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Moments and Inferences based on Generalized Order Statistics from Benktander Type II Distribution

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Abstract In this paper, we employ generalized order statistics to investigate the moment properties of the Benktander Type II distribution. Through this approach, we derive precise and explicit formulas for single moment and establish recurrence relations for single and product moments. Additionally, we present a characterization of the Benktander Type II distribution, accompanied by further implications regarding moments of record values and ordinary order statistics. We estimate the unknown parameters of the Benktander Type II distribution using maximum likelihood estimation for generalized order statistics. Subsequently, we conduct simulation studies encompassing order statistics and record values. The efficacy of the obtained maximum likelihood estimates is evaluated through comprehensive simulation analyses, focusing on various moments and their relative mean squared errors. This research contributes to understanding the Benktander Type II distribution's properties and provides valuable insights into its parameter estimation using generalized order statistics.

Keywords Generalized order statistics, record values, order statistics, single moment, product moments, recurrence relation, Benktander Type II distribution, characterization and maximum likelihood estimator.

AMS 2010 subject classifications 62G30, 62E10, 65C10

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1. Introduction

Numerous real-world situations can be encountered where ordered random variables (*orvs*) are applied. For instance, we might be curious to learn the price trend of any commodity by ordering the prices in rising or falling order of extent, or we might be interested in learning the lifespan of a specific kind of product made using a variety of techniques. When data are arranged in ascending or descending order, order statistics are generated. These statistics can then be used to understand other aspects of the data, such as the maximum and minimum values as well as the range of the data. [1] was the first to develop the idea of generalized order statistics (*gos*), which includes a number of models for random variables (*rvs*) arranged in ascending order, encompassing order statistics, record values, k-th record values, the Pfeifer record model, and more. These models are widely utilized in statistical modeling and inference and are particularly helpful in many statistical applications. The models may be used for a variety of tasks, including the analysis of reliability and conducting goodness-of-fit tests, outlier identification, robust estimation, material strength detection, and flood frequency analysis. Generalized order statistics represent a unifying framework that consolidates multiple models of *orvs*. By providing joint and marginal densities of *gos*. [1] established several distributional characteristics of *gos*.
The *gos* formula may be written mathematically as; Let $U_1, U_2, ..., U_n$ be a series of independent and identically distributed (*iid*) *rvs*, with an absolutely continuous cumulative distribution function (*cdf*) $F(u)$, and the

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probability density function (*pdf*) $f(u)$. Let $n \in N,\ n \geq 2,\ k \geq 1$, and $\tilde{m} = (m_1, m_2, \ldots, m_{n-1}) \in \Re^{n-1},\ M_r = \sum_{j=r}^{n-1} m_j,\ 1 \leq r \leq n-1$, such that

$$\gamma_r = k + n - r + \sum_{j=r}^{n-1} m_j > 0, \text{ for} 1 \leq r \leq n-1.$$

The *rvs* $U(1, n, \tilde{m}, k), U(2, n, \tilde{m}, k), \ldots, U(n, n, \tilde{m}, k)$ are said to be *gos* from a continuous population with *cdf* $F(u)$ and *pdf* $f(u)$ if their joint *pdf* is of the form

$$k \left(\prod_{j=1}^{n-1} \gamma_j \right) \left(\prod_{i=1}^{n-1} \left[\bar{F}(u_i) \right]^{m_i} f(u_i) \right) \left[\bar{F}(u_n) \right]^{k-1} f(u_n). \tag{1}$$

defined on the cone $F^{-1}(0+) < u_1 \leq u_2 \leq \ldots \leq u_n < F^{-1}(1)$ of $\Re^n$, where $\bar{F}(u) = 1 - F(u)$. The different variants of *gos* are specified in the following table through suitable parameter selections.

Table 1. Variants of the *gos*.

		$\gamma_n = k$	γ_r	m_r
1.	Sequential Order Statistics	1	1	-1
2.	Record Values	1	(n-r+1)	0
3.	Order Statistics	α_n	(n-r+1)α_r	$\gamma_r - \gamma_{r+1} - 1$
4.	Pfeifer's Record Values	$R_n + 1$	$n - r + 1 + \sum_{j=r}^{n} R_j$	R_γ
5.	Progressively Type II right censored OS	β_n	β_r	$\beta_r - \beta_{r+1} - 1$

Marginal and joint distributions
The definition of *gos* has considered two cases
Case I: $m_1 = m_2 = \cdots = m_{n-1} = m$ [1].
Case II: $\gamma_i \neq \gamma_j, \quad i \neq j, \quad i, j = 1, 2, \ldots, n-1$ [2].
Here we have considered only Case I.
The *pdf* of r-th *gos* $U(r, n, m, k)$, is provided by

$$f_{U(r,n,m,k)}(u) = \frac{C_{r-1}}{(r-1)!} [\bar{F}(u)]^{\gamma_r - 1} f(u) g_m^{r-1}(F(u)), \tag{2}$$

and the joint *pdf* of $U(r, n, m, k)$ and $U(s, n, m, k), 1 \leq r < s \leq n$, is given by

$$\begin{aligned} f_{U(r,n,m,k),\mathrm{U}(s,n,m,k)}(u,\ v) = & \frac{C_{s-1}}{(r-1)!(s-r-1)!} [\bar{F}(u)]^m g_m^{r-1}(F(u)) \\ & \times [h_m(F(v)) - h_m(F(u))]^{s-r-1} [\bar{F}(v)]^{\gamma_s - 1} f(u) f(v), u < v, \end{aligned} \tag{3}$$

where

$$C_{r-1} = \prod_{i=1}^{r} \gamma_i, \quad \gamma_i = k + (n-i)(m+1),$$

$$h_m(u) = \begin{cases} -\frac{1}{m+1}(1-u)^{m+1} & , \quad m \neq -1 \\ -\ln(1-u) & , \quad m = -1 \end{cases}$$

and

$$g_m(u) = h_m(u) - h_m(0) = \int_0^u (1-t)^m dt, \quad u \in [0,1).$$

This article also focuses on the relationships between the moments used to calculate higher-order moments. In the computation of moments of higher orders, the utility and significance of recurrence relations come to the forefront. They prove advantageous in reducing both the time complexity and the procedural steps necessary to attain a generalized form for any order of the function under investigation. Additionally, they help characterize distributions. Characterization stands as a pivotal technique, facilitating the identification of population distributions through the discernment of sample traits see: for examples; [3] elucidates novel distributional properties of *gos* derived from the two-parameter exponential distribution. Present the minimum variance linear unbiased estimators (*MVLUEs*) for the distribution parameters, enhancing the precision of statistical inference. Furthermore, establish a significant characterization of the exponential distribution. [4] elucidates novel recurrence relations for both moment generating functions and conditional moment generating functions of *gos*. These relations are derived from random samples drawn from the doubly truncated class of distributions, thereby extending the current understanding of distributional properties in truncated scenarios. [5] established recurrence relations for single and product moments of *gos* from doubly truncated Weibull distribution which includes relations for order statistics, k-th record values, sequential order statistics and order statistics with non-integral sample size. [6] presented explicit expressions for single and product moments of *gos* from a general class of distributions. Further, some deductions and particular cases are discussed. [7] obtained the single and product moments of *gos* from linear exponential distribution. [8] obtained moments of Erlang-truncated exponential distribution based on *gos*. [9] obtained the ratio and inverse moments of *gos* from Burr distribution. [10] presented explicit expressions and establishes novel recurrence relations for both marginal and joint moment generating functions of *gos* from the extended Type II generalized logistic distribution. The research further extrapolates these findings to deduce moments of k-th record values and conventional order statistics. [11] presents novel explicit expressions and recurrence relations for single and product moments of lower *gos* derived from the inverse Burr distribution. The formulations are subsequently specialized to encompass order statistics and record values. Parameter estimation is conducted via maximum likelihood methodology. [12] elucidates concise explicit expressions and novel recurrence relations for both single and product moments of *gos* derived from the exponential-Weibull lifetime distribution. These findings are subsequently extrapolated to moments of order statistics and upper record values. [13] derives explicit expressions for single, product, and conditional moments of order statistics and record values from the extended exponential distribution, as well as moments for progressively Type II censored order statistics. [14] derives explicit expressions for single, product, and conditional moments of order statistics and record values from the extended exponential distribution, including moments for progressively Type II censored order statistics. [15] establishes novel recurrence relations for both individual and product moments of *gos* derived from the Erlang-truncated exponential distribution. Leveraging these relations, compute the first four moments for order statistics, record values, and second record values across a range of parameter values. Additionally, utilize the order statistics results to derive best linear unbiased estimators (*BLUEs*) for location and scale parameters, specifically in the context of type-II right-censored samples. This approach provides a robust framework for statistical inference in truncated exponential distributions, with potential applications in reliability analysis and survival studies. [16] have derived relations for marginal and joint moment generating function of Weibull generalized exponential distribution based on *gos*. These derived relations are further reduced to the sub models of *gos* such as order statistics and record values. [17] have elucidated diverse structural characteristics of the Pareto distribution, encompassing the quantile function, explicit moment expressions, mean deviation, Bonferroni and Lorenz curves, and Rényi entropy. [18] have derived relations for marginal and joint moment generating function of Weibull generalized exponential distribution based on *gos*. These derived relations are further reduced to the sub models of *gos* such as order statistics and record values. [19] derives single and product moments of *gos* for the generalized inverse Lindley distribution. [20] presented the exact and explicit expressions for single and product moments of Lindley distribution based on *gos* in terms of Gauss hypergeometric function and Kampe de Feriet series.

These results include the exact expression for the single and product moments of order statistics, progressive Type-II censoring, record values Pfeifer's record value and sequential order statistics from Lindley distribution. [21] have derived the explicit expressions for single and product moments of *gos* from Pareto-Rayleigh distribution using hypergeometric functions. [22] investigates the moments of *gos* using the power linear hazard rate distribution, deriving explicit formulations and relations between moments and provides characterization results through various techniques. [23] have investigated the Bass diffusion model in order to determine a few recurrence relations for the product and single moments within the context of *gos*. For details on the method of recurrence relations and characterization on various statistical distributions, these relations are further reduced in the particular cases of *gos*. [24] derive explicit expressions and recurrence relations for the single and product moments of *gos* from the log-extended exponential–geometric distribution, applying these results to obtain the *BLUEs* for location and scale parameters using progressively Type-II right censored samples. [25] derived and provided exact mathematical formulas for calculating the single, product, triple, and quadruple moments of order statistics arising from the Pareto-Weibull distribution. [26] developed $K-$th-order equilibrium Weibull distribution (KEWD) a model incorporating three parameters, utilizing the method of weighted probabilities extending the traditional Weibull distribution. The study explores various statistical properties of (KEWD), including moments, entropy, and order statistics. It also presents parameter estimation using maximum likelihood, conducts simulation studies, and demonstrates the model's superior fit to real-life data compared to other Weibull extensions.

The purpose of this study is grounded in the examination of the Benktander Type II distribution, a significant distribution in size modeling, particularly within the domains of actuarial science and risk management. The Benktander Type II distribution is essential for simulating heavy-tailed losses commonly observed in non-life/casualty insurance. This distribution was introduced by Gunnar Benktander in 1970 and is characterized by its unique asymptotic resemblance to the mean excess function of the Weibull distribution, distinguishing it from other distributions like the Pareto or lognormal. The empirical mean excess functions that lead to the Benktander distributions effectively bridge the gap between Pareto and exponential distributions, with the type I variant approximating the lognormal distribution.

The reason for selecting the Benktander Type II distribution in the context of *gos* lies in its flexibility and applicability to a broad range of statistical behaviors, especially in modeling extreme values. Generalized order statistics provide a generalized framework that extends traditional order statistics, allowing for more versatile weighting and spacing schemes. This flexibility is particularly relevant when dealing with heavy-tailed distributions like Benktander Type II, where classical order statistics might not adequately capture the distribution's characteristics. By integrating the Benktander Type II distribution with *gos*, this study aims to enhance the theoretical understanding and practical application of these statistical tools, offering more precise methods for parameter estimation and risk assessment in complex, real-world scenarios; see [27], [28], [29]. [30] studied the characterization of Benktander Type II distribution via truncated moments and order statistics.

The Benktander Type II distribution's *pdf* is denoted by

$$f(u) = e^{\frac{a(1-u^b)}{b}} u^{b-2}(au^b - b + 1), \quad u \geq 1, \quad a > 0, \quad 0 < b \leq 1. \tag{4}$$

The corresponding *cdf* is expressed by

$$F(u) = 1 - u^{b-1} e^{\frac{a(1-u^b)}{b}}, \quad u \geq 1, \quad a > 0, \quad 0 < b \leq 1. \tag{5}$$

Therefore, in view of (4) and (5), we have

$$f(u) = \frac{(au^b - b + 1)}{u} \overline{F}(u). \tag{6}$$

The reliability function for the Benktander Type II distribution is

$$R(t) = 1 - F(t) = t^{b-1} e^{\frac{a(1-t^b)}{b}}.$$

The hazard function for the Benktander Type II distribution is

$$h(t) = \frac{f(t)}{R(t)} = \frac{(at^b - b + 1)}{t}.$$

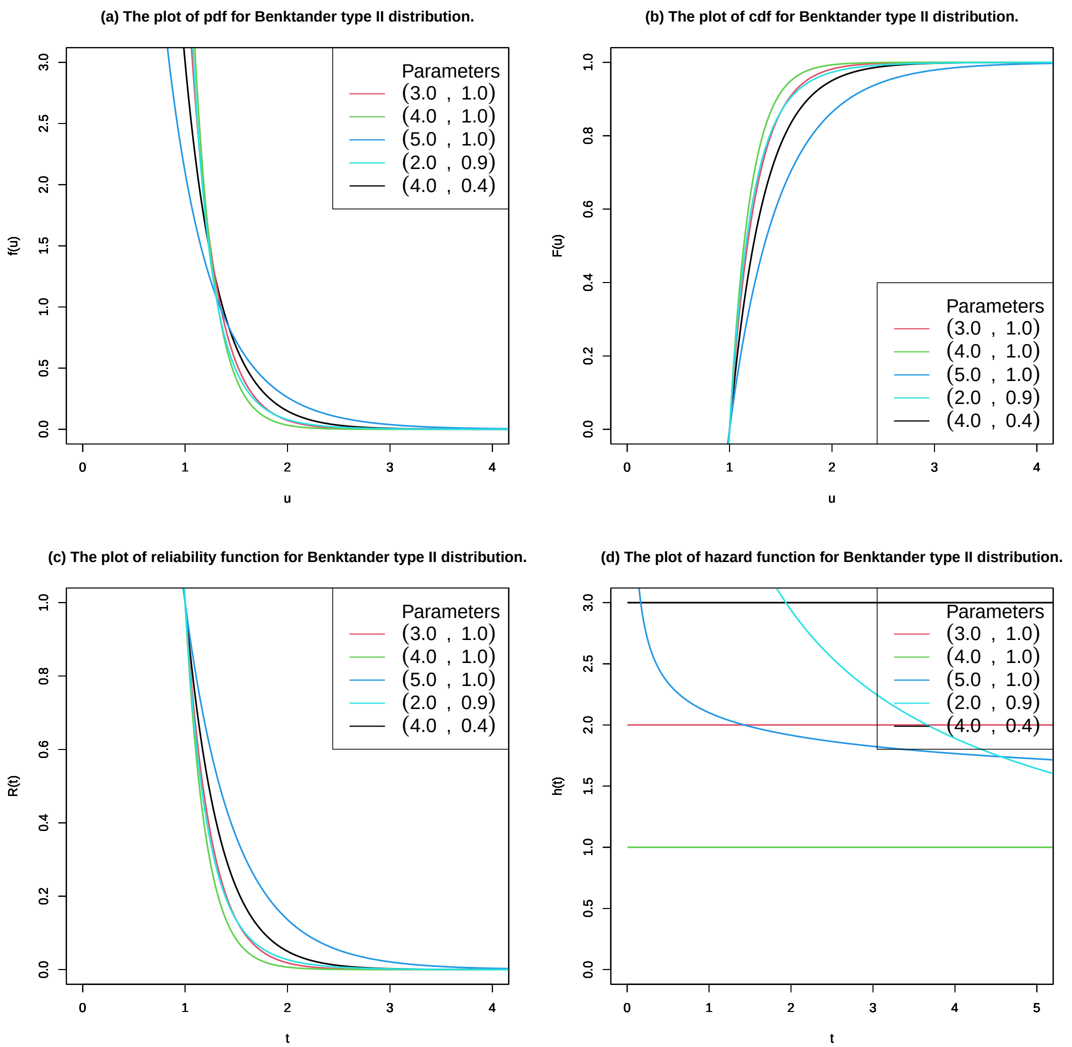

Figure 1. p.d.f., c.d.f., reliability, and hazard function graph at different parameter values.

Figure (a) and (b) displays various possible forms for the density and distribution functions of the Benktander Type II distribution. The suggested distribution can be shown in the figure to be capable of capturing various datasets behaviors. Figures (c) and (d) above show the Benktander Type II distribution's reliability and hazard rate functions behavior.
The paper's structure is as follows. Section 2, presents the derivation of the precise and explicit expression for the single moment and recurrence relations of *gos* from the Benktander Type II distribution. In Section 3, we have shown the characterization of the Benktander Type II distribution. In Section Section 4, we have discussed maximum likelihood for *gos*. In Section 5, we have discussed a simulation study for order statistics, and record values. Results and discussion presented in Section 6. Ultimately, a conclusion is provided within Section 7.

2. Single Moment

Theorem 2.1 For the Benktander Type II distribution as given in (4) for $1 \leq r \leq n,\ \ k \geq 1$ and $p = 1, 2, \ldots$

$$\begin{aligned} &E\left[U^p(r,n,m,k)\right] \\ &= \sum_{u_1=0}^{r-1}(-1)^{u_1}\binom{r-1}{u_1}\frac{C_{r-1}e^{\frac{a\gamma_{r-u_1}}{b}}}{(r-1)!(m+1)^{r-1}}\left[\left(\frac{a}{b}\right)\left(\frac{\gamma_{r-u_1}}{b}\right)^{\frac{-p-(b-1)(\gamma_{r-u_1}+1)-b}{b}}\right. \\ &\times\Gamma\left(\frac{p+(b-1)(\gamma_{r-u_1}+1)+b}{b},\frac{a\gamma_{r-u_1}}{b}\right)+\left(\frac{1-b}{b}\right)\left(\frac{a\gamma_{r-u_1}}{b}\right)^{\frac{-p-(b-1)(\gamma_{r-u_1}+1)}{b}} \\ &\left.\times\Gamma\left(\frac{p+(b-1)(\gamma_{r-u_1}+1)}{b},\frac{a\gamma_{r-u_1}}{b}\right)\right], \qquad m \neq -1. \end{aligned} \tag{7}$$

$$\begin{aligned} &E\left[U^p(r,n,-1,k)\right] \\ &= E\left[(U^{(k)}_{U'_{(r)}})^p\right] = \sum_{v_1=0}^{\infty}\sum_{i=0}^{w+r-1}(-1)^i c_w(r-1)\binom{w+r-1}{i}\frac{k^r e^{\frac{a(k+i)}{b}}}{(r-1)!} \\ &\left[\left(\frac{a(k+i)}{b}\right)^{\frac{-p-(b-1)(k+i+1)-b}{b}}\Gamma\left(\frac{p+(b-1)(k+i+1)+b}{b},\frac{a(k+i)}{b}\right)\right. \\ &\left.+(1-b)\left(\frac{a(k+i)}{b}\right)^{\frac{-p-(b-1)(k+i+1)}{b}}\Gamma\left(\frac{p+(b-1)(k+i+1)}{b},\frac{a(k+i)}{b}\right)\right], \qquad m = -1. \end{aligned} \tag{8}$$

Proof: For **m** $\neq$ **-1**
From (2), we have

$$E\left[U^p(r,n,m,k)\right] = \frac{C_{r-1}}{(r-1)!}\int_1^{\infty} u^p[\bar{F}(u)]^{\gamma_r-1}f(u)g_m^{r-1}(F(u))du. \tag{9}$$

On expanding $g_m^{r-1}(F(u)) = \left[\frac{1}{m+1}(1-\left[\overline{F}(u)\right]^{m+1})\right]^{r-1}$ binomially in (9), we get

$$E\left[U^p(r,n,m,k)\right] = \frac{C_{r-1}}{(r-1)!(m+1)^{r-1}}\sum_{u_1=0}^{r-1}(-1)^{u_1}\binom{r-1}{u_1}\int_1^{\infty} u^p[\bar{F}(u)]^{\gamma_{r-u_1}-1}f(u)du. \tag{10}$$

using (5), (6) in (10) and after simplification, we get

$$E\left[U^p(r,n,m,k)\right] = \frac{C_{r-1}e^{\frac{a\gamma_{r-u_1}}{b}}}{(r-1)!(m+1)^{r-1}}\sum_{u_1=0}^{r-1}(-1)^{u_1}\binom{r-1}{u_1}I_1. \tag{11}$$

where

$$I_1 = a\int_1^{\infty} u^{p+(b-1)(\gamma_{r-u_1}+1)} e^{\frac{-a\gamma_{r-u_1}u^b}{b}} du + (1-b)\int_1^{\infty} u^{p+(b-1)(\gamma_{r-u_1}-1)+b-2} e^{\frac{-a\gamma_{r-u_1}u^b}{b}} du.$$

Setting $t = u^b$ and simplifying using generalized exponential function,

$$E_n(u) = \int_1^{\infty} \frac{e^{-ut}}{t^n} dt = u^{n-1}\Gamma(1-n,\ u).$$

we get,

$$I_1 = \Bigg[\left(\frac{a}{b}\right)\left(\frac{a\gamma_{r-u_1}}{b}\right)^{\frac{-p-(b-1)\gamma_{r-u_1}-b}{b}} \Gamma\left(\frac{p+(b-1)\gamma_{r-u_1}+b}{b}, \frac{a\gamma_{r-u_1}}{b}\right) + \left(\frac{1-b}{b}\right)\left(\frac{a\gamma_{r-u_1}}{b}\right)^{\frac{-p-(b-1)(\gamma_{r-u_1}-1)+1-b}{b}} \Gamma\left(\frac{p+(b-1)\gamma_{r-u_1}}{b}, \frac{a\gamma_{r-u_1}}{b}\right)\Bigg]. \tag{12}$$

Substituting (12) in (11), we get (7).
For **m = -1** we have

$$E\left[U^p(r,n,-1,k)\right] = E\left[(U^{(k)}_{U'_{(r)}})^p\right] = \frac{K^r}{(r-1)!}\int_1^{\infty} u^p[\bar{F}(u)]^{k-1}[-ln(1-F(u))]^{r-1} f(u)du \tag{13}$$

where $E\left[(U^{(k)}_{U'_{(r)}})^p\right]$ denotes the $p-th$ moments of $k-th$ upper record values
Using logarithmic expansion in (13)

$$[-ln(1-z)]^h = \left(\sum_{w=1}^{\infty}\frac{z^w}{w}\right)^h = \sum_{w=0}^{\infty} c_w(h) z^{w+h}$$

where $c_w(h)$ is the coefficient of z^{c+h} in the expansion of $\left(\sum_{w=1}^{\infty}\frac{z^w}{w}\right)^h$.

$$E\left[(U^{(k)}_{U'_{(r)}})^p\right] = \frac{K^r}{(r-1)!}\sum_{w=0}^{\infty} c_w(r-1)\int_1^{\infty} u^p[\bar{F}(u)]^{k-1}[F(u))]^{w+r-1} f(u)du.$$

Using (5) and (6), we get

$$E\left[(U^{(k)}_{U'_{(r)}})^p\right] = \frac{K^r e^{\frac{a(k+i)}{b}}}{(r-1)!}\sum_{w=0}^{\infty}\sum_{i=0}^{w+r-1}(-1)^i c_w(r-1)\binom{w+r-1}{i} I_2 \tag{14}$$

where

$$I_2 = a\int_1^{\infty} u^{p+(b-1)(k+i+1)} e^{\frac{-a(k+i)u^b}{b}} du + (1-b)\int_1^{\infty} u^{p+(b-1)(k+i-1)+b-2} e^{\frac{-a(k+i)u^b}{b}} du.$$

Setting $t = u^b$ and simplifying using generalized exponential function, we get

$$I_2 = \Bigg[\left(\frac{a}{b}\right)\left(\frac{a(k+i)}{b}\right)^{\frac{-p-(b-1)(k+i)-b}{b}} \Gamma\left(\frac{p+(b-1)(k+i)+b}{b}, \frac{a(k+i)}{b}\right) + \left(\frac{1-b}{b}\right)\left(\frac{a(k+i)}{b}\right)^{\frac{-p-(b-1)(k+i)}{b}} \Gamma\left(\frac{p+(b-1)(k+i)}{b}, \frac{a(k+i)}{b}\right)\Bigg]. \tag{15}$$

Substituting I_2 in (14), we get (8).

Remarks 2.1

a. We get single moments of order statistics from Benktander Type II distribution when $m = 0$ and $k = 1$ in (7), obtained as

$$E[U^p_{r:n}] = \sum_{u_1=0}^{r-1} (-1)^{u_1} \binom{r-1}{u_1} C_{r:n} e^{\frac{a(n-r+u_1+1)}{b}} \left[\left(\frac{a}{b}\right) \left(\frac{a(n-r+u_1+1)}{b}\right)^{\frac{-p-(b-1)(n-r+u_1+1)-b}{b}} \right.$$
$$\times\Gamma\left(\frac{p+(b-1)(n-r+u_1+1)+b}{b}, \frac{a(n-r+u_1+1)}{b}\right)$$
$$+\left(\frac{1-b}{b}\right)\left(\frac{a(n-r+u_1+1)}{b}\right)^{\frac{-p-(b-1)(n-r+u_1)+1-b}{b}}$$
$$\left.\Gamma\left(\frac{p+(b-1)(n-r+u_1+1)}{b}, \frac{a(n-r+u_1+1)}{b}\right)\right]. \tag{16}$$

b. We get the moments of upper record values from Benktander Type II distribution, when we put $k = 1$ in (8), obtained as

$$E\left[U^p(r,n,-1,1)\right] = E\left[(U^p_{U'_{(r)}})\right] = \sum_{w=0}^{\infty} \sum_{i=0}^{w+r-1} (-1)^i c_w(r-1) \binom{w+r-1}{i} \frac{e^{\frac{a(i+1)}{b}}}{(r-1)!}$$
$$\left[\left(\frac{a}{b}\right)\left(\frac{a(i+1)}{b}\right)^{\frac{-p-(b-1)(i+1)-b}{b}} \Gamma\left(\frac{p+(b-1)(i+1)+b}{b}, \frac{a(i+1)}{b}\right)\right.$$
$$\left.+\left(\frac{1-b}{b}\right)\left(\frac{a(i+1)}{b}\right)^{\frac{-p-(b-1)(i+1)}{b}} \Gamma\left(\frac{p+(b-1)(i+1)}{b}, \frac{a(i+1)}{b}\right)\right]. \tag{17}$$

Table 2. Means and variances of order statistics for the Benktander Type II distribution for

$n,\ r = 1, 2, 3, 4, 5, 6, 7, 8$ and $(a = 3.5,\ b = 0.50)$.

n	r	1	2	3	4	5	6	7	8
1	Mean	1.2857							
	Variance	0.1050							
2	Mean	1.1339	1.4376						
	Variance	0.0205	0.1433						
3	Mean	1.0873	1.2271	1.5428					
	Variance	0.0085	0.0317	0.1659					
4	Mean	1.0647	1.1549	1.2992	1.6240				
	Variance	0.0045	0.0138	0.0392	0.1817				
5	Mean	1.0514	1.1179	1.2105	1.3583	1.6904			
	Variance	0.0028	0.0077	0.0178	0.0448	0.1939			
6	Mean	1.0427	1.0952	1.1632	1.2578	1.4087	1.7468		
	Variance	0.0019	0.0049	0.0102	0.0208	0.0492	0.2038		
7	Mean	1.0364	1.0799	1.1335	1.2028	1.2990	1.4525	1.7958	
	Variance	0.0014	0.0034	0.0066	0.0121	0.0233	0.0528	0.2121	
8	Mean	1.0318	1.0689	1.1131	1.1676	1.2379	1.3357	1.4914	1.8393
	Variance	0.0011	0.0025	0.0047	0.0080	0.0139	0.0255	0.0559	0.2193

Table 3. Means and variances of *gos* for the Benktander Type II distribution for

$n,\ r = 1, 2, 3, 4, 5, 6, 7, 8$ and $(m = 1,\ a = 3.5,\ b = 0.50,\ k = 2)$.

n	r	1	2	3	4	5	6	7	8
1	Mean	1.1339							
	Variance	0.0205							
2	Mean	1.0647	1.2030						
	Variance	0.0045	0.0269						
3	Mean	1.0427	1.1088	1.2501					
	Variance	0.0019	0.0067	0.0303					
4	Mean	1.0318	1.0752	1.1425	1.2860				
	Variance	0.0010	0.0031	0.0081	0.0326				
5	Mean	1.0252	1.0576	1.1016	1.1698	1.3151			
	Variance	0.0007	0.0018	0.0039	0.0091	0.0343			
6	Mean	1.0211	1.0467	1.0793	1.1238	1.1927	1.3395		
	Variance	0.0005	0.0012	0.0023	0.0045	0.0098	0.0355		
7	Mean	1.0180	1.0393	1.0652	1.0982	1.1430	1.2126	1.3607	
	Variance	0.0003	0.0008	0.0015	0.0027	0.0049	0.0104	0.0366	
8	Mean	1.0158	1.0340	1.0554	1.0816	1.1148	1.1600	1.2301	1.3793
	Variance	0.0003	0.0006	0.0011	0.0018	0.0030	0.0053	0.0109	0.0375

Table 4. Skewness and kurtosis of order statistics for the Benktander Type II distribution for

$n,\ r = 1, 2, 3, 4, 5, 6, 7, 8$ and $(a = 3.5,\ b = 0.50)$.

n	r	1	2	3	4	5	6	7	8
1	Skewness	7.8166							
	Kurtosis	16.8972							
2	Skewness	5.8775	5.5075						
	Kurtosis	12.6863	13.1486						
3	Skewness	5.0354	3.5384	4.7246					
	Kurtosis	12.1118	9.0649	11.8752					
4	Skewness	4.8189	2.9561	2.7666	4.3235				
	Kurtosis	11.4121	7.9099	7.8807	11.2168				
5	Skewness	4.4698	2.7061	2.2052	2.3961	4.0716			
	Kurtosis	12.0599	7.2032	6.8514	7.2554	10.8087			
6	Skewness	4.6740	2.8312	1.9604	1.8343	2.1655	3.9046		
	Kurtosis	9.7888	4.4969	6.3519	6.2733	6.9063	10.5265		
7	Skewness	2.3346	2.5359	1.9475	1.6000	1.6009	2.0022	3.7813	
	Kurtosis	44.1090	6.4948	5.4708	5.8381	5.9768	6.6998	10.3127	
8	Skewness	3.9840	2.9446	1.6847	1.4361	1.3830	1.4716	1.8840	3.6842
	Kurtosis	12.7204	-0.5053	6.3553	5.5329	5.4389	5.6150	6.5361	10.1511

Table 5. Skewness and kurtosis of gos for the Benktander Type II distribution for

$n,\ r = 1,2,3,4,5,6,7,8$ and $(m = 1,\ a = 3.5,\ b = 0.50,\ k = 2)$.

n	r	1	2	3	4	5	6	7	8
1	Skewness	5.8775							
	Kurtosis	12.6863							
2	Skewness	4.8189	3.9922						
	Kurtosis	11.4121	10.0422						
3	Skewness	4.6740	2.9686	3.4100					
	Kurtosis	9.7888	7.0865	9.1344					
4	Skewness	3.9840	2.5834	2.2330	3.1200				
	Kurtosis	12.7204	5.5116	6.5111	8.6680				
5	Skewness	1.7728	1.2236	2.1157	1.9273	2.9181			
	Kurtosis	55.4255	26.9262	3.6147	5.8489	8.4801			
6	Skewness	20.3629	2.1687	1.0756	1.5376	1.6705	2.8113		
	Kurtosis	-226.4224	7.2306	14.6819	5.7002	5.9964	8.2744		
7	Skewness	20.6398	0.3230	1.6518	1.3772	1.3967	1.6562	2.7370	
	Kurtosis	-215.0837	57.3344	5.8316	6.6163	4.8146	4.9989	8.1211	
8	Skewness	2.2052	8.3472	2.0395	1.2176	1.2116	1.2247	1.5209	2.6720
	Kurtosis	291.3899	-119.0091	-0.4338	7.8961	4.9679	4.6527	5.3703	8.0279

Table 6. Moments and characteristics of the Benktander Type II distribution based on record data.

				$a = 30,\ b = 0.25$			
r	$E(U)$	$E(U^2)$	$E(U^3)$	$E(U^4)$	$Variance$	$Skewness$	$Kurtosis$
1	1.0333	1.0690	1.1071	1.1480	0.0012	4.6615	9.1566
2	1.0675	1.1420	1.2245	1.3162	0.0025	2.4449	7.4699
3	1.1025	1.2194	1.3532	1.5071	0.0039	1.7578	6.0281
4	1.1384	1.3012	1.4940	1.7235	0.0054	1.4162	5.2448
5	1.1751	1.3879	1.6481	1.9686	0.0071	1.1825	5.1453
6	1.2127	1.4795	1.8166	2.2458	0.0089	1.0519	4.7753
7	1.2512	1.5763	2.0006	2.5590	0.0109	0.9502	4.6675
8	1.2906	1.6786	2.2015	2.9126	0.0131	0.8791	4.5196
9	1.3309	1.7867	2.4206	3.3112	0.0154	0.8216	4.4019
10	1.3722	1.9008	2.6594	3.7602	0.0179	0.7728	4.3641
1	1.0286	1.0588	1.0908	1.1248	0.0008	3.9782	15.0823
2	1.0576	1.1201	1.1884	1.2629	0.0017	2.1466	8.2071
3	1.0869	1.1841	1.2929	1.4154	0.0027	1.5980	5.0840
4	1.1167	1.2507	1.4050	1.5835	0.0037	1.2351	4.5556
5	1.1469	1.3200	1.5250	1.7686	0.0047	1.0062	4.5738
6	1.1775	1.3922	1.6533	1.9722	0.0058	0.8610	4.4298
7	1.2085	1.4673	1.7903	2.1958	0.0069	0.7695	4.1793
8	1.2399	1.5453	1.9366	2.4409	0.0081	0.6843	4.1908
9	1.2716	1.6264	2.0927	2.7095	0.0093	0.6319	4.0245
10	1.3038	1.7106	2.2589	3.0033	0.0106	0.5839	3.9486

Table 6. (Continued)

				$a = 40,\ b = 0.75$			
r	$E(U)$	$E(U^2)$	$E(U^3)$	$E(U^4)$	$Variance$	$Skewness$	$Kurtosis$
1	1.0250	1.0513	1.0789	1.1079	0.0006	4.0121	10.0967
2	1.0502	1.1041	1.1622	1.2250	0.0013	2.0073	7.4807
3	1.0755	1.1586	1.2503	1.3517	0.0020	1.3871	5.4539
4	1.1009	1.2146	1.3431	1.4887	0.0026	1.0409	5.4705
5	1.1265	1.2724	1.4410	1.6366	0.0033	0.8596	4.7858
6	1.1523	1.3317	1.5440	1.7960	0.0040	0.7396	4.2586
7	1.1782	1.3928	1.6524	1.9676	0.0048	0.6544	3.9173
8	1.2042	1.4556	1.7663	2.1521	0.0055	0.5615	4.1696
9	1.2304	1.5201	1.8860	2.3502	0.0062	0.5139	3.9095
10	1.2567	1.5863	2.0115	2.5627	0.0070	0.4680	3.8344

2.1. Recursive Formulas for Sigle Moment

Theorem 2.2 Let U be a continuous random variable follows the Benktander Type II distribution as given in (4). For $1 \leq r < n,\ p = 1, 2, \ldots$ the following recurrence relations is satisfied

$$\begin{aligned} &E\left[U^p(r,n,m,k)\right] \\ &= \gamma_r\left\{\frac{a}{p+b}E\left[U^{p+b}(r,n,m,k)\right] + \frac{1-b}{p}E\left[U^p(r,n,m,k)\right]\right\} - \frac{(\gamma_r+1)C_{r-1}}{C_{r-1}^{(k+1,m)}} \\ &\left\{\frac{a}{p+b}E\left[U^{p+b}(r-1,n,m,k+1)\right] + \frac{1-b}{p}E\left[U^p(r-1,n,m,k+1)\right]\right\}. \end{aligned} \tag{18}$$

Proof: We have, from (9)

$$E\left[U^p(r,n,m,k)\right] = \frac{C_{r-1}}{(r-1)!}\int_1^\infty u^p[\bar{F}(u)]^{\gamma_r-1}f(u)g_m^{r-1}(F(u))du.$$

Using (6), we get

$$E\left[U^p(r,n,m,k)\right] = \frac{aC_{r-1}}{(r-1)!}I_1(u) + \frac{(1-b)C_{r-1}}{(r-1)!}I_2(u). \tag{19}$$

where

$$I_1(u) = \int_1^\infty u^{p+b-1}[\bar{F}(u)]^{\gamma_r}g_m^{r-1}(F(u))du, \quad I_2(u) = \int_1^\infty u^{p-1}[\bar{F}(u)]^{\gamma_r}g_m^{r-1}(F(u))du.$$

When we integrate in parts, using u^{p+b-1} for integrating and the remaning terms for differentiation, we get

$$\begin{aligned} I_1(u) =& \frac{\gamma_r}{p+b}\int_1^\infty u^{p+b}[\bar{F}(u)]^{\gamma_{r-1}}f(u)g_m^{r-1}(F(u))du \\ &- \frac{r-1}{p+b}\int_1^\infty u^{p+b}[\bar{F}(u)]^{\gamma_r+m}f(u)g_m^{r-2}(F(u))du. \end{aligned}$$

Similarly,

$$I_2(u) = \frac{\gamma_r}{p}\int_1^\infty u^p[\bar{F}(u)]^{\gamma_{r-1}}f(u)g_m^{r-1}(F(u))du - \frac{r-1}{p}\int_1^\infty u^p[\bar{F}(u)]^{\gamma_r+m}f(u)g_m^{r-2}(F(u))du.$$

Substituting $I_1(u),\ I_2(u)$ in (19), we get

$$\begin{aligned}E\left[U^p(r,n,m,k)\right]=&\frac{aC_{r-1}\gamma_r}{(p+b)(r-1)!}\int_1^{\infty}u^{p+b}[\bar{F}(u)]^{\gamma_{r-1}}f(u)g_m^{r-1}(F(u))du\\&-\frac{aC_{r-1}(r-1)}{(p+b)(r-1)!}\int_1^{\infty}u^{p+b}[\bar{F}(u)]^{\gamma_r+m}f(u)g_m^{r-2}(F(u))du\\&+\frac{(1-b)C_{r-1}\gamma_r}{p(r-1)!}\int_1^{\infty}u^{p}[\bar{F}(u)]^{\gamma_{r-1}}f(u)g_m^{r-1}(F(u))du\\&-\frac{(1-b)C_{r-1}(r-1)}{p(r-1)!}\int_1^{\infty}u^{p}[\bar{F}(u)]^{\gamma_r+m}f(u)g_m^{r-2}(F(u))du.\end{aligned}\tag{20}$$

After simplification (20), we get (18).

Special Cases

a. Putting $m=0$ and $k=1$ in equation (18) the recurrence relation is reduces to the recurrence relation for single moment of order statistics.

$$\begin{aligned}E\left[U_{r:n}^p\right]=&(n-r+1)\left\{\frac{a}{p+b}E\left[U_{r:n}^{p+b}\right]+\frac{1-b}{p}E\left[U_{r:n}^p\right]\right\}-\frac{(n-r+1)(n-r+2)}{n+1}\\&\left\{\frac{a}{p+b}E\left[U_{r-1:n+1}^{p+b}\right]+\frac{1-b}{p}E\left[U_{r-1:n+1}^{p}\right]\right\}.\end{aligned}$$

b. The recurrence relation provided in equation (18) becomes the recurrence relation for single moment of the $k-th$ record values when $m=-1$

$$\begin{aligned}E\left[(U_{U'_{(r)}}^{(k)})^p\right]=&k\left\{\frac{a}{p+b}E\left[(U_{U'_{(r)}}^{(k)})^{p+b}\right]+\frac{1-b}{p}E\left[(U_{U'_{(r)}}^{(k)})^p\right]\right\}-\frac{k^r}{(k+1)^{r-1}}\\&\left\{\frac{a}{p+b}E\left[(U_{U'_{(r)}}^{(k+1)})^{p+b}\right]+\frac{1-b}{p}E\left[(U_{U'_{(r)}}^{(k+1)})^p\right]\right\}.\end{aligned}$$

2.2. Recursive Formulas for Product Moments

Theorem 3.1.1 Let U be a continuous random variable follows Benktander Type II distribution as given in (4). For $1\le r<s\le n-1$ and $p,\ j=1,2,\dots$ the following recurrence relation satisfied

$$\begin{aligned}&E\left[U^p(r,n,m,k)U^j(s,n,m,k)\right]\\&=\gamma_s\left\{\frac{a}{b+j}E\left[U^p(r,n,m,k)U^{b+j}(s,n,m,k)\right]+\frac{1-b}{j}\right.\\&\left.E\left[U^p(r,n,m,k)U^j(s,n,m,k)\right]\right\}-\frac{(\gamma_s+1)C_{s-1}}{C_{s-1}^{(k+1,m)}}\\&\left\{\frac{a}{b+j}E\left[U^p(r-1,n,m,k+1)U^{b+j}(r-1,n,m,k+1)\right]\right.\\&\left.+\frac{1-b}{j}E\left[U^p(r-1,n,m,k+1)U^j(r-1,n,m,k+1)\right]\right\}.\end{aligned}\tag{21}$$

Proof: From (3), we have

$$\begin{aligned}&E\left[U^p(r,n,m,k)U^j(s,n,m,k)\right]\\&=\frac{C_{s-1}}{(r-1)!(s-r-1)!}\int_1^{\infty}\int_u^{\infty}u^pv^j[\bar{F}(u)]^m g_m^{r-1}(F(u))\\&\times\left[h_m(F(v))-h_m(F(u))\right]^{s-r-1}\left[\bar{F}(v)\right]^{\gamma_s-1}f(u)f(v)dvdu.\end{aligned}\tag{22}$$

Using (6), we get

$$
\begin{aligned}
&E\left[U^p(r,n,m,k)U^j(s,n,m,k)\right]\\
&=\frac{aC_{s-1}}{(r-1)!(s-r-1)!}\int_1^\infty u^p[\bar{F}(u)]^m f(u)g_m^{r-1}(F(u))I_1(v)du\\
&+\frac{(1-b)C_{s-1}}{(r-1)!(s-r-1)!}\int_1^\infty u^p[\bar{F}(u)]^m f(u)g_m^{r-1}(F(u))I_2(v)du.
\end{aligned}
\tag{23}
$$

where

$$
I_1(v)=\int_u^\infty v^{b+j-1}\left[h_m(F(v))-h_m(F(u))\right]^{s-r-1}\left[\bar{F}(v)\right]^{\gamma_s}dv
$$

and

$$
I_2(v)=\int_u^\infty v^{j-1}\left[h_m(F(v))-h_m(F(u))\right]^{s-r-1}\left[\bar{F}(v)\right]^{\gamma_s}dv.
$$

Integrating by parts treating v^{b+j-1} for integration and rest for differentiation, we get

$$
\begin{aligned}
I_1(v)=&\frac{\gamma_s}{b+j}\int_u^\infty v^{b+j}\left[h_m(F(v))-h_m(F(u))\right]^{s-r-1}\left[\bar{F}(v)\right]^{\gamma_s-1}f(v)dv-\frac{(s-r-1)}{b+j}\\
&\times\int_u^\infty v^{b+j}\left[h_m(F(v))-h_m(F(u))\right]^{s-r-2}\left[\bar{F}(v)\right]^{\gamma_s+m}f(v)dv.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
I_2(v)=&\frac{\gamma_s}{j}\int_u^\infty v^{j}\left[h_m(F(v))-h_m(F(u))\right]^{s-r-1}\left[\bar{F}(v)\right]^{\gamma_s-1}f(v)dv-\frac{(s-r-1)}{j}\\
&\times\int_u^\infty v^{j}\left[h_m(F(v))-h_m(F(u))\right]^{s-r-1}\left[\bar{F}(v)\right]^{\gamma_s+m}f(v)dv.
\end{aligned}
$$

Substituting $I_1(v)$ and $I_2(v)$ in (23), we get

$$
\begin{aligned}
&E\left[U^p(r,n,m,k)U^j(s,n,m,k)\right]\\
&=\frac{aC_{s-1}\gamma_s}{(b+j)(r-1)!(s-r-1)!}\int_1^\infty\int_u^\infty u^p v^{b+j}[\bar{F}(u)]^m g_m^{r-1}(F(u))\\
&\times\left[h_m(F(v))-h_m(F(u))\right]^{s-r-1}\left[\bar{F}(v)\right]^{\gamma_s-1}f(u)f(u)dvdu\\
&-\frac{aC_{s-1}(s-r-1)}{(b+j)(r-1)!(s-r-1)!}\int_1^\infty\int_u^\infty u^p v^{b+j}[\bar{F}(u)]^m g_m^{r-1}(F(u))\\
&\times\left[h_m(F(v))-h_m(F(u))\right]^{s-r-2}\left[\bar{F}(v)\right]^{\gamma_s+m}f(u)f(v)dvdu\\
&+\frac{(1-b)C_{s-1}\gamma_s}{j(r-1)!(s-r-1)!}\int_1^\infty\int_u^\infty u^p v^{j}[\bar{F}(u)]^m g_m^{r-1}(F(u))\\
&\times\left[h_m(F(v))-h_m(F(u))\right]^{s-r-1}\left[\bar{F}(v)\right]^{\gamma_s-1}f(u)f(v)dvdu\\
&-\frac{(1-b)C_{s-1}\gamma_s}{j(r-1)!(s-r-1)!}\int_1^\infty\int_u^\infty u^p v^{j}[\bar{F}(u)]^m g_m^{r-1}(F(u))\\
&\times\left[h_m(F(v))-h_m(F(u))\right]^{s-r-2}\left[\bar{F}(v)\right]^{\gamma_s+m}f(u)f(v)dvdu.
\end{aligned}
\tag{24}
$$

After simplification (24), we get (21).

Special cases:
a. The derivation of the recurrence relation for product moments of order statistics is achieved by establishing $m = 0$ and $k = 1$ in equation (21) instead

$$\begin{aligned}
&E\left[U^p_{r:n}U^j_{s:n}\right]\\
&= (n-s+1)\left\{\frac{a}{b+j}E\left[U^p_{r:n}U^{b+j}_{s:n}\right] + \frac{1-b}{j}E\left[U^p_{r:n}U^j_{s:n}\right]\right\} - \frac{(n-s+1)(n-s+2)}{n+1}\\
&\times\left\{\frac{a}{b+j}E\left[U^p_{r:n+1}U^{b+j}_{s-1:n+1}\right] + \frac{1-b}{j}E\left[U^p_{r:n+1}U^j_{s-1:n+1}\right]\right\}.
\end{aligned}$$

b. Recurrence relation described in (21) is reduces to recurrence relation for product moments of $k-th$ record values when $m = -1$ is used

$$\begin{aligned}
&E\left[(U^{(k)}_{U'_{(r)}})^p(U^{(k)}_{U'_{(s)}})^j\right]\\
&= k\left\{\frac{a}{b+j}E\left[(U^{(k)}_{U'_{(r)}})^p(U^{(k)}_{U'_{(s)}})^{b+j}\right] + \frac{1-b}{j}E\left[(U^{(k)}_{U'_{(r)}})^p(U^{(k)}_{U'_{(s)}})^j\right]\right\} - \frac{k^s}{(k+1)^{s-1}}\\
&\times\left\{\frac{a}{b+j}E\left[(U^{(k+1)}_{U'_{(r)}})^p(U^{(k+1)}_{U'_{(s-1)}})^{b+j}\right] + \frac{1-b}{j}E\left[(U^{(k+1)}_{U'_{(r)}})^p(U^{(k+1)}_{U'_{(s-1)}})^j\right]\right\}.
\end{aligned}$$

3. Characterization

Theorem 4.1 Let U be a continuous random variable as given in (4) and (5) then the following relation is satisfied

$$\begin{aligned}
&E\left[U^p(r,n,m,k)\right]\\
&= \gamma_r\left\{\frac{a}{p+b}E\left[U^{p+b}(r,n,m,k)\right] + \frac{1-b}{p}E\left[U^p(r,n,m,k)\right]\right\} - \frac{(\gamma_r+1)C_{r-1}}{C^{(k+1,m)}_{r-1}}\\
&\left\{\frac{a}{p+b}E\left[U^{p+b}(r-1,n,m,k+1)\right] + \frac{1-b}{p}E\left[U^p(r-1,n,m,k+1)\right]\right\},
\end{aligned} \tag{25}$$

If and only if

$$F(u) = 1 - U^{b-1}e^{\frac{a(1-u^b)}{b}}, \quad u \geq 1, \quad a > 0, \quad 0 < b \leq 1.$$

Proof: From Theorem 2.1, the necessary part immediately follows. Now, if relation in (25) is satisfied, then using (2) in (25), we get

$$\begin{aligned}
&\frac{C_{r-1}}{(r-1)!}\int_1^\infty u^p[\bar{F}(u)]^{\gamma_r-1}f(u)g_m^{r-1}(F(u))du\\
&= \frac{aC_{r-1}\gamma_r}{(p+b)(r-1)!}\int_1^\infty u^{p+b}[\bar{F}(u)]^{\gamma_r-1}f(u)g_m^{r-1}(F(u))du\\
&- \frac{aC_{r-1}(r-1)}{(p+b)(r-1)!}\int_1^\infty u^{p+b}[\bar{F}(u)]^{\gamma_r+m}f(u)g_m^{r-2}(F(u))du\\
&+ \frac{(1-b)C_{r-1}\gamma_r}{p(r-1)!}\int_1^\infty u^p[\bar{F}(u)]^{\gamma_r-1}f(u)g_m^{r-1}(F(u))du\\
&- \frac{(1-b)C_{r-1}(r-1)}{p(r-1)!}\int_1^\infty u^p[\bar{F}(u)]^{\gamma_r+m}f(u)g_m^{r-2}(F(u))du.
\end{aligned} \tag{26}$$

Equation (26) can be written as

$$\begin{aligned}
&\frac{C_{r-1}}{(r-1)!}\int_1^\infty u^p[\bar{F}(u)]^{\gamma_r-1}f(u)g_m^{r-1}(F(u))du\\
&=\frac{aC_{r-1}}{(p+b)(r-1)!}\int_1^\infty u^{p+b}g_m^{r-1}(F(u))\left(-\frac{d}{du}[\bar{F}(u)]^{\gamma_r}\right)du\\
&-\frac{aC_{r-1}(r-1)}{(p+b)(r-1)!}\int_1^\infty u^{p+b}[\bar{F}(u)]^{\gamma_r+m}f(u)g_m^{r-2}(F(u))du\\
&+\frac{(1-b)C_{r-1}}{p(r-1)!}\int_1^\infty u^p g_m^{r-1}(F(u))\left(-\frac{d}{du}[\bar{F}(u)]^{\gamma_r}\right)du\\
&-\frac{(1-b)C_{r-1}(r-1)}{p(r-1)!}\int_1^\infty u^p[\bar{F}(u)]^{\gamma_r+m}f(u)g_m^{r-2}(F(u))du.
\end{aligned}\tag{27}$$

Now, integrating first and third term of R.H.S of (27) and after simplification, we get

$$\frac{C_{r-1}}{(r-1)!}\int_1^\infty u^p[\bar{F}(u)]^{\gamma_r-1}g_m^{r-1}(F(u))\left(f(u)-au^{b-1}\overline{F}(u)-\frac{(1-b)\overline{F}(u)}{u}\right)du=0. \tag{28}$$

Subsequently, employing a generalized form of the Muntz-Szasz theorem [31] in (28), we get

$$f(u)-au^{b-1}\overline{F}(u)-\frac{(1-b)\overline{F}(u)}{u}=0$$

$$f(u)=au^{b-1}\overline{F}(u)+\frac{(1-b)\overline{F}(u)}{u}$$

$$\frac{f(u)}{\overline{F}(u)}=\frac{au^b-b+1}{u}$$

which prove that,

$$F(u)=1-u^{b-1}e^{\frac{a(1-u^b)}{b}},\quad u\geq 1,\quad a>0,\quad 0<b\leq 1.$$

4. Parameter Estimation

4.1. MLEs based on Generalized Order Statistics

In this segment, we derive maximum likelihood estimators (*MLEs*) for the parameters that are not known. Let $U(1,n,m,k),U(2,n,m,k),\dots,U(n,n,m,k)$ be the m-*gos*. In view of (1), the likelihood function is expressed as

$$\begin{aligned}
L(a,\ b/u)=&k\left(\prod_{j=1}^{n-1}\gamma_j\right)\left(\prod_{i=1}^{n-1}\left(u_i^{b-1}e^{\frac{a(1-u_i^b)}{b}}\right)^m e^{\frac{a(1-u_i^b)}{b}}u_i^{b-2}(au_i^b-b+1)\right)\\
&\times\left(\left(u_n^{b-1}e^{\frac{a(1-u_n^b)}{b}}\right)^{k-1}e^{\frac{a(1-u_n^b)}{b}}u_n^{b-2}(au_n^b-b+1)\right)
\end{aligned}$$

After simplification, applying a logarithmic transformation to both sides, the resulting expression is

$$\begin{aligned}
\ln L(a,\ b/u)=&\ln k+\sum_{j=1}^{n-1}\ln\gamma_j+(m(b-1)+b-2)\sum_{i=1}^{n-1}\ln u_i+\frac{a(m+1)(n-1)}{b}-\frac{a(m+1)}{b}\sum_{i=1}^{n-1}u_i^b\\
&+\sum_{i=1}^{n}\ln(au_i^b-b+1)+(bk-k-1)\ln u_n+\frac{ka}{b}-\frac{kau_n^b}{b}.
\end{aligned}\tag{29}$$

$$\frac{\partial \ln L(a,\ b/u)}{\partial a} = \frac{(m+1)(n-1)}{b} - \frac{(m+1)}{b}\sum_{i=1}^{n-1} u_i^b + \sum_{i=1}^{n}\left(\frac{u_i^b}{(au_i^b - b + 1)}\right) + \frac{k}{b} - \frac{ku_n^b}{b}. \tag{30}$$

$$\begin{aligned}\frac{\partial \ln L(a,\ b/u)}{\partial b} =& (m+1)\sum_{i=1}^{n-1} \ln u_i - \frac{a(m+1)(n-1)}{b^2} + \frac{a(m+1)}{b^2}\sum_{i=1}^{n-1} u_i^b - \frac{a(m+1)}{b}\sum_{i=1}^{n-1} u_i^b \ln u_i \\ &+ a\sum_{i=1}^{n}\left(\frac{u_i^b \ln u_i - 1}{(au_i^b - b + 1)}\right) + k\ln u_n - \frac{ka}{b^2} + \frac{kau_n^b}{b^2} - \frac{ka}{b}u_n^b \ln u_n.\end{aligned} \tag{31}$$

Special Cases

a. For $m = 0$ and $k = 1$ the *gos* reduced to the order statistics. Hence, the equations (29 - 31) can be employed to compute the *MLEs* of the parameters within the order statistics framework, expressed as

$$\begin{aligned}\ln L(a,\ b/u) =& \sum_{j=1}^{n-1} \ln(n-j+1) + (b-2)\sum_{i=1}^{n-1} \ln u_i + \frac{a(n-1)}{b} - \frac{a}{b}\sum_{i=1}^{n-1} u_i^b \\ &+ \sum_{i=1}^{n} ln(au_i^b - b + 1) + (b-2)lnu_n + \frac{a}{b} - \frac{au_n^b}{b}.\end{aligned}$$

$$\frac{\partial \ln L(a,\ b/u)}{\partial a} = \frac{n-1}{b} - \frac{1}{b}\sum_{i=1}^{n-1} u_i^b + \sum_{i=1}^{n}\left(\frac{u_i^b}{(au_i^b - b + 1)}\right) + \frac{1}{b} - \frac{u_n^b}{b}.$$

$$\begin{aligned}\frac{\partial \ln L(a,\ b/u)}{\partial b} =& \sum_{i=1}^{n-1} \ln u_i - \frac{a(n-1)}{b^2} + \frac{a}{b^2}\sum_{i=1}^{n-1} u_i^b - \frac{a}{b}\sum_{i=1}^{n-1} u_i^b \ln u_i \\ &+ \sum_{i=1}^{n}\left(\frac{au_i^b \ln u_i - 1}{(au_i^b - b + 1)}\right) + \ln u_n - \frac{a}{b^2} + \frac{au_n^b}{b^2} - \frac{a}{b}u_n^b \ln u_n.\end{aligned}$$

b. For $m = -1$ and $k = 1$ the *gos* reduced to k-th upper record values.Therefore, the equations (29 - 31) find applicability in deriving the *MLEs* of the parameters through the utilization of upper record values.

$$\ln L(a,\ b/u) = -\sum_{i=1}^{n-1} \ln u_i + \sum_{i=1}^{n} \ln(au_i^b - b + 1) + (b-1)\ln u_n + \frac{a}{b} - \frac{au_n^b}{b}.$$

$$\frac{\partial \ln L(a, b/u)}{\partial a} = \sum_{i=1}^{n}\left(\frac{u_i^b}{(au_i^b - b + 1)}\right) + \frac{1}{b} - \frac{u_n^b}{b}.$$

$$\frac{\partial \ln L(a,\ b/u)}{\partial b} = \sum_{i=1}^{n}\left(\frac{au_i^b \ln u_i - 1}{(au_i^b - b + 1)}\right) + \ln u_n - \frac{a}{b^2} + \frac{au_n^b}{b^2} - \frac{a}{b}u_n^b \ln u_n.$$

5. Simulation Study

This section presents a simulation study for all the mathematical findings derived in the preceding section. Numerical computations of mathematical outcomes are documented for specific cases of *gos* like order statistics and record values. To compute numerical outcomes for diverse sample sizes of order statistics and record values across various parameter configurations.

We calculate and display the moments, variances, skewness, and kurtosis of *gos*, order statistics, and record values in Table 2, 3, 4, 5 and 6 respectively. From Table 2 and 3 it can be seen clearly that for large sample size the variance of distribution is decreasing.

Again, it can be seen from Table 6, variance for large sample size, is increasing in case of record data. The contents of Table 7 and 8 pertain to the maximum likelihood (M.L.) estimates of unknown parameters in the context of the Benktander Type II distribution. Specifically, these tables provide information on the estimators employed for the unknown parameters are evaluated in terms of their average estimate (A.E.), average bias (A.B.) and mean squared error (M.S.E.). The algorithm utilized to generate the data in Table 7 and 8 are described as follows.
1) We produce a random sample of size "n" from the Benktander Type II distribution, using a predefined value of an unknown parameter (referred to as "a").
2) The maximum likelihood estimate, denoted as $\hat{a}$, for the unknown parameter a is computed using the random sample generated in step (1) and employing the method described in Section 5.
3) The aforementioned process is iterated 1000 times, resulting in 1000 values of $\hat{a}$, denoted as $\hat{a_1}$, $\hat{a_2}$, $\hat{a_3}$,...,$\hat{a_{1000}}$.
4) The obtained values of $\hat{a_i}$, where i takes values from 1 to 1000, are utilized to calculate the average estimate (A.E.), average bias (A.B.), and mean squared error (M.S.E.) using the following equations.

$$A.E. = \frac{1}{1000}\sum_{i=1}^{1000}\hat{a_i}, \quad A.B. = \frac{1}{1000}\sum_{i=1}^{1000}(\hat{a_i} - a), \quad M.S.E. = \frac{1}{1000}\sum_{i=1}^{1000}(\hat{a_i} - a)^2.$$

where $\hat{a_i}$ is the M.L. estimate of a at i-th repetition.

Table 7 reports A.E., A.B. and M.S.E. of M.L. estimates of unknown parameters based on order statistics for (a, b) $\in \{(1.5,\ 0.50), (2.0,\ 0.5), (2.5,\ 0.5)\}$ and n $\in \{15, 30, 45, 60, 75, 90, 105, 120, 135, 150\}$. Table 8 reports A.E., A.B. and M.S.E. of M.L. estimates of unknown parameters based on record data for (a, b) $\in \{(1.0,\ 0.5), (0.5,\ 0.5), (0.5,\ 0.8)\}$ and n $\in \{3, 4, 5, 6, 7, 8\}$.

Table 7. Estimates of parameters of the Benktander Type II distribution based on order statistics.

n	a			b		
	A.E.	A.B.	M.S.E.	A.E.	A.B.	M.S.E.
$(a,\ b) = (1.5,\ 0.5)$						
15	1.68554	0.19904	0.40757	0.66840	0.17290	0.16522
30	1.59092	0.11342	0.17213	0.61949	0.12699	0.13039
45	1.53988	0.07438	0.09814	0.57812	0.08962	0.10673
60	1.50898	0.05848	0.07156	0.56348	0.07998	0.08953
75	1.49075	0.04775	0.05502	0.56263	0.08163	0.07994
90	1.47001	0.03901	0.04393	0.55340	0.07640	0.06720
105	1.45698	0.03648	0.03715	0.54328	0.06978	0.06010
120	1.45382	0.03482	0.03224	0.53692	0.06392	0.05337
135	1.43852	0.03152	0.02706	0.53052	0.06152	0.04934
150	1.42181	0.02831	0.02320	0.51681	0.05231	0.04344

Table 7. (Continued)

n	a			b		
	A.E.	A.B.	M.S.E.	A.E.	A.B.	M.S.E.
			$(a,\ b) = (2.0,\ 0.5)$			
15	2.23678	0.23878	0.65196	0.67472	0.17522	0.17921
30	2.13305	0.13505	0.27871	0.63079	0.13129	0.14935
45	2.08128	0.08728	0.16158	0.58888	0.09038	0.12824
60	2.05981	0.06581	0.11855	0.57522	0.07672	0.11387
75	2.04806	0.05206	0.09208	0.57664	0.07764	0.10510
90	2.03440	0.04040	0.07448	0.57139	0.07289	0.09323
105	2.02870	0.03670	0.06306	0.56326	0.06526	0.08525
120	2.02914	0.03514	0.05466	0.55878	0.06028	0.07669
135	2.02189	0.02989	0.04662	0.55601	0.05801	0.07215
150	2.01318	0.02518	0.04052	0.54446	0.04746	0.06437
			$(a,\ b) = (2.5,\ 0.5)$			
15	2.77643	0.28143	0.94524	0.67814	0.17914	0.18735
30	2.65838	0.16088	0.40500	0.63655	0.13705	0.16156
45	2.60226	0.10476	0.23663	0.59334	0.09384	0.14236
60	2.57624	0.07874	0.17312	0.58021	0.08071	0.12925
75	2.55934	0.06184	0.13479	0.58165	0.08215	0.12066
90	2.54791	0.04791	0.10909	0.57771	0.07771	0.11093
105	2.54061	0.04311	0.09244	0.57006	0.07056	0.10250
120	2.54108	0.04108	0.08023	0.56535	0.06535	0.09287
135	2.53412	0.03412	0.06887	0.56377	0.06377	0.08907
150	2.52121	0.02871	0.05987	0.55165	0.05315	0.08005

Table 8. Estimates of parameters of the Benktander Type II distribution based on record values.

n	a			b		
	A.E.	A.B.	M.S.E.	A.E.	A.B.	M.S.E.
			$(a,\ b) = (1.0,\ 0.50)$			
3	1.59423	3.41775	0.59424	0.83557	0.21125	0.33557
4	1.29301	1.69618	0.29301	0.77653	0.19174	0.27653
5	1.16439	0.88039	0.16439	0.74338	0.17207	0.24338
6	1.09117	0.63812	0.09118	0.70984	0.15481	0.20984
7	1.04939	0.53379	0.04940	0.68195	0.14328	0.18195
8	1.04050	0.50910	0.04050	0.64193	0.13354	0.14193
			$(a,\ b) = (0.50,\ 0.50)$			
3	0.86711	1.35401	0.36712	0.82396	0.19666	0.32396
4	0.66751	0.63383	0.16752	0.76364	0.17339	0.26364
5	0.58929	0.31196	0.08929	0.72679	0.15142	0.22679
6	0.55193	0.22918	0.05193	0.69508	0.13112	0.19508
7	0.53157	0.19604	0.03157	0.66204	0.11854	0.16204
8	0.53397	0.19266	0.03397	0.62439	0.10708	0.12439
			$(a,\ b) = (0.50,\ 0.80)$			
3	0.77791	0.70439	0.27791	0.87590	0.0738	0.07590
4	0.66963	0.36367	0.16964	0.84786	0.07712	0.04786
5	0.63124	0.21564	0.13124	0.84571	0.06618	0.04571
6	0.60676	0.17309	0.10676	0.84244	0.06101	0.04244
7	0.59274	0.15562	0.09274	0.83199	0.05917	0.03199
8	0.58574	0.15192	0.08574	0.82053	0.0567	0.02053

6. Results and Discussion

Our study applied *gos* to thoroughly investigate the moment properties of the Benktander Type II distribution, providing a comprehensive analysis through various statistical measures and parameters. We systematically examined these properties across different sampling methods, as detailed in a series of tables that captured essential statistical characteristics. Table 1 laid the groundwork by presenting the different variants of *gos*, offering a foundational understanding for the subsequent analyses. Table 2 and 3 focused on the mean and variance of order statistics and *gos* for the Benktander Type II distribution, respectively, using sample sizes (n) and ranks (r) ranging from 1 to 8, with parameters $a = 3.5$ and $b = 0.50$. Additionally, Table 3 incorporated *gos*-specific parameters $m = 1$ and $k = 2$. A key observation from these tables was the decreasing trend in variance with increasing sample size, which aligns with statistical theory and indicates that larger samples yield more precise parameter estimates, thereby enhancing the reliability of our findings. The analysis was extended to higher-order moments in Table 4 and 5, which presented skewness and kurtosis for order statistics and *gos*, respectively. These tables utilized the same parameter values and sample sizes as Table 2 and 3, providing a detailed view of the distribution's shape characteristics across different sample sizes and ranks. Table 6 shifted the focus to the moments and characteristics of the Benktander Type II distribution based on record data. Interestingly, we observed an increasing trend in variance with larger sample sizes in this context, contrasting with the decreasing variance observed for order statistics and *gos*. This divergence underscores the unique properties of record data and highlights the importance of considering different sampling methods when analyzing distributional properties. Finally, Table 7 provided a detailed evaluation of the *MLEs* of the unknown parameters a and b for the Benktander Type II distribution based on order statistics and record values, respectively. We assessed the estimator's performance in terms of their A.E., A.B., and M.S.E..

The observed trends in variance decreasing for order statistics, and increasing for record data provide valuable insights into the behavior of the Benktander Type II distribution under various sampling conditions. The results presented in Table 7 offer practical tools for researchers and practitioners by enabling informed decisions regarding the most suitable estimation approach for specific applications. This study contributes to a deeper understanding of the moment properties of the Benktander Type II distribution and establishes a robust framework for parameter estimation using diverse sampling techniques. Future research could explore the application of these findings in real-world scenarios, such as risk modeling in insurance or finance, where the Benktander Type II distribution may prove particularly useful.

7. Conclusions

In this research paper, we have successfully derived explicit expressions for single moments for *gos*. Specifically, we have obtained these expressions for the general case as well as for special cases involving order statistics and record values. Furthermore, we have developed recurrence relations for calculating higher-order moments of both single and product moments, and these recurrence relations have been explored in the context of order statistics and record values as well. Additionally, we have presented the mathematical formulation required to determine the maximum likelihood estimates of the unknown parameters a and b for the aforementioned cases of order statistics and record values. For the calculation of moments, we employed `MATLAB`, and for the maximum likelihood estimation, we used the `nleqslv` package in **R**. These tools significantly enhanced the precision and robustness of our parameter estimates. Furthermore, we have simplified this formulation for the special cases of interest. Moreover, the paper encompasses diverse findings related to the characterization of the distribution. Notably, we have conducted a comprehensive simulation study to calculate the maximum likelihood estimates, biases, mean squared errors of parameters, and moments of the distribution.This study was performed for different sample sizes and parameter choices. Furthermore, there is an observation that as the sample size increases, the biases and mean squared errors of parameters decrease.

7.1. Contributions

Overall, the results presented in this paper contribute to a deeper understanding of the moment properties and estimation techniques for the Benktander Type II distribution within the context of *gos*, order statistics, and record values. These findings have important implications for the statistical analysis and modeling of distributions with similar characteristics, providing a robust foundation for future research and applications.

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