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On the increase rate of random fields from space $Sub_\varphi(\Omega)$ on unbounded domains

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Abstract
This paper mainly focuses on the estimates for distribution of supremum for the normalized φ-sub-Gaussian random fields defined on the unbounded domain. In particular, we obtain the estimates for distribution of supremum for the normalized solution of the hyperbolic equation of mathematical physics, which will be useful to construct modeless. By using this result, we can approximate the solutions of such equation with given accuracy and reliability in the uniform metric.

Keywords Increase rate of random fields; estimates for distribution of supremum; space $Sub_\varphi(\Omega)$

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1. Introduction

Study the certain classes of stochastic processes, their properties, the conditions of boundedness and distribution of the supremum, as well as construction of mathematical models of the processes are the classical problems in the theory of stochastic processes.

In this paper we shall widely use the notion of φ-sub-Gaussian random variables. It generalizes the notion of sub-Gaussian variables, which were

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introduced in the works by Kahana [3] and [4]. The Banach spaces of sub-Gaussian type, namely the space $Sub_{\varphi}(\Omega)$ of φ-sub-Gaussian random variables and processes, were introduced in the paper by Yu. Kozachenko and E. Ostrovsky [8]. Properties of such spaces, sums of independent random variables from these spaces, random processes from the space $Sub_{\varphi}(\Omega)$, conditions of boundedness and estimates for the distribution of supremum of φ-sub-Gaussian processes for the case when the process is defined on the space equipped with pseudometric generated by this process, have been studied in the monograph [9]. The detailed definition of φ-sub-Gaussian variables and spaces $Sub_{\varphi}(\Omega)$ was given in the paper [6]. The properties of these random variables and their modifications have been also investigated. The upper estimate of overrunning the level specified by the continuous function by $Sub_{\varphi}(\Omega)$ stochastic process was obtained in [16]. The theory of φ-sub-Gaussian processes was successfully applied in the wavelets theory [27], the signal theory [10, 21, 22] and other areas of research [1, 2, 7, 13, 14, 19, 20, 26, 28].

The estimates for distributions of supremum of Gaussian stochastic processes defined on a compact set were investigated in many papers, in particular, in the book [24]. Links to other connected articles can be found there. Estimation of the distributions for a supremum of $Sub_{\varphi}(\Omega)$ stochastic processes defined on compacts were considered in [16].

The increase rate of the random field from the space $Sub_{\varphi}(\Omega)$ defined on an unbounded domain was not considered before. As it is shown in the section 4, the estimates obtained in this paper can be used for investigation the increase rate of the solutions of the mathematical physics problems as $t \to \infty$.

Such results can be used in the different situations. Let, for instance, a differential equation describes some physical process. It is known that if the given process exceeds some level then a disaster occurs. Exceeding of such a level is a rather rare event. If we have the estimates for the increase rate of the process on infinity, then we can estimate probability of the disaster during some time span.

The paper consists of an introduction and three sections. The second section provides the basic information on the theory of $Sub_{\varphi}(\Omega)$ spaces of random variables. The third section presents the estimates for the distribution of supremum of φ-sub-Gaussian random fields at infinity. The fourth section contains an application of these estimates for the solution of a hyperbolic type equation of mathematical physics, where $t \in [0, +\infty)$.

2. Stochastic processes from the space $Sub_{\varphi}(\Omega)$

Definition 1 ([5]). An even continuous convex function $u(x)$, $x \in R^1$ such that $u(0) = 0$, $u(x) > 0$ for $x \neq 0$ and

$$\lim_{x \to 0} \frac{u(x)}{x} = 0, \lim_{x \to \infty} \frac{u(x)}{x} = \infty.$$

is called an N-function.

Definition 2 ([9]). The condition Q is satisfied for an N-function φ, if

$$\liminf_{x \to 0} \frac{\varphi(x)}{x^2} = c > 0.$$

Lemma 1 ([9]). *Let $u(x)$ be an N-function. Then*

1. $u(\alpha x) \leqslant \alpha u(x)$ *for* $0 \leqslant \alpha \leqslant 1$ *and* $x \in R$;
2. $u(\alpha x) \geqslant \alpha u(x)$ *for* $\alpha > 1$ *and* $x \in R$;
3. $u(|x| + |y|) \leqslant u(x) + u(y)$ *for* $x, y \in R$;
4. *the function* $\frac{u(x)}{x}$ *is nondecreasing for* $x > 0$.

Definition 3 ([5]). Let $u(x)$ be an N-function. The function $u^*(x) = \sup_{y \in R}(xy - u(y))$ is called the Young-Fenchel transform of the function $u(x)$.

The function $u^*(x)$ is an N-function as well.

Definition 4 ([6]). Let $\varphi(x)$ be the N-function, for which Q-condition is satisfied. The set of random variables $\xi(\omega)$, $\omega \in \Omega$, is called a space $Sub_{\varphi}(\Omega)$ generated by the N-function $\varphi(x)$ if $E\xi = 0$ and there exists a constant a_{ξ} such that

$$E \exp\{\lambda \xi\} \leqslant \exp\{\varphi(\lambda a_{\xi})\}$$

for all $\lambda \in R^1$.

The space $Sub_{\varphi}(\Omega)$ is a Banach space with respect to the norm [9]

$$\tau_{\varphi}(\xi) = \sup_{\lambda \neq 0} \frac{\varphi^{(-1)}(\ln E \exp\{\lambda \xi\})}{|\lambda|}.$$

Definition 5 ([6]). Let T be a parametric space. A stochastic process $X = \{X(t), \ t \in T\}$ belongs to the space $Sub_{\varphi}(\Omega), (X \in Sub_{\varphi}(\Omega))$ if $X(t) \in Sub_{\varphi}(\Omega)$ for all $t \in T$.

Remark 1 ([8]). *A Gaussian stochastic process $X(t)$ with zero mean belongs to the space $Sub_{\varphi}(\Omega)$, where $\varphi(x) = \frac{x^2}{2}$ and $\tau(X(t)) = (E(X(t))^2)^{1/2}$.*

Lemma 2 ([8]). *If* $\xi \in \; Sub_{\varphi}(\Omega)$ *, then there exists a constant* $C > 0$ *such that* $(E(\xi)^2)^{1/2} \leq C\tau_{\varphi}(\xi)$.

Definition 6 ([6]). A random variable $\xi \in Sub_{\varphi}(\Omega)$ is called strongly $Sub_{\varphi}(\Omega)$, $(SSub_{\varphi}(\Omega))$ random variable if $\tau_{\varphi}(\xi) = \left(E\xi^2\right)^{1/2}$.

Properties and applications of $SSub_{\varphi}(\Omega)$ random variables and stochastic processes from $SSub_{\varphi}(\Omega)$ can be found in [9].

Definition 7 ([15]). A family Δ of random variables ξ from a space $Sub_{\varphi}(\Omega)$ is called $SSub_{\varphi}(\Omega)$ family if

$$\tau_{\varphi}\left(\sum_{i\in I}\lambda_i\xi_i\right) = \left(E\left(\sum_{i\in I}\lambda_i\xi_i\right)^2\right)^{1/2}$$

for all $\lambda_i \in R^1$, where I is at most countable and $\xi_i \in \Delta_i, \quad i \in I$.

Definition 8 ([15]). The stochastic process $\{X(t), \; t \in T\}$ is called as a $SSub_{\varphi}(\Omega)$ process if the family of random variables $\{X(t), \quad t \in T,\}$ is $SSub_{\varphi}(\Omega)$.

Theorem 1 ([15]). *Let* Δ *be a strongly* $Sub_{\varphi}(\Omega)$ *family of random variables. Then the linear closure* $\overline{\Delta}$ *of the family* Δ *in the space* $L_2(\Omega)$ *in the mean square sense is a strongly* $Sub_{\varphi}(\Omega)$ *family.*

Theorem 2 ([17]). *Let* R^k *be the* k*-dimensional space,* $d(t,s) = \max\limits_{1\leq i\leq k}|t_i - s_i|$, $T = \{0 \leq t_i \leq T_i, i = 1, 2, \cdots, k\}$, $T_i > 0$. *Assume that* $X = \{X(t, t \in T)\}$ *is separable and* $X \in Sub_{\varphi}(\Omega)$. *If* $\sup\limits_{d(t,s)\leqslant h} \tau_{\varphi}(X(t) - X(s)) \leqslant \sigma(h)$, *where* $\sigma(h)$ *is a monotone increasing continuous function such that* $\sigma(h) \to 0$ *as* $h \to \infty$, *and*

$$\int\limits_{0+} \Psi\left(\ln\frac{1}{\sigma^{(-1)}(\varepsilon)}\right) d\varepsilon < \infty,$$

where $\Psi(u) = \frac{u}{\varphi^{(-1)}(u)}$. *Then*

$$P\left\{\sup_{t\in T}|X(t)| > u\right\} \leqslant 2\tilde{A}(u, \; \theta),$$

for all $0 < \theta < 1$ *and* $u > \frac{2\mathrm{I}_{\varphi}(\theta\varepsilon_0)}{\theta(1-\theta)}$ *where*

$$\tilde{A}(u, \; \theta) = \exp\left\{-\varphi^*\left(\frac{1}{\tilde{\varepsilon}_0}\left[u(1-\theta) - \frac{2}{\theta}\mathrm{I}_{\varphi}(\theta\tilde{\varepsilon})\right]\right)\right\},$$

$$\tilde{\varepsilon}_0 = \sup_{t \in T} \left(E \left| X(t) \right|^2 \right)^{1/2},$$

$$\mathrm{I}_{\varphi}(\delta) = \int_0^{\delta} \Psi \left(\sum_{i=1}^{k} \ln \left(\frac{T_i}{2\sigma^{(-1)}(\varepsilon)} + 1 \right) \right) d\varepsilon.$$

3. The main result

Theorem 3. *Let $\{\xi(x,t),\ (x,t) \in V\}$, $V = [-A; A] \times [0, +\infty)$ be a separable random field belonging to $Sub_{\varphi}(\Omega)$. Assume also that the following conditions are satisfied.*

1. $[b_k, b_{k+1}]$, $k = 0, 1, \ldots$ *is a family of such segments, that* $-\infty < b_k < b_{k+1} < +\infty$, $k = 0, 1, \ldots$ $V_k = [-A; A] \times [b_k, b_{k+1}]$, $\bigcup_k V_k = V$.
2. *There exist the increasing functions* $\sigma_k(h)$, $0 < h < b_{k+1} - b_k$, *such that* $\sigma_k(h) \longrightarrow 0$ *as* $h \longrightarrow 0$

$$\sup_{\substack{|x-x_1| \le h, \\ |t-t_1| \le h \\ (x,t),(x_1,t_1) \in V_k}} \tau_{\varphi}\left(\xi(x,t) - \xi(x_1,t_1)\right) \leqslant \sigma_k(h) \tag{1}$$

and

$$\int_{0+} \Psi \left(\ln \frac{1}{\sigma_k^{(-1)}(\varepsilon)} \right) d\varepsilon < \infty, \tag{2}$$

where $\Psi(u) = \frac{u}{\varphi^{(-1)}(u)}$, $\sigma_k^{(-1)}(\varepsilon)$ *is an inverse function to* $\sigma_k(\varepsilon)$.
3. $c = \{c(t),\ t \in R\}$ *is some continuous function, such that* $c(t) > 0$, $t \in R$, $c_k = \min_{t \in [b_k, b_{k+1}]} c(t)$.
4. $\sup_k \frac{\varepsilon_k}{c_k} < \infty$, $\sup_k \frac{I_{\varphi}(\theta\varepsilon_k)}{c_k} < \infty$.
5. *The series* $\sum_{k=0}^{\infty} \exp\left\{ -\varphi^* \left(\frac{sc_k(1-\theta)}{2\varepsilon_k} \right) \right\}$ *converges for some* s *in such a way that* $\sup_k \frac{4\varepsilon_k}{c_k(1-\theta)} < s < \frac{u}{2}$, *where* $\varepsilon_k = \sup_{(x,t) \in V_k} \tau_{\varphi}(\xi(x,t))$, $k = 0, 1, \ldots$.

Then

$$P\left\{ \sup_{(x,t) \in V} \frac{|\xi(x,t)|}{c(t)} > u \right\} \leqslant 2 \exp\left\{ -\varphi^*\left(\frac{u}{s} \right) \right\} \cdot \sum_{k=0}^{\infty} \exp\left\{ -\varphi^*\left(\frac{sc_k(1-\theta)}{2\varepsilon_k} \right) \right\} = 2A(u), \tag{3}$$

for $u > \sup\limits_{k} \frac{\tilde{\mathrm{I}}_{\varphi}(\theta\varepsilon_k)}{c_k} \cdot \frac{4}{\theta(1-\theta)}$, *where* $0 < \theta < 1$

$$\tilde{\mathrm{I}}_{\varphi}(\delta) = \int\limits_0^{\delta} \Psi\left[\left(\ln\left(\frac{A}{\sigma_k^{(-1)}(\varepsilon)} + 1\right)\right) + \left(\ln\left(\frac{b_{k+1} - b_k}{2\sigma_k^{(-1)}(\varepsilon)} + 1\right)\right)\right] d\varepsilon, \quad k = 0, 1, \ldots,$$

Proof
Consider

$$P\left\{\sup_{(x,t)\in V} \frac{|\xi(x,t)|}{c(t)} > u\right\} \leqslant \sum_{k=0}^{\infty} P\left\{\sup_{(x,t)\in V_k} \frac{|\xi(x,t)|}{c(t)} > u\right\} \leqslant$$

$$\sum_{k=0}^{\infty} P\left\{\sup_{(x,t)\in V_k} |\xi(x,t)| > uc_k\right\}, \tag{4}$$

where $c_k = \min\limits_{t\in[b_k, b_{k+1}]} c(t)$. It follows from the Theorem 2 that

$$P\left\{\sup_{(x,t)\in V_k} |\xi(x,t)| > uc_k\right\} \leqslant 2\widetilde{A}(uc_k, \theta) \tag{5}$$

for some $u > \frac{2\mathrm{I}_{\varphi}(\theta\varepsilon_k)}{c_k\theta(1-\theta)}$, $0 < \theta < 1$, where $\widetilde{A}(uc_k, \theta) = \exp\{-\varphi^*(\frac{1}{\varepsilon_k}[c_k u(1-\theta) - \frac{2}{\theta}\mathrm{I}_{\varphi}(\theta\varepsilon_{\mathrm{k}})])\}$, $\tilde{\mathrm{I}}_{\varphi}(\delta) = \int\limits_0^{\delta} \Psi\left[\left(\ln\left(\frac{A}{\sigma_k^{(-1)}(\varepsilon)} + 1\right)\right) + \left(\ln\left(\frac{b_{k+2}-b_k}{\sigma_k^{(-1)}(\varepsilon)} + 1\right)\right)\right] d\varepsilon$,
$\varepsilon_k = \sup\limits_{(x,t)\in V_k} \tau_{\varphi}(\xi(x,t))$, $k = 0, 1, \ldots$;

Let $u > \sup\limits_{k} \frac{I_{\varphi}(\theta\varepsilon_k)}{c_k} \cdot \frac{4}{\theta(1-\theta)}$, then

$$\widetilde{A}(uc_k, \theta) = \exp\left\{-\varphi^*\left(\frac{1}{\varepsilon_k}\left[c_k u(1-\theta) - \frac{2}{\theta}\mathrm{I}_{\varphi}(\theta\varepsilon_{\mathrm{k}})\right]\right)\right\} =$$

$$\exp\left\{-\varphi^*\left(\frac{c_k u(1-\theta)}{\varepsilon_k}\left[1 - \frac{2\mathrm{I}_{\varphi}(\theta\varepsilon_{\mathrm{k}})}{c_k u(1-\theta)\theta}\right]\right)\right\} \leqslant \exp\left\{-\varphi^*\left(\frac{c_k u(1-\theta)}{2\varepsilon_k}\right)\right\}.$$

Since $\varphi^*(x)$ is an N-function, then

$$\varphi^*(|x| + |y|) \geqslant \varphi^*(|x|) + \varphi^*(|y|), \quad x,\ y \in R \tag{6}$$

and

$$x \cdot y \geq x + y \tag{7}$$

for $x \geq 2$ and $y \geq 2$.

Thus we obtain from (6) and (7)that

$$\varphi^*\left(\frac{c_k u(1-\theta)}{2\varepsilon_k}\right) = \varphi^*\left(\frac{u}{s}\cdot\frac{sc_k(1-\theta)}{2\varepsilon_k}\right) \geqslant \varphi^*\left(\frac{u}{s}\right) + \varphi^*\left(\frac{sc_k(1-\theta)}{2\varepsilon_k}\right),$$

where s is some number, such that for all $k = 0, 1, \dots$ we have $\inf\limits_k \frac{sc_k(1-\theta)}{2\varepsilon_k} > 2$ and $\frac{u}{s} > 2$.

Since,

$$\exp\left\{-\varphi^*\left(\frac{c_k u(1-\theta)}{2\varepsilon_k}\right)\right\} \leqslant \exp\left\{-\varphi^*\left(\frac{u}{s}\right)\right\} \times \exp\left\{-\varphi^*\left(\frac{sc_k(1-\theta)}{2\varepsilon_k}\right)\right\}. \tag{8}$$

Then from (4), (5) and (8) we have

$$P\left\{\sup_{(x,t)\in V}\frac{|\xi(x,t)|}{c(t)} > u\right\} \leqslant 2A(u),$$

where $A(u) = \exp\left\{-\varphi^*\left(\frac{u}{s}\right)\right\}\cdot\sum\limits_{k=0}^{\infty}\exp\left\{-\varphi^*\left(\frac{sc_k(1-\theta)}{2\varepsilon_k}\right)\right\}.$ □

Remark 2. *In general, the condition 2 can not be essentially better, because, for example, the given condition for the Gaussian stationary process is close to the necessary and sufficient condition of boundedness of this process on an interval with probability one. The inequality (3) for a whole class of processes from $Sub_\varphi(\Omega)$ can be better at the expense of constants, since we always can construct a processes from $Sub_\varphi(\Omega)$ that*

$$P\{|\xi(x,t)| > u\} = c\exp\{-\varphi^*(cu)\}.$$

at every point. It is clear, that we always can construct a function $c(t)$ such that conditions 4) and 5) are satisfied. So, it is desirable to construct a function $c(t)$ increasing as slowly, as possible. It was not the aim of the paper to find a function $c(t)$, which increases in the slowest way, but below we present the functions increasing of which can not be improved.

Corollary 1. If the conditions of Theorem 3 holds, then there is such a random variable $\xi > 0$, $P\{\xi > u\} \le A(u)$, $u > 0$, where $A(u)$ is given by (3), that

$$|\xi(x,t)| \le c(t)\xi$$

for all $(x,t) \in V$ with probability one.

In the next example and Theorem 4 we consider the most interesting examples of spaces $Sub_\varphi(\Omega)$. They are spaces, where $\varphi(x) = \frac{|x|^p}{p}$ for some $p > 1$, $|x| > 1$. These spaces are subspaces of some Orlicz spaces of exponential type [6, 9]. In particular, for $\varphi(x) = \frac{|x|^2}{2}$ these spaces include spaces of centered Gaussian random variables.

Example 1. *Let $\varphi(x)$ be a function such that $\varphi(x) = \frac{|x|^p}{p}$ for some $|x| > 1$ and all $p > 1$. Then $\varphi^*(x) = \frac{|x|^q}{q}$, $\frac{1}{p} + \frac{1}{q} = 1$, $\Psi(x) = \frac{x^{1-\frac{1}{p}}}{p^{\frac{1}{p}}}$ for $x > 1$ and the condition (2) holds for all $\varepsilon > 0$*

$$\int\limits_{0+} \left(\ln \frac{1}{\sigma_k^{(-1)}(u)} \right)^{1-\frac{1}{p}} du < \infty, \tag{9}$$

The condition (9) holds if $\sigma_k(h) = a_k \left| \ln \left(\frac{1}{|h|} + d \right) \right|^{-\delta}$, for $\delta > 1 - \frac{1}{p}$ and $a_k > 0$, $k = 0, 1, \ldots$. In this case, the assumption (1) of the Theorem 3 is satisfied if for $k = 0, 1, 2$ there exist the constants $a_k > 0$ and $d > 1$ such that

$$\sup_{|x-x_1| \le h, |t-t_1| \le h (x,t),(x_1,t_1) \in V_k} \tau_\varphi(\xi(x,t) - \xi(x_1,t_1)) \leqslant \frac{a_k}{\left| \ln \left(\frac{1}{|h|} + d \right) \right|^{\delta}},$$

for $\delta > 1 - \frac{1}{p}$ and some h.

Theorem 4. *Let $\{\xi(x,t),\ (x,t) \in V\}$, $V = [-A; A] \times [0, +\infty)$ be a separable random field belonging to $Sub_\varphi(\Omega)$, where $\varphi(x) = \frac{|x|^p}{p}$ for $|x| > 1$, $p > 1$. Assume the following conditions are satisfied.*

1. *$[b_k, b_{k+1}]$, $k = 0, 1, \ldots$ is a family of such segments, that $-\infty < b_k < b_{k+1} < +\infty$, $k = 0, 1, \ldots$ $V_k = [-A; A] \times [b_k, b_{k+1}]$, $\bigcup\limits_k V_k = V$.*
2. *There exist constants $a_k > 0$ and $d > 1$, such that $A > \frac{1}{d}$, $\frac{b_{k+1} - b_k}{2} > \frac{1}{d}$ and*

$$\sup_{\substack{|x-x_1| \le h, \\ |t-t_1| \le h \\ (x,t),(x_1,t_1) \in V_k}} \tau_\varphi(\xi(x,t) - \xi(x_1,t_1)) \leqslant \frac{a_k}{\left| \ln \left(\frac{1}{|h|} + d \right) \right|^{\alpha}},$$

   for some $|h|$ and $\alpha > 1 - \frac{1}{p}$.
3. *$c = \{c(t),\ t \in R\}$ is some continuous function, such that $c(t) > 0$, $t \in R$, $c_k = \min\limits_{t \in [t_k, t_{k+1}]} c(t)$.*
4. *$\sup\limits_k \frac{\varepsilon_k}{c_k} < \infty$, $\sup\limits_k \frac{(a_k)^{\frac{1}{\alpha q}} (\varepsilon_k)^{1-\frac{1}{\alpha q}}}{c_k} < \infty$, $\sup\limits_k \frac{\varepsilon_k \ln \left(A \cdot \frac{b_{k+1} - b_k}{2} \right)^{\frac{1}{q}}}{c_k} < \infty$.*
5. *The series $\sum\limits_{k=0}^{\infty} \exp \left\{ -\frac{1}{q} \left(\frac{s c_k (1-\theta)}{2 \varepsilon_k} \right)^{\frac{1}{q}} \right\}$ converges for some s, such that, $\sup\limits_k \frac{4 \varepsilon_k}{c_k (1-\theta)} < s < \frac{u}{2}$, where $\varepsilon_k = \sup\limits_{(x,t) \in V_k} \tau_\varphi(\xi(x,t))$, $k = 0, 1, \ldots$*

Then

$$P\left\{\sup_{(x,t)\in V}\frac{|\xi(x,t)|}{c(t)}>u\right\}\leqslant 2\exp\left\{-\frac{1}{q}\left(\frac{u}{s}\right)^{\frac{1}{q}}\right\}\cdot\sum_{k=0}^{\infty}\exp\left\{-\frac{1}{q}\left(\frac{sc_k(1-\theta)}{2\varepsilon_k}\right)^{\frac{1}{q}}\right\}=B(u) \tag{10}$$

for $u>\sup\limits_k\frac{\frac{1}{p^{\frac{1}{p}}}\left(2^{\frac{1}{q}}(a_k)^{\frac{1}{\alpha q}}\frac{(\theta\varepsilon_k)^{1-\frac{1}{\alpha q}}}{1-\frac{1}{\alpha q}}+\theta\varepsilon_k\ln\left(A\cdot\frac{b_{k+1}-b_k}{2}\right)^{\frac{1}{q}}\right)}{c_k}\cdot\frac{4}{\theta(1-\theta)}$, $0<\theta<1$.

Proof
This Theorem follows from the Theorem 3, since in this case

$$\tilde{\mathrm{I}}_\varphi(\delta)=\int\limits_0^\delta\Psi\left[\left(\ln\left(\frac{A}{\sigma_k^{(-1)}(\varepsilon)}+1\right)\right)+\left(\ln\left(\frac{b_{k+1}-b_k}{2\sigma_k^{(-1)}(\varepsilon)}+1\right)\right)\right]d\varepsilon=$$

$$\frac{1}{p^{\frac{1}{p}}}\int\limits_0^\delta\left(\ln\left(A\left(e^{\left(\frac{a_k}{\varepsilon}\right)^{\frac{1}{\alpha}}}-d\right)+1\right)+\ln\left(\frac{b_{k+1}-b_k}{2}\left(e^{\left(\frac{a_k}{\varepsilon}\right)^{\frac{1}{\alpha}}}-d\right)+1\right)\right)^{\frac{1}{q}}d\varepsilon\leq$$

$$\frac{1}{p^{\frac{1}{p}}}\int\limits_0^\delta\left(\ln\left(Ae^{\left(\frac{a_k}{\varepsilon}\right)^{\frac{1}{\alpha}}}\right)+\ln\left(\frac{b_{k+1}-b_k}{2}e^{\left(\frac{a_k}{\varepsilon}\right)^{\frac{1}{\alpha}}}\right)\right)^{\frac{1}{q}}d\varepsilon\leq$$

$$\frac{1}{p^{\frac{1}{p}}}\left(\int\limits_0^\delta\frac{2^{\frac{1}{q}}(a_k)^{\frac{1}{\alpha q}}}{\varepsilon^{\frac{1}{\alpha q}}}d\varepsilon+\delta\ln\left(A\cdot\frac{b_{k+1}-b_k}{2}\right)^{\frac{1}{q}}\right)=$$

$$\frac{1}{p^{\frac{1}{p}}}\left(2^{\frac{1}{q}}(a_k)^{\frac{1}{\alpha q}}\frac{(\delta)^{1-\frac{1}{\alpha q}}}{1-\frac{1}{\alpha q}}+\delta\ln\left(A\cdot\frac{b_{k+1}-b_k}{2}\right)^{\frac{1}{q}}\right).$$

Then series in the conditions 5) of the Theorem 3 will have the following form

$$\sum_{k=0}^{\infty}\exp\left\{-\varphi^*\left(\frac{sc_k(1-\theta)}{2\varepsilon_k}\right)\right\}=\sum_{k=0}^{\infty}\exp\left\{-\frac{1}{q}\left(\frac{sc_k(1-\theta)}{2\varepsilon_k}\right)^{\frac{1}{q}}\right\}.$$

□

Corollary 2. If conditions of the Theorem 4 holds, then there is such a random variable $\xi>0$, $P\{\xi>u\}\leq B(u)$, $u>0$, where $B(u)$ is given by (10), that

$$|\xi(x,t)|\leq c(t)\xi$$

for all $(x,t)\in V$ with probability one.

4. An example

In this section we consider the boundary-value problem of the first kind for a homogeneous hyperbolic equation [25]. The problem is in finding a function $u = (u\,(x, y)\,,\ x \in [0, \pi]\,,\ \ t \in [0, t])$ satisfying the following conditions:

$$\frac{\partial}{\partial x}\left(p(x)\frac{\partial u}{\partial x}\right) - q(x)u - \rho(x)\frac{\partial^2 u}{\partial t^2} = 0;$$

$$x \in [0, \pi]\,,\ t \in [0, +\infty]\,;$$

$$u(0, t) = u(\pi, t) = 0,\ t \in [0, +\infty]\,;$$

$$u(x, 0) = \xi(x),\ \frac{\partial u(x, 0)}{\partial t} = \eta(x),\ x \in [0, \pi]\,.$$

Assume also that $(\xi(x),\ x \in [0, \pi])$ and $(\eta(x),\ x \in [0, \pi])$ are $SSub_\varphi(\Omega)$ stochastic processes defined on a common complete probability space $(\Omega,\ \Im,\ P)$, where $\varphi(x) = \frac{|x|^p}{p}$, $|x| > 1, p > 1$.

Regardless of whether the initial conditions are deterministic or random, the Fourier method consists in searching for a solution of the series

$$u\,(x, t) = \sum_{k=1}^{\infty} X_k(x)\left[A_k cos\sqrt{\lambda_k}t + \frac{B_k}{\sqrt{\lambda_k}} sin\sqrt{\lambda_k}t\right], \tag{11}$$

$$x \in [0, \pi]\,,\ t \in [0, +\infty]\,;$$

where

$$A_k = \int_0^\pi \xi(x)X_k(x)\rho(x)dx,\ k \geq 1,$$

$$B_k = \int_0^\pi \eta(x)X_k(x)\rho(x)dx,\ k \geq 1,$$

and where $\lambda_k,\ k \geq 1$ and $X_k = (X_k(x),\ x \in [0, \pi]),\ k \geq 1$ are eigenvalues and the corresponding orthonormal eigenfunctions (with weight $\rho(\bullet)$) of the following Sturm-Liouville problem

$$\frac{d}{dx}\left(p(x)\frac{dX(x)}{dx}\right) - q(x)X(x) + \lambda\rho(x)X(x) = 0,$$

$$X(0) = X(\pi) = 0.$$

Let $V = [0, \pi] \times [0, +\infty)\,,\ \ [b_k, b_{k+1}]\,,\ k = 0, 1, \ldots$ be a family of such segments, that $-\infty < b_k < b_{k+1} < +\infty,\ k = 0, 1, \ldots\ V_k = [0; \pi] \times [b_k, b_{k+1}]\,,$ $\bigcup\limits_k V_k = V,$

$$S_n(x,t)=\sum_{k=1}^{n}X_k(x)\left[A_k cos\sqrt{\lambda_k}t+\frac{B_k}{\sqrt{\lambda_k}}sin\sqrt{\lambda_k}t\right].$$

According to [18, 11] $u(x,t)$ is the strongly $Sub_\varphi(\Omega)$ random field and the condition

$$\sup_{\substack{|x_k-y_k|\le h\\ |t_k-s_k|\le h\\ (x_k,t_k),(y_k,s_k)\in V_k}}\left|E\left(S_n(x_k,t_k)-S_n(y_k,s_k)\right)^2\right|^{\frac{1}{2}}\le\frac{a}{|\ln|h||^{\delta}}$$

holds, where

$$a=\sum_{k=1}^{\infty}\left(\left(EA_k^2\right)^{\frac{1}{2}}+\frac{\left(EB_k^2\right)^{\frac{1}{2}}}{k}\right)(\ln k)^{\delta}.$$

$$\tilde{\varepsilon}_0=\sup_{\substack{x\in[0,\pi]\\ t\in[0,T]}}\left(E\left(u(x,t)\right)^2\right)^{\frac{1}{2}}\le\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\left[|EA_kA_l|+\frac{|EB_kB_l|}{kl}+\frac{|EA_kB_l|}{l}\right]^{\frac{1}{2}}=\varepsilon_0. \tag{12}$$

Conditions of convergence of the series

$$\sum_{k=1}^{\infty}\left(\left(EA_k^2\right)^{\frac{1}{2}}+\frac{\left(EB_k^2\right)^{\frac{1}{2}}}{k}\right)(\ln k)^{\delta},$$

$$\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\left[|EA_kA_l|+\frac{|EB_kB_l|}{kl}+\frac{|EA_kB_l|}{l}\right]^{\frac{1}{2}}$$

were found in the book [11].

Let $c=\{c(t),\ t\in R\}$ be some continuous function, such that $c(t)>0,\ t\in R$, $c_k=\min_{t\in[t_k,t_{k+1}]}c(t)$. Since $\varepsilon_k=\tilde{\varepsilon}_0$, then conditions of the Theorem 4 hold if

$$\sup_k\frac{1}{c_k}<\infty, \tag{13}$$

$$\sup_k\frac{\ln\left(A\cdot\frac{b_{k+1}-b_k}{2}\right)^{\frac{1}{q}}}{c_k}<\infty, \tag{14}$$

$$\sum_{k=0}^{\infty}\exp\left\{-\frac{1}{q}\left(\frac{sc_k(1-\theta)}{2\varepsilon_0}\right)^{\frac{1}{q}}\right\}<\infty. \tag{15}$$

Then, from 3 we have

$$P\left\{\sup_{(x,t)\in V}\frac{|u(x,t)|}{c(t)}>\omega\right\}\leqslant 2\exp\left\{-\frac{1}{q}\left(\frac{\omega}{s}\right)^{\frac{1}{q}}\right\}\cdot\sum_{k=0}^{\infty}\exp\left\{-\frac{1}{q}\left(\frac{sc_k(1-\theta)}{2\varepsilon_0}\right)^{\frac{1}{q}}\right\}=D(\omega). \tag{16}$$

For $\omega>\sup\limits_{k}\frac{\frac{1}{p^{\frac{1}{p}}}\left(2^{\frac{1}{q}}(a)^{\frac{1}{\alpha q}}\frac{(\theta\varepsilon_0)^{1-\frac{1}{\alpha q}}}{1-\frac{1}{\alpha q}}+\theta\tilde{\varepsilon}_0\ln\left(A\cdot\frac{b_{k+1}-b_k}{2}\right)^{\frac{1}{q}}\right)}{c_k}\cdot\frac{4}{\theta(1-\theta)}$, $0<\theta<1$.

Let us put $b_k=e^k$ and show that the function $c(t)=(\ln t)^{\frac{1}{q}}$ satisfies the conditions (13)–(15). Actually $c_k=(\ln e^k)^{\frac{1}{q}}=k^{\frac{1}{q}}$. It is obviously that the condition (13) holds true. Moreover,

$$\sup_{k\geq 1}\frac{\ln\left(A\left(b_{k+1}-b_k\right)\right)^{\frac{1}{q}}}{c_k}=\sup_{k\geq 1}\frac{\ln\left(Ae^k\left(e-1\right)\right)^{\frac{1}{q}}}{k^{\frac{1}{q}}}<\infty$$

for every $A>0$. Thus, the condition (14) holds for every $A>0$ and

$$\sum_{k=0}^{\infty}\exp\left\{-\frac{1}{q}\left(\frac{sc_k(1-\theta)}{2\varepsilon_0}\right)^{\frac{1}{q}}\right\}=\sum_{k=0}^{\infty}\exp\left\{-\frac{1}{q}s^{\frac{1}{q}}\frac{k^{\frac{1}{q}}(1-\theta)^{\frac{1}{q}}}{2\varepsilon_0}^{\frac{1}{q}}\right\}.$$

Since $\exp\left\{-Dk^{\frac{1}{q}}\right\}\leq\frac{1}{k^2}$ for every constant $D>0$ and sufficiently large k, we deduce that condition (15) holds for every $s>0$.

Corollary 3. There exists such a random variable $\xi>0$, $P\left\{\xi>\omega\right\}\leq B(\omega)$, that

$$|u(x,t)|\leq\xi(\ln t)^{\frac{1}{q}}$$

for all $(x,t)\in V$ with probability one.

Note that for the given example it is not easy to construct the function $c(t)$, which is increasing considerably slower than $(\ln t)^{\frac{1}{q}}$.

5. Conclusion

The estimates for distribution of supremum for normalized φ-sub-Gaussian random fields defined on unbounded domains are found. Received results can be used for investigation of solutions of hyperbolic and parabolic equations of mathematic physics. Using this results one can construct modeless, which approximate solutions of such equations with given accuracy and reliability in the uniform metric.

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