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A weighted full-Newton step primal-dual interior point algorithm for convex quadratic optimization

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Abstract In this paper, a new weighted short-step primal-dual interior point algorithm for convex quadratic optimization (CQO) problems is presented. The algorithm uses at each interior point iteration only full-Newton steps and the strategy of the central path to obtain an ϵ-approximate solution of CQO. This algorithm yields the best currently well-known theoretical iteration bound, namely, $O(\sqrt{n}\log\frac{n}{\epsilon})$ which is as good as the bound for the linear optimization analogue.

Keywords Convex quadratic optimization; weighted interior point methods; short-step primal-dual algorithms; complexity of algorithms

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1. Introduction

Consider the quadratic optimization (QO) problem in standard format:

$$(P) \quad \min_{x}\left\{c^Tx+\frac{1}{2}x^TQx\;:Ax=b,\,x\geq 0\right\}$$

and its dual problem

$$(D) \quad \max_{x,\,y,\,z}\left\{b^Ty-\frac{1}{2}x^TQx:A^Ty+z-Qx=c,z\geq 0\right\},$$

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where Q is a given $(n \times n)$ real symmetric matrix, A is a given $(m \times n)$ real matrix with rank $(A) = m, c \in \mathbf{R}^n$, $b \in \mathbf{R}^m$, $x \in \mathbf{R}^n$, $z \in \mathbf{R}^n$ and $y \in \mathbf{R}^m$.
The QO problems have many important applications in optimization and mathematical programming problems.

There are a variety of solution approaches for CQO which have been studied intensively. Among them, the interior-point methods (IPMs) gained more attention than others methods. Feasible primal-dual path-following methods are the most attractive methods of IPMs [7, 9]. Their derived algorithms achieved important results such as polynomial complexity and numerical efficiency. These algorithms trace approximately the so-called central-path which is a curve that lies in the feasible region of the considered problem and they reach an optimal solution of it. However, in practice these methods don't always find a strictly feasible centered point to starting their derived algorithms. So, it is worth analyzing other cases when the starting points are not centered. Thus leads to the concept of Target-Following IPMs introduced by Jansen et al.,[6] as a generalization of the classical path-following methods. These methods are based on the observation that with every algorithm which follows the central-path we associate a target sequence on the central-path. Weighted path-following methods can be viewed as a particular case of it. These methods were studied extensively by many authors [3, 4, 5, 7, 8] for Linear optimization (LO) and linear complementarity problem (LCP). Recently, Achache and Khebchache [1], introduced a new weighted method for monotone LCP where the complexity of the corresponding short-step algorithm is $O(\sqrt{n} \log \frac{n}{\epsilon})$. Motivated by their work, we propose a new weighted primal-dual path-following algorithm for solving CQO. The algorithm uses at each interior point iteration only weighted full-Newton steps and the strategy of the central path to get an $\epsilon-$approximate solution of CQO. We prove that the short-step algorithm has the following iteration bound $O(\sqrt{n} \log \frac{n}{\epsilon})$ which is as good as the bound for LO [3, 7, 8], CQO [1, 3] and LCP [2, 7], analogue. The algorithm has advantages that no line searches is needed and it can start with a suitable starting point not necessarily centered.

The rest of the paper is built as follows. In Section 2, the weighted-path and the search direction are presented. The generic weighted primal-dual path-following algorithm for CQO is also stated. In Section 3, the analysis of the algorithm and the iteration bound with short-step method are presented. Finally, a conclusion and future remarks follow in Section 4.

The notation used in this paper is as follows. $\mathbf{R}^n$ denotes the space of n-dimensional real vectors and $\mathbf{R}^n_{++}$ is the set of all positive vectors of $\mathbf{R}^n$. Given $x, z \in \mathbf{R}^n_{++}$, their Hadamard product is $xz = (x_1 z_1, \ldots, x_n z_n)^T$. The expressions $\| u\| = \sqrt{u^T u}$ and $\| u\|_\infty = \max_i |u_i|$ denote the Euclidean and the maximum norms for a vector u, respectively. Let $x, z \in \mathbf{R}^n_{++}, \sqrt{x} = (\sqrt{x_1}, \ldots, \sqrt{x_n})^T$, $x^{-1} = (x_1^{-1}, \ldots, x_n^{-1})^T$ and $\frac{x}{z} = (\frac{x_1}{z_1}, \ldots, \frac{x_n}{z_n})^T$. Let $g(x)$ and $f(x)$, be two positive real valued functions, then $g(x) = O(f(x)) \Leftrightarrow g(x) \leq$

$kf(x)$ for some positive constant k. Finally, the vector of all ones and the identity matrix are denoted by e and I, respectively.

2. The weighted-path and the search direction

Throughout the paper, we make the following assumptions for QO.
Assumption 1. Interior Point Condition **(IPC)**. There exists a triplet of vectors (x^0, y^0, z^0) such that:

$$Ax^0 = b,\ x^0 > 0,\ A^T y + z^0 - Qx^0 = c,\ z^0 > 0.$$

Assumption 2. Positive semidefiniteness. The matrix Q is positive semidefinite, i.e., for all $v \in \mathbf{R}^n$, $v^T Qv \geq 0$.
Finding an approximate solution of *(P)* and *(D)* is equivalent to solving the following system of optimality conditions for *(P)* and *(D)*:

$$\begin{cases} Ax & = & b,\ x \geq 0, \\ A^T y + z - Qx & = & c,\ z \geq 0, \\ xz & = & 0. \end{cases} \tag{1}$$

The basic idea behind weighted primal-dual interior-point algorithm is to replace the third equation (*complementarity condition*) in (1) by the parametrized equation $xz = w$ with w is a positive vector in $\mathbf{R}^n$. Thus, we consider the following perturbed system:

$$\begin{cases} Ax & = & b,\ x \geq 0, \\ A^T y + z - Qx & = & c,\ z \geq 0, \\ xz & = & w. \end{cases} \tag{2}$$

Under **Assumption 1** and **Assumption 2**, the system (2) has a unique solution denoted by $(x(w), y(w), z(w))$ for all $w > 0$ [2]. The set

$$\{(x(w), y(w), z(w)) : w > 0\}$$

is called the weighted-path of problems *(P)* and *(D)*. If w goes to zero, then the limit of the weighted-path exists and since the limit point satisfies the complementarity condition, the limit yields an optimal solution for CQO. This limiting property of the weighted-path leads to the main idea of the iterative primal-dual methods for solving (2).

Remark 2.1
If $w = \mu e$ with $\mu > 0$, then the weighted-path coincides with the classical central-path.

Now, we proceed to describe a weighted full-Newton step produced by the algorithm for a given $w > 0$. Applying Newton's method for (2) for a given feasible point (x, y, z) then the Newton direction $(\Delta x, \Delta y, \Delta z)$ at this point is the unique solution of the following linear system of equations:

$$\begin{pmatrix} A & 0 & 0 \\ -Q & A^T & I \\ Z & 0 & X \end{pmatrix} \begin{pmatrix} \Delta x \\ \Delta y \\ \Delta z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ w - Xz \end{pmatrix}, \tag{3}$$

where X :=diag(x), Z :=diag(z).
Again under our assumptions and the fact that rank$(A) = m$, the system (3) has a unique solution $(\Delta x, \Delta y, \Delta z)$. Hence, a new weighted full-Newton iteration is constructed according to:

$$x_+ := x + \Delta x;\ y_+ := y + \Delta y;\ \text{and}\ z_+ = z + \Delta z. \tag{4}$$

To simplify the matters, we define the vectors:

$$v := \sqrt{xz} \text{ and } d := \sqrt{xz^{-1}}.$$

The vector d uses to scale the vectors x and z to the same vector v as

$$d^{-1}x = dz = v \tag{5}$$

and as well as for the original directions to the scaling directions:

$$d_x = d^{-1}\Delta x \text{ and } \ d_z = d\Delta z.$$

It follows that:

$$x\Delta z + z\Delta x = v(d_x + d_z), \tag{6}$$

and

$$d_x d_z = \Delta x \Delta z = \Delta x Q \Delta x \geq 0, \tag{7}$$

since Q is a semidefinite matrix.
Hence, by using (5), (6) and (7), the system (3) becomes:

$$\begin{pmatrix} \bar{A} & 0 & 0 \\ -\bar{Q} & \bar{A}^T & I \\ I & 0 & I \end{pmatrix} \begin{pmatrix} d_x \\ d_y \\ d_z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ p_v \end{pmatrix} \tag{8}$$

where

$$p_v = v^{-1}(w - v^2) \tag{9}$$

and $\bar{A} = DAD$ and $\bar{Q} = DQD$ with D :=diag(d).
In the next sub-section, we describe the generic feasible weighted primal-dual path-following algorithm to solve CQO.

2.1. The Algorithm

Similar to LO case, we define for any positive vector v and in view of (9), a norm-based proximity measure as follows:

$$\delta(v;w) = \frac{\|p_v\|}{2\sqrt{\min(w)}} = \frac{\left\|v^{-1}(w-v^2)\right\|}{2\sqrt{\min(w)}}. \tag{10}$$

One can easily verify that

$$\delta(v;w) = 0 \Leftrightarrow v^2 = w \Leftrightarrow xz = w.$$

Hence the value $\delta(v;w)$ is to measure the distance of a point (x, y, z) to the weighted-path $(x(w), y(w), z(w))$.
Let denote another measure $\sigma_C(w)$ as follows

$$\sigma_C(w) = \frac{\max(w)}{\min(w)}. \tag{11}$$

The role of $\sigma_C(w)$ is to measure the closeness of w to the central path.
Here,

$$\min(w) = \min_i(w_i)$$

and likewise

$$\max(w) = \max_i(w_i).$$

Note that in (11), $\sigma_C(w) \geq 1$, with equality if w is on the central-path.

Now we are ready to describe the generic weighted path-following interior-point algorithm for CQO as follows.

A generic weighted Primal-Dual Path-Following Algorithm for CQO

Input
A threshold parameter $0 < \delta < 1$(default $\delta = \frac{1}{\sqrt{2}}$);
an accuracy parameter $\epsilon > 0$;
a fixed barrier update parameter $0 < \theta < 1$ (default $\theta = \frac{1}{2\sqrt{n}\sigma_C(w^0)}$);
a starting point (x^0, y^0, z^0) and w^0 such that $\delta(x^0 z^0; w^0) \leq \frac{1}{\sqrt{2}}$;
begin
Set $x := x^0; y := y^0; z := z^0; w := w^0$;
while $x^T z \geq \epsilon$ **do**
begin
$w := (1-\theta)w$;
Solve system (3) to obtain the direction $(\Delta x,\ \Delta y,\ \Delta z)$;
Update $x := x + \Delta x,\ y := y + \Delta y, z := z + \Delta z$;
end
end

Algorithm 2.1

In the next section, we will show that Algorithm 2.1 is well-defined for the defaults $\theta = \frac{1}{2\sqrt{n}\sigma_C(w^0)}$ and $\delta \leq \frac{1}{\sqrt{2}}$ and can solve CQO in polynomial-time.

3. Complexity analysis

In the next lemma, we state some useful technical results that will be used later in the analysis of the algorithm.

Lemma 3.1
Let (d_x, d_z) be a solution of (8) and suppose $w > 0$. If $\delta := \delta(v; w)$. Then, one has

$$0 \leq d_x^T d_z \leq 2\delta^2 \min(w), \tag{12}$$

and

$$\|d_x d_z\|_\infty \leq \delta^2 \min(w) \text{ and } \|d_x d_z\| \leq \sqrt{2}\delta^2 \min(w). \tag{13}$$

Proof: Since $0 \leq d_x^T d_z$, the statement in (12) follows immediately from the following equality:

$$\|d_x\|^2 + \|d_z\|^2 + 2d_x^T d_z = \|d_x + d_z\|^2 = \|p_v\|^2 = 4\delta^2 \min(w).$$

For (13), (see Lemma C.4 in [7]), since

$$\|d_x d_z\|_\infty \leq \frac{1}{4} \|p_v\|^2 \text{ and } \|d_x d_z\| \leq \frac{1}{2\sqrt{2}} \|p_v\|^2 .$$

This completes the proof. □

The following lemma shows that the feasibility of the weighted full-Newton step under the condition $\delta := \delta(v; w) < 1$.

Lemma 3.2
Let (x, z) be a strictly feasible primal-dual point. Then $x_+ = x + \Delta x > 0$ and $z_+ = y + \Delta z > 0$ if and only if $w + d_x d_z > 0$.

Proof: For the first statement we have,

$$\begin{aligned} x_+ z_+ &= (x + \Delta x)(z + \Delta z) \\ &= xz + x\Delta z + z\Delta x + \Delta x \Delta z \\ &= xz + (w - xz) + \Delta x \Delta z \\ &= w + \Delta x \Delta z. \end{aligned}$$

Then from equation in (7), we have,

$$\begin{aligned} x_+ z_+ &= w + \Delta x \Delta z \\ &= w + d_x d_z. \end{aligned}$$

If the full-Newton step is strictly feasible $x_+ > 0$ and $z_+ > 0$ then $x_+ z_+ > 0$ and so $w + d_x d_z > 0$.
To show that x_+ and z_+ are positive, we introduce a step length $\alpha \in [0, 1]$ and we define

$$x^\alpha = x + \alpha \Delta x, \quad z^\alpha = z + \alpha \Delta z.$$

So $x^0 = x, x^1 = x_+$ and similar notations for z, hence $x^0 z^0 = xz > 0$. We have,

$$x^\alpha z^\alpha = (x + \alpha \Delta x)(z + \alpha \Delta z) = xz + \alpha(x \Delta z + z \Delta x) + \alpha^2 \Delta x \Delta z.$$

Now by using (6), we get

$$x^\alpha z^\alpha = xz + \alpha(w - xz) + \alpha^2 \Delta x \Delta z.$$

We assume that $w + d_x d_z > 0$, we deduce that $w + \Delta x \Delta z > 0$ which equivalent to $\Delta x \Delta z > -w$. Substitution we obtain

$$\begin{aligned} x^\alpha z^\alpha &> xz + \alpha(w - xz) - \alpha^2 w \\ &= (1 - \alpha) xz + (\alpha - \alpha^2) w \\ &= (1 - \alpha) xz + \alpha(1 - \alpha) w. \end{aligned}$$

Since xz and w are positive it follows that $x^\alpha z^\alpha > 0$ for $\alpha \in [0, 1]$. Hence, none of the entries of x^α and z^α vanish for $\alpha \in [0, 1]$. Since x^0 and z^0 are positive, this implies that $x^\alpha > 0$ and $z^\alpha > 0$ for $\alpha \in [0, 1]$. Hence, by continuity argument, the vectors x^α and z^α must be positive which proves that x_+ and z_+ are positive. This completes the proof. □

Lemma 3.3
If $\delta := \delta(v; w) < 1$. Then, the primal-dual full-Newton step is strictly feasible,i.e., $x_+ > 0$ and $z_+ > 0$.

Proof: In Lemma 3.2, we have seen that:

$$x_+ z_+ > 0 \text{ if } w + d_x d_z > 0.$$

So $w + d_x d_z > 0$ holds if

$$w_i + (d_x)_i (d_z)_i > 0, \text{ for all } i.$$

We have

$$w_i + (d_x)_i (d_z)_i \geq w_i - |(d_x)_i (d_z)_i| \geq \min(w) - \|d_x d_z\|_\infty \text{ for all } i.$$

Now, according to (13), Lemma 3.1, it follows that:

$$\min(w) - \|d_x d_z\|_\infty > \min(w)(1 - \delta^2).$$

Thus $w + d_x d_z > 0$ holds if $\delta < 1$. This completes the proof. $\square$

For convenience, we may write

$$v_+ = \sqrt{x_+ z_+}.$$

Lemma 3.4
If $\delta < 1$. Then

$$\left\|v_+^{-1}\right\| \le \frac{1}{\sqrt{\min(w)(1-\delta^2)}}.$$

Proof: It follows straightforwardly from Lemma 3.3 and since

$$v_+^{-2} = \frac{e}{w + d_x d_z}.$$

In the next lemma, we show the influence of a weighted full-Newton step on the proximity measure.

Lemma 3.5
If $\delta < 1$. Then

$$\delta_+ := \delta(v_+; w) \le \frac{\delta^2}{\sqrt{2(1-\delta^2)}}.$$

Proof: By definition, we have,

$$\begin{aligned} \delta_+ &= \frac{1}{2\sqrt{\min(w)}} \left\|v_+^{-1}(w - v_+^2)\right\| \\ &\le \frac{1}{2\sqrt{\min(w)}} \left\|v_+^{-1}\right\| \left\|w - v_+^2\right\|. \end{aligned}$$

But $w - v_+^2 = -d_x d_z$ and $v_+^{-1} = \frac{e}{\sqrt{w + d_x d_z}}$, then by Lemmas 3.1 and 3.4, we have,

$$\begin{aligned} \delta_+ &= \frac{1}{2\sqrt{\min(w)}} \left\| \frac{d_x d_z}{\sqrt{w + d_x d_z}} \right\| \\ &= \frac{1}{2\sqrt{\min(w)}} \frac{\|d_x d_z\|}{\left\|\sqrt{w + d_x d_z}\right\|} \\ &\le \frac{1}{2\sqrt{\min(w)}} \frac{\sqrt{2}\min(w)\delta^2}{\sqrt{\min(w) - \|d_x d_z\|_\infty}} \\ &\le \frac{1}{2\sqrt{\min(w)}} \frac{\sqrt{2}\min(w)\delta^2}{\sqrt{\min(w)(1-\delta^2)}} \\ &\le \frac{\delta^2}{\sqrt{2(1-\delta^2)}}. \end{aligned}$$

This completes the proof. $\square$

Corollary 3.1
If $\delta < 1$. Then $\delta_+ \leq \delta^2$ which indicates the convergence quadratic of the proximity when iterations are closed to the path. In addition if $\delta \leq \frac{1}{\sqrt{2}}$, then $\delta_+ \leq \frac{1}{2}$.

In the next lemma, we discuss the influence on the proximity measure of the update barrier parameter $w_+ = (1-\theta)\,w$ on the Newton process along the weighted-path.

Lemma 3.6
If $\delta(w;v) < 1$ and $w_+ = (1-\theta)\,w$ where $0 < \theta < 1$. Then

$$\delta(v_+;w_+) \leq \frac{\theta}{2\sqrt{1-\theta}\sqrt{1-\delta^2}}\sqrt{n}\sigma_C(w) + \frac{1}{\sqrt{2(1-\theta)}}\delta_+.$$

In addition, if $\delta \leq \frac{1}{\sqrt{2}}, \theta = \frac{1}{2\sqrt{n\sigma_C(w)}}$ and $n \geq 3$, then we have,

$$\delta(v_+;w_+) \leq \frac{1}{\sqrt{2}}.$$

Proof: Let $\delta(v_+;w_+)$ and $w_+ = (1-\theta)\,w$ with $0 < \theta < 1$. Then, by definition we have,

$$\begin{aligned}
\delta(v_+;w_+) &= \frac{1}{2\sqrt{\min(w_+)}}\left\|v_+^{-1}(w_+ - v_+^2)\right\| \\
&= \frac{1}{2\sqrt{1-\theta}\sqrt{\min(w)}}\left\|v_+^{-1}(w_+ - v_+^2)\right\| \\
&= \frac{1}{2\sqrt{1-\theta}\sqrt{\min(w)}}\left\|v_+^{-1}(w_+ - w + w - v_+^2)\right\| \\
&\leq \frac{1}{2\sqrt{1-\theta}\sqrt{\min(w)}}\left(\left\|v_+^{-1}\right\|\left(\|w_+ - w\| + \left\|w - v_+^2\right\|\right)\right).
\end{aligned}$$

Now since $w - v_+^2 = -d_x d_z$ and $w_+ - w = -\theta w$ and by Lemmas 3.1 and 3.4 and with the fact that $\|w\| \leq \sqrt{n}\,\|w\|_\infty$, we get,

$$\begin{aligned}
\delta(v_+;w_+) &\leq \frac{1}{2\sqrt{1-\theta}\min(w)\sqrt{1-\delta^2}}\left[\|\theta w\| + \|d_x d_z\|\right] \\
&\leq \frac{1}{2\sqrt{1-\theta}\min(w)\sqrt{1-\delta^2}}\left[\|\theta w\| + \min(w)\delta^2\right] \\
&\leq \frac{\theta\,\|w\|}{2\sqrt{1-\theta}\min(w)\sqrt{1-\delta^2}} + \frac{\delta^2}{2\sqrt{1-\theta}\sqrt{1-\delta^2}} \\
&\leq \frac{\theta\sqrt{n}\,\|w\|_\infty}{2\sqrt{1-\theta}\min(w)\sqrt{1-\delta^2}} + \frac{\delta^2}{2\sqrt{1-\theta}\sqrt{1-\delta^2}} \\
&= \frac{\theta\sqrt{n}\max(w)}{2\sqrt{1-\theta}\min(w)\sqrt{1-\delta^2}} + \frac{\delta^2}{2\sqrt{1-\theta}\sqrt{1-\delta^2}}.
\end{aligned}$$

Using Lemma 3.5 and (11), we have,

$$\delta(v_+;w_+) \leq \frac{\theta\sqrt{n}\sigma_C(w)}{2\sqrt{1-\theta}\sqrt{1-\delta^2}} + \frac{\delta_+}{\sqrt{2(1-\theta)}}.$$

If $\theta = \frac{1}{2\sqrt{n}\,\sigma_C(w)}$, and observe that $\sigma_C(w) \geq 1$, and for $n \geq 3$, then $\theta \leq \frac{1}{4}$. Furthermore, if $\delta \leq \frac{1}{\sqrt{2}}$, then from Corollary 3.1, $\delta_+ \leq \frac{1}{2}$. Finally, the above inequalities yield $\delta(v_+;w_+) \leq \frac{1}{\sqrt{2}}$. This completes the proof. □

Note that, in all the iterates produced by Algorithm 2.1, we have $\sigma_C(w) = \sigma_C(w^0)$. Thus, we deduce from Lemma 3.6 that for the default $\theta = \frac{1}{2\sqrt{n}\,\sigma_C(w^0)}$, the conditions $x, y > 0$ and $\delta(v_+;w_+) \leq \frac{1}{\sqrt{2}}$ are maintained during the algorithm. Thus, confirms that Algorithm 2.1, is well-defined.
The upper bound of the duality gap after a weighted full-Newton step is presented in the following lemma.

Lemma 3.7
Let $\delta := \delta\,(v;w) \leq \frac{1}{\sqrt{2}}$ and $x_+ = x + \Delta x$ and $z_+ = z + \Delta z$. Then the duality gap satisfies:

$$x_+^T z_+ \leq (n+1)\max(w).$$

Proof: By Lemma 3.2, we have seen that

$$x_+ z_+ = w + d_x d_z.$$

Hence

$$\begin{aligned} e^T(x_+ z_+) &= e^T w + e^T d_x d_z \\ &= e^T w + d_x^T d_z. \end{aligned}$$

According to (13), Lemma 3.1 and $\delta \leq \frac{1}{\sqrt{2}}$, we deduce that:

$$\begin{aligned} x_+^T z_+ &\leq e^T w + 2\delta^2 \min(w), \\ &\leq e^T w + \min(w). \end{aligned}$$

Now, since $e^T w \leq n\max(w)$, we get

$$x_+^T z_+ \leq (n+1)\max(w).$$

This completes the proof. □

The following lemma gives an upper bound for the total number of iterations produced by Algorithm 2.1.

Lemma 3.8
Let x^{k+1} and z^{k+1} be the $(k+1)-th$ iteration produced by the Algorithm 2.1, with $w := w^k$.Then

$$(x^{k+1})^T z^{k+1} \leq \epsilon$$

if

$$k \geq \left[\frac{1}{\theta} \log \frac{2n \max(w^0)}{\epsilon}\right].$$

Proof: By Lemma 3.7, it follows that:

$$(x^{k+1})^T z^{k+1} \leq (n+1) \max(w^k)$$

with

$$w^k = (1-\theta) w^{k-1} = (1-\theta)^k w^0.$$

Then, we have

$$(x^{k+1})^T z^{k+1} \leq (1-\theta)^k (n+1) \max(w^0) \leq (1-\theta)^k 2n \max(w^0),$$

since $n+1 \leq 2n$ for all $n \geq 1$.
Thus the inequality $(x^{k+1})^T z^{k+1} \leq \epsilon$ is satisfied if

$$(1-\theta)^k 2n \max(w^0) \leq \epsilon.$$

Now taking logarithms, we may write

$$k \log(1-\theta) \leq \log \epsilon - \log 2n \max(w^0)$$

and since $-\log(1-\theta) \geq \theta$ for $0 < \theta < 1$, then the inequality holds if

$$k\theta \geq \log \frac{2n \max(w^0)}{\epsilon}.$$

This completes the proof. □

Theorem 3.1
Suppose that x^0 and z^0 are strictly feasible starting point for CQO, $w^0 = \frac{x^0 z^0}{2\max(x^0 z^0)}$, and such that $\delta(x^0 z^0; w^0) \leq \frac{1}{\sqrt{2}}$ for $n \geq 3$. If $\theta = \frac{1}{2\sqrt{n}\sigma_C(w^0)}$ then, Algorithm 2.1, requires at most $O\left(\sqrt{n}\sigma_C(w^0) \log \frac{n}{\epsilon}\right)$ iterations to obtain an ϵ-approximate solution of CQO.
In particular, if $w^0 = \frac{1}{2}e$, then Algorithm 2.1, requires at most $O\left(\sqrt{n} \log \frac{n}{\epsilon}\right)$ iterations which is the currently best known iteration bound for short-update methods.

Proof: By taking the value of θ and w^0 in Lemma 3.8, the result follows straightforwardly. This completes the proof. □

4. Conclusion and future remarks

In this paper, we have presented a weighted full-Newton step path-following method for CQO. At each interior point iteration, only full-Newton steps are used. The favorable polynomial complexity bound for the algorithm with short-step method is deserved, namely, $O(\sqrt{n}\log\frac{n}{\epsilon})$ which is as good as LO case. Finally, the numerical implementation of this algorithm remains to be investigated.

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