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Estimation of the reliability function for two-parameter exponentiated Rayleigh or Burr type X distribution

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Abstract **Problem Statement:** The two-parameter exponentiated Rayleigh distribution has been widely used especially in the modelling of life time event data. It provides a statistical model which has a wide variety of application in many areas and the main advantage is its ability in the context of life time event among other distributions. The uniformly minimum variance unbiased and maximum likelihood estimation methods are the ways to estimate the parameters of the distribution. In this study, we explore and compare the performance of the uniformly minimum variance unbiased and maximum likelihood estimators of the reliability functions $R(t) = P(X > t)$ and $P = P(X > Y)$ for the two-parameter exponentiated Rayleigh distribution. **Approach:** A new technique of obtaining the estimators of these parametric functions is introduced in which major role is played by the estimators of powers of the parameter(s) and the functional forms of the parametric functions to be estimated are not needed. We explore the performance of these estimators numerically under varying conditions. Through the simulation study a comparison are made on the performance of these estimators with respect to the bias, mean square error (MSE), 95% confidence length and corresponding coverage percentage. **Conclusion:** Based on the results of simulation study, the uniformly minimum variance unbiased estimators of $R(t)$ and 'P' for the two-parameter exponentiated Rayleigh distribution are found to be superior than maximum likelihood estimators of $R(t)$ and 'P'.

Keywords Bootstrap method; Two-parameter exponentiated Rayleigh distribution; Uniformly minimum variance unbiased estimators (UMVUES); Maximum likelihood estimators (MLES)

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1. Introduction

Reliability theory is mainly concerned with the determination of the probability that a system, consisting possibly of several components, will operate adequately for a given period of time in its intended application. The reliability function $R(t)$ is defined as the probability of failure-free operation until time t. Thus, if the random variable (rv) X denotes the lifetime of an item, then $R(t) = P(X > t)$. Another measure of reliability under stress-strength set-up is the probability $P = P(X > Y)$, which represents the reliability of an item of random strength X subject to random stress Y. Many researchers have considered the problems of estimation of $R(t)$ and 'P' for various lifetime distributions and for a brief review, one may refer to [3, 4, 5, 13, 16, 24, 2, 28, 10, 18, 29, 8, 9],etc.

In [7], Burr introduced twelve different forms of cumulative distribution functions for modelling lifetime data. Among those distributions, Burr Type X and Burr Type XII are the most popular ones. Several authors considered different aspects of the Burr Type X and Burr Type XII distributions, see, for example, [21, 23, 30, 11, 12, 1, 19, 25]. For an excellent review for the two distributions the readers are refereed to [15].

[26] (see also [27]) introduced two-parameter Burr Type X distribution and named as the two-parameter exponentiated Rayleigh distribution. The two-parameter exponentiated Rayleigh distribution has the following probability density function (pdf)

$$f(x;\alpha,\lambda) = 2\alpha\lambda^2 x e^{-(\lambda x)^2}(1-e^{-(\lambda x)^2})^{\alpha-1}; \quad x,\alpha,\lambda > 0 \tag{1}$$

and the distribution function

$$F(x;\alpha,\lambda) = \{1-e^{-(\lambda x)^2}\}^{\alpha}; \quad x,\alpha,\lambda > 0. \tag{2}$$

Here, α and λ are the shape and scale parameters, respectively. In [20], the authors observed that for $\alpha \leq 1/2$, the pdf of two-parameter exponentiated Rayleigh distribution is a decreasing function and it is a right skewed unimodal function for $\alpha > 1/2$. They found different forms of the density function. It is also observed that the hazard function of a two-parameter exponentiated Rayleigh distribution can be either bathtub type or an increasing function, depending on the shape parameter α. For $\alpha \leq 1/2$, the hazard function of two-parameter exponentiated Rayleigh distribution is bathtub type and, for $\alpha > 1/2$, it has an increasing hazard function. Surles and Padgett [26] showed that the two-parameter exponentiated Rayleigh distribution can be used quite effectively in modelling strength data and also modelling general lifetime data. Kundu and Raqab [17] proposed different methods of estimation for generalized Rayleigh distribution.

The rest of the study is arranged as follows. In Section 2, we derive the UMVUES of the reliability function $R(t)$ and 'P' assuming α to be unknown but λ known. In Section 3, we obtain the MLES of the reliability function $R(t)$ and 'P', when all the parameters are unknown. In Section 4, simulation study is carried out to investigate the performance of estimators. Finally, In Section 5, discussion is made and followed by conclusion.

2. UMVUES of the powers of α, $R(t)$ and 'P' when λ is known

Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from (1).

Lemma 1
Let $S = -\sum_{i=1}^{n} \ln\{1 - e^{-(\lambda x_i)^2}\}$. Then, S is complete and sufficient for the distribution given at (1). Moreover, the pdf of S is

$$h(s;\alpha) = \frac{\alpha^n}{\Gamma(n)} s^{n-1} \exp(-\alpha s); \quad s > 0.$$

Proof
From (1), the joint pdf of $X_1, X_2, \ldots, X_n$ is

$$\begin{aligned} & h(x_1, x_2, \ldots, x_n; \alpha, \lambda) \\ &= \prod_{i=1}^{n} f(x_i; \alpha, \lambda) \\ &= (2\alpha\lambda^2)^n \left\{\prod_{i=1}^{n} x_i\right\} \exp\left\{-\lambda^2 \sum_{i=1}^{n} x_i^2\right\} \exp\left\{(\alpha - 1) \sum_{i=1}^{n} \ln\left\{1 - e^{-(\lambda x_i)^2}\right\}\right\} \\ &= (2\alpha\lambda^2)^n \left\{\prod_{i=1}^{n} x_i\right\} \exp\left\{-\lambda^2 \sum_{i=1}^{n} x_i^2\right\} \left\{\prod_{i=1}^{n} \frac{1}{\{1 - e^{-(\lambda x_i)^2}\}}\right\} \exp(-\alpha s). \end{aligned} \tag{3}$$

It follows from (3) and Fisher-Neymann factorization theorem [see [22], p. 341] that S is sufficient for the distribution given in (1). In (1), if we make the transformation $Y = \{1 - e^{-(\lambda X)^2}\}$, then the pdf of Y is

$$g(y;\alpha) = \alpha y^{\alpha-1}; \quad 0 < y < 1.$$

Letting $\ln Y = Z$, the pdf of Z is

$$g(z;\alpha) = e^{\alpha z}; \quad -\infty < z < 0.$$

Further, letting, $-2\alpha Z = V$, the pdf of V is

$$g(v;\alpha) = \frac{1}{2} e^{-v/2}; \quad 0 < v < \infty,$$

which is $\chi^2_{(2)}$. Thus, from the additive property of gamma distribution [see [14], p. 170]

$$2\alpha S = -2\alpha \sum_{i=1}^{n} \ln\{1 - e^{-(\lambda X_i)^2} \sim \chi^2_{(2n)}. \tag{4}$$

Hence, the distribution of S follows from (4). Since the distribution of S belongs to exponential family, it is also complete [see [22], p. 170]. □

The following lemma provides the UMVUES of the powers of α.

Theorem 1
For $q \in (-\infty, \infty)$, the UMVUE of α^q is

$$\hat{\alpha}^q = \begin{cases} \dfrac{\Gamma(n)}{\Gamma(n-q)} S^{-q}; & q < n, \\ 0; & \text{otherwise.} \end{cases}$$

Proof
From (4),

$$\begin{aligned} E(2\alpha S)^{-q} &= E[\chi^2_{(2n)}]^{-q} \\ &= \frac{1}{2^n \Gamma(n)} \int_0^\infty e^{-y/2} y^{n-q-1} dy \\ &= \frac{\Gamma(n-q)}{2^q \Gamma(n)}; \quad q < n, \end{aligned}$$

or,

$$E\left[S^{-q} \left\{ \frac{\Gamma(n)}{\Gamma(n-q)} \right\} \right] = \alpha^q.$$

Hence, the theorem follows from Lehmann-Scheffé theorem [see [22], p. 357]. □

In the following lemma, we provide the UMVUE of the sampled pdf (1) at specified point 'x'.

Lemma 2
The UMVUE of $f(x; \alpha, \lambda)$ at a specified point 'x' is

$$\hat{f}(x; \alpha, \lambda) = \begin{cases} 2(n-1)\lambda^2 S^{-1} x e^{-(\lambda x)^2} (1 - e^{-(\lambda x)^2})^{-1} (1 + S^{-1} \ln\{1 - e^{-(\lambda x)^2}\})^{n-2}; \\ \qquad -S < \ln\{1 - e^{-(\lambda x)^2}\}, \\ 0; \quad \text{otherwise.} \end{cases}$$

Proof
Since S is complete and sufficient for the distribution $f(x;\alpha,\lambda)$, any function $H(S)$ of S satisfying $E[H(S)] = f(x;\alpha,\lambda)$ will be the UMVUE of $f(x;\alpha,\lambda)$. From (1) and Lemma 1, we have

$$E[H(S)] = f(x;\alpha,\lambda),$$

or,

$$\int_0^\infty H(s)\frac{\alpha^n}{\Gamma(n)} s^{n-1}e^{-\alpha s}ds = 2\alpha\lambda^2 x e^{-(\lambda x)^2}\{1-e^{-(\lambda x)^2}\}^{\alpha-1},$$

or,

$$\frac{\alpha^n}{\Gamma(n)}\int_0^\infty H(s)s^{n-1}e^{-\alpha s}ds = 2\alpha\lambda^2 x e^{-(\lambda x)^2}(1-e^{-(\lambda x)^2})^{-1}\exp(\alpha\ln\{1-e^{-(\lambda x)^2}\}),$$

or,

$$\begin{aligned}&\frac{\alpha^n}{\Gamma(n)}\int_0^\infty H(s)s^{n-1}\exp[-\alpha(s+\ln\{1-e^{-(\lambda x)^2}\})]ds\\ &\qquad = 2\alpha\lambda^2 x e^{-(\lambda x)^2}\{1-e^{-(\lambda x)^2}\}^{-1}.\end{aligned} \tag{5}$$

Let us choose

$$H(s) = \begin{cases} 2(n-1)\lambda^2 s^{-1} x e^{-(\lambda x)^2}(1-e^{-(\lambda x)^2})^{-1}(1+s^{-1}\ln\{1-e^{-(\lambda x)^2}\})^{n-2}; \\ \qquad\qquad -s < \ln\{1-e^{-(\lambda x)^2}\}, \\ 0; \qquad \text{otherwise.}\end{cases}$$

Then,

$$\begin{aligned}\text{LHS of (2.3)} &= \frac{2(n-1)\alpha^n\lambda^2 x e^{-(\lambda x)^2}\{1-e^{-(\lambda x)^2}\}^{-1}}{\Gamma(n)}\\ &\quad\cdot \int_{-\ln\{1-e^{-(\lambda x)^2}\}}^\infty (s+\ln\{1-e^{-(\lambda x)^2}\})^{n-2}\\ &\quad\cdot \exp[-\alpha(s+\ln\{1-e^{-(\lambda x)^2}\})]ds\\ &= \frac{2(n-1)\alpha^n\lambda^2 x e^{-(\lambda x)^2}\{1-e^{-(\lambda x)^2}\}^{-1}}{\Gamma(n)}\int_0^\infty y^{n-2}\exp(-\alpha y)dy\\ &= \text{RHS}.\end{aligned}$$

Hence the lemma holds. □

Remark 1
We can write (1) as

$$f(x;\alpha,\lambda) = 2\alpha\lambda^2 x e^{-(\lambda x)^2}(1-e^{-(\lambda x)^2})^{-1}\exp(\alpha\ln\{1-e^{-(\lambda x)^2}\})$$
$$= 2\alpha\lambda^2 x e^{-(\lambda x)^2}(1-e^{-(\lambda x)^2})^{-1}\sum_{i=0}^{\infty}\frac{(\ln\{1-e^{-(\lambda x)^2}\})^i}{i!}\alpha^{i+1}. \quad (6)$$

Using (2.4), Theorem 1 and Lemma 1 of Chaturvedi and Tomer (2002), UMVUE of $f(x;\alpha,\lambda)$ at a specified point 'x' is

$$\begin{aligned}
&\hat{f}(x;\alpha,\lambda)\\
&= 2\lambda^2 x e^{-(\lambda x)^2}(1-e^{-(\lambda x)^2})^{-1}\sum_{i=0}^{\infty}\frac{(\ln\{1-e^{-(\lambda x)^2}\})^i}{i!}\hat{\alpha}^{i+1}\\
&= 2\lambda^2 x e^{-(\lambda x)^2}(1-e^{-(\lambda x)^2})^{-1}\sum_{i=0}^{\infty}\frac{(\ln\{1-e^{-(\lambda x)^2}\})^i}{i!}\left\{\frac{\Gamma(n)}{\Gamma(n-i-1)}\right\}S^{-(i+1)}\\
&= 2(n-1)\lambda^2 S^{-1} x e^{-(\lambda x)^2}(1-e^{-(\lambda x)^2})^{-1}\sum_{i=0}^{n-2}\binom{n-2}{i}(S^{-1}\ln\{1-e^{-(\lambda x)^2}\})^i\\
&= \begin{cases} 2(n-1)\lambda^2 S^{-1} x e^{-(\lambda x)^2}(1-e^{-(\lambda x)^2})^{-1}(1+S^{-1}\ln\{1-e^{-(\lambda x)^2}\})^{n-2}; \\ \qquad -S<\ln\{1-e^{-(\lambda x)^2}\}, \\ 0; \qquad \text{otherwise,}\end{cases}
\end{aligned}$$

which coincide with Lemma 2. Thus, the UMVUES of the powers of α can be used to derive the UMVUE of $f(x;\alpha,\lambda)$ at a specified point 'x'.

In the following theorem, we obtain UMVUE of $R(t)$.

Theorem 2
The UMVUE of $R(t)$ is given by

$$\hat{R}(t) = \begin{cases} 1-(1+S^{-1}\ln\{1-e^{-(\lambda t)^2}\})^{n-1}; & -S<\ln\{1-e^{-(\lambda t)^2}\}, \\ 0; & \text{otherwise.}\end{cases}$$

Proof
Since $F(x,s) = f(x;\alpha,\lambda)h(s;\alpha)$ is a continuous function of (X,S) on the rectangle $[t,\infty)\times[0,\infty)$, the conditions of Fubini's theorem [see Bilodeau *et al* ([6], p.207)] are satisfied for the change of order of integration. Let us consider

the expected value of the integral $\int_t^\infty f(x;\alpha,\lambda)dx$ with respect to S, i.e.,

$$\begin{aligned}\int_0^\infty \{\int_t^\infty \hat{f}(x;\alpha,\lambda)dx\}h(s;\alpha)ds &= \int_t^\infty [E_S\{\hat{f}(x;\alpha,\lambda)\}]dx \\ &= \int_t^\infty f(x;\alpha,\lambda)dx \\ &= R(t). \end{aligned} \tag{7}$$

We conclude from (7) that the UMVUE of $R(t)$ can be obtained simply integrating $\hat{f}(x;\alpha,\lambda)$ from t to ∞. Thus, from Lemma 2,

$$\begin{aligned}\hat{R}(t) &= 2(n-1)S^{-1}\int_t^\infty \lambda^2 x e^{-(\lambda x)^2}\{1-e^{-(\lambda x)^2}\}(1+S^{-1}\ln\{1-e^{-(\lambda x)^2}\})^{n-2}dx; \\ &\qquad\qquad -S<\ln\{1-e^{-(\lambda x)^2}\} \\ &= (n-1)\int_{S^{-1}\ln\{1-e^{-(\lambda t)^2}\}}^{0} (1+y)^{n-2}dy\end{aligned}$$

and the theorem follows.

Let X and Y be two independent rv's following the distributions $f_1(x;\alpha_1,\lambda_1)$ and $f_2(y;\alpha_2,\lambda_2)$, respectively, where

$$f_1(x;\alpha_1,\lambda_1) = 2\alpha_1\lambda_1^2 x e^{-(\lambda_1 x)^2}(1-e^{-(\lambda_1 x)^2})^{\alpha_1-1}; \quad x>0,\ \alpha_1,\lambda_1>0$$

and

$$f_2(y;\alpha_2,\lambda_2) = 2\alpha_2\lambda_2^2 y e^{-(\lambda_2 y)^2}(1-e^{-(\lambda_2 y)^2})^{\alpha_2-1}; \quad y>0,\ \alpha_2,\lambda_2>0.$$

Here, we assume that α_1 and α_2 are unknown but λ_1 and λ_2 are known. Let $X_1,X_2,\ldots,X_n$ be a random sample of size n from $f_1(x;\alpha_1,\lambda_1)$ and $Y_1,Y_2,\ldots,Y_m$ be a random sample of size m from $f_2(y;\alpha_2,\lambda_2)$. Let us denote by $S=-\sum_{i=1}^{n}\ln\{1-e^{-(\lambda_1 x_i)^2}\}$ and $T=-\sum_{j=1}^{m}\ln\{1-e^{-(\lambda_2 y_j)^2}\}$.

In what follows, we obtain the UMVUE of 'P'. $\square$

Theorem 3
The UMVUE of 'P' is given by

$$\hat{P} = \begin{cases} 1-(m-1)\displaystyle\int_0^1 (1+S^{-1}\ln\{1-(1-e^{-Tv})^{\lambda_1^2/\lambda_2^2}\}^{n-1}(1-v)^{m-2}dv; \\ \qquad [-\lambda_2^{-2}\ln(1-e^{-T})]^{1/2} > [-\lambda_1^{-2}\ln(1-e^{-S})]^{1/2}, \\ 1-(m-1)\displaystyle\int_0^{-S^{-1}\ln\{1-(1-e^{-S})^{\lambda_2^2/\lambda_1^2}\}} (1+S^{-1}\ln\{1-(1-e^{-Tv})^{\lambda_1^2/\lambda_2^2}\})^{n-1} \\ \qquad \cdot(1-v)^{m-2}dv; \\ \qquad [-\lambda_2^{-2}\ln(1-e^{-T})]^{1/2} < [-\lambda_1^{-2}\ln(1-e^{-S})]^{1/2}. \end{cases}$$

Proof
It follows from Lemma 2 that the UMVUES of $f_1(x;\alpha_1,\lambda_1)$ and $f_2(y;\alpha_2,\lambda_2)$ at specified points 'x' and 'y', respectively, are

$$\hat{f}_1(x;\alpha_1,\lambda_1) = \begin{cases} 2(n-1)\lambda_1^2 S^{-1} x e^{-(\lambda_1 x)^2} \\ \cdot(1-e^{-(\lambda_1 x)^2})^{-1}(1+S^{-1}\ln\{1-e^{-(\lambda_1 x)^2}\})^{n-2}; \\ \qquad -S < \ln\{1-e^{-(\lambda_1 x)^2}\}, \\ 0; \qquad \text{otherwise.} \end{cases} \tag{8}$$

and

$$\hat{f}_2(y;\alpha_2,\lambda_2) = \begin{cases} 2(m-1)\lambda_2^2 T^{-1} x e^{-(\lambda_2 y)^2} \\ \cdot(1-e^{-(\lambda_2 x)^2})^{-1}(1+T^{-1}\ln\{1-e^{-(\lambda_2 y)^2}\})^{m-2}; \\ \qquad -T < \ln\{1-e^{-(\lambda_2 y)^2}\}, \\ 0; \qquad \text{otherwise.} \end{cases} \tag{9}$$

From the arguments similar to those adopted in the proof of Theorem 2, it can be shown that the UMVUE of 'P' is given by

$$\begin{aligned} \hat{P} &= \int_{y=0}^{\infty}\int_{x=y}^{\infty} \hat{f}_1(x;\alpha_1,\lambda_1)\hat{f}_2(y;\alpha_2,\lambda_2)dxdy \\ &= \int_{y=0}^{\infty} \hat{R}_1(y;\alpha_1,\lambda_1)\hat{f}_2(y;\alpha_2,\lambda_2)dy. \end{aligned}$$

Applying Lemma 2 and Theorem 2, we get

$$\hat{P} = 1 - (m-1)\lambda_2 T^{-1} \cdot \int\limits_{y=\max\{[-\lambda_2^{-2}\ln(1-e^{-T})]^{1/2},[-\lambda_1^{-2}\ln(1-e^{-S})]^{1/2}\}}^{\infty} \{1 + S^{-1}\ln\{1 - e^{-(\lambda_1 y)^2}\}\}^{n-1}$$
$$\cdot \lambda_2^2 x e^{-(\lambda_2 y)^2}(1 - e^{-(\lambda_2 x)^2})\{1 + T^{-1}\ln\{1 - e^{-(\lambda_2 y)^2}\}\}^{m-2} dy. \quad (10)$$

Let us first consider the case when $[-\lambda_2^{-2}\ln(1-e^{-T})]^{1/2} > [-\lambda_1^{-2}\ln(1-e^{-S})]^{1/2}$. In this case, from (10),

$$\hat{P} = 1 - (m-1)\int\limits_0^1 (1 + S^{-1}\ln\{1 - (1-e^{-Tv})^{\lambda_1^2/\lambda_2^2}\})^{n-1}(1-v)^{m-2} dv. \quad (11)$$

Now, we consider the case when $[-\lambda_2^{-2}\ln(1-e^{-T})]^{1/2} < [-\lambda_1^{-2}\ln(1-e^{-S})]^{1/2}$. In this case, from (10),

$$\hat{P} = 1 - (m-1) \int\limits_0^{-S^{-1}\ln\{1-(1-e^{-S})^{\lambda_2^2/\lambda_1^2}\}}$$
$$\times (1 + S^{-1}\ln\{1 - (1-e^{-Tv})^{\lambda_1^2/\lambda_2^2}\})^{n-1}(1-v)^{m-2} dv\,. \quad (12)$$

The theorem now follows on combining (11) and (12). $\square$

Corollary 1
In the case when $\lambda_1 = \lambda_2 = \lambda$, say,

$$\hat{P} = \begin{cases} 1 - \sum\limits_{i=0}^{n-1}(-1)^i \frac{(n-1)!(m-1)!}{(n-i-1)!(m+i-1)!}\left(\frac{T}{S}\right)^i; & S < T, \\ 1 - \sum\limits_{j=0}^{m-2}(-1)^j \frac{(n-1)!(m-1)!}{(n+j)!(m-j-2)!}\left(\frac{S}{T}\right)^{j+1}; & S > T. \end{cases}$$

Proof
From Theorem 3, for $S < T$,

$$\hat{P} = 1 - (m-1) \int\limits_{1}^{0} \left(1 - \frac{T}{S}v\right)^{n-1} (1-v)^{m-2} dv$$

$$= 1 - (m-1) \int\limits_{0}^{1} \sum_{i=0}^{n-1} (-1)^i \binom{n-1}{i} \left(\frac{T}{S}v\right)^i (1-v)^{m-2} dv$$

$$= 1 - (m-1) \sum_{i=0}^{n-1} (-1)^i \binom{n-1}{i} \left(\frac{T}{S}\right)^i \int\limits_{0}^{1} v^{i+1-1} (1-v)^{m-2} dv$$

$$= 1 - (m-1) \sum_{i=0}^{n-1} (-1)^i \left(\frac{T}{S}\right)^i B(m-1, i+1)$$

$$= 1 - \sum_{i=0}^{n-1} (-1)^i \frac{(n-1)!(m-1)!}{(n-i-1)!(m+i-1)!} \left(\frac{T}{S}\right)^i$$

and the first assertion follows. From Theorem 3, for $S > T$,

$$\hat{P} = 1 - (m-1) \int\limits_{S/T}^{0} \left(1 - \frac{T}{S}v\right)^{n-1} (1-v)^{m-2} dv$$

$$= 1 - (m-1) \int\limits_{1}^{0} (1-w)^{n-1} \left(1 - \frac{S}{T}w\right)^{m-2} \left(\frac{S}{T}\right) v^{j+1-1} (1-v)^{m-2} dw$$

$$= 1 - (m-1) \int\limits_{1}^{0} (1-w)^{n-1} \sum_{j=0}^{m-2} (-1)^j \binom{m-2}{j} \left(\frac{S}{T}\right)^j (w)^j \left(\frac{S}{T}\right) dw$$

$$= 1 - (m-1) \sum_{j=0}^{m-2} (-1)^j \binom{m-2}{j} \left(\frac{S}{T}\right)^{j+1} B(n, j+1)$$

$$= 1 - \sum_{j=0}^{m-2} (-1)^j \frac{(n-1)!(m-1)!}{(n+j)!(m-j-2)!} \left(\frac{S}{T}\right)^{j+1}$$

and the second assertion follows. □

Remark 2
It follows from Theorem 1 that $V(\hat{\alpha}) = \frac{\alpha^2}{n-2} \to 0$ as $n \to \infty$. Thus, $\hat{\alpha}$ is a

consistent estimator of α. Since $\hat{f}(x;\alpha,\lambda)$, $\hat{R}(t)$ and $\hat{P}$ are continuous functions of consistent estimators, therefore, they are also consistent estimators.

3. MLES of $R(t)$ and 'P' when all the parameters are unknown

Following the lines of derivations in [17], it can be shown that the MLES of α and λ are solutions of

$$\tilde{\alpha}(\lambda) = -\frac{n}{\sum\limits_{i=1}^{n} \ln(1 - e^{-(\lambda x_i)^2})} \tag{13}$$

and

$$\tilde{\lambda} = \left[\frac{\sum\limits_{i=1}^{n} \frac{x_i^2 e^{-(\lambda x_i)^2}}{(1-e^{-(\lambda x_i)^2})}}{\sum\limits_{i=1}^{n} \ln(1 - e^{-(\lambda x_i)^2})} + \frac{1}{n}\sum_{i=1}^{n} x_i^2 + \frac{1}{n}\sum_{i=1}^{n} \frac{x_i^2 e^{-(\lambda x_i)^2}}{(1 - e^{-(\lambda x_i)^2})} \right]^{-1}, \tag{14}$$

respectively.

From one-to-one property, MLES of $f(x;\alpha,\lambda)$, $R(t)$ and 'P' are given, respectively, by $\tilde{f}(x;\alpha,\lambda)$, $\tilde{R}(t)$ and $\tilde{P}$, where

$$\begin{aligned}
\tilde{f}(x;\alpha,\lambda) &= 2\tilde{\alpha}\tilde{\lambda}^2 x e^{-(\tilde{\lambda}x^2)}(1 - e^{-(\tilde{\lambda}x)^2})^{\tilde{\alpha}-1}, \\
\tilde{R}(t) &= \int_t^{\infty} \tilde{f}(x;\alpha,\lambda)dx \\
&= 2\tilde{\alpha}\tilde{\lambda}^2 \int_t^{\infty} x e^{-(\tilde{\lambda}x^2)}(1 - e^{-(\tilde{\lambda}x)^2})^{\tilde{\alpha}-1}dx \quad = \tilde{\alpha} \int_{(1-e^{-(\tilde{\lambda}t)^2})}^{1} u^{\tilde{\alpha}-1}du \\
&= 1 - \{1 - e^{-(\tilde{\lambda}t)^2}\}^{\tilde{\alpha}}
\end{aligned}$$

and

$$\tilde{P} = \int_{y=0}^{\infty}\int_{x=y}^{\infty} \tilde{f}_1(x;\alpha_1,\lambda_1)\tilde{f}_2(y;\alpha_2,\lambda_2)dxdy$$
$$= \int_0^{\infty} (1-\tilde{R}_2(x))\tilde{f}_1(x;\alpha_1,\lambda_1)dx$$
$$= 2\tilde{\alpha}_1\tilde{\lambda}_1^2 \int_0^{\infty} [1-e^{-(\tilde{\lambda}_2 x)^2}]^{\tilde{\alpha}_2} x^2 e^{-(\tilde{\lambda}_1 x^2)}(1-e^{-(\tilde{\lambda}_1 x)^2})^{\tilde{\alpha}_1-1}dx$$
$$= \tilde{\alpha}_1 \int_0^1 [1-(1-u)^{\tilde{\lambda}_2^2/\tilde{\lambda}_1^2}]^{\tilde{\alpha}_2} u^{\tilde{\alpha}_1-1}du$$

It can also be easily verified that when $\lambda_1 = \lambda_2 = \lambda$, say,

$$\tilde{P} = \frac{\tilde{\alpha}_1}{\tilde{\alpha}_1+\tilde{\alpha}_2}.$$

Remarks 1 **(i)** In the literature, the researchers have dealt with the estimation of $R(t)$ and 'P', separately. If we look at the proof of Theorems 2 and 3, we observe that the UMVUE of the sampled pdf is used to obtain the UMVUES of $R(t)$ and 'P', respectively, which is also true for MLES. Thus we have established interrelationship between the two estimation problems. Moreover, in the present approach, one does not require the expressions of $R(t)$ and 'P'.

(ii) Since the UMVUES and MLES of powers of α are obtained under same conditions, we compare their performances. For $q=-1$ the UMVUE and MLE of α are, respectively $\hat{\alpha} = (n-1)(-T)^{-1}$ and $\tilde{\alpha} = (n)(-T)^{-1}$. For these estimators,

$$V(\hat{\alpha}) = \frac{\alpha^2}{n-2} \quad \text{and} \quad V(\tilde{\alpha}) = \frac{n^2\alpha^2}{(n-1)^2(n-2)}.$$

Thus,

$$V(\tilde{\alpha}) - V(\hat{\alpha}) = \frac{(2n-1)}{(n-1)(n-2)}\alpha^2 > 0.$$

Thus, the UMVUE of α is more efficient than its MLE. Similarly, we can compare the performances of these estimators for other powers of α.

4. Numerical Findings

In order to compare the efficiency of the estimators $\hat{\alpha}$ and $\tilde{\alpha}$, when λ is known, we have calculated variances of $\hat{\alpha}$ and $\tilde{\alpha}$, for samples of sizes $n = 5, 10, 20, 30$ and 50 corresponding to $\alpha = 0.80(0.60)4.20$ and these results are reported in Table 1. From Table 1, it is clear that $\hat{\alpha}$ is more efficient than $\tilde{\alpha}$.

Table I

n	**5**		**10**		**20**		**30**		**50**	
α	$v(\hat{\alpha})$	$v(\tilde{\alpha})$	$v(\hat{\alpha})$	$v(\tilde{\alpha})$	$v(\hat{\alpha})$	$v(\tilde{\alpha})$	$v(\hat{\alpha})$	$v(\tilde{\alpha})$	$v(\hat{\alpha})$	$v(\tilde{\alpha})$
0.80	0.2133	0.3333	0.0800	0.0988	0.0356	0.0394	0.0229	0.0245	0.0133	0.0139
1.40	0.6533	1.0208	0.2450	0.3025	0.1089	0.1207	0.0700	0.0749	0.0408	0.0425
2.00	1.3333	2.0833	0.5000	0.6173	0.2222	0.2462	0.1429	0.1529	0.0833	0.0868
2.60	2.2533	3.5208	0.8450	1.0432	0.3756	0.4161	0.2414	0.2584	0.1408	0.1466
3.20	3.4133	5.3333	1.2800	1.5802	0.5689	0.6303	0.3657	0.3914	0.2133	0.2221
3.80	4.8133	7.5208	1.8050	2.2284	0.8022	0.8889	0.5157	0.5519	0.3008	0.3132
4.20	5.8800	9.1875	2.2050	2.7222	0.9800	1.0859	0.6300	0.6742	0.3675	0.3827

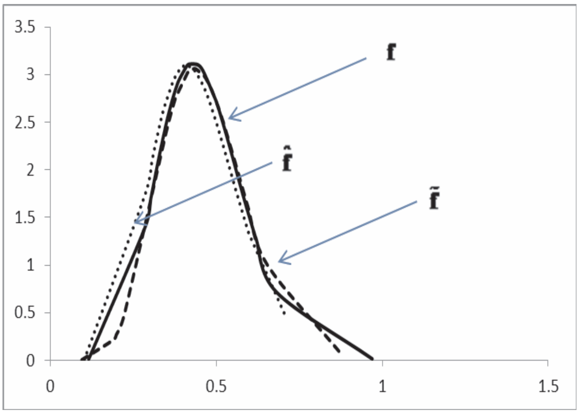

Figure 1. Curves of $f(x;\alpha,\lambda)$, $\hat{f}(x;\alpha,\lambda)$ and $\tilde{f}(x;\alpha,\lambda)$.

In order to verify the consistency of the estimators obtained, we have drawn sample of sizes $n = 30$ from (1), with $\alpha = 4$ and $\lambda = 3$. In Fig. 1, we have plotted $f(x;\alpha,\lambda)$, $\hat{f}(x;\alpha,\lambda)$ and $\tilde{f}(x;\alpha,\lambda)$, respectively, corresponding to this sample. We conclude from Fig. 1 that curves of $\hat{f}(x;\alpha,\lambda)$ and $\tilde{f}(x;\alpha,\lambda)$ overlap to the curve of $f(x;\alpha,\lambda)$ for $n = 30$. This justifies the consistency property of the estimators.

In order to demonstrate the application of the theory developed in Section 3, we generated a sample of size $n = 30$ from (1) for $\alpha = 4$ and $\lambda = 3$. Solving (13) and

Table II. Simulation results for $R(t)$.

t	n	5		10		20		50	
	$R(t)$	$\hat{R}(t)$	$\tilde{R}(t)$	$\hat{R}(t)$	$\tilde{R}(t)$	$\hat{R}(t)$	$\tilde{R}(t)$	$\hat{R}(t)$	$\tilde{R}(t)$
	0.9860	0.9710	0.9539	0.9862	0.9780	0.9840	0.9804	0.9872	0.9858
		-0.0151	-0.0321	2e-04	-0.0080	-0.0020	-0.0056	0.0011	-2e-04
0.21		0.004515932	0.001552421	8.92585e-05	0.0001760942	0.0001253313	0.0001671481	3.53743e-05	3.64641e-05
		0.0792	0.0830	0.0368	0.0417	0.0420	0.0450	0.0228	0.0236
		44.9490	92.3063	89.4167	92.2201	90.6114	92.2691	92.9964	93.3348
	0.8469	0.8195	0.8278	0.8659	0.8650	0.8511	0.8515	0.8485	0.8483
		-0.0274	-0.0191	0.0190	0.0181	0.0041	0.0046	0.0016	0.0013
0.35		0.01147278	0.005809513	0.003760792	0.003122649	0.002368198	0.002151073	0.001233167	0.001281566
		0.3232	0.2760	0.2257	0.2057	0.1870	0.1782	0.1379	0.1406
		88.0252	93.6421	94.5918	94.7441	94.4698	94.4954	95.0602	95.0570
	0.63	0.5396	0.5833	0.5620	0.5829	0.6232	0.6324	0.6351	0.6387
		-0.0904	-0.0467	-0.0680	-0.0471	-0.0067	0.0024	0.0051	0.0087
0.45		0.0172604	0.009738878	0.01090285	0.007991645	0.004080048	0.003829112	0.002163839	0.002168776
		0.3895	0.3516	0.3002	0.2883	0.2489	0.2423	0.1786	0.1767
		93.4056	93.9734	93.8667	94.0039	94.6170	94.6657	94.6296	94.6341
	0.5062	0.4364	0.4850	0.5071	0.5297	0.5007	0.5121	0.4987	0.5032
		-0.0698	-0.0212	8e-04	0.0234	-0.0055	0.0058	-0.0075	-0.0030
0.50		0.01229542	0.007300615	0.005458611	0.005697628	0.002781147	0.002724515	0.001556892	0.00149941
		0.3092	0.2996	0.2847	0.2764	0.2138	0.2112	0.1567	0.1561
		91.0011	91.7419	92.8672	93.0910	93.4982	93.5788	95.3514	95.3630
	0.388	0.3073	0.3536	0.4323	0.4559	0.3366	0.3476	0.3907	0.3952
		-0.0807	-0.0344	0.0444	0.0679	-0.0514	-0.0404	0.0027	0.0073
0.55		0.01107912	0.006118383	0.00668347	0.009234135	0.004913982	0.003954968	0.001140717	0.001191856
		0.2710	0.2814	0.2631	0.2611	0.1873	0.1895	0.1319	0.1322
		91.6221	92.2515	92.9936	93.2361	94.6500	94.6966	94.7881	94.7959
	0.284	0.1920	0.2279	0.2510	0.2605	0.2515	0.2610	0.2753	0.2713
		-0.0920	-0.0561	-0.0331	-0.0236	-0.0326	-0.0230	-0.0088	-0.0127
0.60		0.01069158	0.005908907	0.002508747	0.00203283	0.002159625	0.00167466	0.0008977123	0.0009683548
		0.1980	0.2206	0.1516	0.1549	0.1329	0.1358	0.1173	0.1163
		92.9102	93.3796	94.5525	94.5972	94.6282	94.6683	95.7776	95.7717
	0.1991	0.2473	0.2890	0.1691	0.1842	0.2065	0.2149	0.1943	0.1974
		0.0482	0.0899	-0.0299	-0.0149	0.0074	0.0158	-0.0048	-0.0016
0.65		0.006273621	0.01268783	0.002110118	0.001602851	0.001165438	0.001423038	0.0007551275	0.0007516422
		0.2530	0.2729	0.1395	0.1486	0.1305	0.1341	0.1086	0.1098
		92.7658	93.2679	94.0162	94.1039	94.1657	94.1959	95.1725	95.1831

Here, the first row indicates the estimate, the second row indicates the bias, the third row indicates variance, the fourth row indicates 95% bootstrap confidence length and the fifth row indicates the coverage percentage.

Table III. Simulation results for 'P'.

(α_1, α_2)	**(2,1)**		**(2,3)**		**(2,4)**		**(2,5)**	
P	**0.6666667**		**0.4**		**(0.3333333)**		**(0.2857143)**	
(n, m)	$\hat{P}$	$\tilde{P}$	$\hat{P}$	$\tilde{P}$	$\hat{P}$	$\tilde{P}$	$\hat{P}$	$\tilde{P}$
	0.6528	0.6370	0.4427	0.4493	0.3154	0.3329	0.2657	0.2858
	-0.0139	-0.0297	0.0427	0.0493	-0.0179	-5e-04	-0.0200	1e-04
(5, 5)	0.007232074	0.006763495	0.009493904	0.008494557	0.01030564	0.008504289	0.008154158	0.007051987
	0.3046	0.2804	0.3215	0.2864	0.3986	0.3683	0.3375	0.3214
	93.0223	93.2512	92.6982	92.8389	92.9623	93.6543	93.0178	93.5448
	0.6914	0.6672	0.4054	0.4013	0.3254	0.3271	0.2796	0.2835
	0.0247	5e-04	0.0054	0.0013	-0.0080	-0.0062	-0.0061	-0.0022
(10, 5)	0.006541987	0.005484255	0.006334268	0.005335373	0.005240716	0.004533083	0.008068555	0.00717309
	0.2972	0.2861	0.2921	0.2688	0.2768	0.2582	0.3345	0.3174
	93.8905	94.1889	92.9533	93.0341	93.7132	93.9067	93.2983	93.5995
	0.6590	0.6515	0.3835	0.3892	0.3264	0.3344	0.2732	0.2825
	-0.0077	-0.0152	-0.0165	-0.0108	-0.0069	0.0011	-0.0125	-0.0032
(10, 10)	0.004755181	0.004579777	0.005455311	0.004852003	0.004134199	0.00381056	0.005270595	0.004904695
	0.2652	0.2551	0.2736	0.2620	0.2470	0.2386	0.2724	0.2672
	93.9416	94.0369	94.2555	94.3005	94.4152	94.4906	94.2250	94.3466
	0.6676	0.6638	0.3909	0.3935	0.3315	0.3352	0.2863	0.2906
	9e-04	-0.0029	-0.0091	-0.0065	-0.0018	0.0019	6e-04	0.0049
(20,20)	0.003312423	0.003204202	0.003548195	0.003359474	0.004023466	0.003890517	0.003917004	0.003850783
	0.2276	0.2236	0.2243	0.2196	0.2447	0.2407	0.2392	0.2367
	94.4945	94.5432	94.3049	94.3128	94.5514	94.5784	94.3957	94.4237
	0.6755	0.6724	0.4079	0.4097	0.3247	0.3277	0.2777	0.2813
	0.0088	0.0057	0.0079	0.0097	-0.0086	-0.0056	-0.0080	-0.0045
(25, 25)	0.002752934	0.002639899	0.003348106	0.003261242	0.00354906	0.003422819	0.002955221	0.002866375
	0.1996	0.1971	0.2222	0.2182	0.2283	0.2254	0.2090	0.2072
	94.6152	94.6283	94.4863	94.5019	94.5404	94.546	94.5103	94.5236
	0.6651	0.6636	0.4039	0.4049	0.3358	0.3372	0.2880	0.2898
	-0.0016	-0.0030	0.0039	0.0049	0.0024	0.0039	0.0023	0.0040
(50, 50)	0.001970162	0.001949933	0.002048208	0.002021451	0.002406106	0.002383653	0.002031045	0.002023235
	0.1748	0.1736	0.1797	0.1781	0.1925	0.1913	0.1770	0.1761
	95.0227	95.0298	95.3579	95.3584	95.0372	95.0412	95.0097	95.0157

Here, the first row indicates the estimate, the second row indicates the bias, the third row indicates variance, the fourth row indicates 95% bootstrap confidence length and the fifth row indicates the coverage percentage.

(14) simultaneously, we get $\tilde{\alpha} = 3.906139$ and $\tilde{\lambda} = 2.950872$. It can be seen that $-2\ln L = -36.5843$, $R(0.20) = 0.9916$ and $\tilde{R}(0.20) = 0.9916$.

In order to obtain the MLE of 'P', we have generated one more sample of size $m = 30$ from (1) for $\alpha = 2.5$ and $\lambda = 3$. Solving as above, we get $\tilde{\lambda} = 2.486173$, $\tilde{\lambda} = 2.963452$ and $-2\ln L = -32.0487$. Using this population as Y and above population as X, we get $P = 0.6153846$ and $\tilde{P} = 0.6146800$.

For the case when α is unknown but λ is known, we have conducted simulation experiments using bootstrap re-sampling technique for sample sizes $n = 5, 10, 20$ and 50. The samples are generated from (1), with $\alpha = 3$ and $\lambda = 2.5$. For different values of t, we have computed $\hat{R}(t)$, $\tilde{R}(t)$, their corresponding bias, variance, 95% confidence length and corresponding coverage percentage. All the computations are based on 500 bootstrap replications and results are reported in Table 2.

In order to estimate 'P', for the case when α_1 and α_2 are unknown but other parameters are known, we have conducted simulation experiments using bootstrap re-sampling technique for sample sizes $(n, m) = (5, 5), (10, 5), (10, 10), (20, 20), (25, 25)$ and $(50, 50)$. The samples are generated from (1), with, $\alpha_1 = 2.0$, $\lambda_1 = 2.5$ and $\lambda_2 = 2.5$ and $\alpha_2 = 1, 3(1)5$. The computations are based on 500 bootstrap replications. We have computed $\hat{P}$, $\tilde{P}$, bias, variance, 95% confidence length and corresponding coverage percentage. The results are presented in Table 3.

5. Discussion and Conclusion

In Table 1, we compared UMVUE and MLE of α, keeping λ to be constant for two-parameter exponentiated Rayleigh distribution. The table shows that UMVUE of α is more efficient than MLE of α. From table we observe that as we increase the sample size variance of estimators of α decrease (for both of estimators UMVUE as well as for MLE). Table 1 also shows that as we increase values of the parameter α, variance increases corresponding to both of the estimators.

With the help of Fig. 1, we justified the consistency property of the estimators.

Through Table 2, we compared the efficiency of $\hat{R}(t)$ and $\tilde{R}(t)$. Table 2 shows that UMVUE of $R(t)$ is more efficient than MLE of $R(t)$. It is also clear that as we increase sample size Biasness, MSE and Confidence Length decreases but on the other hand corresponding Coverage Percentage increases. These statements are also true for the estimators $\hat{P}$ and $\tilde{P}$.

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