STATISTICS, OPTIMIZATION AND INFORMATION COMPUTING
Stat., Optim. Inf. Comput., Vol. **2**, December 2014, pp 280–304.
Published online in International Academic Press (www.IAPress.org)

Multiobjective Fractional Programming Problems and Second Order Generalized Hybrid Invexity Frameworks

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Received: 15 July 2014; Accepted: 7 October 2014
Editor: Chen Ling

Abstract In this paper, the parametrically generalized sufficient efficiency conditions for multiobjective fractional programming based on the hybrid $(\Phi, \rho, \eta, \zeta, \theta)-$invexities are developed, and then efficient solutions to the multiobjective fractional programming problems are established. Furthermore, the obtained results on sufficient efficiency conditions are generalized to the case of the $\epsilon-$efficient solutions. The results thus obtained generalize and unify a wider range of investigations on the theory and applications to the multiobjective fractional programming based on the hybrid $(\Phi, \rho, \eta, \zeta, \theta)-$invexity frameworks.

Keywords Generalized invexity, Multiobjective fractional programming, Efficient solutions, $\epsilon-$efficient solutions, Parametric sufficient efficiency conditions

AMS 2010 subject classifications 90C32, 90C45

DOI: 10.19139/soic.v2i4.92

1. Introduction

Mishra and Rueda [11] introduced higher order generalized invexity and duality models in mathematical programming, while Mangasarian [8] focused on the second order duality for a conventional nonlinear programming problem, where the approach is based on constructing a second order dual problem by taking linear and quadratic approximations of the objective and constraint functions for an arbitrary but fixed point leading to the Wolfe dual model for the approximated

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ISSN 2310-5070 (online) ISSN 2311-004X (print)

problem, while letting the fixed point to vary. Verma [24] investigated the second order $(\rho, \eta, \theta)-$invexities to the context of parametric sufficient optimality conditions in semiinfinite discrete minimax fractional programming. Zalmai and Zhang [38] have established a set of necessary efficiency conditions and a fairly large number of global nonparametric sufficient efficiency results under various frameworks for generalized $(\eta, \rho)-$invexity for the semiinfinite discrete minimax fractional programming. Just recently, Verma [22] investigated a general framework for a class of $(\rho, \eta, \theta)-$invex functions to examine some parametric sufficient efficiency conditions for multiobjective fractional programming problems for weakly $\epsilon-$efficient solutions. Inspired by these research developments, we first introduce the hybrid $(\Phi, \rho, \eta, \zeta, \theta)-$invexities as well as the second order hybrid $(\Phi, \rho, \eta, \zeta, \theta)-$invexities, second, introduce some parametrically sufficient efficiency conditions for multiobjective fractional programming, and finally, explore the efficient solutions to multiobjective fractional programming problems. The results established in this communication, not only generalize (and unify) the results on general sufficient efficiency conditions for multiobjective fractional programming problems based on the hybrid invexity of functions, but also generalize second order invexity results to more general settings. We consider, based on the generalized hybrid $(\Phi, \rho, \eta, \zeta, \theta)-$invexities of functions, the following multiobjective fractional programming problem: (P)

$$Minimize\Big(\frac{f_1(x)}{g_1(x)}, \frac{f_2(x)}{g_2(x)}, \cdots, \frac{f_p(x)}{g_p(x)}\Big)$$

subject to $x \in Q = \{x \in X : H_j(x) \leq 0, j \in \{1, 2, \cdots, m\}\}$, where X is an open convex subset of $\Re^n$ (n-dimensional Euclidean space), f_i and g_i for $i \in \{1, \cdots, p\}$, and H_j for $j \in \{1, \cdots, m\}$, are real-valued functions defined on X such that $f_i(x) \geq 0$, $g_i(x) > 0$ for $i \in \{1, \cdots, p\}$ and for all $x \in Q$. Here Q denotes the feasible set of (P).

Next, we observe that problem (P) is equivalent to the nonfractional programming problem: (Pλ)

$$Minimize\Big(f_1(x) - \lambda_1 g_1(x), \cdots, f_p(x) - \lambda_p g_p(x)\Big)$$

subject to $x \in Q$ with

$$\lambda = \Big(\lambda_1, \lambda_2, \cdots, \lambda_p\Big) = \Big(\frac{f_1(x^*)}{g_1(x^*)}, \frac{f_2(x^*)}{g_2(x^*)}, \cdots, \frac{f_p(x^*)}{g_p(x^*)}\Big),$$

where x^* is an efficient solution to (P).

We observe that general mathematical programming problems offer a great opportunity for applications to other fields, for instance, applications to game theory, statistical analysis, engineering design (including design of control systems, design of earthquakes-resistant structures, digital filters, and electronic circuits), random graphs, boundary value problems, wavelet analysis, environmental protection planning, decision and management sciences, optimal control problems, continuum mechanics, robotics, and others. For more details on generalized efficiency and efficiency results and applications, we recommend the reader [1]-[41].

This submission is organized as follows: the introductory section deals with a brief historical development for the multiobjective fractional mathematical programming, while emphasizing the roles of the generalized invex functions. In Section 2, the hybrid $(\Phi, \rho, \eta, \zeta, \theta)-$invex functions of higher orders are introduced, and Section 3 deals with sufficient efficiency conditions leading to the solvability of the problem (P) using the hybrid $(\Phi, \rho, \eta, \zeta, \theta)-$invexities.

2. Hybrid Invexities

In this section, we introduce and develop some concepts and notations for the problem on hand based on the (α, η)-V- invexities introduced by Mond and Zhang [15], and recently generalized by Zalmai [35, 36, 37]. Let X be an open convex subset of $\Re^n$ (n-dimensional Euclidean space). Let $\langle \cdot, \cdot \rangle$ denote the inner product, and let $z \in \Re^n$. Suppose that $f : X \to \Re$ is a real-valued twice continuously differentiable function defined on X, and that $\bigtriangledown f(y)$ and $\nabla^2 f(y)$ denote, respectively, the gradient and Hessian of f at y.

Definition 2.1. A twice differentiable function $f : X \to \Re$ is said to be hybrid $(\Phi, \rho, \eta, \zeta, \theta)-$invex at x^* of second order if there exists a function $\Phi : \Re \to \Re$ such that for each $x \in X$, $\rho : X \times X \to \Re$, $\eta, \zeta, \theta : X \times X \to \Re^n$, and $z \in \Re^n$,

$$\begin{aligned}\Phi\Big(f(x) - f(x^*)\Big) \geq \; & \langle \bigtriangledown f(x^*), \eta(x, x^*)\rangle + \langle \nabla^2 f(x^*)z, \zeta(x, x^*)\rangle \\ & -\tfrac{1}{2}\big\langle \nabla^2 f(x^*)z, z\big\rangle + \rho(x, x^*)\|\theta(x, x^*)\|^2.\end{aligned}$$

Definition 2.2. A twice differentiable function $f : X \to \Re$ is said to be hybrid $(\Phi, \rho, \eta, \zeta, \theta)-$pseudo-invex at x^* of second order if there exists a function $\Phi : \Re \to \Re$ such that for each $x \in X$, $\rho : X \times X \to \Re$, $\eta, \zeta, \theta : X \times X \to \Re^n$, and $z \in \Re^n$,

$$\begin{aligned}& \langle \bigtriangledown f(x^*), \eta(x, x^*)\rangle + \langle \nabla^2 f(x^*)z, \zeta(x, x^*)\rangle - \frac{1}{2}\big\langle \nabla^2 f(x^*)z, z\big\rangle \\ & +\rho(x, x^*)\|\theta(x, x^*)\|^2 \geq 0 \\ & \Rightarrow \Phi\Big(f(x) - f(x^*)\Big) \geq 0.\end{aligned}$$

Definition 2.3. A twice differentiable function $f : X \to \Re$ is said to be strictly hybrid $(\Phi, \rho, \eta, \zeta, \theta)-$ pseudo-invex at x^* of second order if there exists a function

$\Phi : \Re \to \Re$ such that for each $x \in X$, $\rho : X \times X \to \Re$, $\eta, \zeta, \theta : X \times X \to \Re^n$, and $z \in \Re^n$,

$$\begin{aligned}&\langle \triangledown f(x^*), \eta(x,x^*)\rangle + \langle \nabla^2 f(x^*)z, \zeta(x,x^*)\rangle - \frac{1}{2}\Big\langle \nabla^2 f(x^*)z, z\Big\rangle \\ &+\rho(x,x^*)\|\theta(x,x^*)\|^2 \geq 0 \\ &\Rightarrow \Phi\Big(f(x) - f(x^*)\Big) > 0.\end{aligned}$$

Definition 2.4. A twice differentiable function $f : X \to \Re$ is said to be prestrictly hybrid $(\Phi, \rho, \eta, \zeta, \theta)-$pseudo-invex at x^* of second order if there exists a function $\Phi : \Re \to \Re$ such that for each $x \in X$, $\rho : X \times X \to \Re$, $\eta, \zeta, \theta : X \times X \to \Re^n$, and $z \in \Re^n$,

$$\begin{aligned}&\langle \triangledown f(x^*), \eta(x,x^*)\rangle + \langle \nabla^2 f(x^*)z, \zeta(x,x^*)\rangle - \frac{1}{2}\Big\langle \nabla^2 f(x^*)z, z\Big\rangle \\ &+\rho(x,x^*)\|\theta(x,x^*)\|^2 > 0 \\ &\Rightarrow \Phi\Big(f(x) - f(x^*)\Big) \geq 0.\end{aligned}$$

Definition 2.5. A twice differentiable function $f : X \to \Re$ is said to be hybrid $(\Phi, \rho, \eta, \zeta, \theta)-$quasi-invex at x^* of second order if there exists a function $\Phi : \Re \to \Re$ such that for each $x \in X$, $\rho : X \times X \to \Re$, $\eta, \zeta, \theta : X \times X \to \Re^n$, and $z \in \Re^n$,

$$\begin{aligned}&\Phi\big(f(x) - f(x^*)\big) \leq 0 \\ &\Rightarrow \langle \triangledown f(x^*), \eta(x,x^*)\rangle + \langle \nabla^2 f(x^*)z, \zeta(x,x^*)\rangle - \frac{1}{2}\Big\langle \nabla^2 f(x^*)z, z\Big\rangle \\ &+\rho(x,x^*)\|\theta(x,x^*)\|^2 \leq 0.\end{aligned}$$

Definition 2.6. A twice differentiable function $f : X \to \Re$ is said to be strictly hybrid $(\Phi, \rho, \eta, \zeta, \theta)-$ quasi-invex at x^* of second order if there exists a function $\Phi : \Re \to \Re$ such that for each $x \in X$, $\rho : X \times X \to \Re$, $\eta, \zeta, \theta : X \times X \to \Re^n$, and $z \in \Re^n$,

$$\begin{aligned}&\Phi\big(f(x) - f(x^*)\Big) \leq 0 \\ &\Rightarrow \langle \triangledown f(x^*), \eta(x,x^*)\rangle + \langle \nabla^2 f(x^*)z, \zeta(x,x^*)\rangle - \frac{1}{2}\Big\langle \nabla^2 f(x^*)z, z\Big\rangle \\ &+\rho(x,x^*)\|\theta(x,x^*)\|^2 < 0.\end{aligned}$$

Definition 2.7. A twice differentiable function $f : X \to \Re$ is said to be prestrictly hybrid $(\Phi, \rho, \eta, \zeta, \theta)-$ quasi-invex at x^* of second order if there exists a function

$\Phi : \Re \to \Re$ such that for each $x \in X$, $\rho : X \times X \to \Re$, $\eta, \zeta, \theta : X \times X \to \Re^n$, and $z \in \Re^n$,

$$\begin{aligned} & \Phi\big(f(x) - f(x^*)\big) < 0 \\ \Rightarrow \quad & \langle \triangledown f(x^*), \eta(x, x^*)\rangle + \langle \nabla^2 f(x^*)z, \zeta(x, x^*)\rangle - \frac{1}{2}\big\langle \nabla^2 f(x^*)z, z\big\rangle \\ & +\rho(x, x^*)\|\theta(x, x^*)\|^2 \leq 0, \end{aligned}$$

equivalently,

$$\begin{aligned} & \langle \triangledown f(x^*), \eta(x, x^*)\rangle + \langle \nabla^2 f(x^*)z, \zeta(x, x^*)\rangle - \frac{1}{2}\big\langle \nabla^2 f(x^*)z, z\big\rangle \\ & +\rho(x, x^*)\|\theta(x, x^*)\|^2 > 0 \\ \Rightarrow \quad & \Phi\big(f(x) - f(x^*)\big) \geq 0. \end{aligned}$$

Some Examples and Remarks

We observe that Definition 2.1 reduces to Mond and Zhang [15], and Zalmai [37] for $\zeta(x, x^*) = \eta(x, x^*)$, while similar examples hold for Definitions 2.2 - 2.7.

Example 2.1. The function f is said to be second-order $(\Phi, \rho, \eta, \theta)$-V-invex at x^* if there exist functions $\Phi : \Re \to \Re$, $\rho : X \times X \to \Re$ and $\eta, \theta : X \times X \to \Re^n$ such that for each $x \in X,\ z \in \Re^n$,

$$\begin{aligned} \Phi\big(f(x) - f(x^*)\big) \ \geqq \ & \Big\langle [\nabla f(x^*) + \nabla^2 f(x^*)z], \eta(x, x^*)\Big\rangle - \Big\langle \tfrac{1}{2}\nabla^2 f(x^*)z, z\Big\rangle \\ & +\rho(x, x^*)\|\theta(x, x^*)\|^2. \end{aligned}$$

When function f is first-order differentiable, Definition 2.1 unifies most of the well-explored notions of the invexities in the literature.

Example 2.2. A differentiable function $f : X \to \Re$ is said to be $(\Phi, \rho, \eta, \theta)-$invex at x^* of first order if there exists a function $\Phi : \Re \to \Re$ such that for each $x \in X$, $\rho : X \times X \to \Re$, and $\eta, \theta : X \times X \to \Re^n$,

$$\Phi\Big(f(x) - f(x^*)\Big) \geq \Big\langle \triangledown f(x^*), \eta(x, x^*)\Big\rangle + \rho(x, x^*)\|\theta(x, x^*)\|^2.$$

Definition 2.8. A point $x^* \in Q$ is an efficient solution to (P) if there exists no $x \in Q$ such that

$$\frac{f_i(x)}{g_i(x)} \leq \frac{f_i(x^*)}{g_i(x^*)}\, \forall\, i = 1, \cdots, p,$$

$$\frac{f_j(x)}{g_j(x)} < \frac{f_j(x^*)}{g_j(x^*)} \text{ for some } j \in \{1, \cdots, p\}.$$

Next to this context, we have the following auxiliary problem:

(P$\bar{\lambda}$)

$$Minimize_{x \in Q}\, (f_1(x) - \bar{\lambda}_1 g_1(x), \cdots, f_p(x) - \bar{\lambda}_p g_p(x)),$$

subject to $x \in Q$,

where $\bar{\lambda}_i$ for $i \in \{1, \cdots, p\}$ are parameters, and $\bar{\lambda}_i = \frac{f_i(x^*)}{g_i(x^*)}$.

Next, we introduce the efficiency solvability conditions for (P$\bar{\lambda}$) problem.

Definition 2.9. A point $x^* \in Q$ is an efficient solution to (P$\bar{\lambda}$) if there does not exist an $x \in Q$ such that

$$f_i(x) - \bar{\lambda}_i g_i(x) \leq f_i(x^*) - \bar{\lambda}_i g_i(x^*) \, \forall\, i = 1, \cdots, p,$$

$$f_j(x) - \bar{\lambda}_j g_j(x) < f_j(x^*) - \bar{\lambda}_j g_j(x^*) \text{ for some } j \in \{1, \cdots, p\},$$

where $\bar{\lambda}_i = \frac{f_i(x^*)}{g_i(x^*)}$ for $i = 1, \cdots, p.$

Next, we recall the following result (Verma [24]) that provides a set of necessary efficiency conditions for problem (P) to developing some sufficient efficiency conditions for the next section based on second order $(\Phi, \rho, \eta, \zeta, \theta)-$invexities.

Theorem 2.1. [24] Let $x^* \in \mathbb{F}$ and $\lambda^* = \max_{1 \leq i \leq p} f_i(x^*)/g_i(x^*)$ for each $i \in \underline{p}$, and let f_i and g_i be twice continuously differentiable at x^* for each $i \in \underline{p}$. For each $j \in \underline{q}$, let the function $z \to G_j(z, t)$ be twice continuously differentiable at x^* for all $t \in T_j$, and for each $k \in \underline{r}$, let the function $z \to H_k(z, s)$ be twice continuously differentiable at x^* for all $s \in S_k$. If x^* is an efficient solution of (P), if the second order generalized Abadie constraint qualification holds at x^*, and if for any critical direction y, the set cone

$$\begin{aligned} &\{\Big(\nabla G_j(x^*, t), \langle y, \nabla^2 G_j(x^*, t) y\rangle\Big) : t \in \hat{T}_j(x^*), j \in \underline{q}\} \\ + \quad & span\{\Big(\nabla H_k(x^*, s), \langle y, \nabla^2 H_k(x^*, s) y\rangle\Big) : s \in S_k, k \in \underline{r}\}, \\ & \text{where } \hat{T}_j(x^*) \equiv \{t \in T_j \; : \; G_j(x^*, t) = 0\}, \end{aligned}$$

is closed, then there exist $u^* \in U \equiv \{u \in \mathbb{R}^p : u \geq 0,\ \sum_{i=1}^p u_i = 1\}$ and integers ν_0^* and ν^* with $0 \leq \nu_0^* \leq \nu^* \leq n+1$ such that there exist ν_0^* indices j_m with $1 \leq j_m \leq q$ together with ν_0^* points $t^m \in \hat{T}_{j_m}(x^*),\ m \in \underline{\nu_0^*},\ \nu^* - \nu_0^*$ indices k_m with $1 \leq k_m \leq r$ together with $\nu^* - \nu_0^*$ points $s^m \in S_{k_m}$ for $m \in \underline{\nu^*} \backslash \underline{\nu_0^*}$, and ν^*

real numbers v_m^* with $v_m^* > 0$ for $m \in \underline{\nu_0^*}$ with the property that

$$\sum_{i=1}^{p} u_i^*[\nabla f_i(x^*) - \lambda^*(\nabla g_i(x^*)] + \sum_{m=1}^{\nu_0^*} v_m^*[\nabla G_{j_m}(x^*, t^m)$$
$$+ \sum_{m=\nu_0^*+1}^{\nu^*} v_m^* \nabla H_k(x^*, s^m) = 0, \tag{2.1}$$

$$\Big\langle y, \Big[\sum_{i=1}^{p} u_i^*[\nabla^2 f_i(x^*) - \lambda^* \nabla^2 g_i(x^*)] + \sum_{m=1}^{\nu_0^*} v_m^* \nabla^2 G_{j_m}(x^*, t^m)$$
$$+ \sum_{m=\nu_0^*+1}^{\nu^*} v_m^* \nabla^2 H_k(x^*, s^m)\Big] y\Big\rangle \geq 0, \tag{2.2}$$

$$u_i^*[f_i(x^*) - \lambda^* g_i(x^*)] = 0, \ \ i \in \underline{p}, \tag{2.3}$$

where $\underline{\nu} \setminus \underline{\nu_0}$ is the complement of the set $\underline{\nu_0}$ relative to the set $\underline{\nu}$.

3. Efficiency Conditions for Problem (P)

This section deals with some parametrically sufficient efficiency conditions for problem (P) under the hybrid frameworks for $(\Phi, \rho, \eta, \zeta, \theta)-$invexities. We begin with real-valued functions $E_i(., x^*, u^*)$ and $B_j(., v)$ defined by

$$E_i(x, x^*, u^*) = u_i[f_i(x) - \Big(\frac{f_i(x^*)}{g_i(x^*)}\Big) g_i(x)], \, i \in \{1, \cdots, p\}$$

and

$$B_j(., v) = v_j H_j(x), \ j = 1, \cdots, m.$$

Theorem 3.1. Let $x^* \in Q$, let f_i, g_i for $i \in \{1, \cdots, p\}$ with $\frac{f_i(x^*)}{g_i(x^*)} \geq 0$, $g_i(x^*) > 0$ and H_j for $j \in \{1, \cdots, m\}$ be twice continuously differentiable at $x^* \in Q$, and let there exist $u^* \in U = \{u \in \Re^p : u > 0, \Sigma_{i=1}^p u_i = 1\}$ and $v^* \in \Re_+^m$ such that

$$\Sigma_{i=1}^p u_i^*[\triangledown f_i(x^*) - (\frac{f_i(x^*)}{g_i(x^*)}) \triangledown g_i(x^*)] + \Sigma_{j=1}^m v_j^* \triangledown H_j(x^*) = 0, \tag{3.1}$$

$$\Big\langle \Big[\sum_{i=1}^{p} u_i^* [\nabla^2 f_i(x^*) - (\frac{f_i(x^*)}{g_i(x^*)}) \nabla^2 g_i(x^*)] + \sum_{j=1}^{m} v_j^* \nabla^2 H_j(x^*) \Big] z, \zeta(x, x^*) \Big\rangle$$

$$-\frac{1}{2} \Big\langle \Big[\sum_{i=1}^{p} u_i^* [\nabla^2 f_i(x^*) - (\frac{f_i(x^*)}{g_i(x^*)}) \nabla^2 g_i(x^*)]$$

$$+ \sum_{j=1}^{m} v_j^* \nabla^2 H_j(x^*) \Big] z, z \Big\rangle \geq 0, \tag{3.2}$$

and

$$v_j^* H_j(x^*) = 0,\ j \in \{1, \cdots, m\}. \tag{3.3}$$

Suppose, in addition, that any one of the following assumptions holds (for $\rho(x, x^*) \geq 0$):

(i) $E_i(.\,; x^*, u^*)$ $\forall\, i \in \{1, \cdots, p\}$ are hybrid $(\Phi, \rho, \eta, \zeta, \theta)-$pseudo-invex at x^* with $\bar{\Phi}(a) \geq 0 \Rightarrow a \geq 0$, and $B_j(.\,, v^*)$ $\forall\, j \in \{1, \cdots, m\}$ are hybrid $(\tilde{\Phi}, \bar{\rho}, \eta, \zeta, \theta)-$quasi-invex at x^* for $\tilde{\Phi}$ increasing and $\tilde{\Phi}(0) = 0$.

(ii) $E_i(.\,; x^*, u^*)$ $\forall\, i \in \{1, \cdots, p\}$ are prestrictly hybrid $(\Phi, \rho, \eta, \zeta, \theta)-$pseudo-invex at x^* for $\Phi(a) \geq 0 \Rightarrow a \geq 0$, and $B_j(.\,, v^*)$ $\forall\, j \in \{1, \cdots, m\}$ are strictly hybrid $(\tilde{\Phi}, \bar{\rho}, \eta, \zeta, \theta)-$quasi-invex at x^* for $\tilde{\Phi}$ increasing and $\tilde{\Phi}(0) = 0$.

(iii) $E_i(.\,; x^*, u^*)$ $\forall\, i \in \{1, \cdots, p\}$ are prestrictly hybrid $(\Phi, \rho, \eta, \zeta, \theta)-$quasi-invex at x^* for $\Phi(a) \geq 0 \Rightarrow a \geq 0$, and $B_j(.\,, v^*)$ $\forall\, j \in \{1, \cdots, m\}$ are strictly hybrid $(\tilde{\Phi}, \bar{\rho}, \eta, \zeta, \theta)-$quasi-invex at x^* for $\tilde{\Phi}$ increasing and $\tilde{\Phi}(0) = 0$.

(iv) For each $i \in \{1, \cdots, p\}$, f_i is hybrid $(\Phi, \rho_1, \eta, \zeta, \theta)-$invex and $-g_i$ is hybrid $(\Phi, \rho_2, \eta, \zeta, \theta)-$invex at x^* for $\Phi(a) \geq 0 \Rightarrow a \geq 0$, and $H_j(.\,, v^*)$ $\forall\, j \in \{1, \cdots, m\}$ is hybrid $(\bar{\Phi}, \rho_3, \eta, \zeta, \theta)-$quasi-invex at x^* for $\bar{\Phi}$ increasing and $\bar{\Phi}(0) = 0$, and $\Sigma_{j=1}^{m} v_j^* \rho_3(x, x^*) + \rho^* \geq 0$ for $\rho^* = \Sigma_{i=1}^{p} u_i^* (\rho_1(x, x^*) + \phi(x^*) \rho_2(x, x^*))$ and for $\phi(x^*) = \frac{f_i(x^*)}{g_i(x^*)}$.

Then x^* is an efficient solution to (P).

Proof
If (i) holds, and if $x \in Q$, then it follows from (3.1) and (3.2) that

$$\begin{aligned}
&\Big\langle \Sigma_{i=1}^{p} u_i^*[\bigtriangledown f_i(x^*) - (\frac{f_i(x^*)}{g_i(x^*)}) \bigtriangledown g_i(x^*)], \eta(x,x^*)\Big\rangle \\
&+\Big\langle \sum_{i=1}^{p} u_i^*[\nabla^2 f_i(x^*)z - (\frac{f_i(x^*)}{g_i(x^*)})\nabla^2 g_i(x^*)z], \zeta(x,x^*)\Big\rangle \\
&-\frac{1}{2}\Big\langle \sum_{i=1}^{p} u_i^*[\nabla^2 f_i(x^*)z - (\frac{f_i(x^*)}{g_i(x^*)})\nabla^2 g_i(x^*)z], z\Big\rangle \\
&+\Big\langle \Sigma_{j=1}^{m} v_j^* \bigtriangledown H_j(x^*), \eta(x,x^*)\Big\rangle + \Big\langle \Sigma_{j=1}^{m} v_j^* \nabla^2 H_j(x^*)z, \zeta(x,x^*)\Big\rangle \\
&-\frac{1}{2}\Big\langle \Sigma_{j=1}^{m} v_j^* \nabla^2 H_j(x^*)z, z\Big\rangle \geq 0. \qquad (3.4)
\end{aligned}$$

Since $v^* \geq 0$, $x \in Q$ and (3.3) holds, we have

$$\Sigma_{j=1}^{m} v_j^* H_j(x) \leq 0 = \Sigma_{j=1}^{m} v_j^* H_j(x^*),$$

and in light of assumptions on $\tilde{\Phi}$, we find

$$\tilde{\Phi}\Big(\Sigma_{j=1}^{m} v_j^* H_j(x) - \Sigma_{j=1}^{m} v_j^* H_j(x^*)\Big) \leq 0,$$

which applying the hybrid $(\tilde{\Phi}, \bar{\rho}, \eta, \zeta, \theta)-$quasi-invexity of $B_j(., v^*)$ at x^*, results in

$$\begin{aligned}
&\Big\langle \Sigma_{j=1}^{m} v_j^* \bigtriangledown H_j(x^*), \eta(x,x^*)\Big\rangle + \Big\langle \Sigma_{j=1}^{m} v_j^* \nabla^2 H_j(x^*)z, \zeta(x,x^*)\Big\rangle \\
&-\frac{1}{2}\Big\langle \Sigma_{j=1}^{m} v_j^* \nabla^2 H_j(x^*)z, z\Big\rangle + \bar{\rho}(x,x^*)\|\theta(x,x^*)\|^2 \leq 0. \qquad (3.5)
\end{aligned}$$

It follows from (3.4) and (3.5) that

$$\begin{aligned}
&\Big\langle \Sigma_{i=1}^{p} u_i^*[\bigtriangledown f_i(x^*) - (\frac{f_i(x^*)}{g_i(x^*)}) \bigtriangledown g_i(x^*)], \eta(x,x^*)\Big\rangle \\
&+\Big\langle \sum_{i=1}^{p} u_i^*[\nabla^2 f_i(x^*)z - (\frac{f_i(x^*)}{g_i(x^*)})\nabla^2 g_i(x^*)z], \zeta(x,x^*)\Big\rangle \\
&-\frac{1}{2}\Big\langle \sum_{i=1}^{p} u_i^*[\nabla^2 f_i(x^*)z - (\frac{f_i(x^*)}{g_i(x^*)})\nabla^2 g_i(x^*)z], z\Big\rangle \\
\geq\;& \bar{\rho}(x,x^*)\|\theta(x,x^*)\|^2 \geq -\rho(x,x^*)\|\theta(x,x^*)\|^2. \qquad (3.6)
\end{aligned}$$

Since $\rho(x,x^*) \geq 0$, applying the hybrid $(\Phi, \rho, \eta, \zeta, \theta)-$pseudo-invexity at x^* to (3.6) and assumptions on Φ, we have

$$\Phi\Big(\Sigma_{i=1}^{p} u_i^*[f_i(x) - (\frac{f_i(x^*)}{g_i(g^*)})g_i(x)] - \Sigma_{i=1}^{p} u_i^*[f_i(x^*) - (\frac{f_i(x^*)}{g_i(x^*)})g_i(x^*)]\Big) \geq 0,$$

which implies

$$\begin{aligned} & \Sigma_{i=1}^{p} u_i^*[f_i(x) - (\frac{f_i(x^*)}{g_i(x^*)})g_i(x)] \\ \geq \quad & \Sigma_{i=1}^{p} u_i^*[f_i(x^*) - (\frac{f_i(x^*)}{g_i(x^*)})g_i(x^*)]) = 0. \end{aligned}$$

Thus, we have

$$\Sigma_{i=1}^{p} u_i^*[f_i(x) - (\frac{f_i(x^*)}{g_i(x^*)})g_i(x)] \geq 0. \tag{3.7}$$

Since $u_i^* > 0$ for each $i \in \{1, \cdots, p\}$, we conclude that there does not exist an $x \in Q$ such that

$$\frac{f_i(x)}{g_i(x)} - (\frac{f_i(x^*)}{g_i(x^*)}) \leq 0 \ \forall\, i = 1, \cdots, p,$$

$$\frac{f_j(x)}{g_j(x)} - (\frac{f_j(x^*)}{g_j(x^*)}) < 0 \ \text{ for some } \ j \in \{1, \cdots, p\}.$$

Hence, x^* is an efficient solution to (P).

Next, if (ii) holds, and if $x \in Q$, then it follows from (3.1) and (3.2) that

$$\begin{aligned} & \Big\langle \Sigma_{i=1}^{p} u_i^*[\bigtriangledown f_i(x^*) - (\frac{f_i(x^*)}{g_i(x^*)}) \bigtriangledown g_i(x^*)], \eta(x, x^*) \Big\rangle \\ & + \Big\langle \sum_{i=1}^{p} u_i^*[\nabla^2 f_i(x^*)z - (\frac{f_i(x^*)}{g_i(x^*)})\nabla^2 g_i(x^*)z], \zeta(x, x^*) \Big\rangle \\ & - \frac{1}{2} \Big\langle \sum_{i=1}^{p} u_i^*[\nabla^2 f_i(x^*)z - (\frac{f_i(x^*)}{g_i(x^*)})\nabla^2 g_i(x^*)z], z \Big\rangle \\ & + \Big\langle \Sigma_{j=1}^{m} v_j^* \bigtriangledown H_j(x^*), \eta(x, x^*) \Big\rangle + \Big\langle \Sigma_{j=1}^{m} v_j^* \nabla^2 H_j(x^*)z, \zeta(x, x^*) \Big\rangle \\ & - \frac{1}{2} \Big\langle \Sigma_{j=1}^{m} v_j^* \nabla^2 H_j(x^*)z, z \Big\rangle \geq 0. \end{aligned} \tag{3.8}$$

Since $v^* \geq 0$, $x \in Q$ and (3.3) holds, we have

$$\Sigma_{j=1}^{m} v_j^* H_j(x) \leq 0 = \Sigma_{j=1}^{m} v_j^* H_j(x^*),$$

which results (using assumptions on $\tilde{\Phi}$) in

$$\tilde{\Phi}\Big(\Sigma_{j=1}^{m} v_j^* H_j(x) - \Sigma_{j=1}^{m} v_j^* H_j(x^*)\Big) \leq 0.$$

Now, in light of the strictly hybrid $(\tilde{\Phi}, \bar{\rho}, \eta, \zeta, \theta)-$quasi-invexity of $B_j(., v^*)$ at x^*, we find

$$\begin{aligned} & \Big\langle \Sigma_{j=1}^m v_j^* \bigtriangledown H_j(x^*), \eta(x, x^*)\Big\rangle + \Big\langle \Sigma_{j=1}^m v_j^* \nabla^2 H_j(x^*) z, \zeta(x, x^*)\Big\rangle \\ - & \quad \frac{1}{2}\Big\langle \Sigma_{j=1}^m v_j^* \nabla^2 H_j(x^*) z, z\Big\rangle + \bar{\rho}(x, x^*)\|\theta(x, x^*)\|^2 < 0. \end{aligned} \tag{3.9}$$

It follows from (3.8) and (3.9) that

$$\begin{aligned} & \Big\langle \Sigma_{i=1}^p u_i^* [\bigtriangledown f_i(x^*) - (\frac{f_i(x^*)}{g_i(x^*)}) \bigtriangledown g_i(x^*)], \eta(x, x^*)\Big\rangle \\ & +\Big\langle \sum_{i=1}^p u_i^* [\nabla^2 f_i(x^*) z - (\frac{f_i(x^*)}{g_i(x^*)})\nabla^2 g_i(x^*) z], \zeta(x, x^*)\Big\rangle \\ & -\frac{1}{2}\Big\langle \sum_{i=1}^p u_i^* [\nabla^2 f_i(x^*) z - (\frac{f_i(x^*)}{g_i(x^*)})\nabla^2 g_i(x^*) z], z\Big\rangle \\ > & \quad \bar{\rho}(x, x^*)\|\theta(x, x^*)\|^2 > -\rho(x, x^*)\|\theta(x, x^*)\|^2. \end{aligned} \tag{3.10}$$

As a result, since $\rho(x, x^*) \geq 0$, applying the prestrictly hybrid $(\Phi, \rho, \eta, \zeta, \theta)-$pseudo-invexity at x^* to (3.10) and assumptions on Φ, we have

$$\Phi\Big(\Sigma_{i=1}^p u_i^* [f_i(x) - (\frac{f_i(x^*)}{g_i(g^*)}) g_i(x)] - \Sigma_{i=1}^p u_i^* [f_i(x^*) - (\frac{f_i(x^*)}{g_i(x^*)}) g_i(x^*)]\Big) \geq 0,$$

which implies

$$\begin{aligned} & \Sigma_{i=1}^p u_i^* [f_i(x) - (\frac{f_i(x^*)}{g_i(x^*)}) g_i(x)] \\ \geq & \quad \Sigma_{i=1}^p u_i^* [f_i(x^*) - (\frac{f_i(x^*)}{g_i(x^*)}) g_i(x^*)]) = 0. \end{aligned}$$

Thus, we have

$$\Sigma_{i=1}^p u_i^* [f_i(x) - (\frac{f_i(x^*)}{g_i(x^*)}) g_i(x)] \geq 0. \tag{3.11}$$

Since $u_i^* > 0$ for each $i \in \{1, \cdots, p\}$, we conclude that there does not exist an $x \in Q$ such that

$$\frac{f_i(x)}{g_i(x)} - (\frac{f_i(x^*)}{g_i(x^*)}) \leq 0 \;\forall\, i = 1, \cdots, p,$$

$$\frac{f_j(x)}{g_j(x)} - (\frac{f_j(x^*)}{g_j(x^*)}) < 0 \,\text{for some}\, j \in \{1, \cdots, p\}.$$

Hence, x^* is an efficient solution to (P). The proof applying (iii) is similar to that of (ii), and we just need to include the proof using (iv) as follows: since $x \in Q$, it follows that $H_j(x) \leq H_j(x^*)$, which implies $\bar{\Phi}\Big(H_j(x) - H_j(x^*)\Big) \leq 0$. Then applying the hybrid $(\bar{\Phi}, \rho_3, \eta, \zeta, \theta)-$quasi-invexity of H_j at x^* and $v^* \in R_+^m$, we have

$$\Big\langle \Sigma_{j=1}^m v_j^* \bigtriangledown H_j(x^*), \eta(x,x^*)\Big\rangle + \Big\langle \Sigma_{j=1}^m v_j^* \nabla^2 H_j(x^*) z, \zeta(x,x^*)\Big\rangle$$
$$-\frac{1}{2}\Big\langle \Sigma_{j=1}^m v_j^* \nabla^2 H_j(x^*) z, z\Big\rangle + \Sigma_{j=1}^m v_j^* \rho_3(x,x^*)\|\theta(x,x^*)\|^2 \leq 0. \quad (3.12)$$

Since $u^* \geq 0$ and $\frac{f_i(x^*)}{g_i(x^*)} \geq 0$, it follows from the hybrid $(\Phi, \rho_1, \eta, \zeta, \theta)-$invexity and $(\Phi, \rho_2, \eta, \zeta, \theta)-$ invexity assumptions on f_i and g_i, respectively, that

$$\begin{aligned}
& \Phi\Big(\Sigma_{i=1}^p u_i^*[f_i(x) - (\frac{f_i(x^*)}{g_i(x^*)})g_i(x)]\Big) \\
= \quad & \Phi\Big(\Sigma_{i=1}^p u_i^*\{[f_i(x) - f_i(x^*)] - (\frac{f_i(x^*)}{g_i(x^*)})[g_i(x) - g_i(x^*)]\}\Big) \\
\geq \quad & \Sigma_{i=1}^p u_i^*\{\langle \bigtriangledown f_i(x^*) - (\frac{f_i(x^*)}{g_i(x^*)}) \bigtriangledown g_i(x^*), \eta(x,x^*)\rangle\} \\
& + \langle \Sigma_{i=1}^p u_i^*[\nabla^2 f_i(x^*) z - (\frac{f_i(x^*)}{g_i(x^*)})\nabla^2 g_i(x^*) z, \zeta(x,x^*)\rangle] \\
& -\frac{1}{2}\langle \Sigma_{i=1}^p u_i^*[\nabla^2 f_i(x^*) z - (\frac{f_i(x^*)}{g_i(x^*)})\nabla^2 g_i(x^*) z, z\rangle] \\
& + \Sigma_{i=1}^p u_i^*[\rho_1(x,x^*) + \phi(x^*)\rho_2(x,x^*)]\|\theta(x,x^*)\|^2 \\
\geq \quad & -\big[\langle \Sigma_{j=1}^m v_j^* \bigtriangledown H_j(x^*), \eta(x,x^*)\rangle + \Big\langle \Sigma_{j=1}^m v_j^* \nabla^2 H_j(x^*) z, \zeta(x,x^*)\Big\rangle \\
& -\frac{1}{2}\langle \Sigma_{j=1}^m v_j^* \nabla^2 H_j(x^*) z, z\rangle\Big] + \Sigma_{i=1}^p u_i^*[\rho_1(x,x^*) + \phi(x^*)\rho_2(x,x^*)]\|\theta(x,x^*)\|^2 \\
\geq \quad & (\Sigma_{j=1}^m v_j^* \rho_3(x,x^*) + \Sigma_{i=1}^p u_i^*[\rho_1(x,x^*) + \phi(x^*)\rho_2(x,x^*)])\|\theta(x,x^*)\|^2 \\
= \quad & (\Sigma_{j=1}^m v_j^* \rho_3(x,x^*) + \rho^*)\|\theta(x,x^*)\|^2 \\
\geq \quad & 0,
\end{aligned}$$

where $\phi(x^*) = \frac{f_i(x^*)}{g_i(x^*)}$ and $\rho^* = \Sigma_{i=1}^p u_i^*(\rho_1(x,x^*) + \phi(x^*)\rho_2(x,x^*))$. This implies that

$$\Phi\Big(\Sigma_{i=1}^p u_i^*[f_i(x) - (\frac{f_i(x^*)}{g_i(x^*)})g_i(x)]\Big) \geq 0.$$

□

Some Specializations

First, if we consider when $\zeta(x,x^*) = \eta(x,x^*)$ in Theorem 3.1, which means, hybrid $(\Phi, \rho, \eta, \theta)-$ invexities, we have the following result which generalizes (Zalmai [35], Theorem 3.2).

Theorem 3.2. Let $x^* \in Q$, let f_i, g_i for $i \in \{1, \cdots, p\}$ with $\frac{f_i(x^*)}{g_i(x^*)} \geq 0$, $g_i(x^*) > 0$ and H_j for $j \in \{1, \cdots, m\}$ be twice continuously differentiable at $x^* \in Q$, and let there exist $u^* \in U = \{u \in \Re^p : u > 0, \Sigma_{i=1}^p u_i = 1\}$ and $v^* \in \Re_+^m$ such that

$$\Sigma_{i=1}^p u_i^* [\bigtriangledown f_i(x^*) - (\frac{f_i(x^*)}{g_i(x^*)}) \bigtriangledown g_i(x^*)] + \Sigma_{j=1}^m v_j^* \bigtriangledown H_j(x^*) = 0, \tag{3.13}$$

$$\begin{aligned}&\Big\langle \Big[\sum_{i=1}^p u_i^* [\nabla^2 f_i(x^*) - (\frac{f_i(x^*)}{g_i(x^*)}) \nabla^2 g_i(x^*)] + \sum_{j=1}^m v_j^* \nabla^2 H_j(x^*) \Big] z, \eta(x, x^*) \Big\rangle \\ &-\frac{1}{2} \Big\langle \Big[\sum_{i=1}^p u_i^* [\nabla^2 f_i(x^*) - (\frac{f_i(x^*)}{g_i(x^*)}) \nabla^2 g_i(x^*)] \\ &+ \sum_{j=1}^m v_j^* \nabla^2 H_j(x^*) \Big] z, z \Big\rangle \geq 0,\end{aligned} \tag{3.14}$$

and

$$v_j^* H_j(x^*) = 0, \; j \in \{1, \cdots, m\}. \tag{3.15}$$

Suppose, in addition, that any one of the following assumptions holds (for $\rho(x, x^*) \geq 0$):

(i) $E_i(.\,; x^*, u^*)$ $\forall\, i \in \{1, \cdots, p\}$ are $(\Phi, \rho, \eta, \theta)-$pseudo-invex at x^*. $B_j(.\,, v^*)$ $\forall\, j \in \{1, \cdots, m\}$ are $(\Phi, \bar{\rho}, \eta, \theta)-$quasi-invex at x^*.

(ii) $E_i(.\,; x^*, u^*)$ $\forall\, i \in \{1, \cdots, p\}$ are prestrictly hybrid $(\Phi, \rho, \eta, \theta)-$pseudo-invex at x^* for $\Phi(a) \geq 0 \Rightarrow a \geq 0$, and $B_j(.\,, v^*)$ $\forall\, j \in \{1, \cdots, m\}$ are strictly hybrid $(\tilde{\Phi}, \rho, \eta, \theta)-$quasi-invex at x^* for $\tilde{\Phi}$ increasing and $\tilde{\Phi}(0) = 0$.

(iii) $E_i(.\,; x^*, u^*)$ $\forall\, i \in \{1, \cdots, p\}$ are prestrictly hybrid $(\Phi, \rho, \eta, \theta)-$quasi-invex at x^* for $\Phi(a) \geq 0 \Rightarrow a \geq 0$, and $B_j(.\,, v^*)$ $\forall\, j \in \{1, \cdots, m\}$ are strictly hybrid $(\tilde{\Phi}, \bar{\rho}, \eta, \theta)-$quasi-invex at x^* for $\tilde{\Phi}$ increasing and $\tilde{\Phi}(0) = 0$.

(iv) For each $i \in \{1, \cdots, p\}$, f_i is hybrid $(\Phi, \rho_1, \eta, \theta)-$invex and $-g_i$ is hybrid $(\Phi, \rho_2, \eta, \theta)-$invex at x^* for $\Phi(a) \geq 0 \Rightarrow a \geq 0$, $H_j(.\,, v^*)$ $\forall\, j \in \{1, \cdots, m\}$ is hybrid $(\bar{\Phi}, \rho_3, \eta, \theta)-$quasi-invex at x^* for $\bar{\Phi}$ increasing and $\bar{\Phi}(0) = 0$, and $\Sigma_{j=1}^m v_j^* \rho_3(x, x^*) + \rho^* \geq 0$ for $\rho^* = \Sigma_{i=1}^p u_i^* (\rho_1(x, x^*) + \phi(x^*) \rho_2(x, x^*))$ and for $\phi(x^*) = \frac{f_i(x^*)}{g_i(x^*)}$.

Then x^* is an efficient solution to (P).

Next we consider the case when the functions are first-order differentiable, Theorem 3.1 reduces to the result which is similar to results of Zalmai and Zhang [37].

Theorem 3.3. Let $x^* \in Q$, let f_i, g_i for $i \in \{1, \cdots, p\}$ with $\frac{f_i(x^*)}{g_i(x^*)} \geq 0$, $g_i(x^*) > 0$ and H_j for $j \in \{1, \cdots, m\}$ be differentiable at $x^* \in Q$, and let there exist $u^* \in U = \{u \in \Re^p : u > 0, \Sigma_{i=1}^p u_i = 1\}$ and $v^* \in \Re_+^m$ such that

$$\left\langle \Sigma_{i=1}^p u_i^* [\nabla f_i(x^*) - (\frac{f_i(x^*)}{g_i(x^*)}) \nabla g_i(x^*)] + \Sigma_{j=1}^m v_j^* \nabla H_j(x^*), \eta(x, x^*) \right\rangle \geq 0 \tag{3.16}$$

and

$$v_j^* H_j(x^*) = 0,\ j \in \{1, \cdots, m\}. \tag{3.17}$$

Suppose, in addition, that any one of the following assumptions holds (for $\rho(x, x^*) \geq 0$):

(i) $E_i(.\,; x^*, u^*)\ \ \forall i \in \{1, \cdots, p\}$ are first-order hybrid $(\Phi, \rho, \eta, \theta)-$pseudo-invex at x^* for $\Phi(a) \geq 0 \Rightarrow a \geq 0$, and $B_j(.\,, v^*)\ \forall j \in \{1, \cdots, m\}$ are first-order hybrid $(\bar{\Phi}, \bar{\rho}, \eta, \theta)-$quasi-invex at x^* for $\bar{\Phi}$ increasing and $\bar{\Phi}(0) = 0$.

(ii) $E_i(.\,; x^*, u^*)\ \ \forall i \in \{1, \cdots, p\}$ are first-order hybrid prestrictly $(\Phi, \rho, \eta, \theta)-$pseudo-invex at x^* for $\Phi(a) \geq 0 \Rightarrow a \geq 0$, and $B_j(.\,, v^*)$ $\forall j \in \{1, \cdots, m\}$ are first-order strictly hybrid $(\bar{\Phi}, \bar{\rho}, \eta, \theta)-$quasi-invex at x^* for $\bar{\Phi}$ increasing and $\bar{\Phi}(0) = 0$.

(iii) $E_i(.\,; x^*, u^*)\ \ \forall i \in \{1, \cdots, p\}$ are first-order prestrictly hybrid $(\Phi, \rho, \eta, \theta)-$quasi-invex at x^* $\Phi(a) \geq 0 \Rightarrow a \geq 0$, and $B_j(.\,, v^*)$ $\forall j \in \{1, \cdots, m\}$ are first-order strictly hybrid $(\bar{\Phi}, \bar{\rho}, \eta, \theta)-$quasi-invex at x^* for $\bar{\Phi}$ increasing and $\bar{\Phi}(0) = 0$.

(iv) For each $i \in \{1, \cdots, p\}$, f_i is first-order hybrid $(\Phi, \rho_1, \eta, \theta)-$invex and $-g_i$ is first-order hybrid $(\Phi, \rho_2, \eta, \theta)-$invex at x^* for $\Phi(a) \geq 0 \Rightarrow a \geq 0$. $H_j(.\,, v^*)\ \ \forall j \in \{1, \cdots, m\}$ is hybrid $(\bar{\Phi}, \bar{\rho_3}, \eta, \theta)-$quasi-invex at x^*, and $\Sigma_{j=1}^m v_j^* \bar{\rho_3}(x, x^*) + \rho^* \geq 0$ for $\bar{\Phi}$ increasing and $\bar{\Phi}(0) = 0$, $\rho^* = \Sigma_{i=1}^p u_i^*(\rho_1(x, x^*) + \phi(x^*)\rho_2(x, x^*))$ and for $\phi(x^*) = \frac{f_i(x^*)}{g_i(x^*)}$.

Then x^* is an efficient solution to (P).

Proof

Although the proof is similar to that of Theorem 3.1), we include for the sake of the completeness. If we consider (i), then proceeding as in Theorem 3.1 (and using the first-order hybrid $(\Phi,\rho,\eta,\theta)-$invexity assumptions instead), we arrive at

$$\begin{aligned}&\langle\Sigma_{i=1}^{p}u_i^*[\bigtriangledown f_i(x^*)-(\frac{f_i(x^*)}{g_i(x^*)})\bigtriangledown g_i(x^*)],\eta(x,x^*)\rangle\\ \geq\quad&\rho(x,x^*)\|\theta(x,x^*)\|^2.\end{aligned}\tag{3.18}$$

Since $\rho(x,x^*)\geq 0$, applying the hybrid $(\Phi,\rho,\eta,\theta)-$pseudo-invexity at x^* to (3.18) and assumptions on Φ, we have

$$\Phi\Big(\Sigma_{i=1}^{p}u_i^*[f_i(x)-(\frac{f_i(x^*)}{g_i(g^*)})g_i(x)]-\Sigma_{i=1}^{p}u_i^*[f_i(x^*)-(\frac{f_i(x^*)}{g_i(x^*)})g_i(x^*)]\Big)\geq 0,$$

which implies

$$\begin{aligned}&\Sigma_{i=1}^{p}u_i^*[f_i(x)-(\frac{f_i(x^*)}{g_i(x^*)})g_i(x)]\\ \geq\quad&\Sigma_{i=1}^{p}u_i^*[f_i(x^*)-(\frac{f_i(x^*)}{g_i(x^*)})g_i(x^*)])\\ =\quad&0.\end{aligned}$$

Thus, we have

$$\Sigma_{i=1}^{p}u_i^*[f_i(x)-(\frac{f_i(x^*)}{g_i(x^*)})g_i(x)]\geq 0.\tag{3.19}$$

Since $u_i^*>0$ for each $i\in\{1,\cdots,p\}$, we conclude that there does not exist an $x\in Q$ such that

$$\frac{f_i(x)}{g_i(x)}-(\frac{f_i(x^*)}{g_i(x^*)})\leq 0\ \forall\, i=1,\cdots,p,$$

$$\frac{f_j(x)}{g_j(x)}-(\frac{f_j(x^*)}{g_j(x^*)})<0\ \text{ for some }\ j\in\{1,\cdots,p\}.$$

Hence, x^* is an efficient solution to (P).
The proofs for (ii)-(iv) are similar to that of Theorem 3.1. □

Next, we note that Theorem 3.1 can be specialized to the context of second order $(\rho,\eta,\theta)-$ invexities as follows:

Theorem 3.4. Let $x^*\in Q$. Let f_i, g_i for $i\in\{1,\cdots,p\}$ with $\frac{f_i(x^*)}{g_i(x^*)}\geq 0, g_i(x^*)>0$ and H_j for $j\in\{1,\cdots,m\}$ be twice continuously differentiable at $x^*\in Q$, and let there exist $u^*\in U=\{u\in\Re^p: u>0,\Sigma_{i=1}^{p}u_i=1\}$ and $v^*\in\Re_+^m$ such that

$$\Sigma_{i=1}^{p}u_i^*[\bigtriangledown f_i(x^*)-(\frac{f_i(x^*)}{g_i(x^*)})\bigtriangledown g_i(x^*)]+\Sigma_{j=1}^{m}v_j^*\bigtriangledown H_j(x^*)=0\tag{3.20}$$

$$\left\langle \eta(x,x^*), \Big[\sum_{i=1}^{p} u_i^*[\nabla^2 f_i(x^*) - (\frac{f_i(x^*)}{g_i(x^*)})\nabla^2 g_i(x^*)] + \sum_{j=1}^{m} v_j^* \nabla^2 H_j(x^*)\Big] z \right\rangle \geq 0, \tag{3.21}$$

and

$$v_j^* H_j(x^*) = 0,\ j \in \{1, \cdots, m\}. \tag{3.22}$$

Suppose, in addition, that any one of the following assumptions holds (for $\rho(x,x^*) \geq 0$):

(i) $E_i(.\,; x^*, u^*)\ \forall i \in \{1, \cdots, p\}$ are hybrid $(\rho, \eta, \theta)-$pseudo-invex at x^*, and $B_j(.\,, v^*)\ \forall j \in \{1, \cdots, m\}$ are hybrid $(\rho, \eta, \theta)-$quasi-invex at x^*.

(ii) $E_i(.\,; x^*, u^*)\ \forall i \in \{1, \cdots, p\}$ are prestrictly hybrid $(\rho, \eta, \theta)-$pseudo-invex at x^*, and $B_j(.\,, v^*)\ \forall j \in \{1, \cdots, m\}$ are hybrid $(\rho, \eta, \theta)-$strictly-quasi-invex at x^*.

(iii) $E_i(.\,; x^*, u^*)\ \forall i \in \{1, \cdots, p\}$ are strictly hybrid $(\rho, \eta, \theta)-$pseudo-invex at x^*, and $B_j(.\,, v^*)\ \forall j \in \{1, \cdots, m\}$ are strictly hybrid $(\rho, \eta, \theta)-$quasi-invex at x^*.

(iv) For each $i \in \{1, \cdots, p\}$, f_i is hybrid $(\rho_1, \eta, \theta)-$invex and $-g_i$ is $(\rho_2, \eta, \theta)-$invex at x^*. $H_j(.\,, v^*)\ \ \forall j \in \{1, \cdots, m\}$ is hybrid $(\rho_3, \eta, \theta)-$quasi-invex at x^*, and $\Sigma_{j=1}^{m} v_j^* \rho_3(x,x^*) + \rho^* \geq 0$ for $\rho^* = \Sigma_{i=1}^{p} u_i^*(\rho_1(x,x^*) + \phi(x^*)\rho_2(x,x^*))$ and for $\phi(x^*) = \frac{f_i(x^*)}{g_i(x^*)}$.

Then x^* is an efficient solution to (P).

Proof
The proof is similar to that of Theorem 3.1 based on the hybrid $(\rho, \eta, \theta)-$ invexity assumptions. □

We observe that Theorem 3.1 can be further generalized to the case of the $\epsilon-$efficient conditions based on the hybrid $(\Phi, \rho, \eta, \zeta, \theta)-$invexity frameworks. As a matter of fact, we generalize the $\epsilon-$efficient solvability conditions for problem (P) based on the work of Verma [22], and Kim, Kim and Lee [6], where they have investigated the $\epsilon-$efficiency as well as the weak $\epsilon-$efficiency conditions for multiobjective fractional programming problems under constraint qualifications. To the best of our knowledge, the results established in this communication (Theorem 3.1 and Theorem 3.5) generalize and unify most of the results on the multiobjective fractional programming to the context of the generalized invexities in the literature. We recall some auxiliary concepts (for the hybrid $(\Phi, \rho, \eta, \zeta, \theta)-$invexity) crucial to the problem on hand.

Definition 3.1. A point $x^* \in Q$ is an $\epsilon-$efficient solution to (P) if there does not exist an $x \in Q$ such that

$$\frac{f_i(x)}{g_i(x)} \leq \frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i \,\forall\, i = 1, \cdots, p,$$

$$\frac{f_j(x)}{g_j(x)} < \frac{f(_j x^*)}{g_j(x^*)} - \epsilon_j \text{ for some } j \in \{1, \cdots, p\},$$

where ϵ_i=$(\epsilon_1, \cdots, \epsilon_p)$ is with $\epsilon_i \geq 0$ for $i = 1, \cdots, p$.

For $\epsilon = 0$, Definition 3.1 reduces to the case that $x^* \in Q$ is an efficient solution to (P).

Next, we start with real-valued functions $E_i(., x^*, u^*)$ and $B_j(., v)$ defined by

$$E_i(x, x^*, u^*) = u_i[f_i(x) - \Big(\frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i\Big)g_i(x)],\, i \in \{1, \cdots, p\}$$

and

$$B_j(., v) = v_j H_j(x),\; j = 1, \cdots, m.$$

Theorem 3.5. Let $x^* \in Q$, let f_i, g_i for $i \in \{1, \cdots, p\}$ with $f_i(x^*) \geq \epsilon_i g_i(x^*)$, $g_i(x^*) > 0$ and H_j for $j \in \{1, \cdots, m\}$ be twice continuously differentiable at $x^* \in Q$, and let there exist $u^* \in U = \{u \in \Re^p : u > 0, \Sigma_{i=1}^p u_i = 1\}$, $v^* \in \Re_+^m$ and $z \in \Re^n$ such that

$$\Sigma_{i=1}^p u_i^*[\nabla f_i(x^*) - \Big(\frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i\Big) \bigtriangledown g_i(x^*)] + \Sigma_{j=1}^m v_j^* \bigtriangledown H_j(x^*) = 0, \quad (3.23)$$

$$\Big\langle \Big[\sum_{i=1}^p u_i^*[\nabla^2 f_i(x^*) - (\frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i)\nabla^2 g_i(x^*)] + \sum_{j=1}^m v_j^* \nabla^2 H_j(x^*)\Big]z, \zeta(x, x^*)\Big\rangle$$
$$-\frac{1}{2}\Big\langle \Big[\sum_{i=1}^p u_i^*[\nabla^2 f_i(x^*) - (\frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i)\nabla^2 g_i(x^*)]$$
$$+\sum_{j=1}^m v_j^* \nabla^2 H_j(x^*)\Big]z, z\Big\rangle \geq 0, \quad (3.24)$$

and

$$v_j^* H_j(x^*) = 0,\, j \in \{1, \cdots, m\}. \quad (3.25)$$

Suppose, in addition, that any one of the following assumptions holds (for $\rho(x,x^*)\geq 0$):

(i) $E_i(.\,;x^*,u^*)\ \ \forall\, i\in\{1,\cdots,p\}$ are hybrid $(\Phi,\rho,\eta,\zeta,\theta)-$pseudo-invex at x^* for $\Phi(a)\geq 0\Rightarrow a\geq 0$, and $B_j(.\,,v^*)\ \ \forall\, j\in\{1,\cdots,m\}$ are hybrid $(\tilde{\Phi},\bar{\rho},\eta,\zeta,\theta)-$quasi-invex at x^* for $\tilde{\Phi}$ increasing and $\tilde{\Phi}(0)=0$.

(ii) $E_i(.\,;x^*,u^*)\ \ \forall\, i\in\{1,\cdots,p\}$ are prestrictly hybrid $(\Phi,\rho,\eta,\zeta,\theta)-$pseudo-invex at x^* for $\Phi(a)\geq 0\Rightarrow a\geq 0$, and $B_j(.\,,v^*)\ \ \forall\, j\in\{1,\cdots,m\}$ are strictly hybrid $(\tilde{\Phi},\bar{\rho},\eta,\zeta,\theta)-$quasi-invex at x^* for $\tilde{\Phi}$ increasing and $\tilde{\Phi}(0)=0$.

(iii) $E_i(.\,;x^*,u^*)\ \ \forall\, i\in\{1,\cdots,p\}$ are strictly hybrid $(\Phi,\rho,\eta,\zeta,\theta)-$pseudo-invex at x^* for $\Phi(a)\geq 0\Rightarrow a\geq 0$, and $B_j(.\,,v^*)\ \ \forall\, j\in\{1,\cdots,m\}$ are strictly hybrid $(\bar{\Phi},\bar{\rho},\eta,\zeta,\theta)-$quasi-invex at x^* for $\bar{\Phi}$ increasing and $\bar{\Phi}(0)=0$.

(iv) For each $i\in\{1,\cdots,p\}$, f_i is hybrid $(\Phi,\rho_1,\eta,\zeta,\theta)-$invex and $-g_i$ is $(\Phi,\rho_2,\eta,\zeta,\theta)-$invex at x^* for $\Phi(a)\geq 0\Rightarrow a\geq 0$, and $H_j(.\,,v^*)\ \ \forall\, j\in\{1,\cdots,m\}$ is hybrid $(\bar{\Phi},\rho_3,\eta,\zeta,\theta)-$quasi-invex at x^* for $\bar{\Phi}$ increasing and $\bar{\Phi}(0)=0$ and $\Sigma_{j=1}^m v_j^*\rho_3(x,x^*)+\rho^*\geq 0$ for $\rho^*=\Sigma_{i=1}^p u_i^*(\rho_1(x,x^*)+\phi(x^*)\rho_2(x,x^*))$, where $\phi(x^*)=\frac{f_i(x^*)}{g_i(x^*)}-\epsilon_i$.

Then x^* is an $\epsilon-$efficient solution to (P).

Proof
If (i) holds, and if $x\in Q$, then it follows from (3.23) and 3.24) that

$$\begin{aligned}&\Big\langle \Sigma_{i=1}^p u_i^*[\bigtriangledown f_i(x^*)-(\frac{f_i(x^*)}{g_i(x^*)}-\epsilon_i)\bigtriangledown g_i(x^*)],\eta(x,x^*)\Big\rangle\\ &+\Big\langle \sum_{i=1}^p u_i^*[\nabla^2 f_i(x^*)z-(\frac{f_i(x^*)}{g_i(x^*)}-\epsilon_i)\nabla^2 g_i(x^*)z],\zeta(x,x^*)\Big\rangle\\ &-\frac{1}{2}\Big\langle \sum_{i=1}^p u_i^*[\nabla^2 f_i(x^*)z-(\frac{f_i(x^*)}{g_i(x^*)}-\epsilon_i)\nabla^2 g_i(x^*)z],z\Big\rangle\\ &+\Big\langle \Sigma_{j=1}^m v_j^*\bigtriangledown H_j(x^*),\eta(x,x^*)\Big\rangle+\Big\langle \Sigma_{j=1}^m v_j^*\nabla^2 H_j(x^*)z,\zeta(x,x^*)\Big\rangle\\ &-\frac{1}{2}\Big\langle \Sigma_{j=1}^m v_j^*\nabla^2 H_j(x^*)z,z\Big\rangle\geq 0. \qquad (3.26)\end{aligned}$$

Since $v^*\geq 0$, $x\in Q$ and (3.25) holds, we have

$$\Sigma_{j=1}^m v_j^* H_j(x)\leq 0=\Sigma_{j=1}^m v_j^* H_j(x^*),$$

which implies

$$\Sigma_{j=1}^m v_j^* H_j(x)-\Sigma_{j=1}^m v_j^* H_j(x^*)\leq 0,$$

so using the assumptions on $\tilde{\Phi}$ it results in

$$\tilde{\Phi}\Big(\Sigma_{j=1}^m v_j^* H_j(x) - \Sigma_{j=1}^m v_j^* H_j(x^*)\Big) \leq 0,$$

which, in light of the hybrid $(\tilde{\Phi}, \bar{\rho}, \eta, \zeta, \theta)-$ quasi-invexity of $B_j(., v^*)$ at x^*, implies

$$\begin{aligned} \langle \triangledown H_j(x^*), \eta(x,x^*)\rangle + \langle \zeta(x,x^*), \nabla^2 H_j(x^*)z\rangle - \tfrac{1}{2}\langle \nabla^2 H_j(x^*)z, z\rangle \\ + \bar{\rho}(x,x^*)\|\theta(x,x^*)\|^2 \leq 0. \end{aligned} \tag{3.27}$$

It follows from (3.26) and (3.27) that

$$\begin{aligned} & \Big\langle \Sigma_{i=1}^p u_i^*[\triangledown f_i(x^*) - (\frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i) \triangledown g_i(x^*)], \eta(x,x^*)\Big\rangle \\ & + \Big\langle \sum_{i=1}^p u_i^*[\nabla^2 f_i(x^*)z - (\frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i)\nabla^2 g_i(x^*)z], \zeta(x,x^*)\Big\rangle \\ & -\frac{1}{2}\Big\langle \sum_{i=1}^p u_i^*[\nabla^2 f_i(x^*)z - (\frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i)\nabla^2 g_i(x^*)z], z\Big\rangle \\ \geq \quad & \bar{\rho}(x,x^*)\|\theta(x,x^*)\|^2 \geq -\rho(x,x^*)\|\theta(x,x^*)\|^2. \end{aligned} \tag{3.28}$$

As a result, since $\rho(x,x^*) \geq 0$, applying the hybrid $(\Phi, \rho, \eta, \zeta, \theta)-$ pseudo-invexity at x^* to (3.28) and assumptions on Φ, we have

$$\begin{aligned} & \Phi\Big(\Sigma_{i=1}^p u_i^*[f_i(x) - (\frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i)g_i(x)] \\ & -\Sigma_{i=1}^p u_i^*[f_i(x^*) - \Big(\frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i)g_i(x^*)]\Big) \geq 0, \end{aligned}$$

which implies

$$\begin{aligned} & \Sigma_{i=1}^p u_i^*[f_i(x) - (\frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i)g_i(x)] \\ & \geq \Sigma_{i=1}^p u_i^*[f_i(x^*) - (\frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i)g_i(x^*)] \\ & \geq \Sigma_{i=1}^p u_i^*[f_i(x^*) - (\frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i)g_i(x^*)] \\ & -\Sigma_{i=1}^p u_i^* \epsilon_i g_i(x^*) = 0. \end{aligned}$$

Thus, we have

$$\Sigma_{i=1}^p u_i^*[f_i(x) - (\frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i)g_i(x)] \geq 0. \tag{3.29}$$

Since $u_i^* > 0$ for each $i \in \{1, \cdots, p\}$, we conclude that there does not exist an $x \in Q$ such that

$$\sum_{i=1}^{p} \big[\frac{f_i(x)}{g_i(x)} - (\frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i)\big] \leq 0 \;\forall\, i = 1, \cdots, p,$$

$$\sum_{j=1}^{p} \big[\frac{f_j(x)}{g_j(x)} - (\frac{f_j(x^*)}{g_j(x^*)} - \epsilon_j)\big] < 0 \text{ for some } j \in \{1, \cdots, p\}.$$

Hence, x^* is an $\epsilon-$efficient solution to (P).
If (ii) holds, and if $x \in Q$, then it follows from (3.23) and (3.24) that

$$\begin{aligned}
&\Big\langle \Sigma_{i=1}^{p} u_i^* [\bigtriangledown f_i(x^*) - (\frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i) \bigtriangledown g_i(x^*)], \eta(x, x^*)\Big\rangle \\
&+\Big\langle \sum_{i=1}^{p} u_i^* [\nabla^2 f_i(x^*) z - (\frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i) \nabla^2 g_i(x^*) z], \zeta(x, x^*)\Big\rangle \\
&-\frac{1}{2}\Big\langle \sum_{i=1}^{p} u_i^* [\nabla^2 f_i(x^*) z - (\frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i) \nabla^2 g_i(x^*) z], z\Big\rangle \\
&+\Big\langle \Sigma_{j=1}^{m} v_j^* \bigtriangledown H_j(x^*), \eta(x, x^*)\Big\rangle + \Big\langle \Sigma_{j=1}^{m} v_j^* \nabla^2 H_j(x^*) z, \zeta(x, x^*)\Big\rangle \\
&-\frac{1}{2}\Big\langle \Sigma_{j=1}^{m} v_j^* \nabla^2 H_j(x^*) z, z\Big\rangle \geq 0.
\end{aligned} \tag{3.30}$$

Since $v^* \geq 0$, $x \in Q$ and (3.25) holds, we have

$$\Sigma_{j=1}^{m} v_j^* H_j(x) \leq 0 = \Sigma_{j=1}^{m} v_j^* H_j(x^*),$$

or

$$\Sigma_{j=1}^{m} v_j^* H_j(x) - \Sigma_{j=1}^{m} v_j^* H_j(x^*) \leq 0,$$

which implies based on assumptions on $\tilde{\Phi}$ that

$$\tilde{\Phi}\Big(\Sigma_{j=1}^{m} v_j^* H_j(x) - \Sigma_{j=1}^{m} v_j^* H_j(x^*)\Big) \leq 0.$$

Next, in light of the strict $(\tilde{\Phi}, \bar{\rho}, \eta, \zeta, \theta)-$quasi-invexity of $B_j(., v^*)$ at x^* with $\tilde{\Phi}$ increasing and $\tilde{\Phi}(0) = 0$, we find

$$\begin{aligned}
\langle \bigtriangledown H_j(x^*), \eta(x, x^*)\rangle + \langle \zeta(x, x^*), \nabla^2 H_j(x^*) z\rangle - \tfrac{1}{2}\langle \nabla^2 H_j(x^*) z, z\rangle \\
+\bar{\rho}(x, x^*)\|\theta(x, x^*)\|^2 < 0.
\end{aligned} \tag{3.31}$$

It follows from (3.30) and (3.31) that

$$
\begin{aligned}
&\Big\langle \Sigma_{i=1}^{p} u_i^*[\bigtriangledown f_i(x^*) - (\frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i) \bigtriangledown g_i(x^*)], \eta(x,x^*)\Big\rangle \\
&+\Big\langle \sum_{i=1}^{p} u_i^*[\nabla^2 f_i(x^*)z - (\frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i)\nabla^2 g_i(x^*)z], \zeta(x,x^*)\Big\rangle \\
&-\frac{1}{2}\Big\langle \sum_{i=1}^{p} u_i^*[\nabla^2 f_i(x^*)z - (\frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i)\nabla^2 g_i(x^*)z], z\Big\rangle \\
> \quad & \bar{\rho}(x,x^*)\|\theta(x,x^*)\|^2 > -\rho(x,x^*)\|\theta(x,x^*)\|^2. \qquad (3.32)
\end{aligned}
$$

As a result, since $\rho(x,x^*) \geq 0$, applying the prestrictly hybrid $(\Phi,\rho,\eta,\zeta,\theta)-$pseudo-invexity at x^* to (3.32) and assumptions on Φ, we have

$$
\begin{aligned}
&\Phi\Big(\Sigma_{i=1}^{p} u_i^*[f_i(x) - (\frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i)g_i(x)] \\
&-\Sigma_{i=1}^{p} u_i^*[f_i(x^*) - (\frac{f_i(x^*)}{g_i(x^*)}) - \epsilon_i)g_i(x^*)]\Big) \geq 0,
\end{aligned}
$$

which implies

$$
\begin{aligned}
&\Sigma_{i=1}^{p} u_i^*[f_i(x) - (\frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i)g_i(x)] \\
\geq \quad & \Sigma_{i=1}^{p} u_i^*[f_i(x^*) - (\frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i)g_i(x^*)] \\
\geq \quad & \Sigma_{i=1}^{p} u_i^*[f_i(x^*) - (\frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i)g_i(x^*)] - \Sigma_{i=1}^{p} u_i^*\epsilon_i g_i(x^*) \\
= \quad & 0.
\end{aligned}
$$

Thus, we have

$$
\Sigma_{i=1}^{p} u_i^*[f_i(x) - (\frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i)g_i(x)] \geq 0. \qquad (3.33)
$$

Since $u_i^* > 0$ for each $i \in \{1,\cdots,p\}$, we conclude that there does not exist an $x \in Q$ such that

$$
\frac{f_i(x)}{g_i(x)} - (\frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i) \leq 0 \ \forall\, i = 1,\cdots,p,
$$

$$
\frac{f_j(x)}{g_j(x)} - (\frac{f_j(x^*)}{g_j(x^*)} - \epsilon_j) < 0 \, \text{for some}\, j \in \{1,\cdots,p\}.
$$

Hence, x^* is an $\epsilon-$efficient solution to (P).

The proof applying (iii) is similar to that of (ii), and we just prove using (iv) as follows: since $x \in Q$, it follows that $H_j(x) \leq H_j(x^*)$. Then applying the $(\bar{\Phi}, \rho_3, \eta, \zeta, \theta)-$quasi-invexity of H_j at x^* and $v^* \in R^m_+$, we have

$$\langle \Sigma_{j=1}^m v_j^* \bigtriangledown H_j(x^*), \eta(x, x^*)\rangle + \Big\langle \zeta(x, x^*), \Sigma_{j=1}^m v_j^* \nabla^2 H_j(x^*) z\Big\rangle$$
$$-\frac{1}{2}\Big\langle \Sigma_{j=1}^m v_j^* \nabla^2 H_j(x^*) z, z\Big\rangle \leq -\Sigma_{j=1}^m v_j^* \rho_3 \|\theta(x, x^*)\|^2.$$

Since $u^* \geq 0$ and $f_i(x^*) \geq \epsilon_i g_i(x^*)$, it follows from $(\Phi, \rho_1, \eta, \zeta, \theta)-$invexity and $(\Phi, \rho_2, \eta, \zeta, \theta)-$invexity assumptions that

$$\begin{aligned}
& \Phi\Big(\Sigma_{i=1}^p u_i^*[f_i(x) - (\frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i) g_i(x)]\Big) \\
= \;& \Phi\Big(\Sigma_{i=1}^p u_i^* \{[f_i(x) - f_i(x^*)] - (\frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i)[g_i(x) - g_i(x^*)] + \epsilon_i g_i(x^*)\}\Big) \\
\geq \;& \Sigma_{i=1}^p u_i^* \{\langle \bigtriangledown f_i(x^*) - (\frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i) \bigtriangledown g_i(x^*), \eta(x, x^*)\rangle\} \\
& + \langle \zeta(x, x^*), \Sigma_{i=1}^p u_i^* [\nabla^2 f_i(x^*) z - (\frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i) \nabla^2 g_i(x^*) z \rangle] \\
& - \frac{1}{2}\langle \Sigma_{i=1}^p u_i^* [\nabla^2 f_i(x^*) z - (\frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i) \nabla^2 g_i(x^*) z, z\rangle \\
& + [\rho_1(x, x^*) + \phi(x^*) \rho_2(x, x^*)] \|\theta(x, x^*)\|^2 + \Sigma_{i=1}^p u_i^* \epsilon_i g_i(x^*) \\
\geq \;& -\big[\langle \Sigma_{j=1}^m v_j^* \bigtriangledown H_j(x^*), \eta(x, x^*)\rangle + \Big\langle \zeta(x, x^*), \Sigma_{j=1}^m v_j^* \nabla^2 H_j(x^*) z\Big\rangle \\
& - \frac{1}{2}\Big\langle \Sigma_{j=1}^m v_j^* \nabla^2 H_j(x^*) z, z\Big\rangle\big] \\
& + \Sigma_{i=1}^p u_i^* [\rho_1(x, x^*) + \phi(x^*) \rho_2(x, x^*)] \|\theta(x, x^*)\|^2 + \Sigma_{i=1}^p u_i^* \epsilon_i g_i(x^*) \\
\geq \;& (\Sigma_{j=1}^m v_j^* \rho_3(x, x^*) + \Sigma_{i=1}^p u_i^* [\rho_1(x, x^*) + \phi(x^*) \rho_2(x, x^*)]) \|\theta(x, x^*)\|^2 \\
& + \Sigma_{i=1}^p u_i^* \epsilon_i g_i(x^*) \\
= \;& (\Sigma_{j=1}^m v_j^* \rho_3(x, x^*) + \rho^*) \|\theta(x, x^*)\|^2 + \Sigma_{i=1}^p u_i^* \epsilon_i g_i(x^*) \\
\geq \;& (\Sigma_{j=1}^m v_j^* \rho_3(x, x^*) + \rho^*) \|\theta(x, x^*)\|^2 \geq 0.
\end{aligned}$$

Therefore, we have

$$\Sigma_{i=1}^p u_i^* [f_i(x) - (\frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i) g_i(x)] \geq 0. \tag{3.34}$$

Thus, we conclude that there does not exist an $x \in Q$ such that

$$\sum_{i=1}^p \big[\frac{f_i(x)}{g_i(x)} - (\frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i)\big] \leq 0 \;\forall\, i = 1, \cdots, p,$$

$$\sum_{j=1}^{p}\Big[\frac{f_j(x)}{g_j(x)} - \Big(\frac{f_j(x^*)}{g_j(x^*)} - \epsilon_j\Big)\Big] < 0 \text{ for some } j \in \{1, \cdots, p\}.$$

Hence, x^* is an $\epsilon-$efficient solution to (P).

□

4. Concluding Remarks

We established several results on multiobjective fractional programming problems based on the generalized hybrid $(\Phi, \rho, \eta, \zeta, \theta)-$invexities and on efficient solutions to the multiobjective fractional programming problems. We observe that the obtained results in this communication can be generalized to the case of multiobjective fractional programming with generalized hybrid invex functions of higher orders (including the exponential type generalized invexities), for instance, based on the work of Mishra and Rueda [11], Mishra, Laha and Verma [13], and Zalmai and Zhang [38] to the case of the efficiency as well as to the $\epsilon-$efficiency conditions relating to the minimax fractional programming problems involving generalized invex functions. Furthermore, the hybrid $(\Phi, \rho, \eta, \zeta, \theta)-$ invexities can effectively be applied generalizing/unifying the first-order sufficient efficiency condition results [35], first-order parametric duality model results [36] as well as second-order duality model results (Zalmai [37]) on Hanson-Antczak-type generalized V-invex functions in semiinfinite multiobjective fractional programming. Based on new duality models and suitable constraint structures, the weak, strong, and strict converse duality theorems can be established using appropriate hybrid $(\Phi, \rho, \eta, \zeta, \theta)-$ invexities.

Acknowledgment

The author is greatly indebted to the reviewers for their highly constructive comments and valuable suggestions leading the revised version.

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