810 3.6041	0.3497 1.0023	
GKW	0.2313 0.1352	5.3315 5.5272	0.2361 0.2353	14.1719 8.1867	16.7160 22.4393	
GKLLo	1.0286 0.3616	14.8795 10.4339	0.9234 0.3560	4.1345 1.2608		1.1759 0.3278
GKGPGo	0.1072 0.2590	8.7072 20.0322	0.7154 0.9566	1.1696 1.7497	4.1875 3.9062	4.6645 4.1758
KPL	0.2959 0.3878	4.7106 7.4043		12.9367 13.2159	1.4222 3.1438	
EKPL	2.1282 2.3606	46.4631 50.6943	0.5816 0.8831	0.8938 3.3480	2.8439 4.8422	
TIIGTPL	0.0541 0.0082	2.4037 0.5260		31.8540 0.0205	21.9429 0.8527	
EGPL	4.4897 1.8161	1.2795 0.6850		3.5544 1.1701	1.9241 0.6019	
PL				4.2126 0.4845	6.9022 1.3404	
GKL	4.9420 3.1502	452.5376 953.0150	0.2243 0.4037		1.2463 2.0667	
KL	4.4539 1.0254	400.1597 1070.0771			0.8715 0.5950	
L					2.2194 0.2542	

The Wingo data used in Refs. [29], [17], and [1] for examining the fitting performance of the odd-Burr generalized Frechet (OB-F), the generalized odd log-logistic reciprocal Weibull (GOLLRW), the generalized odd log-logistic reciprocal Rayleigh (GOLLRR), and the generalized Odd Log-Logistic Frechet (GOLLF) distributions. The GOF measures results for these studies are presented in Table 10.

By comparing the results in Table 10 with the finding of our current study in Table 9, we observed that the GKPL distribution is a more accurate model for modeling the relief time data (Wingo data).

Table 9. The GOF measures for Wingo data.

Model	A*	W*	K-S	p-value
GKPL	0.4055	0.0514	0.0805	0.9019
GKW	0.4424	0.0528	0.0840	0.8726
GKLLo	0.4204	0.0521	0.0972	0.7317
GKGPGo	0.4266	0.0589	0.0882	0.8312
KPL	0.4007	0.0538	0.0836	0.8758
EKPL	0.4408	0.0623	0.0905	0.8077
TIIGTPL	0.4643	0.0633	0.0960	0.7460
EGPL	0.5068	0.0636	0.1055	0.6343
PL	0.4148	0.0516	0.0830	0.8814
GKL	0.4285	0.0506	0.0917	0.7941
KL	0.4296	0.0520	0.0866	0.8473
L	11.2192	2.2591	0.3796	0.0000

Table 10. The GOF measures of the OB-F, GOLLRW, GOLLRR, and GOLLF for Wingo data.

Models	A*	W*	K-S	p-value	Ref.
OB-F	0.8163	0.1038	0.1043	0.6482	[29]
GOLLRW	0.8163	0.1038	0.1043	0.6482	[17]
GOLLRR	1.3498	0.19551	0.11008	0.5797	[17]
GOLLF	1.1276	0.15615	0.10476	0.6427	[1]

5. Conclusion

The generalized Kumaraswamy power Lindley (GKPL) distribution is developed in this research by combining the PL distribution with the GK-G family. The GKPL distribution is flexible and useful in modeling real-life data such as growth hormone, the annual mortality rate due to Alzheimer's disease in India, and the relief time datasets. The GKPL distribution is better than the other competing distributions (which are considered in this study, such as KPL, EKPL, TIIGTPL, EGPL, GKW, GKLLo, GKGPGo, OB-F, GOLLRW, GOLLRR, and GOLLF) in modeling such types of real-life data. The GKPL distribution parameters were estimated using ML, LS, AD, and CVM estimation methods. Several mathematical properties related to the GKPL distribution were derived, including moments, quantile function, order statistics, entropy, and mean residual life function.

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Conflicts of interest

The authors declare that they have no conflicts of interest related to this work.

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