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Liu-Type Estimator for the Poisson-Inverse Gaussian Regression Model: Simulation and Practical Applications

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Abstract The Poisson-Inverse Gaussian regression model (PIGRM) is commonly used to analyze count datasets with over-dispersion. While the maximum likelihood estimator (MLE) is a standard choice for estimating PIGRM parameters, its performance may be suboptimal in the presence of correlated explanatory variables. To overcome this limitation, we introduce a novel Liu-type estimator for PIGRM. Our analysis includes an examination of the matrix mean square error (MMSE) and scalar mean square error (SMSE) properties of the proposed estimator, comparing them with those of the MLE, ridge, and Liu estimators. We also present several parameters of the Liu-type estimator for PIGRM. We evaluated the performance of the proposed estimator through a simulation study and application to real-life data, using SMSE as the primary evaluation criterion. Our results demonstrate that the proposed estimators outperform the MLE, ridge, and Liu estimators in both simulated and real-world scenarios.

Keywords Biased estimator, Liu estimator, Liu-type estimator, Maximum Likelihood Estimator, Matrix Mean Square Error, Multicollinearity, Over-dispersion, Parameter estimation, Poisson-Inverse Gaussian regression model, Ridge estimator, Scalar Mean Square Error

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1. Introduction

In regression analysis, Count regression models (CRMs) are a class of statistical models designed to analyze data where the response variable represents counts of events or occurrences. These models are particularly useful when dealing with data that naturally lend themselves to count-based representation, such as the number of accidents in a given period, customer complaints, or items sold. Common types of CRMs include the Poisson regression model (PRM), which assumes that the counts follow a Poisson distribution with a mean equal to variance, and the Negative Binomial regression model (NBRM), which relaxes the equality assumption and allows for over-dispersion. Conway-Maxwell-Poisson regression model (CMPRM) is an extension of the traditional PRM, designed to address over- and underdispersion in count data, and the Poisson-inverse Gaussian regression model (PIGRM) is an alternative to the NBRM because the PIG distribution has slightly longer tails and larger kurtosis [13, 46, 49]. The PRM is often used with count datasets, assuming equal dispersion where the mean and variance are equivalent. However, this assumption is commonly unmet in practice, leading to over-dispersion, where the variance surpasses the mean [20]. Several factors, such as data heterogeneity and excess zeros generated by the model, can contribute to over-dispersion.

To address overdispersion in count data, various regression models can be employed. Common approaches include NBRM, CMPRM, and PIGRM [13], which are used as alternatives to NBRM. Willmot [49] used the

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PIGRM as an alternative to the NBRM, as it is particularly well-suited for highly skewed count datasets with heterogeneity due to its capability to model long-tailed data, making it preferable over the NBRM in such cases. Putri et al. [38] compared the PIGRM and the NBRM to handle the overdispersion in the data. Saraiva et al. [44] used PIGRM to handle overdispersion in overdispersed dengue data. Husna and Azizah [25] used PIGRM to analyze cases of dengue hemorrhagic fever in central Java.

The PIGRM commonly utilizes the MLE to estimate its coefficients. The explanatory variables must be independent in multiple regression models, as the MLE performs optimally under this assumption [48]. However, practical applications often reveal linear relationships among explanatory variables, leading to multicollinearity issues. As multicollinearity increases, the ability to identify the separate effects of individual explanatory variables on the response variable diminishes, making it challenging to disentangle their effects. This issue can lead to high variance in the estimated coefficients, reducing the precision of estimation and potentially causing coefficients to appear to have the wrong sign. Important variables may be dropped from the model and some estimated regression coefficients may be highly sensitive to changes in data points or the exclusion of an explanatory variable lacking statistical significance. First described by Frisch [17], multicollinearity poses challenges in parameter estimation using MLE, including large variances in coefficient estimates that can undermine result efficiency [16, 6].

Several methods have been proposed in the literature to tackle multicollinearity. One of the most common and attractive approaches is the ridge regression, which was initially developed by Hoerl and Kennard [22]. Several studies have aimed to determine the optimal shrinkage parameter value for ridge regression, particularly within the Linear Regression Model (LRM) context. Key contributions in this area include the works of Hoerl and Kennard [23], Khalaf and Shukur [26], Dorugade [14], Golub et al. [18], Alkhamisi et al. [7], and others. Segerstedt [45] a ridge regression method within the Generalized Linear Model (GLM) framework. Building on Segerstedt's work, statisticians have extended the ridge regression approach to various models. For example, Månsson and Shukur [35], Månsson [33], Saleh et al.[41], Kibria et al. [28], Sami et al. [42], Amin [8], and Ashraf [10]. The Liu estimator [30] has also emerged as a valuable approach for addressing multicollinearity. Unlike the ridge parameter k, which is a non-linear function, the key advantage of the Liu estimator lies in its property where the shrinkage parameter d is a linear function of the estimates. Several studies have explored the Liu estimator's performance in the Linear Regression Model (LRM) context. For instance, Akdeniz and Kaciranlar [1], Alheety and Kibria [5]. Kibria [27], and Qasim et al. [39]. Kurtoğlu [29] initially defined the Liu estimator for the Generalized Linear Model (GLM), after which it has been applied in various models. For instance, Qasim et al. [40], Akram [4], Månsson [34], and Bulut [12].

Several studies have proposed new estimators based on two parameters for various regression models to mitigate multicollinearity. These include works by Akram [3], Lukman et al. [31], Akram et al. [2], Akram et al. [4], Sami et al. [43], Asar and Algamal [9], and Tanış and Asar [47].

The strengths of the ridge and Liu estimators are combined in this article to create a Liu-type estimator for the PIGRM. The suggested estimator applies innovative strategies to find the best values for the shrinkage parameters (k and d), which are inspired by the work of Huang [24] and Akram et al. [3]. These shrinkage parameters must be chosen carefully to maximize the performance of the Liu-type biased estimating techniques. The scalar mean squared error (SMSE) is used as the benchmark in a Monte Carlo simulation to assess the performance of the suggested estimator. The results demonstrate the superiority of the proposed estimator compared to the existing methods. In addition, the article presents the practical utility of the proposed estimator through two real-world applications. These applications illustrate the improved accuracy and robustness of the Liu-type estimator in multicollinearity.

The article is organized into five sections. Section 2 introduces the PIGRM and reviews biased estimation methods for the PIGRM, including the ridge, Liu estimators, and the proposed estimator. It also provides a theoretical comparison of these methods based on MSE criteria and discusses the selection of biasing parameters. Section 3 outlines the design and results of a Monte Carlo simulation to evaluate the performance of the proposed Liu-type estimator. Section 4 demonstrates the practical utility of the proposed estimator through real-world applications. Finally, Section 5 summarizes the main findings and discusses the implications of the investigation.

2. Methodology

The Poisson-Inverse Gaussian distribution (PIG) is a composite of the Poisson and Inverse Gaussian distributions. Let Y be distributed as ν following a Poisson distribution with a mean $\mu\nu$. The probability mass function for Y is given by:

$$f(y|\mu,\nu) = \frac{(\mu\nu)^y}{y!}\exp(-\mu\nu) \tag{1}$$

Here, ν follows an Inverse Gaussian distribution with a mean of 1 and a dispersion parameter ϕ (i.e., $\nu \sim \text{IG}(1,\phi)$). The probability mass function for ν is given by:

$$g(\nu) = \left(\frac{\phi}{2\pi\nu^3}\right)^{\frac{1}{2}} \exp\left(\frac{-\phi(\nu-1)^2}{2\nu}\right) \tag{2}$$

The marginal probability mass function for Y, denoted by the PIG distribution, is then given by:

$$P(Y=y|\mu) = \int f(y|\mu,\nu)g(\nu)d\nu = \left(\frac{2\phi}{\pi}\right)^{\frac{1}{2}} \frac{\exp(\phi)\mu^y \mathbf{K}_s(\alpha)}{y!\,(\alpha/\phi)^s}, \qquad y = 0,1,2,... \tag{3}$$

where $\alpha^2 = \phi^2\left(1+2\frac{\mu}{\phi}\right)$, $s = y - 0.5$, and $\mathbf{K}_s(\alpha)$ represents the modified Bessel function of the second kind [21, 50], both parameters μ and ϕ are constrained to be non-negative. The mean and variance of Equation (3) are $E(Y) = \mu$ and $V(Y) = \mu + \mu^3/\phi$, respectively.

In the PIGRM, the relationship $g(\mu) = \eta_i$ holds, where η_i denotes the linear predictor of the explanatory variables and is defined as $\eta_i = x_i'\beta$. Here, $\beta = (\beta_1, \beta_2, \ldots, \beta_p)'$ represents the vector of regression coefficients, and $x_i = (1, x_{i1}, \ldots, x_{i(p-1)})'$ vector of p explanatory variables for $i = 1, 2, \ldots, (n-1)$. The estimation parameters for PIGRM is done using the Poisson-Inverse Gaussian maximum likelihood method [11]. The log-likelihood function for PIGRM is given as

$$\ell(\mu_i,\phi) = \sum_{i=1}^{n} y_i \log(\mu_i) + \phi - \log(y_i!) - \frac{1}{2}\log\left(\frac{1}{\phi}\right) - \frac{2y_i-1}{4}\log\left(1+2\frac{\mu_i}{\phi}\right) + \log \mathbf{K}_{y-0.5}\phi\left(1+2\frac{\mu_i}{\phi}\right) \tag{4}$$

Also, we have

$$\ell(\beta) = \sum_{i=1}^{n} y_i x_i'\beta + \phi - \log(y_i!) - \frac{1}{2}\log\left(\frac{1}{\phi}\right) - \frac{2y_i-1}{4}\log\left(1+2\frac{x_i'\beta}{\phi}\right) + \log \mathbf{K}_{y-0.5}\phi\left(1+2\frac{x_i'\beta}{\phi}\right) \tag{5}$$

To derive the MLE, we can take the first derivative of Equation (5) with respect to the parameter of interest and set it equal to zero, we have

$$\frac{\partial \ell}{\partial \beta_j} = \sum_{i=1}^{n}\left[x_i\left(y_i - \frac{K_{y_i-\frac{1}{2}}(\alpha)}{\sqrt{(1+\frac{2}{\phi}\exp(x_i'\beta)}}\exp(x_i'\beta)\right)\right] = 0, \tag{6}$$

$$\frac{\partial \ell}{\partial \phi} = \sum_{i=1}^{n}\left(-\phi^2 - y_i\phi + \frac{\phi^2 K_{y_i-\frac{1}{2}}(\alpha)\left(1+\frac{\exp(x_i'\beta)}{\phi}\right)}{\sqrt{(1+\frac{2}{\phi}\exp(x_i'\beta)}}\right) = 0, \tag{7}$$

where i=1,..,n

Given that Eqs. (6) and (7) are non-linear in β, the Maximum Likelihood Estimate (MLE) needs to be computed using numerical methods like the iteratively reweighted least squares algorithm or the Newton-Raphson algorithm.

Following the final iteration, the MLE estimate is obtained as:

$$\hat{\beta}_{\text{MLE}} = (X'\hat{V}X)^{-1}X'\hat{V}\hat{u}, \tag{8}$$

where $\hat{V} = \text{diag}(\mu_i + \alpha\mu_i^3)$ and $\hat{u}_i = \log(\mu) + \frac{y_i - \mu_i}{\mu_i + \mu^3/\phi}$, which is known as the adjusted response variable. Both $\hat{V}$ and $\hat{u}_i$ are evaluated using the Fisher scoring procedure [11].

The MMSE and the SMSE of any estimator can be determined using the spectral decomposition of the matrix $X'\hat{V}X$, where $X'\hat{V}X = \mathcal{Q}\Lambda\mathcal{Q}'$. Here, $\Lambda = \text{diag}(\lambda_1, \lambda_2, ..., \lambda_p)$ is the diagonal matrix of the eigenvalues of $X'\hat{V}X$, and $\mathcal{Q}$ is the orthogonal matrix whose the columns are the eigenvectors of $X'\hat{V}X$.

The bias vector for $\hat{\beta}_{\text{MLE}}$ is 0, and its covariance matrix is given by $\text{Cov}(\hat{\beta}_{\text{MLE}}) = \hat{\phi}(X'\hat{V}X)$. Consequently, the MMSE of $\hat{\beta}_{\text{MLE}}$ defined as follows:

$$\text{MMSE}(\hat{\beta}_{\text{MLE}}) = \hat{\phi}\mathcal{Q}\Lambda^{-1}\mathcal{Q}' \tag{9}$$

and the SMSE of $\hat{\beta}_{\text{MLE}}$ is defined as

$$\text{SMSE}(\hat{\beta}_{\text{MLE}}) = \sum_{j=1}^{p} \frac{\hat{\phi}}{\lambda_j} \tag{10}$$

where $\hat{\phi}$ is the estimated dispersion parameter calculated as: $\hat{\phi} = \left(\sum_{i=1}^{n} \frac{(y_i - \hat{\mu}_i)^2}{V(\mu_i)}\right)/(n-p)$.

2.1. Poisson-Inverse Gaussian Ridge Regression Estimator

In response to multicollinearity challenges, Segerstedt [45] introduced a ridge regression method within the Generalized Linear Model (GLM) as an alternative to Maximum Likelihood Estimation (MLE). This method has been further developed and applied by researchers such as Månsson and Shukur [35], Månsson [33], Kibria et al. [28], Sami et al. [42], Amin [8], and Ashraf [10]. Building on this foundation, Batool et al. [11] introduced the Ridge estimator to mitigate multicollinearity in the Poisson-Inverse Gaussian Regression Model (PIGRM). They named it the Poisson-Inverse Gaussian Ridge Regression Estimator (PIGRRE) and defined it as follows:

$$\hat{\beta}_{\text{PIGRRE}} = (X'\hat{V}X + k_r I)^{-1}X'\hat{V}X\hat{\beta}_{\text{MLE}}, \tag{11}$$

where k_r is the non-negative shrinkage ridge parameter, and I represents the identity matrix of order $p \times p$. When $k_r = 0$, the PIGRRE ($\hat{\beta}_{\text{PIGRRE}}$) reduces to the MLE ($\hat{\beta}_{\text{MLE}}$).

The bias vector and covariance matrix for PIGRRE are defined as follows:

$$\text{Bias}(\hat{\beta}_{\text{PIGRRE}}) = E(\hat{\beta}_{\text{PIGRRE}}) - \beta = -k_r\Lambda_k^{-1}\beta. \tag{12}$$

$$\text{Cov}(\hat{\beta}_{\text{PIGRRE}}) = E\left(\left[\hat{\beta}_{\text{PIGRRE}} - E(\hat{\beta}_{\text{PIGRRE}})\right]\left[\hat{\beta}_{\text{PIGRRE}} - E(\hat{\beta}_{\text{PIGRRE}})\right]'\right) = \hat{\phi}(\mathcal{Q}\Lambda_k^{-1}\Lambda_k\Lambda_k^{-1}\mathcal{Q}') \tag{13}$$

The MMSE and the SMSE of PIGRRE are

$$\begin{aligned} \text{MMSE}(\hat{\beta}_{\text{PIGRRE}}) =& \text{Cov}(\hat{\beta}_{\text{PIGRRE}}) + \text{Bias}(\hat{\beta}_{\text{PIGRRE}})\text{Bias}(\hat{\beta}_{\text{PIGRRE}})' \\ =& \hat{\phi}(\mathcal{Q}\Lambda_k^{-1}\Lambda_k\Lambda_k^{-1}\mathcal{Q}') + b_{\text{PIGRRE}}b'_{\text{PIGRRE}}, \end{aligned} \tag{14}$$

where $b_{\text{PIGRRE}} = \text{Bias}(\hat{\beta}_{\text{PIGRRE}}) = -k_r\mathcal{Q}\Lambda_k^{-1}\alpha$ and $\Lambda_k = \text{diag}(\lambda_1 + k_r, \lambda_2 + k_r, ..., \lambda_p + k_r)$.

$$\begin{aligned} \text{SMSE}(\hat{\beta}_{\text{PIGRRE}}) &= \text{tr}\left(\text{MMSE}(\hat{\beta}_{\text{PIGRRE}})\right) \\ &= \hat{\phi}\sum_{j=1}^{p} \frac{\lambda_j}{(\lambda_j + k_r)^2} + \sum_{j=1}^{p} \frac{k_r^2\alpha_j^2}{(\lambda_j + k_r)^2}, \end{aligned} \tag{15}$$

where $\alpha = \mathcal{Q}'\beta_{\text{MLE}}$.

2.2. *Poisson-Inverse Gaussian Liu estimator*

Liu [30] introduced an estimator known as the Liu estimator as an alternative to the Ridge estimator. Following Qasim et al. [40], Akram [4], Månsson [34], and Bulut [12] the Liu estimator is defined to PIGRM as follows:

$$\hat{\beta}_d = (X'X + I)^{-1}(X'X + d_l I)\hat{\beta} \tag{16}$$

The Liu estimator incorporates advantages from the Ridge and Stein-type estimators. It is also a linear function of the shrinkage parameter d_l. Given these characteristics, the Liu estimator can be a compelling alternative to the Ridge estimator. The Liu estimator for the PIGRM is defined as:

$$\hat{\beta}_{\text{PIGLE}} = (X'\hat{V}X + I)^{-1}(X'\hat{V}X + d_l I)\hat{\beta}_{\text{MLE}} \tag{17}$$

where d_l is a Liu shrinkage parameter that should be in the range of $(0, 1)$. If $d_l = 1$ then $\hat{\beta}_{\text{PIGLE}} = \hat{\beta}_{\text{MLE}}$. The bias vector and MMSE of PIGLE are expressed as:

$$\text{Bias}(\hat{\beta}_{\text{PIGLE}}) = E(\hat{\beta}_{\text{PIGLE}}) - \beta = (d_l - 1)\Lambda_l^{-1}\beta. \tag{18}$$

$$\text{Cov}(\hat{\beta}_{\text{PIGLE}}) = E\left(\left[\hat{\beta}_{\text{PIGLE}} - E(\hat{\beta}_{\text{PIGLE}})\right]\left[\hat{\beta}_{\text{PIGLE}} - E(\hat{\beta}_{\text{PIGLE}})\right]'\right) = \hat{\phi}(\mathcal{Q}\Lambda_l^{-1}\Lambda_d\Lambda^{-1}\Lambda_d\Lambda_l^{-1}\mathcal{Q}') \tag{19}$$

$$\begin{aligned}\text{MMSE}(\hat{\beta}_{\text{PIGLE}}) =& \text{Cov}(\hat{\beta}_{\text{PIGLE}}) + \text{Bias}(\hat{\beta}_{\text{PIGLE}})\text{Bias}(\hat{\beta}_{\text{PIGLE}})' \\ =& \hat{\phi}(\mathcal{Q}\Lambda_l^{-1}\Lambda_d\Lambda^{-1}\Lambda_d\Lambda_l^{-1}\mathcal{Q}') + b_{\text{PIGLE}}b'_{\text{PIGLE}},\end{aligned} \tag{20}$$

where $b_{\text{PIGLE}} = \text{Bias}(\hat{\beta}_{\text{PIGLE}}) = (d_l - 1)\mathcal{Q}\Lambda_l^{-1}\alpha$ and $\Lambda_d = \text{diag}(\lambda_1 + d_l, \lambda_2 + d_l, ..., \lambda_p + d_l)$, $\Lambda_l = \text{diag}(\lambda_1 + 1, \lambda_2 + 1, ..., \lambda_p + 1)$.

Therefore, the SMSE of PIGLE defined as:

$$\begin{aligned}\text{SMSE}(\hat{\beta}_{\text{PIGLE}}) &= \text{tr}\left(\text{MMSE}(\hat{\beta}_{\text{PIGLE}})\right) \\ &= \hat{\phi}\sum_{j=1}^{p}\frac{(\lambda_j + d_l)^2}{\lambda_j(\lambda_j + 1)^2} + \sum_{j=1}^{p}\frac{(d_l - 1)^2\alpha_j^2}{(\lambda_j + 1)^2},\end{aligned} \tag{21}$$

The optimal value for d_l can be obtained by minimizing Eq. (21) in the following manner:

$$d_l = \frac{\sum_{j=1}^{p}\frac{\alpha_j - \hat{\phi}}{(\lambda_j + 1)^2}}{\sum_{j=1}^{p}\frac{\hat{\phi} + \lambda_j\alpha_j}{\lambda_j(\lambda_j + 1)^2}}.$$

2.3. *Proposed estimator*

Building on the research by Özkale and Kaciranlar [37], Akram et al. [3] introduced a Liu-type estimator for the inverse Gaussian regression model. This study introduces a Poisson-Inverse Gaussian Liu-type Estimator (PIGLTE) with two shrinkage parameters k_t and d_t, specifically for the Poisson-Inverse Gaussian Regression Model (PIGRM), aimed at addressing multicollinearity. The estimator is defined as follows:

$$\hat{\beta}_{\text{PIGLTE}} = (X'\hat{V}X + k_t I)^{-1}(X'\hat{V}X + k_t d_t I)\hat{\beta}_{\text{MLE}} \tag{22}$$

where $k_t > 0$, $0 < d_t < 1$ are the Liu-type parameters. It is important to note that $\hat{\beta}_{\text{PIGLTE}}$ is the PIGLTE that encompasses the MLE, the PIGLE and the PIGRRE as special instances:

1. If $k_t = 0$, then $\hat{\beta}_{\text{PIGLTE}} = \hat{\beta}_{\text{MLE}}$.
2. If $k_t = 1$, then $\hat{\beta}_{\text{PIGLTE}} = \hat{\beta}_{\text{PIGLE}}$.

3. If $d_t = 0$, then $\hat{\beta}_{\text{PIGLTE}} = \hat{\beta}_{\text{PIGRRE}}$.

The bias vector and the covariance matrix of the PIGLTE are expressed as

$$\text{Bias}(\hat{\beta}_{\text{PIGLTE}}) = E(\hat{\beta}_{\text{PIGLTE}}) - \beta = k_t(d_t - 1)\Lambda_{k_t}^{-1}\beta. \tag{23}$$

$$\begin{aligned}\text{Cov}(\hat{\beta}_{\text{PIGLTE}}) =& E\left(\left[\hat{\beta}_{\text{PIGLTE}} - E(\hat{\beta}_{\text{PIGLTE}})\right]\left[\hat{\beta}_{\text{PIGLTE}} - E(\hat{\beta}_{\text{PIGLTE}})\right]'\right) \\ =& \hat{\phi}(\mathcal{Q}\Lambda_{k_t}^{-1}\Lambda_{k_t d_t}\Lambda^{-1}\Lambda_{k_t d_t}\Lambda_{k_t}^{-1}\mathcal{Q}')\end{aligned} \tag{24}$$

Thus, MMSE and SMSE of the PIGLTE are obtained using the covariance matrix and bias vector of PIGLTE as follows

$$\begin{aligned}\text{MMSE}(\hat{\beta}_{\text{PIGLTE}}) =& \text{Cov}(\hat{\beta}_{\text{PIGLTE}}) + \text{Bias}(\hat{\beta}_{\text{PIGLTE}})\text{Bias}(\hat{\beta}_{\text{PIGLTE}})' \\ =& \hat{\phi}(\mathcal{Q}\Lambda_{k_t}^{-1}\Lambda_{k_t d_t}\Lambda^{-1}\Lambda_{k_t d_t}\Lambda_{k_t}^{-1}\mathcal{Q}') + b_{\text{PIGLTE}}b'_{\text{PIGLTE}},\end{aligned} \tag{25}$$

where $b_{\text{PIGLE}} = \text{Bias}(\hat{\beta}_{\text{PIGLTE}}) = k_t(d_t - 1)\mathcal{Q}\Lambda_{k_t}^{-1}\alpha$, $\Lambda_{k_t} = \text{diag}(\lambda_1 + k_t, \lambda_2 + k_t, ..., \lambda_p + k_t)$, and $\Lambda_{k_t d_t} = \text{diag}(\lambda_1 + k_t d_t, \lambda_2 + k_t d_t, ..., \lambda_p + k_t d_t)$.

$$\begin{aligned}\text{SMSE}(\hat{\beta}_{\text{PIGLTE}}) &= \text{tr}\left(\text{MMSE}(\hat{\beta}_{\text{PIGLTE}})\right) \\ &= \hat{\phi}\sum_{j=1}^{p}\frac{(\lambda_j + k_t d_t)^2}{\lambda_j(\lambda_j + k_t)^2} + \sum_{j=1}^{p}\frac{(d_t - 1)^2 k_t^2\alpha_j^2}{(\lambda_j + k_t)^2}\end{aligned} \tag{26}$$

2.4. The superiority of the proposed estimator

The following lemma provides theorems that illustrate how the proposed estimator outperforms other estimators.

Lemma 1
Assuming B is a positive definite (p.d.) matrix, c is a positive constant, and γ is a vector of nonzero constants, the expression $cB - \gamma\gamma' > 0$ holds true iff $\gamma B\gamma' < c$ [15].

Theorem 2.1
Within the context of the PIGRM, $\hat{\beta}_{\text{PIGLTE}}$ is superior to $\hat{\beta}_{\text{MLE}}$ if and only if $\text{MMSE}(\hat{\beta}_{\text{MLE}}) - \text{MMSE}(\hat{\beta}_{\text{PIGLTE}}) > 0$, where $k_t > 0$, $0 < d_t < 1$, and $b_{\text{PIGLTE}} = \text{Bias}(\hat{\beta}_{\text{PIGLTE}})$.

Proof
The disparity in MMSE functions between $\hat{\beta}_{\text{MLE}}$ and $\hat{\beta}_{\text{PIGLTE}}$, as computed using Eq. (9) and Eq. (25), is derived as:

$$\text{MMSE}(\hat{\beta}_{\text{MLE}}) - \text{MMSE}(\hat{\beta}_{\text{PIGLTE}}) = \hat{\phi}\mathcal{Q}(\Lambda^{-1} - \Lambda_{k_t}^{-1}\Lambda_{k_t d_t}\Lambda^{-1}\Lambda_{k_t d_t}\Lambda_{k_t}^{-1})\mathcal{Q}' - b_{\text{PIGLTE}}b'_{\text{PIGLTE}} \tag{27}$$

Equation (27) can alternatively be formulated using the SMSE as:

$$\text{SMSE}(\hat{\beta}_{\text{MLE}}) - \text{SMSE}(\hat{\beta}_{\text{PIGLTE}}) = \hat{\phi}\mathcal{Q}\text{diag}\left(\frac{1}{\lambda_j} - \frac{(\lambda_j + k_t d_t)^2}{\lambda_j(\lambda_j + k_t)^2}\right)\mathcal{Q}' - b_{\text{PIGLTE}}b'_{\text{PIGLTE}} \tag{28}$$

The matrix $(\Lambda^{-1} - \Lambda_{k_t}^{-1}\Lambda_{k_t d_t}\Lambda^{-1}\Lambda_{k_t d_t}\Lambda_{k_t}^{-1})$ is positive definite if $(\lambda_j + k_t)^2 - (\lambda_j + k_t d_t)^2 > 0$, which equivalent to $[(\lambda_j + k_t) + (\lambda_j + k_t d_t)][(\lambda_j + k_t) - (\lambda_j + k_t d_t)] > 0$. We obtain $k_t(1 - d_t)(2\lambda_j + (d_t + 1)k_t) > 0$. Therefore, if $0 < d_t < 1$ and $k_t > 0$, which completed the proof by Lemma 1. □

Theorem 2.2
Within the context of the PIGRM, $\hat{\beta}_{\text{PIGLTE}}$ is superior to $\hat{\beta}_{\text{PIGRRE}}$ if and only if $\text{MMSE}(\hat{\beta}_{\text{PIGRRE}}) - \text{MMSE}(\hat{\beta}_{\text{PIGLTE}}) > 0$, where $k_r, k_t > 0$, $0 < d_t < 1$, $b_{\text{PIGLTE}} = \text{Bias}(\hat{\beta}_{\text{PIGLTE}})$, and $b_{\text{PIGRRE}} = \text{Bias}(\hat{\beta}_{\text{PIGRRE}})$.

Proof
The disparity in MMSE functions between $\hat{\beta}_{\text{PIGRRE}}$ and $\hat{\beta}_{\text{PIGLTE}}$, as computed using Eq. (14) and Eq. (25), is derived as:

$$\begin{aligned}\text{MMSE}(\hat{\beta}_{\text{MLE}}) - \text{MMSE}(\hat{\beta}_{\text{PIGLTE}}) = \hat{\phi}\mathcal{Q}(\Lambda_k^{-1}\Lambda_k\Lambda_k^{-1} - \Lambda_{k_t}^{-1}\Lambda_{k_t d_t}\Lambda^{-1}\Lambda_{k_t d_t}\Lambda_{k_t}^{-1})\mathcal{Q}' \\ + b_{\text{PIGRRE}}b'_{\text{PIGRRE}} - b_{\text{PIGLTE}}b'_{\text{PIGLTE}}\end{aligned} \tag{29}$$

Equation (29) can alternatively be formulated using the SMSE as:

$$\begin{aligned}\text{SMSE}(\hat{\beta}_{\text{PIGRRE}}) - \text{SMSE}(\hat{\beta}_{\text{PIGLTE}}) = \hat{\phi}\mathcal{Q}\text{diag}\left(\frac{\lambda_j}{(\lambda_j + k_r)^2} - \frac{(\lambda_j + k_t d_t)^2}{\lambda_j(\lambda_j + k_t)^2}\right)\mathcal{Q}' \\ + b_{\text{PIGRRE}}b'_{\text{PIGRRE}} - b_{\text{PIGLTE}}b'_{\text{PIGLTE}}\end{aligned} \tag{30}$$

The matrix $(\Lambda_k^{-1}\Lambda_k\Lambda_k^{-1} - \Lambda_{k_t}^{-1}\Lambda_{k_t d_t}\Lambda^{-1}\Lambda_{k_t d_t}\Lambda_{k_t}^{-1})$ is positive definite if $\lambda_j^2(\lambda_j + k_t)^2 - (\lambda_j + k_r)^2(\lambda_j + k_t d_t)^2 > 0$, which equivalent to $[\lambda_j(\lambda_j + k_t) + (\lambda_j + k_r)(\lambda_j + k_t d_t)][\lambda_j(\lambda_j + k_t) - (\lambda_j + k_r)(\lambda_j + k_t d_t)] > 0$, then $\lambda_j^2(\lambda_j + k_t)^2 - (\lambda_j + k_r)^2(\lambda_j + k_t d_t)^2 > 0$. If $0 < d_t < 1$ and $k_r, k_t > 0$, which completed the proof by Lemma 1. □

Theorem 2.3
Within the context of the PIGRM, $\hat{\beta}_{\text{PIGLTE}}$ is superior to $\hat{\beta}_{\text{PIGLE}}$ if and only if $\text{MMSE}(\hat{\beta}_{\text{PIGLE}}) - \text{MMSE}(\hat{\beta}_{\text{PIGLTE}}) > 0$, where $k_t > 0$, $0 < d_l, d_t < 1$, $b_{\text{PIGLTE}} = \text{Bias}(\hat{\beta}_{\text{PIGLTE}})$, and $b_{\text{PIGLE}} = \text{Bias}(\hat{\beta}_{\text{PIGLE}})$.

Proof
The disparity in MMSE functions between $\hat{\beta}_{\text{PIGLE}}$ and $\hat{\beta}_{\text{PIGLTE}}$, as computed using Eq. (20) and Eq. (25), is derived as:

$$\begin{aligned}\text{MMSE}(\hat{\beta}_{\text{MLE}}) - \text{MMSE}(\hat{\beta}_{\text{PIGLTE}}) = \hat{\phi}\mathcal{Q}(\Lambda_l^{-1}\Lambda_d\Lambda^{-1}\Lambda_d\Lambda_l^{-1} - \Lambda_{k_t}^{-1}\Lambda_{k_t d_t}\Lambda^{-1}\Lambda_{k_t d_t}\Lambda_{k_t}^{-1})\mathcal{Q}' \\ + b_{\text{PIGLE}}b'_{\text{PIGLE}} - b_{\text{PIGLTE}}b'_{\text{PIGLTE}}\end{aligned} \tag{31}$$

Equation (31) can alternatively be formulated using the SMSE as:

$$\begin{aligned}\text{SMSE}(\hat{\beta}_{\text{PIGRRE}}) - \text{SMSE}(\hat{\beta}_{\text{PIGLTE}}) = \hat{\phi}\mathcal{Q}\text{diag}\left(\frac{(\lambda_j + d_l)^2}{\lambda_j(\lambda_j + 1)^2} - \frac{(\lambda_j + k_t d_t)^2}{\lambda_j(\lambda_j + k_t)^2}\right)\mathcal{Q}' \\ + b_{\text{PIGLE}}b'_{\text{PIGLE}} - b_{\text{PIGLTE}}b'_{\text{PIGLTE}}\end{aligned} \tag{32}$$

The matrix $(\Lambda_l^{-1}\Lambda_d\Lambda^{-1}\Lambda_d\Lambda_l^{-1} - \Lambda_{k_t}^{-1}\Lambda_{k_t d_t}\Lambda^{-1}\Lambda_{k_t d_t}\Lambda_{k_t}^{-1})$ is positive definite if $(\lambda_j + d_l)^2(\lambda_j + k_t)^2 - (\lambda_j + 1)^2(\lambda_j + k_t d_t)^2 > 0$, which equivalent to $(\lambda_j + d_l)^2(\lambda_j + k_t)^2 > (\lambda_j + 1)^2(\lambda_j + k_t d_t)^2$, then $(\lambda_j + d_l)^2(\lambda_j + k_t)^2 - (\lambda_j + 1)^2(\lambda_j + k_t d_t)^2 > 0$. If $0 < d_l, d_t < 1$ and $k_t > 0$, which completed the proof by Lemma 1. □

2.5. Selection of Shrinkage Parameters k_t and d_t for PIGLTE

The optimal values for k_t and d_t can be obtained by minimizing Eq. (26) as follows:

$$k_{tj} = \frac{\lambda_j\hat{\phi}}{\alpha_j^2\lambda_j(1 - d_t) - d_t\hat{\phi}}, \tag{33}$$

and

$$d_{tj} = \frac{k_t\alpha_j^2\lambda_j - \lambda_j\hat{\phi}}{\hat{\phi}k_t + k_t\alpha_j^2\lambda_j}. \tag{34}$$

Following the approach of Akram et al. [3] and Huang [24], the algorithm selects the best values for the shrinkage parameters d_t and k_t using the following steps:

Step 1: Start with an initial value of d_t as $\hat{d} = \min\left(\frac{\hat{\alpha}_j^2}{\frac{\hat{\phi}}{\lambda_j}+\hat{\alpha}_j^2}\right)$, for $j = 1, \ldots, p$.

Step 2: Estimate the shrinkage parameter k_t using d_t value from step 1 as follows:

$$\hat{k}_{t1} = p\min\left(\frac{\lambda_j\hat{\phi}}{\max(\hat{\alpha}_j^2)\lambda_j(1-\hat{d}_t)-\hat{d}_t\hat{\phi}}\right), \tag{35}$$

$$\hat{k}_{t2} = \text{mean}\left(\frac{\lambda_j\hat{\phi}}{\max(\hat{\alpha}_j^2)\lambda_j(1-\hat{d}_t)-\hat{d}_t\hat{\phi}}\right), \tag{36}$$

$$\hat{k}_{t3} = \max\left(\frac{\lambda_j\hat{\phi}}{\max(\hat{\alpha}_j^2\lambda_j(1-\hat{d}_t)-\hat{d}_t\hat{\phi})}\right), \tag{37}$$

$$\hat{k}_{t4} = \frac{1}{p}\max\left(\frac{\lambda_j\hat{\phi}}{\max(\hat{\alpha}_j^2)\lambda_j(1-\hat{d}_t)-\hat{d}_t\hat{\phi}}\right). \tag{38}$$

Step 3: Estimate the parameters d_t using the k_t values obtained in step 2:

$$\hat{d}_{tm} = \left(\frac{\max(\hat{\alpha}_j^2)\lambda_j\hat{k}_{tr}-\lambda_j\hat{\phi}}{\hat{\phi}\hat{k}_{tr}+\hat{k}_{tr}\alpha_j^2\lambda_j}\right), \quad m = 1,2,3,4. \tag{39}$$

Step 4: If the estimated value of $\hat{d}_{tm}$ in Step 3 does not lie between 0 and 1, it should be replaced with $\hat{d}$ from Step 1.

3. Simulation study

In this section, we numerically evaluate the proposed estimator by comparing it with the MLE, Ridge estimator, and Liu estimator through a Monte Carlo simulation study and two real-world applications.

3.1. Simulation layout

We conduct simulations considering several key factors such as the number of explanatory variables (p), the sample size (n), dispersion parameters (ϕ), and levels of multicollinearity (ρ) as detailed in Table 1.

The simulation is summarized in the following steps:

1. Choose the regression parameters β such that $\sum_{j=1}^{p}\beta_j^2 = 1$, a common restriction in the field.
2. Generate the response variable for the PIGRM from a PIG(μ, ϕ) distribution, where $\mu = \exp(x_i'\beta)$.
3. Generate the correlated explanatory variables x_{ij} using the formula $x_{ij} = \sqrt{1-\rho^2}G_{ij} + \rho G_{i(j+1)}$, where G_{ij} represents independent standard normal pseudo-random numbers. Repeat this process for $i = 1, \ldots, n$ and $j = 1, \ldots, p$.
4. Estimate the regression parameters using the 'gamlss' package in R, selecting the PIG family.
5. Repeat the data generation process for various combinations of n, p, ρ, and ϕ 1000 times.
6. Evaluate the performance of the proposed estimator by calculating the SMSE criterion:

$$\text{SMSE}(\hat{\beta}) = \frac{\sum_{i=1}^{L}(\hat{\beta}_i-\beta)'(\hat{\beta}_i-\beta)}{L},$$

where $\hat{\beta}_i - \beta$ represents the difference between the true parameter and the estimated vectors of the proposed and other considered estimators at the i-th replication, and L represents the number of replications.

The biasing parameter estimation for PIGLE and PIGRRE in both the simulation and application rely on the approaches detailed in the works of Hoerl and Kennard [22], and Batool et al. [11], respectively. Following Hoerl and Kennard [22], we utilize $\hat{k}_{r_1} = \frac{p\hat{\phi}}{\sum_{j=1}^{p}\hat{\alpha}_j^2}$ and $\hat{k}_{r_2} = p\sum_{j=1}^{p}\left(\frac{\hat{\alpha}_j^2}{\frac{\hat{\phi}}{\lambda_j}+2\hat{\alpha}_j^2}\right)$ as estimates of k in the PIGRRE.

Additionally, based on the approach of Akram et al. [3], we estimate the Liu parameter d in PIGLE using $\hat{d}_{l_1} = \max\left(0, \min\left(\frac{\hat{\alpha}_j^2-\hat{\phi}}{\frac{\hat{\phi}}{\lambda_j}+\hat{\alpha}_j^2}\right)\right)$ and $\hat{d}_{l_2} = \max\left(0, \frac{\sum_{j=1}^{p}\left(\frac{\hat{\alpha}_j^2-\hat{\phi}}{(\lambda_j+1)^2}\right)}{\sum_{j=1}^{p}\left(\frac{\hat{\alpha}_j^2\lambda_j+\hat{\phi}}{\lambda_j(\lambda_j+1)^2}\right)}\right)$.

Table 1. Different factor for simulation study.

Factor	**Notation**	**Values**
Number of Explanatory Variables	p	4,7,10
Dispersion Parameter	ϕ	2, 4, 6
Degree of Correlation	ρ	0.80, 0.90, 0.95, 0.99
Sample Size	n	30,50, 100, 150, 200, 300,400
Number of replicates	L	1000

Table 2. Simulated SMSE for the estimators when p=4 and $\phi = 2$.

ρ	n	ML	PIGRRE		PIGLE		PIGLTE			
		-	k_{r_1}	k_{r_2}	d_{l_1}	d_{l_2}	$(\hat{k}_{t1}, \hat{d}_{t1})$	$(\hat{k}_{t2}, \hat{d}_{t2})$	$(\hat{k}_{t3}, \hat{d}_{t3})$	$(\hat{k}_{t4}, \hat{d}_{t4})$
0.80	30	1.6642	1.3659	1.2142	1.1200	1.0427	**0.6657**	0.7311	0.7134	0.6584
	50	0.8644	0.7285	0.6622	0.6230	0.6442	**0.3876**	0.4400	0.4296	0.3918
	100	0.3742	0.3367	0.3196	0.3158	0.3266	**0.2216**	0.2438	0.2395	0.2227
	150	0.2871	0.2613	0.2520	0.2496	0.2560	**0.1771**	0.1940	0.1916	0.1779
	200	0.1603	0.1545	0.1524	0.1493	0.1515	**0.1261**	0.1332	0.1321	0.1262
	300	0.1196	0.1148	0.1132	0.1128	0.1140	**0.0925**	0.0979	0.0971	0.0926
	400	0.0897	0.0873	0.0866	0.0859	0.0866	**0.0741**	0.0778	0.0773	0.0742
0.90	30	3.1134	2.4173	2.1042	1.8677	1.4588	**0.8569**	0.9621	0.9366	0.8588
	50	1.0898	0.9704	0.8860	0.7807	0.7933	**0.5438**	0.5832	0.5725	0.5478
	100	0.7345	0.6681	0.6182	0.5436	0.5613	**0.3841**	0.4232	0.4131	0.3861
	150	0.4627	0.4418	0.4272	0.3777	0.3978	**0.3075**	0.3369	0.3303	0.3088
	200	0.3196	0.2988	0.2881	0.2708	0.2817	**0.1990**	0.2194	0.2155	0.2005
	300	0.2187	0.2128	0.2101	0.1942	0.2005	**0.1748**	0.1834	0.1821	0.1750
	400	0.1651	0.1552	0.1514	0.1504	0.1526	**0.1103**	0.1194	0.1176	0.1104
0.95	30	6.2448	4.6564	4.1268	3.5892	2.1703	**1.2592**	1.3851	1.3419	1.2659
	50	2.5252	1.7735	1.3968	1.1797	1.1545	**0.4419**	0.4920	0.4812	0.4470
	100	1.2226	0.9170	0.7373	0.6976	0.7480	**0.2869**	0.3348	0.3240	0.2894
	150	0.8201	0.7591	0.7076	0.5781	0.6167	**0.4250**	0.4683	0.4581	0.4267
	200	0.6419	0.5776	0.5349	0.4637	0.4936	**0.3066**	0.3540	0.3428	0.3079
	300	0.4185	0.3822	0.3605	0.3325	0.3503	**0.2218**	0.2454	0.2409	0.2223
	400	0.3179	0.2848	0.2690	0.2631	0.2716	**0.1592**	0.1784	0.1746	0.1594
0.99	30	30.2799	21.4294	19.6137	16.3248	6.1309	**4.3509**	4.3653	4.3256	4.3627
	50	12.0689	10.0219	9.1699	7.8936	4.3380	**3.5441**	3.6801	3.6402	3.5626
	100	6.5675	4.9204	4.1565	3.2151	1.8498	**1.0256**	1.1081	1.0844	1.0294
	150	5.4217	4.3329	3.6987	2.8992	1.9358	**1.1407**	1.2023	1.1870	1.1438
	200	2.7977	1.9725	1.5186	1.1969	1.1190	**0.3952**	0.4464	0.4341	0.3964
	300	2.3691	1.7088	1.3038	1.0329	1.0354	**0.3179**	0.3673	0.3529	0.3195
	400	1.5597	1.1675	0.9099	0.7568	0.8113	**0.2542**	0.3053	0.2930	0.2569

Table 3. Simulated SMSE for the estimators when p=4 and $\phi = 4$.

ρ	n	ML	PIGRRE		PIGLE		PIGLTE			
		-	k_{r_1}	k_{r_2}	d_{l_1}	d_{l_2}	$(\hat{k}_{t1}, \hat{d}_{t1})$	$(\hat{k}_{t2}, \hat{d}_{t2})$	$(\hat{k}_{t3}, \hat{d}_{t3})$	$(\hat{k}_{t4}, \hat{d}_{t4})$
0.80	30	2.4914	2.0401	1.8072	1.6274	1.3638	**0.8715**	0.9641	0.9284	0.8994
	50	1.3506	1.0181	0.8678	0.8518	0.8746	**0.4909**	0.5316	0.5213	0.4942
	100	0.5552	0.4702	0.4303	0.4424	0.4577	**0.2858**	0.3059	0.3011	0.2865
	150	0.4196	0.3565	0.3272	0.3444	0.3545	**0.2104**	0.2315	0.2262	0.2112
	200	0.2741	0.2640	0.2588	0.2423	0.2492	**0.2033**	0.2166	0.2140	0.2039
	300	0.1868	0.1687	0.1627	0.1711	0.1728	**0.1160**	0.1253	0.1237	0.1163
	400	0.1337	0.1262	0.1236	0.1251	0.1262	**0.0932**	0.1014	0.1002	0.0937
0.90	30	4.9579	3.3209	2.8414	2.5060	1.7414	**0.9002**	0.9853	0.9492	0.9192
	50	2.6935	2.0519	1.7326	1.4920	1.3126	**0.7020**	0.7811	0.7559	0.7129
	100	1.1723	0.8607	0.6961	0.7169	0.7646	**0.3130**	0.3552	0.3419	0.3158
	150	0.7306	0.6196	0.5539	0.5165	0.5508	**0.2911**	0.3325	0.3242	0.2929
	200	0.4865	0.4251	0.3922	0.3771	0.3951	**0.2205**	0.2522	0.2454	0.2216
	300	0.3409	0.2802	0.2548	0.2841	0.2909	**0.1354**	0.1525	0.1491	0.1360
	400	0.2556	0.2287	0.2167	0.2229	0.2272	**0.1308**	0.1473	0.1437	0.1314
0.95	30	8.9631	5.8188	5.0565	4.1425	2.1367	**1.1748**	1.2388	1.2147	1.1813
	50	4.5557	2.5057	1.9018	1.5872	1.3534	**0.4776**	0.4836	0.4812	0.4777
	100	2.1332	1.3465	1.0206	0.9774	0.9948	**0.3294**	0.3619	0.3526	0.3317
	150	1.3824	0.9668	0.7520	0.7291	0.7739	**0.2630**	0.3070	0.2952	0.2656
	200	0.9106	0.7635	0.6560	0.5871	0.6206	**0.2893**	0.3432	0.3266	0.2920
	300	0.6426	0.5284	0.4564	0.4509	0.4811	**0.2039**	0.2386	0.2297	0.2049
	400	0.4929	0.4088	0.3629	0.3748	0.3909	**0.1653**	0.1915	0.1851	0.1667
0.99	30	42.2512	31.2117	28.6260	23.8785	8.2699	6.6411	6.5540	6.5655	**6.5005**
	50	23.0478	16.8579	15.1164	12.2381	4.8788	3.6585	3.7098	3.6793	**3.6548**
	100	12.3980	7.3788	6.0171	4.1632	1.7847	**0.9049**	0.9083	0.9091	0.9056
	150	6.6824	4.6985	3.8325	2.8025	1.7263	**0.8433**	0.8863	0.8735	0.8438
	200	4.5514	2.5753	1.8957	1.3498	1.1291	**0.3193**	0.3494	0.3416	0.3196
	300	2.8494	2.1725	1.7214	1.2896	1.1499	**0.4602**	0.5172	0.5046	0.4617
	400	2.4422	1.7139	1.2897	1.0072	0.9872	**0.2728**	0.3303	0.3182	0.2746

Table 4. Simulated SMSE for the estimators when p=4 and $\phi = 6$.

ρ	n	ML	PIGRRE		PIGLE		PIGLTE			
		-	k_{r_1}	k_{r_2}	d_{l_1}	d_{l_2}	$(\hat{k_{t1}}, \hat{d}_{t1})$	$(\hat{k_{t2}}, \hat{d}_{t2})$	$(\hat{k_{t3}}, \hat{d}_{t3})$	$(\hat{k_{t4}}, \hat{d}_{t4})$
0.80	30	3.2547	2.5776	2.2665	2.0348	1.5840	**0.9268**	1.0608	1.0098	0.9479
	50	1.7477	1.2386	1.0326	1.0142	1.0249	**0.5944**	0.6131	0.6032	0.5949
	100	0.6704	0.5672	0.5182	0.5238	0.5418	**0.3498**	0.3766	0.3701	0.3509
	150	0.4559	0.4213	0.4015	0.3736	0.3895	**0.2671**	0.3009	0.2931	0.2690
	200	0.3640	0.3433	0.3314	0.3091	0.3211	**0.2311**	0.2574	0.2521	0.2325
	300	0.2352	0.2168	0.2089	0.2113	0.2146	**0.1475**	0.1621	0.1592	0.1479
	400	0.1723	0.1604	0.1560	0.1589	0.1606	**0.1112**	0.1239	0.1222	0.1141
0.90	30	6.3618	4.1962	3.6190	3.1548	2.0104	**1.0636**	1.0899	1.0757	1.0669
	50	3.1274	2.0387	1.6141	1.4441	1.3323	**0.6218**	0.6618	0.6476	0.6258
	100	1.3272	0.8934	0.7065	0.7741	0.8047	**0.3356**	0.3550	0.3506	0.3364
	150	0.8005	0.6591	0.5773	0.5552	0.5901	**0.2952**	0.3353	0.3250	0.2983
	200	0.6146	0.5157	0.4626	0.4537	0.4800	**0.2379**	0.2715	0.2644	0.2395
	300	0.4085	0.3267	0.2941	0.3294	0.3378	**0.1496**	0.1669	0.1637	0.1502
	400	0.3167	0.2731	0.2535	0.2682	0.2741	**0.1323**	0.1583	0.1545	0.1388
0.95	30	12.2614	7.7455	6.7906	5.6549	2.5966	**1.4003**	1.4564	1.4038	1.4127
	50	5.6835	4.4225	3.8392	3.1616	2.0591	**1.2095**	1.2717	1.2481	1.2195
	100	2.9964	2.2046	1.7683	1.4521	1.2679	**0.5484**	0.5990	0.5844	0.5540
	150	1.8618	1.4134	1.1352	0.9432	0.9620	**0.3661**	0.4387	0.4145	0.3706
	200	1.2870	1.0169	0.8284	0.7298	0.7806	**0.3066**	0.3539	0.3408	0.3087
	300	0.7361	0.5407	0.4438	0.5025	0.5259	**0.1768**	0.2067	0.1994	0.1785
	400	0.6081	0.4797	0.4090	0.4359	0.4567	**0.1662**	0.1942	0.1870	0.1684
0.99	30	56.0575	40.9718	37.5924	31.3325	10.3380	**7.2779**	7.8503	7.9737	8.0670
	50	46.7279	32.2220	29.1155	22.8062	6.7367	5.4401	5.3413	5.3744	**5.4207**
	100	16.0235	12.1427	10.5777	8.1726	3.2562	**2.1475**	2.1800	2.1622	2.1841
	150	8.4176	4.8075	3.7920	2.6166	1.5742	0.6622	0.6801	0.6780	**0.6603**
	200	6.0786	3.2202	2.3868	1.6089	1.1886	**0.3138**	0.3322	0.3284	0.3143
	300	3.9097	2.8232	2.2117	1.5945	1.2689	**0.4679**	0.5098	0.4984	0.4692
	400	3.0894	2.0682	1.5302	1.1531	1.1096	0.3114	0.3456	0.3368	**0.3096**

Table 5. Simulated SMSE for the estimators when p=7 and $\phi = 2$.

ρ	n	ML	PIGRRE		PIGLE		PIGLTE			
		-	k_{r_1}	k_{r_2}	d_{l_1}	d_{l_2}	$(\hat{k}_{t1}, \hat{d}_{t1})$	$(\hat{k}_{t2}, \hat{d}_{t2})$	$(\hat{k}_{t3}, \hat{d}_{t3})$	$(\hat{k}_{t4}, \hat{d}_{t4})$
0.80	30	3.4454	3.1615	2.8999	2.5669	2.1533	**1.3005**	1.5565	1.5163	1.3790
	50	1.7075	1.5355	1.3907	1.2573	1.2217	**0.6551**	0.7725	0.7496	0.6658
	100	0.7224	0.6774	0.6452	0.6000	0.6081	**0.3920**	0.4330	0.4262	0.3941
	150	0.4759	0.4631	0.4542	0.4226	0.4284	**0.3304**	0.3606	0.3557	0.3315
	200	0.3357	0.3267	0.3219	0.3072	0.3099	**0.2413**	0.2606	0.2582	0.2419
	300	0.2199	0.2152	0.2130	0.2075	0.2086	**0.1686**	0.1794	0.1782	0.1689
	400	0.1655	0.1626	0.1614	0.1583	0.1589	**0.1321**	0.1398	0.1389	0.1322
0.90	30	8.1162	6.9656	6.3159	5.4733	3.1185	**1.8282**	1.9793	1.9366	1.8398
	50	3.5635	2.9317	2.4540	2.0884	1.8084	**0.6609**	0.7846	0.7555	0.6739
	100	1.5484	1.3893	1.2398	1.0700	1.0773	**0.5051**	0.6218	0.5955	0.5125
	150	0.9860	0.9248	0.8668	0.7570	0.7700	**0.4628**	0.5167	0.5060	0.4660
	200	0.6064	0.5837	0.5651	0.5113	0.5194	**0.3712**	0.3976	0.3926	0.3725
	300	0.4485	0.4353	0.4257	0.3938	0.3991	**0.2996**	0.3176	0.3141	0.3002
	400	0.3294	0.3214	0.3166	0.2990	0.3020	**0.2392**	0.2522	0.2500	0.2393
0.95	30	12.3605	10.5043	9.4422	7.8304	3.8799	**1.929**1	2.2422	2.1549	1.9781
	50	7.9989	6.7022	5.7357	4.4771	2.7309	**1.1689**	1.3138	1.2644	1.1889
	100	3.0344	2.7517	2.4095	1.9252	1.6547	**0.8201**	0.9451	0.9107	0.8315
	150	1.8690	1.7214	1.5405	1.2521	1.2285	**0.6355**	0.7159	0.6969	0.6424
	200	1.3566	1.2713	1.1705	0.9570	0.9662	**0.5578**	0.6198	0.6060	0.5609
	300	0.9695	0.9172	0.8620	0.7418	0.7618	**0.4667**	0.4999	0.4922	0.4680
	400	0.6540	0.6172	0.5870	0.5398	0.5494	0.3202	0.3477	0.3414	**0.3193**
0.99	30	106.8064	86.9934	81.0617	68.6691	17.0434	14.0801	14.0818	14.0843	**14.0736**
	50	29.6234	25.6250	22.8967	17.8424	4.8615	**3.5434**	3.7379	3.6688	3.5844
	100	12.0980	9.4090	7.4847	4.7805	2.2593	**0.6009**	0.6273	0.6189	0.6037
	150	8.7064	7.4911	6.2862	4.4432	2.2181	**0.7701**	0.8452	0.8219	0.7748
	200	6.6548	5.4872	4.3705	2.8556	1.8964	**0.4558**	0.5134	0.4943	0.4594
	300	3.9331	3.5157	2.9529	2.0358	1.7038	**0.5858**	0.6723	0.6471	0.5891
	400	3.2088	2.8082	2.2947	1.6406	1.5610	0.4625	0.5257	0.5067	**0.4605**

Table 6. Simulated SMSE for the estimators when p=7 and $\phi = 4$.

ρ	n	ML	PIGRRE		PIGLE		PIGLTE			
		-	k_{r_1}	k_{r_2}	d_{l_1}	d_{l_2}	$(\hat{k_{t1}}, \hat{d_{t1}})$	$(\hat{k_{t2}}, \hat{d_{t2}})$	$(\hat{k_{t3}}, \hat{d_{t3}})$	$(\hat{k_{t4}}, \hat{d_{t4}})$
0.80	30	8.6810	7.0838	6.4104	5.7621	3.0102	**1.4457**	1.7858	1.7124	1.5168
	50	2.6701	2.2479	1.9150	1.7201	1.6233	**0.6990**	0.8384	0.7996	0.7110
	100	1.0336	0.9517	0.8816	0.8096	0.8239	**0.4479**	0.5261	0.5109	0.4527
	150	0.6666	0.6291	0.6018	0.5646	0.5735	**0.3590**	0.3984	0.3924	0.3611
	200	0.5072	0.4807	0.4634	0.4470	0.4520	**0.2826**	0.3183	0.3127	0.2840
	300	0.3233	0.3148	0.3096	0.2967	0.2989	**0.2225**	0.2433	0.2401	0.2232
	400	0.2423	0.2345	0.2304	0.2267	0.2278	**0.1628**	0.1777	0.1757	0.1635
0.90	30	16.0978	13.4493	12.3058	10.2811	4.3283	2.8101	2.9752	2.9117	**2.7767**
	50	5.4521	4.1163	3.3122	2.7490	2.1145	**0.6386**	0.7357	0.6979	0.6465
	100	2.4251	2.1719	1.8698	1.5085	1.4179	**0.6205**	0.7252	0.6995	0.6319
	150	1.4978	1.3335	1.1618	1.0309	1.0493	**0.4482**	0.5118	0.4970	0.4528
	200	0.8946	0.8452	0.7973	0.7043	0.7208	**0.4144**	0.4616	0.4526	0.4176
	300	0.6930	0.6549	0.6238	0.5678	0.5787	**0.3350**	0.3806	0.3722	0.3369
	400	0.4720	0.4469	0.4294	0.4109	0.4154	0.2454	0.2698	0.2649	**0.2446**
0.95	30	31.3985	26.1215	24.1207	19.8594	6.4250	**4.3483**	4.8221	4.7260	4.6087
	50	10.5166	8.5547	7.3592	5.6932	2.8215	**1.2957**	1.4803	1.4198	1.3285
	100	4.1088	3.1692	2.4036	1.9029	1.7442	**0.4303**	0.4925	0.4741	0.4370
	150	2.9620	2.2461	1.6724	1.4914	1.4716	**0.3146**	0.3791	0.3593	0.3192
	200	2.1174	1.9253	1.6716	1.2842	1.2877	**0.5199**	0.6075	0.5844	0.5259
	300	1.2985	1.2036	1.0883	0.9079	0.9375	**0.4332**	0.4922	0.4770	0.4365
	400	0.9433	0.8663	0.7915	0.7177	0.7343	0.3326	0.3711	0.3615	**0.3311**
0.99	30	160.0399	123.6640	113.7517	93.7264	16.4526	12.7525	**12.6495**	12.7720	12.7980
	50	55.3306	43.6810	37.8247	27.0462	4.1756	**2.4555**	2.4630	2.4585	2.4591
	100	19.0255	14.5341	11.7532	7.4832	2.4063	**0.7297**	0.7788	0.7563	0.7336
	150	13.9299	11.1717	9.0413	5.8064	2.2799	**0.6029**	0.6653	0.6448	0.6084
	200	9.3550	7.8184	6.3593	4.1230	2.1492	**0.5801**	0.6541	0.6271	0.5855
	300	6.7659	5.5206	4.2808	2.7295	1.9486	**0.3776**	0.4223	0.4077	0.3805
	400	4.5330	3.8431	2.9901	1.9813	1.7485	0.2846	0.3311	0.3147	**0.2825**

Table 7. Simulated SMSE for the estimators when p=7 and $\phi = 6$.

ρ	n	ML	PIGRRE		PIGLE		PIGLTE			
		-	k_{r_1}	k_{r_2}	d_{l_1}	d_{l_2}	$(\hat{k_{t1}}, \hat{d}_{t1})$	$(\hat{k_{t2}}, \hat{d}_{t2})$	$(\hat{k_{t3}}, \hat{d}_{t3})$	$(\hat{k_{t4}}, \hat{d}_{t4})$
0.80	30	11.5691	9.0499	8.1683	7.2195	3.0997	**1.4194**	1.6600	1.5659	1.4358
	50	3.3648	2.7140	2.2428	1.9886	1.8548	**0.7190**	0.8455	0.8058	0.7305
	100	1.3691	1.1531	0.9950	0.9987	1.0124	**0.4225**	0.4725	0.4622	0.4262
	150	0.9559	0.8903	0.8255	0.7486	0.7607	**0.3858**	0.4500	0.4370	0.3896
	200	0.6519	0.6120	0.5824	0.5530	0.5602	**0.3256**	0.3684	0.3616	0.3277
	300	0.4541	0.4365	0.4244	0.4036	0.4078	**0.2675**	0.3043	0.2980	0.2689
	400	0.3004	0.2883	0.2816	0.2772	0.2787	0.1877	0.2055	0.2028	**0.1872**
0.90	30	20.1629	16.6076	15.0803	12.3779	4.5658	**2.7058**	2.9694	2.8865	2.7125
	50	6.7997	5.2106	4.2255	3.2981	2.3899	**0.7996**	0.9258	0.8782	0.8136
	100	2.8921	2.2655	1.7758	1.6077	1.5693	**0.4598**	0.5156	0.5002	0.4653
	150	1.7737	1.5882	1.3766	1.1569	1.1723	**0.4722**	0.5450	0.5273	0.4779
	200	1.3357	1.1479	0.9848	0.9388	0.9563	**0.3348**	0.3930	0.3809	0.3387
	300	0.7150	0.6749	0.6395	0.5872	0.5966	**0.3410**	0.3699	0.3632	0.3424
	400	0.5482	0.5129	0.4864	0.4629	0.4693	**0.2516**	0.2851	0.2791	0.2565
0.95	30	39.4136	31.9700	29.1489	23.2483	6.4944	**4.1654**	4.4238	4.3135	4.2244
	50	12.0185	9.7663	8.2968	6.1741	3.0443	**1.2089**	1.2825	1.2519	1.2190
	100	5.8626	4.3808	3.3241	2.4757	1.9605	**0.4596**	0.5347	0.5107	0.4714
	150	3.5614	2.8155	2.1631	1.7326	1.6345	**0.3982**	0.4810	0.4582	0.4050
	200	2.3197	2.0479	1.7084	1.3481	1.3407	**0.4556**	0.5420	0.5201	0.4632
	300	1.5923	1.4473	1.2688	1.0465	1.0672	**0.4237**	0.4801	0.4665	0.4269
	400	1.1511	1.0385	0.9210	0.8304	0.8517	**0.3100**	0.3714	0.3609	0.3280
0.99	30	202.9988	164.9131	151.6796	123.8422	18.2901	14.0911	14.1115	**13.9672**	14.1701
	50	73.6802	57.9718	50.7108	36.3066	5.7738	**3.9360**	4.0425	4.0508	3.9508
	100	29.4230	20.8077	16.4645	9.8899	2.2894	0.8255	0.8422	0.8299	**0.8248**
	150	14.8910	11.9815	9.6623	6.0173	2.2103	**0.6106**	0.6497	0.6409	0.6188
	200	12.3708	10.3309	8.4232	5.3425	2.1134	**0.5303**	0.6056	0.5877	0.5361
	300	6.6411	5.6243	4.5133	2.8434	1.8685	**0.3931**	0.4521	0.4359	0.3992
	400	5.4888	4.5003	3.4071	2.1606	1.7797	0.3394	0.3835	0.3690	**0.3361**

Table 8. Simulated SMSE for the estimators when p=10 and $\phi = 2$.

ρ	n	ML	PIGRRE		PIGLE		PIGLTE			
		-	k_{r_1}	k_{r_2}	d_{l_1}	d_{l_2}	$(\hat{k}_{t1}, \hat{d}_{t1})$	$(\hat{k}_{t2}, \hat{d}_{t2})$	$(\hat{k}_{t3}, \hat{d}_{t3})$	$(\hat{k}_{t4}, \hat{d}_{t4})$
0.80	30	10.9661	9.8733	9.2642	8.6389	4.3147	2.6531	2.9960	2.8821	**2.6265**
	50	2.5835	2.3777	2.1030	1.9505	1.8461	**1.0337**	1.0945	1.0668	1.0367
	100	1.1838	1.1373	1.0483	0.9868	0.9888	**0.6406**	0.6593	0.6507	0.6409
	150	0.6735	0.6478	0.6084	0.5997	0.6025	0.5197	0.4929	0.4980	**0.5184**
	200	0.5154	0.4991	0.4703	0.4701	0.4742	0.4259	0.3999	0.4036	**0.4248**
	300	0.3257	0.3193	0.3081	0.3073	0.3091	0.2898	0.2753	0.2779	**0.2894**
	400	0.2402	0.2362	0.2342	0.2296	0.2301	0.1787	0.1913	0.1900	**0.1784**
0.90	30	9.7519	9.2142	8.6652	7.7375	4.4753	3.2490	3.4858	3.4174	**3.2198**
	50	4.8008	4.2558	3.7074	3.2095	2.6424	**1.2182**	1.2649	1.2405	1.2195
	100	2.0790	1.9076	1.6453	1.5150	1.4911	0.8112	0.8257	0.8179	**0.8111**
	150	1.5663	1.4005	1.1674	1.1739	1.1859	0.7426	0.7052	0.7158	**0.7403**
	200	0.9908	0.9524	0.8742	0.8343	0.8413	**0.5619**	0.5710	0.5670	0.5620
	300	0.6906	0.6641	0.6114	0.6062	0.6134	0.4483	0.4451	**0.4448**	0.4478
	400	0.5541	0.5349	0.5155	0.5004	0.5026	0.2734	0.2951	0.2902	**0.2728**
0.95	30	41.1251	34.6600	31.2726	25.0182	6.5055	**3.4976**	3.7242	3.5604	3.5444
	50	14.0191	12.4331	11.1453	9.0553	4.5080	**2.1170**	2.1985	2.1535	2.1218
	100	4.3505	3.5879	2.8237	2.3940	2.2481	0.9892	**0.9758**	0.9797	0.9875
	150	2.5245	2.4100	2.1843	1.8248	1.6952	**0.8504**	0.8995	0.8769	0.8523
	200	2.1181	1.9044	1.5642	1.4551	1.4596	0.7385	0.7371	**0.7356**	0.7380
	300	1.3171	1.2691	1.1544	1.0433	1.0581	**0.5953**	0.6124	0.6040	0.5957
	400	0.9459	0.9153	0.8812	0.7842	0.7947	0.4804	0.5136	0.5062	**0.4792**
0.99	30	159.9455	139.3635	128.0745	104.4589	15.7353	**13.2867**	13.8503	13.4291	13.2932
	50	50.3154	43.2732	37.9591	27.5802	5.6801	**3.0555**	3.1072	3.0793	3.0565
	100	23.1099	20.0231	16.9258	11.6518	3.4749	**1.2844**	1.3222	1.3141	1.2850
	150	12.5915	11.0629	9.2632	6.2460	3.1975	**1.1746**	1.1972	1.1880	1.1757
	200	8.9443	8.1639	6.9727	4.7307	2.8263	**0.9343**	0.9530	0.9439	0.9350
	300	6.2297	5.3279	4.1746	2.9014	2.4924	**0.8507**	0.8563	0.8526	0.8508
	400	4.8353	4.1980	3.2408	2.3140	2.1899	**0.3426**	0.3861	0.3738	0.3447

Table 9. Simulated SMSE for the estimators when p=10 and $\phi = 4$.

ρ	n	ML	PIGRRE		PIGLE		PIGLTE			
		-	k_{r_1}	k_{r_2}	d_{l_1}	d_{l_2}	$(\hat{k_{t1}}, \hat{d_{t1}})$	$(\hat{k_{t2}}, \hat{d_{t2}})$	$(\hat{k_{t3}}, \hat{d_{t3}})$	$(\hat{k_{t4}}, \hat{d_{t4}})$
0.80	30	15.7776	13.6566	12.7046	11.4487	5.0909	2.7401	3.0440	2.9338	**2.7139**
	50	4.0583	3.5393	3.0015	2.6708	2.4174	1.2092	1.2253	1.2083	**1.2044**
	100	1.6656	1.5054	1.3149	1.2834	1.2805	0.8251	**0.8001**	0.8031	0.8223
	150	1.0343	0.9770	0.8852	0.8683	0.8739	0.5957	0.5811	0.5821	**0.5944**
	200	0.8155	0.7852	0.7286	0.7054	0.7126	0.4700	0.4754	0.4698	**0.4695**
	300	0.5008	0.4794	0.4411	0.4573	0.4604	0.4231	**0.3850**	0.3934	0.4219
	400	0.3883	0.3806	0.3752	0.3617	0.3628	0.2620	0.2845	0.2816	**0.2615**
0.90	30	30.9378	26.2881	24.0452	19.9609	6.4171	**3.4187**	3.8478	3.6363	3.4340
	50	11.9651	9.9439	8.4932	6.6198	3.8797	1.5897	1.6137	1.5904	**1.5859**
	100	3.3771	2.9133	2.4001	2.1505	2.0588	1.0039	0.9927	0.9925	**1.0015**
	150	2.1680	2.0321	1.7882	1.5431	1.5083	**0.7973**	0.8087	0.7996	0.7974
	200	1.3322	1.2076	1.0165	1.0477	1.0582	0.7323	**0.7076**	0.7146	0.7308
	300	0.9098	0.8596	0.7697	0.7674	0.7757	0.5406	**0.5262**	0.5296	0.5398
	400	0.6950	0.6682	0.6442	0.6032	0.6069	0.3401	0.3719	0.3660	**0.3388**
0.95	30	39.4981	34.8003	31.9657	26.7156	7.5097	**4.0315**	4.3997	4.1597	3.9863
	50	17.3916	14.4535	12.3204	8.9386	3.9645	1.5140	1.5337	1.5221	**1.5130**
	100	6.1920	5.5983	4.7952	3.6025	2.7861	**1.0697**	1.0951	1.0818	1.0714
	150	3.5049	3.0803	2.4952	2.1195	2.0415	0.9015	0.8992	0.8996	**0.9009**
	200	2.7896	2.5084	2.0843	1.8199	1.7894	0.8631	0.8584	0.8592	**0.8626**
	300	1.9541	1.8344	1.5849	1.3841	1.3955	**0.6603**	0.6746	0.6660	0.6605
	400	1.3246	1.2468	1.1457	1.0028	1.0144	**0.4104**	0.4597	0.4454	0.4112
0.99	30	178.8157	151.1036	146.5920	103.3936	9.4365	7.6332	7.7569	**7.5432**	7.6608
	50	95.2678	79.8644	69.3930	48.1834	6.1445	3.7745	3.7537	3.7645	**3.7700**
	100	34.0360	26.6485	21.2873	12.5040	3.3091	1.3289	**1.3065**	1.3150	1.3253
	150	19.2731	17.2043	14.6978	9.7090	3.3516	**1.1980**	1.2359	1.2126	1.1994
	200	13.8266	11.9101	9.6760	6.0311	3.0816	**1.0545**	1.0608	1.0580	1.0547
	300	10.0314	8.4616	6.6230	4.0770	2.8177	0.9207	0.9184	**0.9180**	0.9203
	400	6.4357	5.7360	4.5792	2.7346	2.2572	0.3644	0.4113	0.3975	**0.3616**

Table 10. Simulated SMSE for the estimators when p=10 and $\phi = 6$.

ρ	n	ML	PIGRRE		PIGLE		PIGLTE			
		-	k_{r_1}	k_{r_2}	d_{l_1}	d_{l_2}	$(\hat{k}_{t1}, \hat{d}_{t1})$	$(\hat{k}_{t2}, \hat{d}_{t2})$	$(\hat{k}_{t3}, \hat{d}_{t3})$	$(\hat{k}_{t4}, \hat{d}_{t4})$
0.80	30	20.8139	17.3929	16.1348	14.2333	8.2384	2.3867	2.6825	2.5611	**2.3517**
	50	5.3011	4.4460	3.7111	3.1979	2.8128	1.3524	**1.3404**	1.3358	1.3594
	100	2.0174	1.8420	1.5939	1.4958	1.4821	0.8643	**0.8505**	0.8508	0.8670
	150	1.2682	1.1895	1.0547	1.0275	1.0357	0.6916	**0.6679**	0.6709	0.6939
	200	1.0362	0.9906	0.8979	0.8702	0.8825	0.5594	**0.5592**	0.5541	0.5603
	300	0.6696	0.6363	0.5781	0.5947	0.5991	0.5500	**0.4942**	0.5088	0.5522
	400	0.4443	0.4317	0.4224	0.4086	0.4099	0.2612	0.2883	0.2845	**0.2604**
0.90	30	33.1064	26.6156	23.4844	18.2441	5.7688	7.5543	15.4660	9.5919	**7.1375**
	50	11.7665	9.8015	8.4034	6.4839	3.8702	**1.6940**	1.7158	1.7046	1.6961
	100	5.0201	4.2743	3.4801	2.8257	2.4844	1.0834	**1.0798**	1.0812	1.0848
	150	2.3658	2.0386	1.6408	1.6151	1.6097	0.9401	**0.9084**	0.9190	0.9435
	200	1.9752	1.7432	1.4205	1.4225	1.4332	0.8531	**0.8258**	0.8349	0.8553
	300	1.1341	1.0760	0.9557	0.9298	0.9403	0.5911	**0.5845**	0.5851	0.5917
	400	0.8242	0.7850	0.7449	0.6945	0.7004	**0.3336**	0.3762	0.3688	0.3379
0.95	30	49.1590	42.0846	38.4271	31.0901	7.8787	4.1245	4.2830	4.1842	**4.0776**
	50	22.9521	18.2841	15.3564	10.8063	4.0265	**1.6218**	1.6341	1.6255	1.6259
	100	8.0655	6.3281	4.8583	3.5681	2.8569	1.1222	**1.1002**	1.1097	1.1252
	150	5.2348	4.4738	3.5776	2.7366	2.4658	1.0271	**1.0173**	1.0206	1.0285
	200	3.8637	3.2702	2.5171	2.1486	2.1235	0.9088	**0.8914**	0.8981	0.9107
	300	2.4549	2.0840	1.6253	1.6040	1.6234	0.8863	**0.8602**	0.8711	0.8880
	400	1.6570	1.5458	1.3680	1.1840	1.2172	0.4203	0.4670	0.4545	**0.4065**
0.99	30	219.0152	185.1343	168.3919	129.6787	15.3331	11.1499	**11.0454**	11.0997	11.2049
	50	110.4431	91.4097	79.9789	55.5758	6.6075	4.0507	**4.0062**	4.0315	4.0640
	100	42.8823	35.3426	28.9429	17.5752	3.4125	1.5104	**1.5012**	1.5053	1.5134
	150	23.8422	17.5461	13.0341	6.8060	2.9512	1.2025	**1.1777**	1.1901	1.2061
	200	19.3136	15.6858	12.2169	6.8557	3.0717	1.1361	**1.1294**	1.1326	1.1373
	300	10.3844	8.3924	6.2426	3.6362	2.8003	0.9842	**0.9772**	0.9806	0.9849
	400	8.0394	7.0835	5.5365	3.3102	2.4452	**0.3018**	0.3373	0.3269	0.3056

3.2. Simulation results

Tables 2–10 display the estimated SMSEs for various estimators under different influencing factors, such as the degree of multicollinearity (ρ), sample size (n), number of explanatory variables (p), and dispersion parameters (ϕ). The simulated results consistently show that the SMSE values of the PIGLTE estimator with different simulated factors are consistently lower compared to other estimators across all evaluated scenarios, and MLE estimator with different simulated factors are consistently lower compared to other estimators due to the effect of multicollinearity.

When the sample size (n), number of explanatory variables (p), and dispersion parameter (ϕ) are held constant, increasing the degree of multicollinearity leads to higher SMSE values for the PIGRM estimators. This relationship holds across all simulation scenarios, with the MLE method being particularly affected under severe multicollinearity conditions.

An analysis based on sample size while holding other factors constant reveals a decreasing trend in simulated SMSE values as the sample size increases. Furthermore, increasing the number of explanatory variables (p) and the dispersion parameter (ϕ) results in higher estimated SMSE values for the estimators. However, PIGLTE consistently outperforms other estimation methods across various conditions, especially with $(\hat{k}_1, \hat{d}_1)$ to $(\hat{k}_4, \hat{d}_4)$.

Comparing the estimated SMSE of MLE to that of PIGLTE shows that the former is three times larger than the latter, highlighting the robustness of PIGLTE in the presence of severe multicollinearity. While PIGLTE with $(\hat{k}_1, \hat{d}_1)$ and $(\hat{k}_2, \hat{d}_2)$ exhibits smaller MSE compared to MLE, PIGRRE, and PIGLE.

Overall, the analysis in Tables 2–10 indicates that the preferred choice is PIGLTE with $(\hat{k}_1, \hat{d}_1)$, followed by $(\hat{k}_4, \hat{d}_4)$, across various conditions considered in the simulation study.

In conclusion, the overall performance of PIGLTE with $(\hat{k}_1, \hat{d}_1)$ is superior in terms of minimizing MSE. Therefore, it is recommended to use PIGLTE with shrinkage parameter $(\hat{k}_1, \hat{d}_1)$ for obtaining reliable statistical inferences.

4. Real-world dataset application

In this section, we compare our proposed estimator to the MLE, Ridge, and Liu estimators, demonstrating its benefits with real-life examples and evaluating its performance using the SMSE criterion.

4.1. HIV cases datasets

The performance of the proposed estimators is assessed using a real dataset on HIV cases taken from Herindrawati et al. [19]. This dataset is sourced from the Java Province Health Profile and the 2015 Health Pocket Book published by the Central Java Provincial Health Service, as well as the Publication of Social Survey Results on the National Economy (Susenas) issued by the Central Agency of Statistics (BPS) for the Central Java Province. It pertains to 2015 and encompasses research units at the Regency/City level within the Central Java Province, consisting of 29 Regencies and 6 Cities.

The dataset comprises 35 observations with one response variable and seven explanatory variables. The response variable is the number of new HIV cases in each district or city in Central Java Province in 2015 (Y). The description of the explanatory variables of the HIV cases dataset is given in Table 11. We examined whether the response variable in the HIV cases dataset is suitable for the PIGRM. The Chi-square (χ^2) test yielded a χ^2 value of 0.42436 and a p-value of 0.8088, suggesting that the PIGRM fits the dataset well.

We aim to evaluate how well the proposed PIGLTE method performs within the PIGRM framework for addressing multicollinearity. The dataset has a Conditional Number (CN) estimate of 1426.034, and an analysis of the correlation matrix in Table 12 shows severe multicollinearity among the explanatory variables.

Table 11. Description of explanatory variables for HIV cases dataset.

Explanatory Variable	**Description**
X1	Percentage of poor people in each district/city in Central Java Province in 2015
X2	Percentage of the population with at least a high school education in each district/city in 2015
X3	Percentage of couples of childbearing age (PUS) using condom contraceptives in each district/city in Central Java Province in 2015
X4	Ratio of the number of health workers per 100,000 population in each district/city in Central Java Province in 2015
X5	Ratio of the number of health facilities per 100,000 population in each district/city in Central Java Province in 2015
X6	Percentage of urban areas in each district/city in Central Java Province in 2015
X7	Percentage of the population aged 25-34 years in each district/city in Central Java Province in 2015

Table 12. Correlation matrix between the explanatory variables in the HIV cases dataset.

	x1	x2	x3	x4	x5	x6	x7
x1	1.0000						
x2	-0.5267	1.0000					
x3	-0.2787	0.7810	1.0000				
x4	-0.3273	0.7721	0.7119	1.0000			
x5	-0.2515	0.5958	0.6236	0.8497	1.0000		
x6	-0.4195	0.7990	0.8231	0.8143	0.7669	1.0000	
x7	-0.4474	0.6125	0.4594	0.3250	0.1235	0.5742	1.0000

Table 13. Coefficients estimated and SMSEs of the different estimators for the HIV cases dataset.

Parameters	Estimators								
	MLE	PIGRRE		PIGLE		PIGLTE			
	-	$\hat{k}_1$	$\hat{k}_2$	$\hat{d}_1$	$\hat{d}_2$	$\hat{k}_{t1}, \hat{d}_{t1}$	$\hat{k}_{t2}, \hat{d}_{t2}$	$\hat{k}_{t3}, \hat{d}_{t3}$	$\hat{k}_{t4}, \hat{d}_{t4}$
β_0	0.44730	0.04733	0.04549	0.14689	0.14689	0.05689	0.05068	0.00452	0.04189
β_1	-0.00596	0.01822	0.02137	-0.00401	-0.00401	0.00613	0.01022	0.10822	0.01803
β_2	-0.02776	0.02349	0.03018	-0.02624	-0.02624	-0.00333	0.00591	0.11858	0.02307
β_3	0.40292	0.29067	0.27584	0.40121	0.40121	0.35044	0.32982	0.02079	0.29164
β_4	0.00778	0.00315	0.00261	0.00751	0.00751	0.00542	0.00462	-0.00053	0.00318
β_5	-0.15250	-0.06272	-0.05472	-0.13599	-0.13599	-0.09806	-0.08516	0.02826	-0.06285
β_6	-0.02933	-0.02658	-0.02610	-0.02992	-0.02992	-0.02851	-0.02786	-0.01336	-0.02663
β_7	0.48661	0.41255	0.39745	0.51564	0.51564	0.47282	0.45236	0.09039	0.41411
SMSE	10.76408	0.23420	0.23990	0.76543	0.76543	0.22502	0.22552	0.53093	0.23290

The PIGRM estimates obtained through MLE, PIGRRE, PIGLE, and PIGLTE are derived from Eqs. (8), (11), (16), and (22), respectively. Correspondingly, the SMSEs for MLE, PIGRRE, PIGLE, and PIGLTE are calculated using Eqs. (10), (15), (21), and (26), respectively.

The results in Table 13 indicate that the PIGLTE outperforms both MLE and the other two biased estimators. The inferior performance of MLE is attributed to the presence of multicollinearity among the explanatory variables. Furthermore, when comparing MLE with PIGRRE and PIGLE, it is evident that the ridge and Liu estimators significantly outperform MLE. However, the performance of PIGLTE, particularly with parameters $(\hat{k}_{t1}, \hat{d}_{t1})$, surpasses all others due to its minimal SMSE.

4.2. Mussels dataset

To assess the proposed estimator's performance practically, we utilize the Mussels dataset, originally compiled by Sepkoski and Rex (1974) and later utilized by Batool et al. (2023). This dataset includes information on the count of mussel species in a river, comprising a sample size of $n = 44$ and featuring one response variable and eight explanatory variables. The objective of this dataset is to identify the factors influencing the count of mussel species (y), such as Area $(x1)$, the number of stepping stones to four major species-source river systems (Alabama-Coosa $(x2)$, Apalachicola $(x3)$, St. Lawrence $(x4)$, and Savannah $(x5)$), Nitrate Concentration, solid residue $(x7)$, and Hydronium concentration (10^{-pH}) $(x8)$. Before conducting statistical modeling, we examine whether the PIGRM is suitable for the response variable in the mussels dataset. Using the Chi-square (χ^2) test, we found a χ^2 value of 18.436 with a p-value of 0.299, indicating that the PIGRM fits the Mussels dataset well.

Our goal is to evaluate the performance of the proposed PIGLTE within the PIGRM framework for handling multicollinearity. The Conditional Number (CN) estimated for this dataset is 414220.42 and the analysis of the correlation matrix analysis in Table 14 reveals significant multicollinearity among the explanatory variables.

Table 14. Correlation matrix between the explanatory variables in the Mussels dataset.

	$x1$	$x2$	$x3$	$x4$	$x5$	$x6$	$x7$	$x8$
$x1$	1.0000							
$x2$	0.1425	1.0000						
$x3$	0.1404	0.9938	1.0000					
$x4$	-0.0535	-0.9844	-0.9720	1.0000				
$x5$	-0.0274	0.6572	0.6742	-0.6536	1.0000			
$x6$	0.1172	0.3756	0.3629	-0.3533	0.2318	1.0000		
$x7$	-0.1421	-0.2728	-0.2997	0.2852	-0.1571	0.0809	1.0000	
$x8$	-0.1989	0.0253	0.0138	-0.0468	-0.0568	0.1481	-0.1986	1.0000

Table 15. Coefficients estimated and SMSEs of the different estimators for the Mussels dataset.

Parameters	Estimators								
	MLE	PIGRR		PIGLE		PIGLTE			
	-	$\hat{k}_1$	$\hat{k}_2$	$\hat{d}_1$	$\hat{d}_2$	$\hat{k}_{t1}, \hat{d}_{t1}$	$\hat{k}_{t2}, \hat{d}_{t2}$	$\hat{k}_{t3}, \hat{d}_{t3}$	$\hat{k}_{t4}, \hat{d}_{t4}$
β_1	0.00002	0.00003	0.00003	0.00002	0.00002	0.00003	0.00003	0.00032	0.00003
β_2	0.13005	0.05672	0.11002	0.12958	0.12958	0.06386	0.05740	-0.08723	0.10131
β_3	-0.09511	-0.00906	-0.07144	-0.09456	-0.09456	-0.01702	-0.00943	0.06388	-0.06115
β_4	0.07710	0.07811	0.07767	0.07712	0.07712	0.07864	0.07874	-0.05164	0.07791
β_5	0.01022	0.00827	0.00934	0.01020	0.01020	0.00771	0.00754	-0.00682	0.00896
β_6	0.04899	0.02958	0.04618	0.04893	0.04893	0.03641	0.03463	-0.03288	0.04499
β_7	-0.00349	-0.00299	-0.00339	-0.00349	-0.00349	-0.00311	-0.00307	0.00275	-0.00334
β_8	-0.04041	-0.03228	-0.03899	-0.04038	-0.04038	-0.03483	-0.03411	0.02714	-0.03839
SMSE	0.01554	0.01593	0.01193	0.01542	0.01542	0.01100	0.01097	0.01090	0.01344

Once again, Table 15 highlights the benefits of the proposed estimator. Notably, there is a substantial reduction in the SMSE in favor of the PIGLTE $(\hat{k}_{t3}, \hat{d}_{t3})$. Specifically, the PIGRRE and PIGLE demonstrate lower SMSE values than the MLE.

In summary, based on the results of both applications and simulations, it can be concluded that the proposed estimator outperforms the MLE and other existing estimators in multicollinearity scenarios, thereby delivering superior performance.

5. Conclusion

In the PIGRM, estimating regression coefficients typically relies on the MLE method. However, multicollinearity issues can arise if explanatory variables correlate, impacting inference and estimation. We propose a novel PIGLTE method to address this and suggest several techniques to determine the optimal shrinkage parameter $\hat{k}_t$. Our research uses simulations across a range of criteria to evaluate the effectiveness of our suggested shrinkage estimators with current techniques. These include sample size, dispersion parameter, degree of collinearity, and number of explanatory variables. According to the results, our PIGLTE approach works better than the MLE and other biased estimators. Specifically, among all the estimators, the PIGLTE with parameters $\hat{k}_{t1}, \hat{d}_{t1})$ yields the lowest SMSE. Furthermore, we illustrate the efficacy of our suggested estimator with two practical applications. Based on these findings, we recommend using the PIGLTE with the shrinkage parameter $(\hat{k}_{t1}, \hat{d}_{t1})$ to address severe multicollinearity in PIGRM. Future research could explore applying our estimator to other regression models, such as the zero-inflated negative binomial regression model, the zero-inflated Poisson regression model, and CMPRM. Extending our work to provide a robust biased estimation of PIGRM, following approaches by Lukman et al. [32], Omara [36], and Lukman et al. [32], could be beneficial.

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