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Recurrence relations for moments of multiply type-II censored order statistics from Lindley distribution with applications to inference

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Abstract In this paper, we derive the recurrence relations for the moments of function of single and two order statistics from Lindley distribution. We also consider the maximum likelihood estimation (MLE) of the parameter of the distribution based on multiply type-II censoring. The maximum likelihood estimator is comupted numerically because it does not have an explicit form for the parameter. Then, a Monte Carlo simulation study is carried out to evaluate the performance of the MLE obtained from multiply type-II censored sample.

Keywords Order statistics, recurrence relations, Lindley distribution, multiply type-II censoring, Monte Carlo simulation.

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1. Introduction

A random variable X is said to have Lindley distribution if its probability density function (pdf) is given by

$$f(x) = \frac{\theta^2}{1+\theta}(1+x)\, e^{-\theta x}; \quad x > 0, \theta > 0, \tag{1}$$

and it was introduced by Lindley (1952). The corresponding cumulative distribution function (cdf) is given by

$$F(x) = 1 - \frac{1+\theta+\theta x}{1+\theta}\, e^{-\theta x}; \quad x > 0, \theta > 0. \tag{2}$$

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The Lindley distribution gives a better model where the exponential distribution is not good fit. Since the equation $F(x) = u$, where $u \sim U(0,1)$, cannot be solved explicitly in terms of x, the inversion method for generation random data from the Lindley distribution fails. However, one can use the fact that the distribution is a special mixture of Exponential(θ) and Gamma(2, θ) distributions as

$$f(x) = pf_1(x) + (1-p)f_2(x); x > 0, \theta > 0,$$

where $p = \theta/(1+\theta)$, $f_1(x) = \theta e^{-\theta x}$ and $f_2(x) = \theta^2 x e^{-\theta x}$.
To generate random data, X_i, $i = 1, 2, \cdots, n$ from the Lindley distribution with parameter θ one may follow the acceptance-rejection method which can be given by the following algorithm:

i. Generate $U_i \sim U(0,1)$, $i = 1, 2, \cdots, n$
ii. Generate $E_i \sim Exp(\theta)$, $i = 1, 2, \cdots, n$
iii. Generate $G_i \sim Gamma(2, \theta)$, $i = 1, 2, \cdots, n$
iv. If $U_i \leq p$, then set $X_i = G_i$, otherwise, set $X_i = E_i$, $i = 1, 2, \cdots, n$, where p is as before.

Ghitany, et al. (2008) developed different properties of Lindley distribution. The main aim of this paper is to develop recurrence relations of moments of order statistics for the function of single and two order statistics. Also develop a maximum likelihood estimation procedure of the parameter θ by Monte Carlo simulation from multiply type-II censored sample. Then a comparison study will be made between maximum likelihood estimates (Ghitany, et al., 2008) and MLE from Monte Carlo study from multiply type-II censored sample.

Let $X_1, X_2, \cdots, X_n$ be a random sample of size n from the pdf (1) corresponding to the cdf (2). Then $X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{n:n}$ denote ordered statistics for the above sample. Assume that n items are put on a life test, but only r_1-th, $\cdots, r_k$-th failures are observed, the rest are unobserved, where $r_1, \cdots, r_k$ are considered to be fixed. This is the multiply Type-II censoring, for more details see e.g. Jang et al. (2001) and Schenk et al. (2011). Multiply Type-II censoring is a generalization of Type-II censoring where only the first k failure times are observed. In this paper, we let $0 \leq X_{r_1:n} \leq X_{r_2:n} \leq \cdots \leq X_{r_k:n} < \infty$ to be a multiply Type-II censored sample from a population with pdf (1) and cdf (2) for $\theta \in R^q$, where, $1 \leq r_1 < r_2 < \cdots < r_n \leq n$.

The motivation behind using multiply type-II censoring is made clear in the particularl case if one fails to record the failure time of every subject, only several failure times and the number of failures between them are recorded, see Kong and Fei (1996).

2. Recurrence relation for Moments from function of single order statistic

The pdf of r-th order statistic $X_{r:n}$, $(1 \leq r \leq n)$ is given by

$$f_{r:n}(x) = \frac{n!}{(r-1)!\,(n-r)!}[F(x)]^{r-1}[1-F(x)]^{n-r}f(x), \quad x \in (0,\infty) \tag{3}$$

Let g(x) be a Borel measurable function of x in the interval $x \in (0,\infty)$, then

$$\begin{aligned} E\left[g\left(X_{r:n}\right)\right] &= C_{r:n}\int_0^\infty g(x)\,[F(x)]^{r-1}[1-F(x)]^{n-r}f(x)\,dx. \\ &= p\theta C_{r:n}\int_0^\infty g(x)\left[1-(1+px)e^{-\theta x}\right]^{r-1}[1+px]^{n-r} \\ &\quad \times (1+x)e^{-(n-r+1)\theta x}\,dx, \end{aligned} \tag{4}$$

where $C_{r:n} = n!/((r-1)!\,(n-r)!)$.

Theorem 1
For $1 \leq r \leq n$; $n = 1, 2, \cdots$

$$\begin{aligned} E\left[g\left(X_{r:n}\right)\right] - E\left[g\left(X_{r-1:n-1}\right)\right] &= \binom{n-1}{r-1}\sum_{i=0}^{r-1}(-1)^{i-1-i}\binom{r-1}{i} \\ &\times \int_0^\infty g^{'}(x)(1+px)^{n-i}e^{-(n-i)\theta x}\,dx, \end{aligned} \tag{5}$$

where $p = \theta/(1+\theta)$ and $q = 1/(1+\theta)$.

Proof. From (4), we have

$$\begin{aligned} E\left[g\left(X_{r:n}\right)\right] - E\left[g\left(X_{r-1:n-1}\right)\right] &= \binom{n-1}{r-1}\theta p\int_0^\infty g(x)\left[1-(1+px)e^{-\theta x}\right]^{r-1} \\ &\times [1+px]^{n-r}\left[\frac{n-r+1-n\,(1+px)\,e^{-\theta x}}{1-(1+px)e^{-\theta x}}\right] \\ &\times (1+x)e^{-(n-r+1)\theta x}dx. \end{aligned}$$

Let $q(x) = -\left[1-(1+px)e^{-\theta x}\right]^{r-1}[1+px]^{n-r+1}e^{-(n-r+1)\theta x}$, then we have

$$E\left[g\left(X_{r:n}\right)\right] - E\left[g\left(X_{r-1:n-1}\right)\right] = \binom{n-1}{r-1}\int_0^\infty g(x)\,q'(x)dx,$$

which on integration by parts gives

$$\begin{aligned} E\left[g\left(X_{r:n}\right)\right] - E\left[g\left(X_{r-1:n-1}\right)\right] &= \binom{n-1}{r-1}\int_0^\infty g'(x)\left[1-(1+px)e^{-\theta x}\right]^{r-1} \\ &\times [1+px]^{n-r+1}(1+x)e^{-(n-r+2)\theta x}dx. \end{aligned} \tag{6}$$

Now expanding $\left[1-(1+px)e^{-\theta x}\right]^{r-1}$ binomially to get,

$$\left[1-(1+px)e^{-\theta x}\right]^{r-1} = \sum_{i=0}^{r-1} (-1)^{r-1-i} \left(\begin{array}{c} r-1 \\ i \end{array} \right) (1+px)^{r-1-i} e^{-(r-1-i)\theta x}.$$

Putting this result in (6) and on algebraic simplification gives the required proof of the result in (5).

Theorem 2
For $1 \leq r \leq n; n = 1, 2, \cdots$

$$\begin{aligned} E\left[g\left(X_{r:n}\right)\right] - E\left[g\left(X_{r-1:n}\right)\right] &= \binom{n}{r-1} \sum_{i=0}^{r-1} (-1)^{r-1-i} \binom{r-1}{i} \\ &\quad \times \int_0^\infty g'(x)(1+px)^{n-i} e^{-(n-i)\theta x} dx. \end{aligned}$$

Proof. The proof of this theorem is same as that of Theorem 1. Also, for further details we refer to Ali and Khan (1998a).

Theorem 3
For $1 \leq r \leq n; n = 1, 2, \cdots$

$$\begin{aligned} \left[E\left[g\left(X_{r-1:n-1}\right)\right] - E\left[g\left(X_{r-1:n}\right)\right]\right. &= \binom{n-1}{r-2} \sum_{i=0}^{r-1} (-1)^{r-1-i} \binom{r-1}{i} \\ &\quad \times \int_0^\infty g'(x)(1+px)^{n-i} e^{-(n-i)\theta x}\, dx. \end{aligned}$$

Proof. The proof is same as that of Theorem 2 Also, we refer to Ali and Khan (1998*a*).

It is important to note that the above theorems lead to establish the well-known relation given in David and Nagaraja (2003), pp. 45.

Corollary 1
If $g(x) = x^k$, for $1 \leq r \leq n; n = 1, 2, \cdots$, then

$$\mu_{r:n}^{(k)} - \mu_{r-1:n-1}^{(k)} = C \text{ (constant)},$$

where

$$C = k \left(\begin{array}{c} n-1 \\ r-1 \end{array} \right) \sum_{i=0}^{r-1} \sum_{j=0}^{n-i} (-1)^{r-1-i} \left(\begin{array}{c} r-1 \\ i \end{array} \right) \left(\begin{array}{c} n-i \\ j \end{array} \right) p^{n-i-j} \frac{\Gamma(n+k-i-j)}{[(n-i)\,\theta]^{n+k-i-j}}.$$

Proof. Putting $g(x) = x^k$ in Theorem 1, to get

$$\begin{aligned} & \mu_{r:n}^{(k)} - \mu_{r-1:n-1}^{(k)} \\ = \; & k\binom{n-1}{r-1}\sum_{i=0}^{r-1}(-1)^{r-1-i}\binom{r-1}{i} \\ & \times \int_0^\infty x^{k-1}(1+px)^{n-i}e^{-(n-i)\theta x}\,dx. \\ = \; & k\binom{n-1}{r-1}\sum_{r=0}^{r-1}\sum_{j=0}^{n-i}(-1)^{r-1-i}\binom{r-1}{i}\binom{n-j}{j}p^{n-i-j} \\ & \times \int_0^\infty x^{n+k-i-j-1}e^{-(n-i)\theta x}\,dx, \end{aligned}$$

using the gamma function to the integrand to get the require result.

Corollary 2
If $g(x) = x^k$, for $1 \le r \le n$; $n = 1, 2, \cdots$, then

$$\mu_{r:n}^{(k)} - \mu_{r-1:n}^{(k)} = C \text{ (constant)},$$

where

$$C = k\left(\begin{array}{c} n-1 \\ r-2 \end{array}\right)\sum_{i=0}^{r-1}\sum_{j=0}^{n-i}(-1)^{r-1-i}\left(\begin{array}{c} r-1 \\ i \end{array}\right)\left(\begin{array}{c} n-i \\ j \end{array}\right)p^{n-i-j}\frac{\Gamma(n+k-i-j)}{[(n-i)\,\theta]^{n+k-i-j}}.$$

Proof. Putting $g(x) = x^k$ in Theorem 2 and the rest is similar to Corollary 1.

Corollary 3
If $g(x) = x^k$, for $1 \le r \le n$; $n = 1, 2, \cdots$, then

$$\mu_{r-1:n-1}^{(k)} - \mu_{r-1:n}^{(k)} = C \text{ (constant)},$$

where

$$C = k\left(\begin{array}{c} n-1 \\ r-2 \end{array}\right)\sum_{i=0}^{r-1}\sum_{j=0}^{n-i}(-1)^{r-1-i}\left(\begin{array}{c} r-1 \\ i \end{array}\right)\left(\begin{array}{c} n-i \\ j \end{array}\right)p^{n-i-j}\frac{\Gamma(n+k-i-j)}{[(n-i)\,\theta]^{n+k-i-j}}.$$

Proof. Putting $g(x) = x^k$ in Theorem 3 and the rest is similar to Corollary 1.

3. Recurrence relation for Moments from the function of two order statistics

The joint pdf of $X_{r:n} = x$ and $X_{s:n} = y$, $1 \le r \le s \le n$, is given by

$$f_{r,s:n}(x,y) = C_{r,s:n}[F(x)]^{r-1}[F(y)-F(x)]^{s-r-1}[1-F(y)]^{n-s}f(x)f(y),$$

where $0 \leq x < y < \infty,\ C_{r,s:n} = n!/((r-1)!\ (s-r-1)!\ (n-s)!)$.
If g is a Borel measurable function from $\mathbb{R}^2$ to $\mathbb{R}$, then

$$\begin{aligned} & E\left[g\left(X_{r:n}, X_{s:n}\right)\right] \\ = \ & C_{r,s:n} \iint\limits_{0\leq x<y<\infty} \Big\{ g(x,y)[F(x)]^{r-1}[F(y)-F(x)]^{s-r-1} \\ & \times [1-F(y)]^{n-s} f(x) f(y) \Big\} dxdy. \\ = \ & C_{r,s:n}\, \theta^2 p^2 \iint\limits_{0\leq x<y<\infty} \Big\{ \sum_{i=0}^{r-1} (-1)^{r-1-i} \binom{r-1}{i} g(x,y)(1+px)^{r-1-i} \\ & \times \left[e^{-\theta x} - e^{-\theta y} + pxe^{-\theta x} - pye^{-\theta y}\right]^{s-r-1} [1+py]^{n-s} \\ & \times\ (1+x)\,(1+y)\, e^{-\theta[(r-i)x+(n-s+1)y]} \Big\} dxdy. \end{aligned} \tag{7}$$

Theorem 4
For $1 \leq r \leq n;\ n = 1, 2, \cdots$

$$\begin{aligned} & E\left[g\left(X_{r:n}, X_{s:n}\right)\right] - E\left[g\left(X_{r:n}, X_{s-1:n}\right)\right] \\ = \ & \frac{C_{r,s:n}}{(n-s+1)} \iint\limits_{0\leq x<y<\infty} \Big\{ \sum_{i=0}^{r-1} (-1)^{r-1-i} \binom{r-1}{i}\ g'(x,y)(1+px)^{r-1-i} \\ \times \ & \left[e^{-\theta x} - e^{-\theta y} + pxe^{-\theta x} - pye^{-\theta y}\right]^{s-r-1} [1+py]^{n-s+1}\ (1+x)\,(1+y) \\ \times \ & e^{-\theta[(r-i)x+(n-s+2)y]} dxdy \Big\}, \end{aligned} \tag{8}$$

where $g'(x,y) = \frac{d}{dy}\, g(x,y)$.

Proof. From (7), we have

$$\begin{aligned} & E\left[g\left(X_{r:n}, X_{s:n}\right)\right] - E\left[g\left(X_{r:n}, X_{s-1:n}\right)\right] \\ = \ & \frac{C_{r,s:n}}{(n-s+1)} \iint\limits_{0\leq x<y<\infty} \sum_{i=0}^{r-1} (-1)^{r-1-i} \binom{r-1}{i} g(x,y)(1+px)^{r-1-i} \Big[e^{-\theta x} - e^{-\theta y} \\ & + pxe^{-\theta x} - pye^{-\theta y s-r-2}\Big] [1+py]^{n-s}\ (1+x)\,(1+y)\, e^{-\theta[(r-i)x+(n-s+1)y]} \\ & [(n-r)\left\{1-(1+py)\,e^{-\theta y}\right\} - (n-s+1)\,\theta p\,(1+x)\,e^{-\theta x} - (n-s+1)]dx\, dy. \end{aligned} \tag{9}$$

Let

$$K(x,y) = -\left[e^{-\theta x} - e^{-\theta y} + pxe^{-\theta x} - pye^{-\theta y}\right]^{s-r-1} [1+py]^{n-s+1} e^{-(n-s+1)\theta y}.$$

Then the right hand side of (9) becomes,

$$\frac{\theta p\, C_{r,s:n}}{(n-s+1)} \iint_{0\le x<y<\infty} \sum_{i=0}^{r-1} (-1)^{r-1-i} \binom{r-1}{i} g(x,y)(1+px)^{r-1-i} e^{-\theta(r-1)x} K'(x,y)dx\, dy$$

where, $K'(x,y) = \frac{d}{dy} g(x,y)$ and hence the theorem (Ali and Khan, 1998).

The usual technique is establishing the recurrence relations will be to express $[1-F(x)]$ or $F(x)$ as a function of x and $f(x)$ and then obtain the general form relations.

Corollary 4
If $g(x,y) = x^i y^j$ and $1 \le r \le n; n = 1, 2, 3, \cdots$

$$\mu^{(i,j)}_{r,r+1:n} - \mu^{(i+j)}_{r:n}$$
$$= \sum_{m=0}^{r-1} \sum_{t=0}^{r-1-m} \sum_{u=0}^{n-r} (-1)^{r-1-m} \binom{r-1}{m} \binom{r-1-m}{t} \binom{n-r}{u} p^{n-m-t-u-1}$$
$$\times \iint_{0\le x<y\infty} x^{r-1-m-t+i} y^{n-r-u+j-i} (1+x)\, e^{-(r-m)\theta x} e^{-(n-r)\theta y} dxdy.$$

Proof. Set $s = r+1$ and $g(x,y) = x^i y^j$ in Theorem 4 and on algebraic simplification in the same line of Corollary 1 to get the required proof.

4. MLE for θ from multiply type-II censored sample

The likelihood function of θ based on the multiply type-II censored sample as

$$\begin{aligned} L(x;\theta) &= L\left(x_{r_1:n} \le x_{r_2:n} \le \cdots \le x_{r_k:n};\theta\right) \\ &= C[F(x_{r_1:n};\theta)]^{r_1-1} \prod_{i=2}^{k} \left[F(x_{r_i:n};\theta) - F\left(x_{r_{i-1}:n};\theta\right)\right]^{r_i-r_{i-1}-1} \\ &\quad \times [1-F(x_{r_k:n};\theta)]^{n-r_k} \prod_{i=1}^{k} f(x_{r_i:n};\theta) \qquad (10) \end{aligned}$$

Where, C is a constant free from θ. Taking $\log$ on both sides to get,

$$\begin{aligned} & \log L(x;\theta) \\ = \; & constant + (r_1 - 1)\log\left[F(x_{r_1:n};\theta)\right] \\ & + \sum_{i=2}^{k}(r_i - r_{i-1} - 1)\log\left[F(x_{r_i:n};\theta) - F\left(x_{r_{i-1}:n};\theta\right)\right] \\ & + (n - r_k)\log\left[1 - F(x_{r_k:n};\theta)\right. + \sum_{i=1}^{k}\log f(x_{r_i:n};\theta) \end{aligned} \tag{11}$$

Differentiating (11) with respect to θ and equate it to zero yields the likelihood equation for θ

$$\begin{aligned} h(\theta) = \; & \frac{\delta \log L}{\delta\theta} = (r_1 - 1)\,\frac{f(x_{r_1:n};\theta)}{F(x_{r_1:n};\theta)}\frac{\delta}{\delta\theta}f(x_{r_1:n};\theta) \\ & + \sum_{i=2}^{k}(r_i - r_{i-1} - 1)\,\frac{\left[f(x_{r_i:n};\theta) - f\left(x_{r_{i-1}:n};\theta\right)\right]}{\left[F(x_{r_i:n};\theta) - F\left(x_{r_{i-1}:n};\theta\right)\right]} \\ & \left[\frac{\delta}{\delta\theta}f(x_{r_i:n};\theta) - \frac{\delta}{\delta\theta}f\left(x_{r_{i-1}:n};\theta\right)\right] \\ & + (n - r_k)\,\frac{\left[-f(x_{r_k:n};\theta)\right.}{\left[1 - F(x_{r_k:n};\theta)\right.}\frac{\delta}{\delta\theta}f(x_{r_k:n};\theta) + \sum_{i=1}^{k}\frac{\frac{\delta}{\delta\theta}f(x_{r_k:n};\theta)}{f(x_{r_i:n};\theta)} \\ = \; & 0, \end{aligned} \tag{12}$$

where

$$\frac{\delta}{\delta\theta}f(x;\theta) = \left(\frac{2}{\theta} - x - \frac{1}{\theta + 1}\right)f(x;\theta).$$

The solution of (12) will be consistent, asymptotically normal and asymptotically efficient under some conditions.

For the multiply type-II censored data, let the gap between $x_{r_{i-1}:n}$ and $x_{r_i:n}$ is $(r_i - r_{i-1} - 1)$ and it is equal to the number of unobserved failures. Let maximum gap g, where $g = \max_i(r_i - r_{i-1} - 1)$.

Condition 1: For all most all x, the derivatives

$$\frac{\partial^i \log f(x,\theta)}{\partial\theta^i},\; i = 1, 2 \text{ and } \frac{\partial^{i+1}\log f(x,\theta)}{\partial x\,\partial\theta^i}, i = 1, 2, 3$$

exists, and are piecewise continuous for all θ belongs to a non-degenerate interval I and $x \in [0, \infty)$.

Condition 2: There exist nonnegative numbers a_1, a_2, λ_{ij}, $i = 1, 2$, $j = 1, 2, \cdots, 5$, such that when θ is in some neighborhood of true value θ_0, and

x is large enough,

$$\left|\frac{\partial^i \log f(x,\theta)}{\partial\theta^i}\right| \leq a_1 x^{\lambda_{1i}}, \quad i = 1, 2,$$

$$\left|\frac{\partial^{i+1} \log f(x,\theta)}{\partial x\, \partial\theta^i}\right| \leq a_1 x^{\lambda_{1(i+2)}} \ i = 1, 2, 3,$$

and when x is small enough and close enough to zero,

$$\left|\frac{\partial^i \log f(x,\theta)}{\partial\theta^i}\right| \leq a_2 x^{-\lambda_{2i}}, \ i = 1, 2,$$

$$\left|\frac{\partial^{i+1} \log f(x,\theta)}{\partial x\, \partial\theta^i}\right| \leq a_2 x^{-\lambda_{2(i+2)}}, \ i = 1, 2, 3.$$

Also assume that there exists a function G(x) for each $\theta \in \mathcal{R}$,

$$\left|\frac{\partial^3 \log f(x,\theta)}{\partial\theta^3}\right| \leq G(x), \ \text{for} - \infty < x < \infty$$

and there exists K independent of θ such that

$$\int_{-\infty}^{\infty} G(x) f(x,\theta) dx \leq K < \infty.$$

Condition 3: There exist positive numbers δ_1 and δ_2 for large enough x such that

$$f(x,\theta) \geq \delta_1 [1 - F(x,\theta)]^{\delta_2}$$

There exist positive numbers δ_3 and δ_4 for small enough x such that

$$f(x,\theta) \geq \delta_3 [F(x,\theta)]^{\delta_4}.$$

Condition 4: For each θ in I, the integral,

$$0 < \int_{-\infty}^{\infty} \left(\frac{\partial \log f(x,\theta)}{\theta}\right)^2 f(x,\theta)\, dx < \infty.$$

The conditions may be modified for multiple parameter case. Thus if the maximum gap g is always bounded the likelihood equation (12) has a solution converging in probability to the true value θ_0 as $n \to \infty$, i.e., $\lim_{n\to\infty} \theta_n \xrightarrow{p} \theta_0$. If the maximum gap g is always bounded the solution of (12) is an asymptotically normal and asymptotically efficient estimate of the true value θ_0.
Under the conditions mentioned above and let $\lim_{n\to\infty} n\, e^{-(n/g)\varepsilon} \to 0$ for any $\varepsilon > 0$. We assume $\frac{r_{i+1}-r_i-1}{r_i-1}$ on the left tail or $\frac{r_{i+1}-r_i-1}{n-r_{i+1}-1}$ on the right tail bounded at two tails of order statistics then the MLE is consistent, see Kong and Fei (1996).

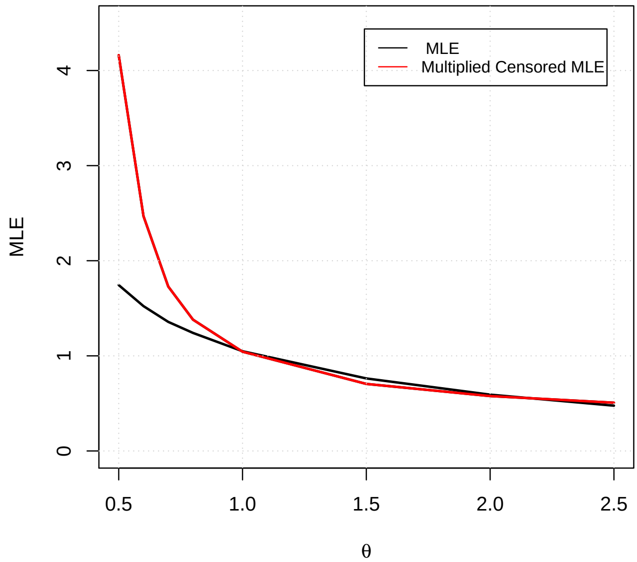

Figure 1. Simulated MLE curves for $n = 30$ and $m = 5$.

5. Simulation Study

Usual algebraic solution for the equation (12) is not working due to the properties of transcendental equation. Therefore, fixed point iteration can be used to solve the above equation. For an initial value $\theta^{(1)}$, the $(i+1)th$ iterate $\theta^{(i+1)}$ can be obtained from the ith iterate $\theta^{(i+1)}$ using $\theta^{(i+1)} = h(\theta^{(i)})$. The iterative procedure can be stopped if $|\theta^{(i+1)} - \theta^{(i)}| < \varepsilon$, where ε is a pre-assigned small positive number. The procedure of estimation is repeated 5000 times for each value of θ with sample size, $n = 30$, multiply type-II censored sample size, $m = 5, 7, 9, 12, 15, 20$ are presented in Table 5.2 and with $n = 100$, $m = 5, 7, 9, 12, 15, 20$ are presented in Table 5.3. A graphical comparison between MLE obtained from the complete case and MLE obtained from multiply censored sample are presented in Figure 1 and Figure 2 for sample size $n = 30$ and 100 respectively.

From Figure 1, it is observed that MLE and MLE from multiply censored sample of θ are biased except for $\theta = 1.0$. For $\theta < 1.0$, both estimates are positively biased where as MLE from censored sample is highly biased than MLE.

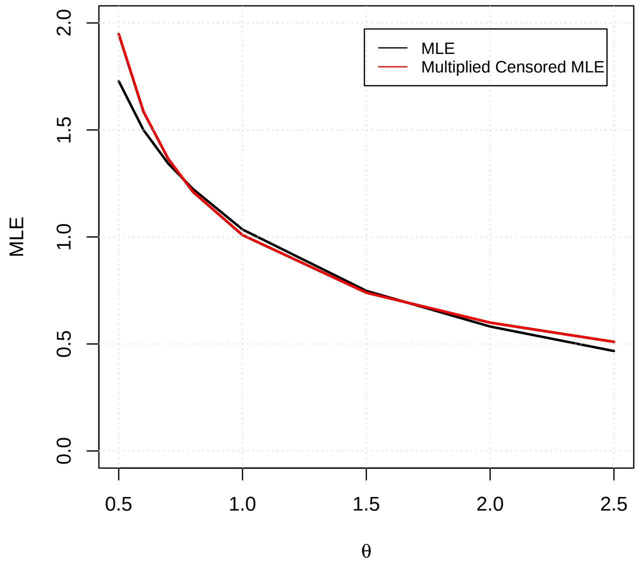

Figure 2. Simulated MLE curves for $n = 100$ and $m = 5$.

On the other hand when $\theta > 1.0$, both estimators are negatively biased with almost same value. For $\theta > 2.0$, the MLE from multiplied censored sample admits the better performance than MLE in complete case. From Figure 2, it is also observed that MLE and MLE from multiply censored sample of θ are biased except for $\theta = 1.0$. For $\theta < 1.0$, both estimates are positively biased where as MLE from censored sample is moderately higher biased than MLE. On the other hand when $\theta > 0.7$, both estimators gives the same estimate and gradually goes to unbiased at $\theta = 1.0$. After then increase the negatively bias. Performance of MLE from multiply censored sample goes to superior than MLE as increase the sample size $n(= 100)$.

Table I. Estimated values of θ for $n = 30$ and $m = 5, 7, 9, 12, 15, 20$.

n	θ	MLE $(\widehat{\theta}_n)$	S.E. $(\widehat{\theta}_n)$	m	MLE $(\widehat{\theta}_m)$	S.E. $(\widehat{\theta}_m)$
30	0.5	1.74424	0.23863	5	4.16115	108.29687
				7	5.19627	125.29512
	0.6	1.52288	0.21213	5	2.46926	142.30428
				7	3.09672	151.48994
				9	3.68743	168.32375
				12	4.07592	174.96544
	0.7	1.35711	0.18284	5	1.72878	122.41613
				7	2.00455	131.08574
				9	2.46257	145.54077
				12	2.81153	159.97457
	0.8	1.24109	0.17249	5	1.38062	097.49470
				7	1.49723	105.62920
				9	1.66574	118.70461
				12	2.00986	147.58269
				15	2.24177	150.52720
				20	2.27611	152.30640
	1.0	1.04838	0.15706	5	1.04366	73.009340
				7	1.04757	73.213190
				9	1.07673	73.741700
				12	1.09524	78.071560
				15	1.14241	84.400380
				20	1.15221	85.118260
	1.5	0.76252	0.13409	5	0.70500	48.633980
				7	0.67805	46.252740
				9	0.61697	44.903430
				12	0.57415	39.684760
				15	0.50341	34.864700
				20	0.45744	24.723070
	2.0	0.59196	0.11506	5	0.57714	35.824450
				7	0.53536	34.860380
				9	0.48571	33.270660
				12	0.42460	29.088130
				15	0.35889	24.479550
				20	0.29724	18.663180
	2.5	0.47623	0.10041	5	0.50720	26.022670
				7	0.48070	24.274150
				9	0.40256	23.434160
				12	0.34855	21.704760
				15	0.29850	18.815920
				20	0.22257	14.651580

Table II. Estimated values of θ for $n = 30$ and $m = 5, 7, 9, 12, 15, 20$.

n	θ	MLE $(\widehat{\theta}_n)$	S.E. $(\widehat{\theta}_n)$	m	MLE $(\widehat{\theta}_m)$	S.E. $(\widehat{\theta}_m)$
100	0.5	1.72839	0.13082	5	1.94828	135.31219
				7	2.11653	146.56967
	0.6	1.49979	0.11160	5	1.58527	110.17368
				7	1.65039	114.19110
				9	1.73228	119.50743
				12	1.87486	128.15120
	0.7	1.34292	0.09419	5	1.36471	95.611090
				7	1.39678	98.057820
				9	1.42720	99.721370
				12	1.49337	103.41175
	0.8	1.22417	0.09121	5	1.21048	85.17485
				7	1.22851	86.56835
				9	1.24142	87.13272
				12	1.26858	88.58547
				15	1.29882	90.13348
				20	1.35520	93.59640
	1.0	1.03487	0.08526	5	1.00874	70.41334
				7	1.00868	70.26557
				9	1.00869	70.30250
				12	1.01051	69.96724
				15	1.00845	69.70745
				20	1.01126	69.59897
	1.5	0.74844	0.06983	5	0.73956	51.45035
				7	0.73035	50.81887
				9	0.72242	50.19667
				12	0.70901	49.10274
				15	0.69596	47.99910
				20	0.67256	46.18182
	2.0	0.58144	0.06097	5	0.59953	41.67866
				7	0.58909	41.34870
				9	0.58000	40.57883
				12	0.56468	39.30672
				15	0.55145	38.23132
				20	0.52870	36.21664
	2.5	0.46687	0.05465	5	0.50991	27.84197
				7	0.49940	34.61033
				9	0.49127	33.90684
				12	0.47541	32.56191
				15	0.46303	31.67288
				20	0.44056	29.90937

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