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Solving Burger's equation by semi-analytical method and implicit method

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Abstract In this work, the modified Laplace Adomian decomposition method (LADM) is applied to solve the Burgers' equation. In addition, example that illustrate the pertinent features of this method is presented, and the results of the study is discussed. We prove the convergence of LADM applied to the Burgers' equation. Also, Burgers' equation has some discontinuous solutions because of effects of viscosity term. These discontinuities raise phenomenon of shock waves. Some explicit and implicit numerical methods have been experimented on Burgers' equation but these schemes have not been seen proper in this case because of their conditional stability and existence of viscosity term. We consider two types of box schemes and implement on Burgers' equation to get better results with no artificial wiggles.

Keywords Modified Laplace Adomian decomposition method (LADM), Burgers' equation, explicit and implicit numerical methods

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1. Introduction

In the recent decade, the study of nonlinear partial differential equations modelling physical phenomena, has become an important tool. Nonlinear phenomena are of fundamental importance in various fields of science and engineering. Most nonlinear phenomena are models of our real–life problems. The investigation of the travelling wave solutions plays an important role in nonlinear science.

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A variety of powerful methods have been presented, such as the inverse scattering transform [1], sine-cosine method [2], homotopy perturbation method [3], variational iteration method [4, 14], homotopy analysis method [6, 7], tanh-function method [8, 9], tanh-coth method [10], Bäcklund transformation [11], $(\frac{G'}{G})$-expansion method [12] and so on.

In 1980, George Adomian introduced a new method to solve nonlinear functional equations. This method has since been termed the Adomian decomposition method (ADM) and has been the subject of many investigations [13, 14, 15]. The ADM involves separating the equation under investigation into linear and nonlinear portions. This method generates a solution in the form of a series whose terms are determined by a recursive relation using the Adomian polynomials. Some fundamental works on various aspects of modifications of the Adomian's decomposition method are given by Andrianov [16], Venkatarangan [17, 18] and Wazwaz [19]. The modified form of Laplace decomposition method has been introduced by Khuri [20, 21]. Agadjanov [22] solved the Duffing equation by this method. This numerical technique basically illustrates how the Laplace transform may be used to approximate the solutions of the nonlinear partial differential equations by manipulating the decomposition method. Elgasery [23] applied the Laplace decomposition method for the solution of Falkner–Skan equation. Hussain and Khan in [24] the modified Laplace decomposition method have applied for solving some PDEs. The Burgers' equation [25, 26]

$$u_t + uu_x = u_{xx}, \tag{1}$$

is a nonlinear partial differential equation of second order which appears in various areas of applied mathematics, such as modelling of fluid dynamics, turbulence, boundary layer behavior, shock wave formation, and traffic flow [27]. Burgers' equation is parabolic when the viscous term is included. If the viscous term is neglected, the remaining equation is hyperbolic. If the viscous term is dropped from the Burgers' equation the nonlinearity allows discontinuous solutions to develop. In Burgers' equation discontinuities may appear in finite time, even if the initial condition is smooth. They give rise to the phenomenon of shock waves with important applications in physics [28]. These properties make Burgers' equation a proper model for testing numerical algorithms in flows where severe gradients or shocks are anticipated. Discretization methods are well-known techniques for solving Burgers' equation. One of the most simple one is leap-frog explicit scheme [29] which was proposed in the 1960s. This explicit scheme is very easy to formulate but fails to give a correct solution when the viscosity is too small. To avoid these unstable conditions, implicit methods such as CranckNicolson type scheme is presented, but this scheme cannot be used for very small viscosities.

A variety of powerful methods has been presented, such as the homotopy analysis method [32, 33], homotopy perturbation method [34], the Exp-function method [36], variational iteration method [35] and the Adomian decomposition method [35]. By using the LADM we obtain analytical solutions for the integro-differential equations. Our aim in this paper is to obtain the numerical and analytical solutions by using the modified Laplace Adomian decomposition method and explicit and implicit numerical methods. The remainder of the paper is organized as follows: In Sections 2 and 3, a brief discussions for the modified Laplace Adomian decomposition method and application of this method are presented and approximate solution for one example is obtained. In Section 4 and 5, numerical methods and numerical results are discussion. Also, in Section 6 we will study the convergence analysis. Also a conclusion is given in Section 7. Section 8 ends this paper with a brief conclusion.

2. MLADM

The purpose of this section is to discuss the use of modified Laplace decomposition algorithm for the integro-differential equations. We consider the general form of second order nonlinear partial differential equations with initial conditions in the form

$$\mathrm{Lu(x,t)} + \mathrm{Ru(x,t)} + \mathrm{Nu(x,t)} = \mathrm{h(x,t)}, \quad \mathrm{u(x,0)} = \mathrm{f(x)}, \quad \mathrm{u_t(x,0)} = \mathrm{g(x)}, \tag{2}$$

where L is the second order differential operator $\mathrm{L_{xx}} = \frac{\partial^n}{\partial x^n}$, R is the remaining linear operator, N represents a general non-linear differential operator and $\mathrm{h(x,t)}$ is the source term. Applying Laplace transform (denoted by $\mathcal{L}$) on both sides of Eq. (2) we have

$$\mathcal{L}[\mathrm{Lu(x,t)}] + \mathcal{L}[\mathrm{Ru(x,t)}] + \mathcal{L}[\mathrm{Nu(x,t)}] = \mathcal{L}[\mathrm{h(x,t)}],$$

and by using the differentiation property of Laplace transform we obtain:

$$s^2\mathcal{L}[\mathrm{u(x,t)}] - \mathrm{sf(x)} - \mathrm{g(x)} + \mathcal{L}[\mathrm{Ru(x,t)}] + \mathcal{L}[\mathrm{Nu(x,t)}] = \mathcal{L}[\mathrm{h(x,t)}],$$

and so:

$$\mathcal{L}[\mathrm{u(x,t)}] = \frac{\mathrm{f(x)}}{s} + \frac{\mathrm{g(x)}}{s^2} - \frac{1}{s^2}\mathcal{L}[\mathrm{Ru(x,t)}] - \frac{1}{s^2}\mathcal{L}[\mathrm{Nu(x,t)}] + \frac{1}{s^2}\mathcal{L}[\mathrm{h(x,t)}]. \tag{3}$$

The next step in Laplace decomposition method is representing the solution as an infinite series given below

$$\mathrm{u(x,t)} = \sum_{n=0}^{\infty} \mathrm{u_n(x,t)}. \tag{4}$$

The nonlinear operator is decomposed as

$$\mathrm{Nu(x,t)} = \sum_{n=0}^{\infty} \mathrm{A_n(x,t)}, \tag{5}$$

where for every $n \in N$ A_n is the Adomian polynomial given below

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[N\left(\sum_{i=0}^{\infty}\lambda^i u_i\right)\right]_{\lambda=0}.$$

Using (3), (4) and (5) we get

$$\sum_{n=0}^{\infty}\mathcal{L}[u_n(x,t)] = \frac{f(x)}{s} + \frac{g(x)}{s^2} - \frac{1}{s^2}\mathcal{L}[Ru(x,t)] - \frac{1}{s^2}\mathcal{L}\left[\sum_{n=0}^{\infty}A_n(x,t)\right] + \frac{1}{s^2}\mathcal{L}[h(x,t)]. \quad (6)$$

Comparing both sides of (6) we have

$$\mathcal{L}[u_0(x,t)] = k_1(x,s), \quad (7)$$

$$\mathcal{L}[u_1(x,t)] = k_2(x,s) - \frac{1}{s^2}\mathcal{L}[R_0u(x,t)] - \frac{1}{s^2}\mathcal{L}\left[A_0(x,t)\right], \quad (8)$$

$$\mathcal{L}[u_{n+1}(x,t)] = -\frac{1}{s^2}\mathcal{L}[R_nu(x,t)] - \frac{1}{s^2}\mathcal{L}\left[A_n(x,t)\right], \qquad n \geq 1, \quad (9)$$

where $k_1(x,s)$ and $k_2(x,s)$ are Laplace transform of $k_1(x,t)$ and $k_2(x,t)$ respectively. Applying the inverse Laplace transform to Eqs. (7)–(9) gives our required recursive relation as follows

$$u_0(x,t) = k_1(x,t), \quad (10)$$

$$u_1(x,t) = k_2(x,t) - \mathcal{L}^{-1}\left[\frac{1}{s^2}\mathcal{L}[R_0u(x,t)] - \frac{1}{s^2}\mathcal{L}\left[A_0(x,t)\right]\right], \quad (11)$$

$$u_{n+1}(x,t) = -\mathcal{L}^{-1}\left[\frac{1}{s^2}\mathcal{L}[R_nu(x,t)] - \frac{1}{s^2}\mathcal{L}\left[A_n(x,t)\right]\right], \qquad n \geq 1. \quad (12)$$

The solution through the modified Adomian decomposition method highly depends upon the choice of $k_0(x,t)$ and $k_1(x,t)$, where $k_0(x,t)$ and $k_1(x,t)$ represent the terms arising from the source term and prescribed initial conditions.

3. Application of the modified Adomian decomposition method

In this section we give one example to illustrate this method for the Burgers' equation.

Example 1. Consider a nonlinear partial differential equation

$$u_t + uu_x = u_{xx}, \qquad u(x,0) = \frac{1}{2} - \frac{1}{2}\tanh\left(\frac{x}{4}\right). \quad (13)$$

Applying the Laplace transform (denoted by $\mathcal{L}$) we have

$$\mathrm{su}(\mathrm{x},\mathrm{s})-\mathrm{u}(\mathrm{x},0)=-\mathcal{L}(\mathrm{uu_x})+\mathcal{L}(\mathrm{u_{xx}}),$$

or

$$\mathrm{u}(\mathrm{x},\mathrm{s})=\frac{1}{\mathrm{s}}\mathrm{u}(\mathrm{x},0)-\frac{1}{\mathrm{s}}\mathcal{L}(\mathrm{uu_x})+\frac{1}{\mathrm{s}}\mathcal{L}(\mathrm{u_{xx}}).$$

Using initial condition (13) becomes

$$\mathrm{u}(\mathrm{x},\mathrm{s})=\frac{1}{2\mathrm{s}}-\frac{1}{2\mathrm{s}}\tanh\left(\frac{\mathrm{x}}{4}\right)-\frac{1}{\mathrm{s}}\mathcal{L}(\mathrm{uu_x})+\frac{1}{\mathrm{s}}\mathcal{L}(\mathrm{u_{xx}}).$$

Applying the inverse Laplace transform we get

$$\mathrm{u}(\mathrm{x},\mathrm{t})=\frac{1}{2}-\frac{1}{2}\tanh\left(\frac{\mathrm{x}}{4}\right)+\mathcal{L}^{-1}\left[-\frac{1}{\mathrm{s}}\mathcal{L}(\mathrm{uu_x})+\frac{1}{\mathrm{s}}\mathcal{L}(\mathrm{u_{xx}})\right]. \tag{14}$$

We decompose the solution as an infinite sum given below

$$\mathrm{u}(\mathrm{x},\mathrm{t})=\sum_{\mathrm{n}=0}^{\infty}\mathrm{u_n}(\mathrm{x},\mathrm{t}). \tag{15}$$

Using (15) on (14) we get

$$\sum_{\mathrm{n}=0}^{\infty}\mathrm{u_n}(\mathrm{x},\mathrm{t})=\frac{1}{2}-\frac{1}{2}\tanh\left(\frac{\mathrm{x}}{4}\right)+\mathcal{L}^{-1}\left[-\frac{1}{\mathrm{s}}\mathcal{L}\left(\sum_{\mathrm{n}=0}^{\infty}\mathrm{A_n}(\mathrm{t})\right)+\frac{1}{\mathrm{s}}\mathcal{L}\sum_{\mathrm{n}=0}^{\infty}\mathrm{u_{n,xx}}(\mathrm{x},\mathrm{t})\right],$$

in which $\mathrm{A_n}=\sum_{\mathrm{j}=0}^{\mathrm{n}}\mathrm{u_j}\mathrm{u_{n-j,x}}$. The recursive relation is given below

$$\mathrm{u_0}(\mathrm{x},\mathrm{t})=\frac{1}{2}-\frac{1}{2}\tanh\left(\frac{\mathrm{x}}{4}\right),$$

$$\mathrm{u_1}(\mathrm{x},\mathrm{t})=\frac{1}{16}\mathrm{t}\left[1-\tanh^2\left(\frac{\mathrm{x}}{16}\right)\right],$$

$$\mathrm{u_2}(\mathrm{x},\mathrm{t})=\frac{1}{128}\mathrm{t}^2\left[\tanh\left(\frac{\mathrm{x}}{4}\right)-\tanh^3\left(\frac{\mathrm{x}}{4}\right)\right],$$

and so on. We use an 8-term approximation and set

$$\mathrm{app7}:=\mathrm{u_0}+\mathrm{u_1}+\mathrm{u_2}+...+\mathrm{u_7}.$$

the maximum error occurs in the x-interval $(-2,2)$, so we have tabulated the absolute errors for various times on this interval, in Table 1.

Our approximation has one more interesting property, we expand app7 using Taylors expansion about (0,0) we would have

$$\mathrm{app}(\mathrm{x},\mathrm{t})\cong\frac{1}{2}-\frac{1}{8}\mathrm{x}+\frac{1}{16}\mathrm{t}+\frac{1}{384}\mathrm{x}^3-\frac{1}{256}\mathrm{tx}^2+\frac{1}{512}\mathrm{t}^2\mathrm{x}-\frac{1}{3072}\mathrm{t}^3+\frac{1}{491520}\mathrm{t}^5-$$

$$\frac{1}{49152}\mathrm{xt}^4 + \frac{1}{12288}\mathrm{x}^2\mathrm{t}^3 - \frac{1}{6144}\mathrm{t}^2\mathrm{x}^3 + ...,$$

using the Taylor series gives the exact solution

$$\mathrm{u}(\mathrm{x},\mathrm{t}) = \frac{1}{2} - \frac{1}{2}\tanh\left[\frac{1}{4}\left(\mathrm{x} - \frac{1}{2}\mathrm{t}\right)\right]. \tag{16}$$

Table I. Absolute errors of app7 for Burgers' equation using LADM.

x	$t = 0.1$	$t = 0.5$	$t = 1.0$	$t = 1.5$	$t = 2.0$
0.1	1.39×10^{-18}	4.14×10^{-13}	6.50×10^{-11}	6.30×10^{-10}	3.87×10^{-9}
0.5	6.47×10^{-18}	2.43×10^{-12}	5.92×10^{-10}	1.43×10^{-8}	1.34×10^{-7}
1.0	9.03×10^{-18}	3.52×10^{-12}	8.99×10^{-10}	2.28×10^{-8}	2.25×10^{-7}
1.5	6.89×10^{-18}	2.75×10^{-12}	7.25×10^{-10}	1.90×10^{-8}	1.94×10^{-7}
2.0	2.43×10^{-18}	1.03×10^{-12}	2.89×10^{-10}	8.07×10^{-8}	8.71×10^{-8}

4. Numerical methods

4.1. Some discretization methods

The Burgers' equation is given by

$$\frac{\partial \mathrm{u}}{\partial \mathrm{t}} = -\mathrm{u}\frac{\partial \mathrm{u}}{\partial \mathrm{x}} + \mathrm{c}\frac{\partial^2 \mathrm{u}}{\partial \mathrm{x}^2}, \tag{17}$$

we consider this equation in the case of initial condition with periodic boundary condition. There are several numerical methods for solving Burgers' equation based on discritization on a fixed grid for both space and time variables.

$$\mathrm{u_i}^{\mathrm{n}+1} = \mathrm{u_i}^{\mathrm{n}-1} + \frac{\mu}{6}(\mathrm{u}^{\mathrm{n}}_{\mathrm{i}-1} + \mathrm{u}^{\mathrm{n}}_{\mathrm{i}} + \mathrm{u}^{\mathrm{n}}_{\mathrm{i}+1})(\mathrm{u}^{\mathrm{n}}_{\mathrm{i}-1} - \mathrm{u}^{\mathrm{n}}_{\mathrm{i}+1}) + \frac{\mathrm{c}}{\Delta \mathrm{x}}(\mathrm{u}^{\mathrm{n}}_{\mathrm{i}-1} - \mathrm{u}^{\mathrm{n}}_{\mathrm{i}} + \mathrm{u}^{\mathrm{n}}_{\mathrm{i}+1}), \tag{18}$$

In the above formula the artificial boundary at $\mathrm{n} = 0$ is approximated by $\mathrm{u_i}^{-1} = \mathrm{u_i}^1$. By approximating the artificial boundary with extrapolation methods along characteristics, better results can be extracted. In this scheme $\mathrm{x} = \mathrm{x_i} = \mathrm{i}\Delta\mathrm{x}, \mathrm{t} = \mathrm{t_n} = \mathrm{n}\Delta\mathrm{t}$ and $\mu = \frac{\Delta \mathrm{t}}{\Delta \mathrm{x}}$, by

$$\mathrm{u}(\mathrm{x}, 0) = \sin(\pi \mathrm{x}),\ \mathrm{c} = 10^{-3},\ \Delta\mathrm{x} = 0.005,\ \Delta\mathrm{t} = 0.01.$$

This method is conditionally stable [1]. Another numerical scheme is the implicit CrankNicolson method, its formulation is

$$\frac{\Delta \mathrm{u_j}^{\mathrm{n}+1}}{\Delta \mathrm{t}} = -\frac{\mathrm{L_x}(\mathrm{u_j}^{\mathrm{n}} + \mathrm{u_j}^{\mathrm{n}+1})}{2} + \mathrm{c}\frac{\mathrm{L_{xx}}(\mathrm{u_j}^{\mathrm{n}} + \mathrm{u_j}^{\mathrm{n}+1})}{2}, \tag{19}$$

where $\Delta \mathrm{u_j}^{\mathrm{n+1}} = \mathrm{u_j}^{\mathrm{n+1}} - \mathrm{u_j}^{\mathrm{n}}$ and $\mathrm{L_x} = \frac{(-1,0,1)}{2\Delta\mathrm{x}}$, $\mathrm{L_{xx}} = \frac{(1,-2,1)}{(\Delta\mathrm{x})^2}$. By reducing these equations to a system of linear equations, we can overcome on the effect of nonlinear terms appeared in (19) and then we can find the solutions. In the case of very small viscosity, wiggles appear and cause the solution to be perturbed. In this situation other numerical schemes must be used. In the following we show the $c = 10^{-4}$ at the top and the bottom of the shock wiggles. This implicit scheme is unconditionally stable by Von Neumann criteria and has a truncation error of order $\mathrm{O(h^2, k^2)}$. As we see there are some problems in the long time solution of Burgers' equation which depend on the viscosity term. In this case we examine some symplectic and multisymplectic box methods on Burgers' equation. These methods are very high quality schemes for the long time integration of nonlinear, conservative partial differential equations [1,2]. These numerical schemes are constructed on Finite Difference Discretization (FDD) which are represented as explicit and implicit discretization methods. Among these methods the semi-explicit symplectic box method is very effective because of ensuring that no artificial wiggles appear in the approximate solution.

4.2. Multisymplectic box scheme for Burgers' Equation

This type of compact discretization method in both x and t is centered at a cell (box), whose corners are

$$(\mathrm{x_i}, \mathrm{t_n}),\ \ (\mathrm{x_i}, \mathrm{t_{n+1}}),\ \ (\mathrm{x_{i+1}}, \mathrm{t_n}),\ \ (\mathrm{x_{i+1}}, \mathrm{t_{n+1}}).$$

Based on this compact scheme some multisymplectic box and fully implicit narrow box schemes can be constructed. We apply two 12-point and 8-point multisymplectic schemes on Burgers' equation. For applying these discretization methods on Burgers' equation, Burgers' equation can be represented by the following formula:

$$\mathrm{u_t} = -\mathrm{u}\mathrm{u_x} + \mathrm{c}\mathrm{u_{xx}} = -\mathrm{V}'(\mathrm{u}) + \mathrm{c}\mathrm{u_{xx}},\ \ \mathrm{V}(\mathrm{u}) = \frac{\mathrm{u}^3}{6}. \tag{20}$$

By applying the 8-point multismyplectic schemes on Burgers' equation we have the following:

$$\begin{aligned} u_t &= \frac{1}{8\Delta t}\begin{bmatrix} 1 & 3 & 3 & 1 \\ -1 & -3 & -3 & -1 \end{bmatrix} u \\ &= \frac{1}{2\Delta x}\begin{bmatrix} -1 & 0 & 1 \end{bmatrix} V'\left(\frac{1}{4}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} u\right) + \frac{1}{2(\Delta x)^2}\begin{bmatrix} 1 & -2 & 1 \\ 1 & -2 & 1 \end{bmatrix} u. \end{aligned}$$

Also by applying the 12-point multisymplectic scheme on Burgers' equation the following discretization method with stencil notation is resulted:

$$u_t = \frac{1}{16\Delta t}\begin{bmatrix} 1 & 3 & 3 & 1 \\ 0 & 0 & 0 & 0 \\ -1 & -3 & -3 & -1 \end{bmatrix} u$$

$$= -\frac{1}{4\Delta x}\begin{bmatrix} -1 & 0 & 1 \\ -1 & 0 & 1 \end{bmatrix} V'\left(\frac{1}{4}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} u\right) + \frac{1}{4(\Delta x)^2}\begin{bmatrix} 1 & -2 & 1 \\ 2 & -4 & 2 \\ 1 & -2 & 1 \end{bmatrix} u.$$

As we saw already in constructing the two above schemes, when using the 8-point scheme for initializing 12- point scheme identical results up to round off error level are obtained [1].

Table II. Error results of Example 1 for $t = 0.05$.

Method	Infinity-norm	2-Norm
8-Point multisymplectic box	7×10^{-2}	3×10^{-1}
12-Point multisymplectic box	10^{-4}	55×10^{-5}
Crank–Nicolson	5×10^{-1}	55×10^{-2}

One example of the Burgers' equation is considered in this section by multisymplectic box methods. The multisymplectic scheme is fully implicit scheme and is more accurate than explicit and semi-explicit methods and has stability for large time steps and different parameters [38, 39]. The one above example showed that if we want to get more accurate solution in longer times the finer mesh is required. In the latter case the set of equations will be very large and it takes more time and memory for solving, which we did not examine it in this article. In the case of steady state and dispersion the stability analysis of these schemes can be found in [38].

5. Numerical results

We have examined some well-known numerical methods and two 8-point and 12-point multisymplectic box methods on Burgers' equation.

Example 1. Consider Burgers' equation $u_t = uu_x + cu_{xx}$ with the following exact solution:

$$u(x,t) = \frac{\sinh(\frac{x}{2c})}{\cosh(\frac{x}{2c}) + \exp(-\frac{t}{4c})}.$$

The viscosity term in this equation is $c = 0.00075$ and $x \in [0,1], t \geq 0$. We can find that the 12- point multisymplectic box method is more accurate than 8-point multisymplectic box method and in both cases the discretization error is decreasing exponentially when x grows. We have examined this algorithm for large time and variety of viscosity terms too. The errors with $h = 0.05, k = 0.01$ are presented in Table 1.

6. Convergence analysis

Here, we will study the convergence analysis as same manner in [37] of the LADM applied to the Burgers' equation. Let us consider the Hilbert space H which may define by $\mathrm{H} = \mathrm{L}^2((\alpha, \beta)\mathrm{X}[0, \mathrm{T}])$ the set of applications:

$$\mathrm{u} : (\alpha, \beta)\mathrm{X}[0, \mathrm{T}] \rightarrow \mathrm{R} \quad \text{with} \quad \int_{(\alpha,\beta)\mathrm{x}[0,\mathrm{T}]} \mathrm{u}^2(\mathrm{x}, \mathrm{s})\mathrm{d}\mathrm{s}\mathrm{d}\tau < +\infty.$$

Now we consider the Burgers' equation in the light of above assumptions and let us denote

$$\mathrm{L}(\mathrm{u}) = \frac{\partial}{\partial \mathrm{t}}\mathrm{u},$$

then the Burgers' equation become in a operator form

$$\mathrm{L}(\mathrm{u}) = -\mathrm{u}\frac{\partial}{\partial \mathrm{x}}\mathrm{u} + \mathrm{c}\frac{\partial^2}{\partial \mathrm{x}^2}\mathrm{u},$$

The LADM is convergence if the following two hypotheses are satisfied:
(H1) $(\mathrm{L}(\mathrm{u}) - \mathrm{L}(\mathrm{v}), \mathrm{u} - \mathrm{v}) \geq \mathrm{k} \parallel \mathrm{u} - \mathrm{v} \parallel^2; \mathrm{k} > 0, \forall \mathrm{u}, \mathrm{v} \in \mathrm{H}$
(H2) whatever may be $\mathrm{M} > 0$, there exist a constant $\mathrm{C}(\mathrm{M}) > 0$ such that for $\mathrm{u}, \mathrm{v} \in \mathrm{H}$ with $\parallel \mathrm{u} \parallel \leq \mathrm{M}, \parallel \mathrm{v} \parallel \leq \mathrm{M}$ we have: $(\mathrm{L}(\mathrm{u}) - \mathrm{L}(\mathrm{v}), \mathrm{u} - \mathrm{v}) \leq \mathrm{C}(\mathrm{M}) \parallel \mathrm{u} - \mathrm{v} \parallel \parallel \mathrm{w} \parallel$ for every $\mathrm{w} \in \mathrm{H}$. (see, [37] and the references therein).

Theorem 1. (Sufficient condition of convergence for example 1). The Laplace Adomian method applied to the Burgers' equation as follows

$$\mathrm{L}(\mathrm{u}) = \frac{\partial}{\partial \mathrm{t}}\mathrm{u} = -\mathrm{u}\frac{\partial}{\partial \mathrm{x}}\mathrm{u} + \mathrm{c}\frac{\partial^2}{\partial \mathrm{x}^2}\mathrm{u},$$

without initial condition, converges towards a particular solution.

Proof. Now, we will verify the conditions (H1) and (H2) of convergence. We will start to verify the convergence hypotheses (H1) for the operator $\mathrm{L}(\mathrm{u})$: i.e., $\exists \mathrm{k} > 0, \forall \mathrm{u}, \mathrm{v} \in \mathrm{H}$, we have:

$$\mathrm{L}(\mathrm{u}) - \mathrm{L}(\mathrm{v}) = -\left[\mathrm{u}\frac{\partial}{\partial \mathrm{x}}\mathrm{u} - \mathrm{v}\frac{\partial}{\partial \mathrm{x}}\mathrm{v}\right] + \mathrm{c}\frac{\partial^2}{\partial \mathrm{x}^2}(\mathrm{u} - \mathrm{v}) = -\frac{1}{2}\frac{\partial}{\partial \mathrm{x}}[\mathrm{u}^2 - \mathrm{v}^2] + \mathrm{c}\frac{\partial^2}{\partial \mathrm{x}^2}(\mathrm{u} - \mathrm{v}).$$

Then we get

$$(\mathrm{L}(\mathrm{u}) - \mathrm{L}(\mathrm{v}), \mathrm{u} - \mathrm{v}) = \frac{1}{2}\left(-\frac{\partial}{\partial \mathrm{x}}(\mathrm{u}^2 - \mathrm{v}^2), \mathrm{u} - \mathrm{v}\right) + \mathrm{c}\left(\frac{\partial^2}{\partial \mathrm{x}^2}(\mathrm{u} - \mathrm{v}), \mathrm{u} - \mathrm{v}\right). \tag{21}$$

Since $\frac{\partial}{\partial x}$ and $\frac{\partial^2}{\partial x^2}$ are differential operators in H, then there exists constants λ_1 and λ_2, such that

$$\left(\frac{\partial^2}{\partial \mathrm{x}^2}(\mathrm{u} - \mathrm{v}), \mathrm{u} - \mathrm{v}\right) \leq \lambda_1 \parallel \mathrm{u} - \mathrm{v} \parallel^2, \tag{22}$$

and according the Schwartz inequality, we get

$$\left(\frac{\partial}{\partial \mathrm{x}}(\mathrm{u}^2-\mathrm{v}^2), \mathrm{u}-\mathrm{v}\right) \leq \lambda_2 \parallel \mathrm{u}^2-\mathrm{v}^2 \parallel\parallel \mathrm{u}-\mathrm{v} \parallel.$$

Now we use the mean value theorem, then we have

$$\left(\frac{\partial}{\partial \mathrm{x}}(\mathrm{u}^2-\mathrm{v}^2), \mathrm{u}-\mathrm{v}\right) \leq \lambda_2 \parallel \mathrm{u}^2-\mathrm{v}^2 \parallel\parallel \mathrm{u}-\mathrm{v} \parallel = 2\lambda_2\eta \parallel \mathrm{u}-\mathrm{v} \parallel^2 \leq 2\lambda_2 \mathrm{M} \parallel \mathrm{u}-\mathrm{v} \parallel^2,$$

where $\mathrm{u} < \eta < \mathrm{v}$ and $\parallel \mathrm{u} \parallel, \parallel \mathrm{v} \parallel \leq \mathrm{M}$. Therefore:

$$\left(\frac{\partial}{\partial \mathrm{x}}(\mathrm{u}^2-\mathrm{v}^2), \mathrm{u}-\mathrm{v}\right) \leq 2\lambda_2 \mathrm{M} \parallel \mathrm{u}-\mathrm{v} \parallel^2 \Leftrightarrow$$

$$\left(-\frac{\partial}{\partial \mathrm{x}}(\mathrm{u}^2-\mathrm{v}^2), \mathrm{u}-\mathrm{v}\right) \geq 2\lambda_2 \mathrm{M} \parallel \mathrm{u}-\mathrm{v} \parallel^2 \tag{23}$$

Substituting (21) and (22) into (23) we get

$$(\mathrm{L}(\mathrm{u})-\mathrm{L}(\mathrm{v}), \mathrm{u}-\mathrm{v}) \geq (\lambda_1+\lambda_2 \mathrm{M}) \parallel \mathrm{u}-\mathrm{v} \parallel^2 = \mathrm{k} \parallel \mathrm{u}-\mathrm{v} \parallel^2,$$

where $\mathrm{k}=\lambda_1+\lambda_2\mathrm{M}$. Hence, we find the hypothesis (H1). Now we verify the convergence hypotheses (H2) for the operator $\mathrm{L}(\mathrm{u})$ which is for every $\mathrm{M}>0$, there exist a constant $\mathrm{C}(\mathrm{M})>0$ such that for $\mathrm{u}, \mathrm{v} \in \mathrm{H}$ with $\parallel \mathrm{u} \parallel \leq \mathrm{M}, \parallel \mathrm{v} \parallel \leq \mathrm{M}$ we have $(\mathrm{L}(\mathrm{u})-\mathrm{L}(\mathrm{v}), \mathrm{u}-\mathrm{v}) \leq \mathrm{C}(\mathrm{M}) \parallel \mathrm{u}-\mathrm{v} \parallel\parallel \mathrm{w} \parallel$ for every $\mathrm{w} \in \mathrm{H}$. For that we have:

$$\begin{aligned}(\mathrm{L}(\mathrm{u})-\mathrm{L}(\mathrm{v}), \mathrm{w}) &= \frac{1}{2}\left(-\frac{\partial}{\partial \mathrm{x}}(\mathrm{u}^2-\mathrm{v}^2), \mathrm{w}\right) + \mathrm{c}\left(\frac{\partial^2}{\partial \mathrm{x}^2}(\mathrm{u}-\mathrm{v}), \mathrm{w}\right). \\ &\leq (\mathrm{M} \parallel \mathrm{u}-\mathrm{v} \parallel\parallel \mathrm{w} \parallel + \parallel \mathrm{u}-\mathrm{v} \parallel\parallel \mathrm{w} \parallel) \\ &= (1+\mathrm{M}) \parallel \mathrm{u}-\mathrm{v} \parallel\parallel \mathrm{w} \parallel \\ &= \mathrm{C}(\mathrm{M}) \parallel \mathrm{u}-\mathrm{v} \parallel\parallel \mathrm{w} \parallel,\end{aligned}$$

where $\mathrm{C}(\mathrm{M})=1+\mathrm{M}$ and therefore (H2) is hold. The proof is complete.

7. Conclusion

The main idea of this work was to give a simple method for solving the Burgers' equation. We carefully applied a reliable modification of Laplace decomposition method for this equation. The main advantage of this method is the fact that it gives the analytical solution. Also, two types of multisymplectic box methods were considered and implemented on Burgers' equation. These methods are fully implicit methods which are more accurate than explicit and semi-explicit methods. In both cases the artificial wiggle which is appeared in usual discretization methods is diminished. Table showed their advantages on well-known usual discretization methods. In the above example we observed that the LADM with the initial approximation obtained from initial conditions yield a good approximation to the exact solution only in a few iterations. It is also worth noting that the advantage of the decomposition methodology displays a fast convergence of the solutions. The illustrations show the dependence of the rapid convergence depend on the character and behavior of the solutions just as in a closed form solutions.

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