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# Second order duality involving second order cone arcwise connected functions and their generalizations in vector optimization problem over cones

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**Abstract** In this paper, we introduce second order cone arcwise connected function and its generalizations. Further, we study the interrelations among these functions also. Mond Weir type second order dual is formulated and duality results are proved using these functions.

**Keywords** Vector optimization over cones, second order cone arcwise connected functions, second order duality

**AMS 2010 subject classifications** 90C29, 90C46, 49N15

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## 1. Introduction

An arcwise connected function defined on an arcwise connected set is the generalization of a convex function. Ortega and Rheinboldt [8] introduced such type of functions. Later on Mukherjee [7] introduce arcwise connected functions over cones. After that authors studied different types of arcwise connected functions like Suneja and Sharma [11] introduced arcwise connected $d$-type I functions and its generalizations over cones and proved optimality and duality results for vector optimization problem over cones involving these functions and their generalizations. Recently, Chaudhary and Kapoor [1] introduced a new class of arcwise connected functions called arcwise $\rho$-$K$-connected functions and its generalizations. They established necessary and sufficient optimality conditions for a vector optimization problem over cones by involving these functions. Wolfe type dual and Mond-Weir type duals are formulated and corresponding duality results are also proved using these functions. The aim of this paper is to introduce second order cone arcwise connected function and its generalizations and use these functions in proving second order duality results.

The second order duality has a significant importance due to the computational advantage over first order duality as it provides tighter bounds for the value of the objective function when approximations are used. This type of duality was firstly formulated by Mangasarian [5] that involved second order derivatives of the functions constituting the primal problem and derived the duality results. Later on authors like Mishra [6] and Srivastava and Govil [9] defined different types of second order type I functions and their generalizations and applied these functions to obtain second order duality results for several mathematical programming problems. Ivanov [2] defined new first and second-order duals of the nonlinear programming problem with inequality constraints. He further derived that the first-order duality results are satisfied in the second-order case. Suneja et al. [10, 12] formulated second order Mond-Weir type dual and proved the duality results for a vector optimization problem

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over cones using second-order cone convex functions involving twice differentiable functions and second-order cone convex functions involving second-order directional derivatives and their generalizations. Further Dash and Dalai [4] introduced several kinds of second order invexity and duality problems in non-linear programming problem. Recently, Zhang and Lin [13] formulated second order and higher order duals for a mathematical program with complimentarity constraints (MPCC). They then proposed several duality theorems for second and higher order duality in MPCC problems under suitable generalized convexity assumptions. Kassem and Alshanbari [3] established and studied six new types of higher-order duality models and programs for multiple objective nonlinear programming problems and proved several duality theorems under generalized higher-order type-I functions and higher-order psuedo convexity type-I functions.

In this paper, second order cone arcwise connected, second order cone arcwise pseudoconnected, second order strongly cone arcwise pseudoconnected and second order cone arcwise quasiconnected functions are introduced. The relations among these functions are discussed. Further Mond-Weir type second order dual for a vector minimization problem over cones is formulated to obtain weak and strong duality theorems under these new concepts of second order cone arcwise connected functions and its generalizations.

## 2. Definitions and Preliminaries

Let $X$ be a nonempty subset of $R^n$ and $K \subseteq R^m$ be a closed convex pointed cone with nonempty interior. The positive dual cone $K^+$ of $K$ is defined as

$$K^+ = \{y^* \in R^m : x^T y^* \geq 0, \text{ for all } x \in K\}$$

The interior of $K$ is denoted by $\operatorname{int} K$

**Definition 2.1** ([8]). A subset $X \subseteq R^n$ is said to be an arcwise connected set if for every $\bar{x}, x \in X$, there exists a continuous vector valued function

$$H_{\bar{x},x} : [0,1] \to X$$

called an arc such that

$$H_{\bar{x},x}(0) = \bar{x} \quad \text{and} \quad H_{\bar{x},x}(1) = x\,.$$

**Definition 2.2** ([8]). Let $f$ be a real valued function defined on an arcwise connected set $X \subseteq R^n$. Then $f$ is said to be arcwise connected function if for every $\bar{x}, x \in X$, there exists an arc $H_{\bar{x},x}$ such that

$$f(H_{\bar{x},x}(\theta)) \leq (1-\theta) f(\bar{x}) + \theta f(x), \quad \text{for all } 0 \leq \theta \leq 1.$$

The function $f$ is called arcwise connected function at $\bar{x}$ on $X$ if the above inequality holds for all $x \in X$.

If $f$ is differentiable function on an open convex set $X \subseteq R^n$, then $f$ is said to be arcwise connected function iff for all $\bar{x}, x \in X$

$$f(x) - f(\bar{x}) \geq (x-\bar{x})^T \nabla f(H_{\bar{x},x}(0)).$$

We now introduce the definitions of second order cone arcwise connected fucntions and their generalizations. Let $f_i$, $i = 1, 2, \ldots, m$ be twice continuously differentiable real valued functions defined on an arcwise connected set $X$ of $R^n$ and $f = (f_1, f_2, \ldots, f_m)$ be a vector valued function defined on an arcwise connected set $X \subseteq R^n$.

**Definition 2.3.** $f$ is said to be second order $K$-arcwise connected (SKACN) at $\bar{x} \in X$ with respect to $p \in R^n$ if for every $x \in X$

$$\Big[ f_1(x) - f_1(\bar{x}) - (x-\bar{x})^T(\nabla f_1(H_{\bar{x},x}(0)) + \nabla^2 f_1(H_{\bar{x},x}(0))p) + \frac{1}{2} p^T \nabla^2 f_1(H_{\bar{x},x}(0))p, \ldots, f_m(x) - f_m(\bar{x})$$

$$- (x-\bar{x})^T(\nabla f_m(H_{\bar{x},x}(0)) + \nabla^2 f_m(H_{\bar{x},x}(0))p) + \frac{1}{2} p^T \nabla^2 f_m(H_{\bar{x},x}(0))p \Big] \in K\,.$$

We now give an example of a second order $K$-arcwise connected function.

**Example 2.4.** Let $K = \{(x_1, x_2)|x_1 \geq 0, x_2 \leq x_1\}$ be a cone in $R^2$.
Define $X \subseteq R^2$ as

$$X = \{(x_1, x_2)^T : x_1^2 + x_2^2 \geq 1,\ x_1 > 0,\ x_2 > 0\}.$$

Then $X$ is an arcwise connected set with respect to an arc $H_{\bar{x},x} : [0,1] \to X$ given by

$$H_{\bar{x},x}(\theta) = (((1-\theta)\bar{x}_1^2 + \theta x_1^2)^{\frac{1}{2}}, ((1-\theta)\bar{x}_2^2 + \theta x_2^2)^{\frac{1}{2}}), \quad \text{for all } \theta \in [0,1]$$

where $\bar{x} = (\bar{x}_1, \bar{x}_2)^T$, $x = (x_1, x_2)^T$.
Define $f : X \to R^2$ as $f = (f_1, f_2)$ where

$$f_1(x_1, x_2) = \begin{cases} x_1^2 + x_2^2, & \text{if } x_1 > 1,\ x_2 > 1, \\ 2, & \text{otherwise,} \end{cases}$$

$$f_2(x_1, x_2) = \begin{cases} -x_1^2, & \text{if } x_1 > 1,\ x_2 > 1, \\ -1, & \text{otherwise.} \end{cases}$$

Let $\bar{x} = (2,2)^T$. Then $f_1(H_{\bar{x},x}(0)) = 8$, $f_2(H_{\bar{x},x}(0)) = -4$.

$$\nabla f_1 = \begin{cases} \begin{pmatrix} 2x_1 \\ 2x_2 \end{pmatrix}, & \text{if } x_1 > 1,\ x_2 > 1 \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, & \text{otherwise} \end{cases}$$

$$\nabla^2 f_1 = \begin{cases} \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, & \text{if } x_1 > 1,\ x_2 > 1 \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, & \text{otherwise} \end{cases}$$

$$\nabla f_2 = \begin{cases} \begin{pmatrix} -2x_1 \\ 0 \end{pmatrix}, & \text{if } x_1 > 1,\ x_2 > 1 \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, & \text{otherwise} \end{cases}$$

$$\nabla^2 f_2 = \begin{cases} \begin{pmatrix} -2 & 0 \\ 0 & 0 \end{pmatrix}, & \text{if } x_1 > 1,\ x_2 > 1 \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, & \text{otherwise} \end{cases}$$

Then $f$ is second order $K$-arcwise connected at $\bar{x} = (2,2)$ and $p = (p_1, p_2) = (-2,-2)$ because

$$\Big[ f_1(x) - f_1(\bar{x}) - (x-\bar{x})^T \nabla f_1(H_{\bar{x},x}(0)) - (x-\bar{x})^T \nabla^2 f_1(H_{\bar{x},x}(0))p$$
$$+ \frac{1}{2} p^T \nabla^2 f_1(H_{\bar{x},x}(0))p, f_2(x) - f_2(\bar{x}) - (x-\bar{x})^T \nabla f_2(H_{\bar{x},x}(0))$$
$$- (x-\bar{x})^T \nabla^2 f_2(H_{\bar{x},x}(0))p + \frac{1}{2} p^T \nabla^2 f_2(H_{\bar{x},x}(0))p \Big]$$

$$= [(x_1-2)^2+(x_2-2)^2+(p_1+2)^2+(p_2+2)^2-2p_1x_1-2p_2x_2-8, \\ -(x_1-2)^2-(p_1+2)^2+2p_1x_1+4] \in K.$$

**Definition 2.5.** $f$ is said to be second order $K$-arcwise pseudoconnected (SKAPCN) at $\bar{x} \in X$ with respect to $p \in R^n$ if for every $x \in X$

$$[-(x-\bar{x})^T(\nabla f_1(H_{\bar{x},x}(0))+\nabla^2 f_1(H_{\bar{x},x}(0))p), \dots, \\ -(x-\bar{x})^T(\nabla f_m(H_{\bar{x},x}(0))+\nabla^2 f_m(H_{\bar{x},x}(0))p)] \notin \operatorname{int} K$$

$$\Rightarrow \quad \Bigg[-(f_1(x)-f_1(\bar{x})+\frac{1}{2}p^T\nabla^2 f_1(H_{\bar{x},x}(0))p), \dots, \\ -(f_m(x)-f_m(\bar{x})+\frac{1}{2}p^T\nabla^2 f_m(H_{\bar{x},x}(0))p)\Bigg] \notin \operatorname{int} K\,.$$

**Remark 2.6.** Every second order $K$-arcwise connected function at a point is second order $K$-arcwise pseudoconnected at the same point. But the converse is not true as can be seen from the following example.

**Example 2.7.** Let $K = \{(x_1, x_2) \mid x_2 \leq 0,\ x_1 \geq x_2\}$ be a cone in $R^2$.
Define $X \subseteq R^2$ as $X = \{(x_1, x_2)^T : x_1^2 + x_2^2 \geq 1,\ x_1 > 0,\ x_2 > 0\}$.
Then $X$ is an arcwise connected set with respect to an arc $H_{\bar{x},x} : [0,1] \to X$ given by

$$H_{\bar{x},x}(\theta) = (((1-\theta)\bar{x}_1^2+\theta x_1^2)^{\frac{1}{2}}, ((1-\theta)\bar{x}_2^2+\theta x_2^2)^{\frac{1}{2}}), \quad \text{for all } \theta \in [0,1]$$

where $\bar{x} = (\bar{x}_1, \bar{x}_2)^T$, $x = (x_1, x_2)^T$.
Define $f : X \to R^2$ as $f = (f_1, f_2)$ where

$$f_1(x_1,x_2) = \begin{cases} -x_1^3, & \text{if } x_1 > 1,\ x_2 > 1, \\ 1, & \text{otherwise}, \end{cases}$$

$$f_2(x_1,x_2) = \begin{cases} -x_1^3 - x_2, & \text{if } x_1 > 1,\ x_2 > 1, \\ 1, & \text{otherwise}. \end{cases}$$

Let $\bar{x} = (2,2)^T$. Then $f_1(H_{\bar{x},x}(0)) = -8$, $f_2(H_{\bar{x},x}(0)) = -10$.

$$\nabla f_1 = \begin{cases} \begin{pmatrix} -3x_1^2 \\ 0 \end{pmatrix}, & \text{if } x_1 > 1,\ x_2 > 1, \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, & \text{otherwise}, \end{cases}$$

$$\nabla^2 f_1 = \begin{cases} \begin{pmatrix} -6x_1 & 0 \\ 0 & 0 \end{pmatrix}, & \text{if } x_1 > 1,\ x_2 > 1, \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, & \text{otherwise}, \end{cases}$$

$$\nabla f_2 = \begin{cases} \begin{pmatrix} -3x_1^2 \\ -1 \end{pmatrix}, & \text{if } x_1 > 1,\ x_2 > 1, \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, & \text{otherwise}, \end{cases}$$

$$\nabla^2 f_2 = \begin{cases} \begin{pmatrix} -6x_1 & 0 \\ 0 & 0 \end{pmatrix}, & \text{if } x_1 > 1,\ x_2 > 1, \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, & \text{otherwise.} \end{cases}$$

The $f$ is second order $K$-arcwise pseudoconnected at $\bar{x} = (2,2)$ for any $p = (p_1, p_2) \in R^2$. But $f$ is not second order $K$-arcwise connected at $\bar{x} = (2,2)$, because for $x = (1,1)$

$$\Big[f_1(x) - f_1(\bar{x}) - (x-\bar{x})^T(\nabla f_1(H_{\bar{x},x}(0)) + \nabla^2 f_1(H_{\bar{x},x}(0))p) + \frac{1}{2}p^T\nabla^2 f_1(H_{\bar{x},x}(0))p,$$
$$f_2(x) - f_2(\bar{x}) - (x-\bar{x})^T(\nabla f_2(H_{\bar{x},x}(0)) + \nabla^2 f_2(H_{\bar{x},x}(0))p) + \frac{1}{2}p^T\nabla^2 f_2(H_{\bar{x},x}(0))p\Big]$$
$$= (9, 11) \notin K\,.$$

**Definition 2.8.** $f$ is said to be second order strongly $K$-arcwise pseudoconnected (SSKAPCN) at $\bar{x} \in X$ with respect to $p \in R^n$ if for every $x \in X$

$$[-(x-\bar{x})^T(\nabla f_1(H_{\bar{x},x}(0)) + \nabla^2 f_1(H_{\bar{x},x}(0))p), \ldots,$$
$$-(x-\bar{x})^T(\nabla f_m(H_{\bar{x},x}(0)) + \nabla^2 f_m(H_{\bar{x},x}(0))p)] \notin \operatorname{int} K$$
$$\Rightarrow \left[f_1(x) - f_1(\bar{x}) + \frac{1}{2}p^T\nabla^2 f_1(H_{\bar{x},x}(0))p, \ldots, f_m(x) - f_m(\bar{x}) + \frac{1}{2}p^T\nabla^2 f_m(H_{\bar{x},x}(0))p\right] \in K$$

**Remark 2.9.** Every second order strongly $K$-arcwise pseudoconnected function at a point is second order $K$-arcwise pseudoconnected at the same point. But the converse is not true as can be seen from Example 2.7, where $f$ is second order $K$-arcwise pseudoconnected at $\bar{x} = (2,2)$ but not second order strongly $K$-arcwise pseudoconnected at $\bar{x} = (2,2)$ because for $x = (1,1)$

$$[-(x-\bar{x})^T(\nabla f_1(H_{\bar{x},x}(0)) + \nabla^2 f_1(H_{\bar{x},x}(0))p), -(x-\bar{x})^T(\nabla f_2(H_{\bar{x},x}(0)) + \nabla^2 f_2(H_{\bar{x},x}(0))p)]$$
$$= (0,0) \notin \operatorname{int} K$$

and

$$\left[f_1(x) - f_1(\bar{x}) + \frac{1}{2}p^T\nabla^2 f_1(H_{\bar{x},x}(0))p, f_2(x) - f_2(\bar{x}) + \frac{1}{2}p^T\nabla^2 f_2(H_{\bar{x},x}(0))p\right] = (9,11) \notin K.$$

**Definition 2.10.** $f$ is said to be second order $K$-arcwise quasiconnected (SKAQCN) at $\bar{x} \in X$ with respect to $p \in R^n$ if for every $x \in X$

$$\left[f_1(x) - f_1(\bar{x}) + \frac{1}{2}p^T\nabla^2 f_1(H_{\bar{x},x}(0))p, \ldots, f_m(x) - f_m(\bar{x}) + \frac{1}{2}p^T\nabla^2 f_m(H_{\bar{x},x}(0))p\right] \notin \operatorname{int} K$$
$$\Rightarrow [-(x-\bar{x})^T(\nabla f_1(H_{\bar{x},x}(0)) + \nabla^2 f_1(H_{\bar{x},x}(0))p), \ldots$$
$$-(x-\bar{x})^T(\nabla f_m(H_{\bar{x},x}(0)) + \nabla^2 f_m(H_{\bar{x},x}(0))p)] \in K$$

**Remark 2.11.** Every second order $K$-arcwise connected function at a point may not be second order $K$-arcwise quasiconnected at the same point. Also there exist functions which are second order $K$-arcwise quaisconnected at a point but not second order $K$-arcwise connected at the same point, as can be seen from the following example.

**Example 2.12.** Let $K = \{(x_1, x_2) \mid x_1 \leq 0,\, x_1 \leq x_2\}$.
Define $X \subseteq R^2$ as $X = \{(x_1, x_2)^T : x_1^2 + x_2^2 \geq 1,\, x_1 > 0,\, x_2 > 0\}$.
Then $X$ is an arcwise connected set with respect to an arc $H_{\bar{x},x} : [0,1] \to X$ as defined in Example 2.7.
Define $f : X \to R^2$ as $f = (f_1, f_2)$ where

$$f_1(x_1, x_2) = \begin{cases} x_1^2 x_2^2, & \text{if } x_1 > 1,\, x_2 > 1, \\ 1, & \text{otherwise,} \end{cases}$$

$$f_2(x_1, x_2) = \begin{cases} x_2^2, & \text{if } x_1 > 1,\, x_2 > 1, \\ 1, & \text{otherwise.} \end{cases}$$

Let $\bar{x} = (1, 1)^T$. Then $f_1(H_{\bar{x},x}(0)) = 1 = f_2(H_{\bar{x},x}(0))$

$$\nabla f_1 = \begin{cases} \begin{pmatrix} 2x_1 x_2^2 \\ 2x_1^2 x_2 \end{pmatrix}, & \text{if } x_1 > 1,\ x_2 > 1, \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, & \text{otherwise,} \end{cases}$$

$$\nabla^2 f_1 = \begin{cases} \begin{pmatrix} 2x_2^2 & 4x_1 x_2 \\ 4x_1 x_2 & 2x_1^2 \end{pmatrix}, & \text{if } x_1 > 1,\ x_2 > 1, \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, & \text{otherwise,} \end{cases}$$

$$\nabla f_2 = \begin{cases} \begin{pmatrix} 0 \\ 2x_2 \end{pmatrix}, & \text{if } x_1 > 1,\ x_2 > 1, \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, & \text{otherwise,} \end{cases}$$

$$\nabla^2 f_2 = \begin{cases} \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}, & \text{if } x_1 > 1,\ x_2 > 1, \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, & \text{otherwise.} \end{cases}$$

Then $f$ is second order $K$-arcwise quasiconnected at $\bar{x} = (1, 1)$ for any $p = (p_1, p_2) \in R^2$. But $f$ is not second order $K$-arcwise connected at $\bar{x} = (1, 1)$, because for $x = (3, 3)$

$$\Big[ f_1(x) - f_1(\bar{x}) - (x - \bar{x})^T(\nabla f_1(H_{\bar{x},x}(0)) + \nabla^2 f_1(H_{\bar{x},x}(0))p) + \frac{1}{2} p^T \nabla^2 f_1(H_{\bar{x},x}(0))p,$$
$$f_2(x) - f_2(\bar{x}) - (x - \bar{x})^T(\nabla f_2(H_{\bar{x},x}(0)) + \nabla^2 f_2(H_{\bar{x},x}(0))p) + \frac{1}{2} p^T \nabla^2 f_2(H_{\bar{x},x}(0))p \Big]$$
$$= (80, 8) \notin K\,.$$

**Remark 2.13.** There exists functions which are second order $K$-arcwise quasiconnected at a point but not second order $K$-arcwise pseudoconnected at the same point. For example the function considered in Example 2.12 is second order $K$-arcwise quasiconnected at $\bar{x} = (1, 1)$ but $f$ is not second order $K$-arcwise pseudoconnected at

$\bar{x} = (1, 1)$, because for $x = (2, 2)$

$$[-(x-\bar{x})^T(\nabla f_1(H_{\bar{x},x}(0)) + \nabla^2 f_1(H_{\bar{x},x}(0))p), -(x-\bar{x})^T(\nabla f_2(H_{\bar{x},x}(0)) + \nabla^2 f_2(H_{\bar{x},x}(0))p)]$$
$$= (0, 0) \notin \operatorname{int} K$$

and

$$\left[-\left(f_1(x) - f_1(\bar{x}) + \frac{1}{2}p^T\nabla^2 f_1(H_{\bar{x},x}(0))p\right), -\left(f_2(x) - f_2(\bar{x}) + \frac{1}{2}p^T\nabla^2 f_2(H_{\bar{x},x}(0))p\right)\right]$$
$$= (-15, -3) \notin K.$$

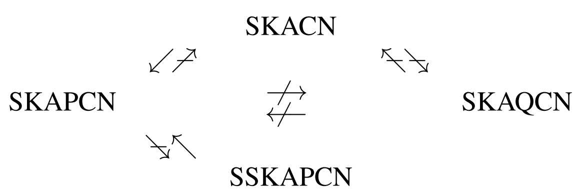
**Remark 2.14.** Note that there exists functions which are second order $K$-arcwise pseudoconnected at a point but not second order $K$-arcwise quasiconnected at the same point. For example the function considered in Example 2.7 is second order $K$-arcwise pseudoconnected at $\bar{x} = (2, 2)$ but $f$ is not second order $K$-arcwise quasiconnected at $\bar{x} = (2, 2)$, because for $x = (\frac{3}{2}, 3)$ and $p = (\frac{1}{3}, 1)$

$$\left[f_1(x) - f_1(\bar{x}) + \frac{1}{2}p^T\nabla^2 f_1(H_{\bar{x},x}(0))p, f_2(x) - f_2(\bar{x}) + \frac{1}{2}p^T\nabla^2 f_2(H_{\bar{x},x}(0))p\right]$$
$$= \left(\frac{95}{24}, \frac{71}{24}\right) \notin \operatorname{int} K$$

and

$$[-(x-\bar{x})^T(\nabla f_1(H_{\bar{x},x}(0)) + \nabla^2 f_1(H_{\bar{x},x}(0))p), -(x-\bar{x})^T(\nabla f_2(H_{\bar{x},x}(0)) + \nabla^2 f_2(H_{\bar{x},x}(0))p)]$$
$$= (-20, -19) \notin K$$

Interrelations between second order $K$-arcwise connected functions and its generalizations.




**Remark 2.15.**

(i) If $H_{\bar{x},x} = \bar{x}$, then the definition of second order $K$-arcwise connected function at $\bar{x}$ reduces to second order $K$-convex at $\bar{x}$ given by Suneja et al. [10]. Similarly, we have the notions of second order $K$-pseudoconvex, second order strongly $K$-pseudoconvex and second order $K$-quasiconvex at $\bar{x}$ if $H_{\bar{x},x} = \bar{x}$, in the definitions of SKAPCN, SSKAPCN and SKAQCN respectively.

(ii) If $p = 0$, then a second order $K$-arcwise connected function at $\bar{x}$ reduces to arcwise connected function over cones at $\bar{x}$ given by Mukherjee [7].

## 3. Duality

Consider the following vector optimization problem

$$\text{(VP)} \quad K\text{-minimize } f(x)$$
$$\text{subject to } \quad -g(x) \in Q$$

where $f_i,\ g_j,\ i=1,\dots,m,\ j=1,\dots,\ell$ are real valued twice differentiable functions defined on an arcwise connected set $X$ of $R^n$, with respect to the arc $H_{\bar{x},x}:[0,1]\to X$, where $\bar{x}\in X$, $x\in X$ and $f=(f_1,f_2,\dots,f_m)$ and $g=(g_1,g_2,\dots,g_l)$. $K$ and $Q$ are closed convex pointed cones with non empty interiors in $R^m$ and $R^\ell$ respectively. Let $X_0=\{x\in X|-g(x)\in Q\}$ denote the feasible set of (VP).

We associate the following second order dual problem with (VP)

$$\text{(SD)}\quad K\text{-maximize}\left(f_1(u)-\frac{1}{2}p^T\nabla^2 f_1(H_{u,x}(0))p,\dots,f_m(u)-\frac{1}{2}p^T\nabla^2 f_m(H_{u,x}(0))p\right)$$

$$\text{subject to}\ \ (x-u)^T\nabla(\tau^T f(H_{u,x}(0))+\lambda^T g(H_{u,x}(0))) + (x-u)^T\nabla^2(\tau^T f(H_{u,x}(0))+\lambda^T g(H_{u,x}(0)))p\geq 0,\quad \text{for all } x\in X_0 \tag{3.1}$$

$$\lambda^T g(u)-\frac{1}{2}p^T\nabla^2(\lambda^T g)(H_{u,x}(0))p\geq 0 \tag{3.2}$$

where $0\neq\tau\in K^+$, $\lambda\in Q^+$, $p\in R^n$, $u\in X$.

Now, we will establish the weak duality relation between feasible points of the primal (VP) and the second order dual (SD).

*Theorem 3.1* (Weak Dualtiy)
If $x$ is feasible for (VP) and $(u,\tau,\lambda,p)$ is feasible for (SD), $f$ is second order $K$-arcwise connected at $u\in X$ and $g$ is second order $Q$-arcwise connected at $u\in X$, with respect to the same arc $H_{u,x}$, for every $x\in X$, then

$$\left[f_1(u)-\frac{1}{2}p^T\nabla^2 f_1(H_{u,x}(0))p-f_1(x),\dots,f_m(u)-\frac{1}{2}p^T\nabla^2 f_m(H_{u,x}(0))p-f_m(x)\right]\notin \operatorname{int} K.$$

*Proof*
Suppose that

$$\left[f_1(u)-\frac{1}{2}p^T\nabla^2 f_1(H_{u,x}(0))p-f_1(x),\dots,f_m(u)-\frac{1}{2}p^T\nabla^2 f_m(H_{u,x}(0))p-f_m(x)\right]\in \operatorname{int} K. \tag{3.3}$$

Since $f$ is second order $K$-arcwise connected and $g$ is second order $Q$-arcwise connected at $u\in X$, we get

$$\Big[f_1(x)-f_1(u)-(x-u)^T(\nabla f_1(H_{u,x}(0))+\nabla^2 f_1(H_{u,x}(0))p) + \frac{1}{2}p^T\nabla^2 f_1(H_{u,x}(0))p,\dots,f_m(x)-f_m(u)-(x-u)^T(\nabla f_m(H_{u,x}(0))+\nabla^2 f_m(H_{u,x}(0))p) + \frac{1}{2}p^T\nabla^2 f_m(H_{u,x}(0))p\Big]\in K \tag{3.4}$$

and

$$\Big[g_1(x)-g_1(u)-(x-u)^T(\nabla g_1(H_{u,x}(0))+\nabla^2 g_1(H_{u,x}(0))p) + \frac{1}{2}p^T\nabla^2 g_1(H_{u,x}(0))p,\dots,g_\ell(x)-g_\ell(x)-(x-u)^T(\nabla g_\ell(H_{u,x}(0))+\nabla^2 g_\ell(H_{u,x}(0))p) + \frac{1}{2}p^T\nabla^2 g_\ell(H_{u,x}(0))p\Big]\in Q \tag{3.5}$$

Adding (3.3) and (3.4), we get

$$[-(x-u)^T(\nabla f_1(H_{u,x}(0)) + \nabla^2 f_1(H_{u,x}(0))p), \dots, \\ (x-u)^T(\nabla f_m(H_{u,x}(0)) + \nabla^2 f_m(H_{u,x}(0))p)] \in \operatorname{int} K$$

Since $0 \neq \tau \in K^+$, we get

$$(x-u)^T \nabla(\tau^T f)(H_{u,x}(0)) + (x-u)^T \nabla^2(\tau^T f)(H_{u,x}(0))p < 0.$$

Now feasibility of $(u, \tau, \lambda, p)$ for (SD) gives

$$(x-u)^T \nabla(\lambda^T g)(H_{u,x}(0)) + (x-u)^T \nabla^2(\lambda^T g)(H_{u,x}(0))p > 0 \tag{3.6}$$

From (3.5), since $\lambda \in Q^+$, we get

$$\lambda^T g(x) - \lambda^T g(u) - (x-u)^T(\nabla(\lambda^T g)(H_{u,x}(0)) \\ + \nabla^2(\lambda^T g)(H_{u,x}(0))p) + \frac{1}{2}p^T \nabla^2(\lambda^T g)(H_{u,x}(0))p \geq 0 \tag{3.7}$$

Adding (3.6) and (3.7), we get

$$\lambda^T g(x) - \lambda^T g(u) + \frac{1}{2}p^T \nabla^2(\lambda^T g)(H_{u,x}(0))p > 0$$

which is equivalent to

$$\lambda^T g(u) - \frac{1}{2}p^T \nabla^2(\lambda^T g)(H_{u,x}(0))p < \lambda^T g(x) \leq 0$$

This contradicts (3.2). Hence

$$\left[f_1(u) - \frac{1}{2}p^T \nabla^2 f_1(H_{u,x}(0))p - f_1(x), \dots, f_m(u) - \frac{1}{2}p^T \nabla^2 f_m(H_{u,x}(0))p - f_m(x)\right] \notin \operatorname{int} K.$$

□

*Theorem 3.2* (Weak Duality)
If $x$ is feasible for (VP) and $(u, \tau, \lambda, p)$ is feasible for (SD) and $f$ is second order $K$-arcwise pseudoconnected at $u \in X$ and $g$ is second order $Q$-arcwise quasiconnected at $u \in X$, then

$$\left[f_1(u) - \frac{1}{2}p^T \nabla^2 f_1(H_{u,x}(0))p - f_1(x), \dots, f_m(u) - \frac{1}{2}p^T \nabla^2 f_m(H_{u,x}(0))p - f_m(x)\right] \notin \operatorname{int} K.$$

*Proof*
Since $x$ is feasible for (VP) and $(u, \tau, \lambda, p)$ is feaisble for (SD), we get

$$\lambda^T g(x) - \lambda^T g(u) + \frac{1}{2}p^T \nabla^2(\lambda^T g)(H_{u,x}(0))p \leq 0 \tag{3.8}$$

Now we claim that

$$(x-u)^T(\nabla(\lambda^T g)(H_{u,x}(0)) + \nabla^2(\lambda^T g)(H_{u,x}(0))p) \leq 0 \tag{3.9}$$

If $\lambda = 0$, then (3.9) trivially holds. If $\lambda \neq 0$, then from (3.8), we get

$$\left[g_1(x) - g_1(u) + \frac{1}{2}p^T \nabla^2 g_1(H_{u,x}(0))p, \dots, g_\ell(u) - g_\ell(u) + \frac{1}{2}p^T \nabla^2 g_\ell(H_{u,x}(0))p\right] \notin \operatorname{int} Q$$

Now $g$ is second order $Q$-arcwise quasiconnected at $u \in X$, therefore we get

$$[-(x-u)^T(\nabla g_1(H_{u,x}(0)) + \nabla^2 g_1(H_{u,x}(0))p), \ldots, -(x-u)^T(\nabla g_\ell(H_{u,x}(0)) + \nabla^2 g_\ell(H_{u,x}(0))p)] \in Q$$

which implies that (3.9) holds. On using (3.1) and (3.9), we get

$$(x-u)^T(\nabla(\tau^T f)(H_{u,x}(0)) + \nabla^2(\tau^T f)(H_{u,x}(0))p) \geq 0$$

Now $0 \neq \tau \in K^+$ gives that

$$\begin{aligned}&[-(x-u)^T(\nabla f_1(H_{u,x}(0)) + \nabla^2 f_1(H_{u,x}(0))p), \ldots \\ &-(x-u)^T(\nabla f_m(H_{u,x}(0)) + \nabla^2 f_m(H_{u,x}(0))p)] \notin \operatorname{int} K\end{aligned}$$

Since $f$ is second order $K$-arcwise pseudoconnected at $u \in X$, therefore it follows that

$$\left[f_1(u) - f_1(x) - \frac{1}{2}p^T\nabla^2 f_1(H_{u,x}(0))p, \ldots, f_m(u) - f_m(x) - \frac{1}{2}p^T\nabla^2 f_m(H_{u,x}(0))p)\right] \notin \operatorname{int} K.$$

□

We shall be using the following constraint qualifications given by Suneja et al. [10] for proving the Strong Duality Theorems for (SD).

**Definition 3.3.** The function $g$ is said to satisfy the Slater type constraint qualification at $\bar{x}$.

(CQ1) if $g$ is $Q$-convex at $\bar{x}$ and there exists $x^* \in X$ such that $-g(x^*) \in \operatorname{int} Q$.
(CQ2) if $g$ is strongly $Q$-pseudoconvex at $\bar{x}$ and there exists $x^* \in X$ such that $-g(x^*) \in \operatorname{int} Q$.

The following lemma gives generalized form of Fritz John optimality conditions for a point to be a weak minimum of (VP), established by Suneja et al. [10].

*Lemma 3.4*
If $\bar{x}$ is a weak minimum of (VP), then there exist $\bar{\tau} \in K^+$, $\bar{\lambda} \in Q^+$ not both zero such that

$$(x-\bar{x})^T(\bar{\tau}^T\nabla f(H_{\bar{x},x}(0)) + \bar{\lambda}^T\nabla g(H_{\bar{x},x}(0))) \geq 0, \quad \text{for all } x \in X \tag{3.10}$$

and

$$\bar{\lambda}^T g(\bar{x}) = 0. \tag{3.11}$$

In order to prove the strong duality theorem, we shall now prove generalized form of Kuhn-Tucker type necessary optimality condition for (VP).

*Theorem 3.5*
If $\bar{x}$ is a weak minimum of (VP), then there exist $\bar{\tau} \in K^+$, $\bar{\lambda} \in Q^+$ not both zero such that conditions (3.10) and (3.11) of Lemma 3.4 hold. If Slater's type constraint qualification (CQ1) holds at $\bar{x}$ then $\bar{\tau} \neq 0$.

*Proof*
We claim that $\bar{\tau} \neq 0$. On the contrary suppose that $\bar{\tau} = 0$, then $\bar{\lambda} \neq 0$ and from (3.10), we get

$$(x-\bar{x})^T(\bar{\lambda}^T\nabla g(H_{\bar{x},x}(0))) \geq 0, \quad \text{for all } x \in X. \tag{3.12}$$

Since the Slater type constraint qualification (CQ1) is satisfied, it follows that there exists $x^* \in X$ such that

$$-g(x^*) \in \operatorname{int} Q$$

Now $0 \neq \bar{\lambda} \in Q^+$ gives that

$$\bar{\lambda}^T g(x^*) < 0 \tag{3.13}$$

Also since $g$ is $Q$-convex at $\bar{x}$, we get

$$[g_1(x) - g_1(\bar{x}) - (x-\bar{x})^T \nabla g_1(H_{\bar{x},x}(0)), \ldots, g_\ell(x) - g_\ell(\bar{x}) \\ - (x-\bar{x})^T \nabla g_\ell(H_{\bar{x},x}(0))] \in Q, \text{ for all } x \in X$$

Since $\bar{\lambda} \in Q^+$, we get

$$\bar{\lambda}^T g(x) - \bar{\lambda}^T g(\bar{x}) - (x-\bar{x})^T \bar{\lambda}^T \nabla g(H_{\bar{x},x}(0)) \geq 0, \text{ for all } x \in X$$

Using (3.11) and (3.12), we obtain

$$\bar{\lambda}^T g(x) \geq 0, \text{ for all } x \in X .$$

In particular for $x = x^*$

$$\bar{\lambda}^T g(x^*) \geq 0$$

which contradicts (3.13). Hence $\bar{\tau} \neq 0$. □

*Theorem 3.6*
If $\bar{x}$ is a weak minimum of (VP), then there exist $\bar{\tau} \in K^+$, $\bar{\lambda} \in Q^+$ not both zero such that conditions (3.10) and (3.11) of Lemma 3.4 hold. If Slater's type constraint qualification (CQ2) holds at $\bar{x}$ then $\bar{\tau} \neq 0$.

*Proof*
We assert that $\bar{\tau} \neq 0$. On the contrary suppose that $\bar{\tau} = 0$, then $\bar{\lambda} \neq 0$ and from (3.10), we get

$$(x-\bar{x})^T(\bar{\lambda}^T \nabla g(H_{\bar{x},x}(0))) \geq 0, \text{ for all } x \in X.$$

Since $0 \neq \bar{\lambda} \in Q^+$, we get

$$[-(x-\bar{x})^T \nabla g_1(H_{\bar{x},x}(0)), \ldots, -(x-\bar{x})^T \nabla g_\ell(H_{\bar{x},x}(0))] \notin \operatorname{int} Q, \text{ for all } x \in X.$$

Now since Slater type constraint qualification (CQ2) holds, therefore $g$ is strongly $Q$-pseudoconvex at $\bar{x}$, so we get

$$g(x) - g(\bar{x}) \in Q, \text{ for all } x \in X,$$

which gives that

$$\bar{\lambda}^T(g(x) - g(\bar{x})) \geq 0, \text{ for all } x \in X. \tag{3.14}$$

Since the Slater type constraint qualification (CQ2) is satisfied at $\bar{x}$, it follows that there exists $x^* \in X$ such that

$$-g(x^*) \in \operatorname{int} Q$$

which gives that

$$\bar{\lambda}^T g(x^*) < 0.$$

Using (3.11), we get

$$\bar{\lambda}^T(g(x^*) - g(\bar{x})) < 0$$

which contradicts (3.14). Hence $\bar{\tau} \neq 0$. □

*Theorem 3.7* (Strong Duality)
Let $\bar{x}$ be a weak minimum for (VP) at which the Slater type constraint qualification (CQ1) is satisfied. Then there exist $0 \neq \bar{\tau} \in K^+$ and $\bar{\lambda} \in Q^+$ such that $(\bar{x}, \bar{p} = 0, \bar{\tau}, \bar{\lambda})$ is feasible for the second order dual problem (SD) and the values of both the objective functions are equal. Moreover, if $f$ is second order $K$-arcwise connected and $g$ is second order $Q$-arcwise connected on $X$, then $(\bar{x}, \bar{p} = 0, \bar{\tau}, \bar{\lambda})$ is weak maximum for (SD).

*Proof*
Since all the conditions of Theorem 3.5 hold, there exist $0 \neq \bar{\tau} \in K^+$, $\bar{\lambda} \in Q^+$ such that (3.10) and (3.11) hold.

Hence we get that $(\bar{x}, \bar{p} = 0, \bar{\tau}, \bar{\lambda})$ is a feasible solution for (SD). Both the objective functions coincide as $\bar{p} = 0$. Suppose that $(\bar{x}, \bar{p} = 0, \bar{\tau}, \bar{\lambda})$ is not a weak maximum for (SD), then there exists a feasible solution $(u, p, \tau, \lambda)$ of (SD) such that

$$\Bigg[f_1(u) - \frac{1}{2}p^T\nabla^2 f_1(H_{u,x}(0))p - f_1(\bar{x}) + \frac{1}{2}\bar{p}^T\nabla^2 f_1(H_{\bar{x},x}(0))\bar{p}, \ldots,$$
$$f_m(u) - \frac{1}{2}p^T\nabla^2 f_m(H_{u,x}(0))p - f_m(\bar{x}) + \frac{1}{2}\bar{p}^T\nabla^2 f_m(H_{\bar{x},x}(0))\bar{p}\Bigg] \in \operatorname{int} K$$

Since $\bar{p} = 0$, we get

$$\left[f_1(u) - \frac{1}{2}p^T\nabla^2 f_1(H_{u,x}(0))p - f_1(\bar{x}), \ldots, f_m(u) - \frac{1}{2}p^T\nabla^2 f_m(H_{u,x}(0))p - f_m(\bar{x})\right] \in \operatorname{int} K$$

which contradicts Weak Duality Theorem 3.1, for the feasible solution $\bar{x}$ of (VP) and $(u, p, \tau, \lambda)$ of (SD). Hence $(\bar{x}, \bar{p} = 0, \bar{\tau}, \bar{\lambda})$ is a weak maximum for (SD). □

*Theorem 3.8* (Strong Duality)
Let $\bar{x}$ be a weak minimum for (VP) at which the Slater type constraint qualification (CQ2) is satisfied. Then there exist $0 \neq \bar{\tau} \in K^+$ and $\bar{\lambda} \in Q^+$ such that $(\bar{x}, \bar{p} = 0, \bar{\tau}, \bar{\lambda})$ is feasible for the second order dual problem (SD) and both the objective functions are equal. Moreover, if $f$ is second order $K$-arcwise pseudoconnected and $g$ is second order $Q$-arcwise quasiconnected on $X$, then $(\bar{x}, \bar{p} = 0, \bar{\tau}, \bar{\lambda})$ is weak maximum for (SD).

*Proof*
Since all the conditions of Theorem 3.6 hold, there exist $0 \neq \bar{\tau} \in K^+$, $\bar{\lambda} \in Q^+$ such that (3.10) and (3.11) hold.

Rest of the proof is on the lines of Theorem 3.7 except that we use Weak Duality Theorem 3.2 instead of Theorem 3.1. □

## 4. Conclusion

In this paper, we investigate the sufficient optimality conditions for the vector optimization problem (VP) over cones using second order cone arcwise connected functions and their generalizations. We also examined Mond-Weir type dual and demonstrate the duality results for weak minimum between the primal problem (VP) and the corresponding dual problem (SD).

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