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Filtering Problem for Stationary Sequences with Missing Observations

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Abstract This paper deals with the mean-square optimal linear estimation of linear functionals, which depend on the unknown values of a stationary stochastic sequence from observations with a stationary noise sequence. If spectral densities of the sequences are exactly known, we derive the formulas for calculating the mean-square errors and the spectral characteristics of the optimal linear estimates of functionals. The minimax (robust) method of estimation is applied in the case of spectral uncertainty, where spectral densities are not known exactly while sets of admissible spectral densities are given. Formulas that determine the least favorable spectral densities and the minimax spectral characteristics are proposed for some special sets of admissible densities.

Keywords Stationary sequence, mean square error, minimax-robust estimate, least favorable spectral density, minimax spectral characteristic

AMS 2010 subject classifications. Primary: 60G10, 62M20, 60G35, Secondary: 93E10, 93E11

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1. Introduction

Estimation of unknown values of stochastic sequences is of a great interest both in the theory of random processes and applications of this theory to the data analysis. Such problems arise in such areas of science as oceanography, meteorology, astronomy, radio physics, statistical hydromechanics etc. Effective methods of solution of estimation problems (interpolation, extrapolation and filtering) of stationary sequences were developed by A. N. Kolmogorov (see selected works by Kolmogorov [15]). Detailed description and further development of the methods can be found in books by Yu. A. Rozanov [35] and E. J. Hannan [10]. A significant contribution to the theory of forecasting was made by H. Wold [39, 40], T. Nakazi [31]. Constructive methods of solution of the estimation problems for stationary stochastic processes were proposed by N. Wiener [38] and A. M. Yaglom [41, 42].

The crucial assumption of most of the methods of estimation of the unobserved values of stochastic processes is that the spectral densities of the considered stochastic processes are exactly known. However, in practice, complete information on the spectral densities is impossible in most cases. In this situation, one finds a parametric or nonparametric estimate of the unknown spectral density and then apply one of the traditional estimation methods provided that the selected density is the true one. This procedure can result in significant increasing of the value of error as K. S. Vastola and H. V. Poor [37] have demonstrated with the help of some examples. To avoid this effect one can search the estimates which are optimal for all densities from a certain class of admissible spectral densities. These estimates are called minimax since they minimize the maximum value of the error. The paper by

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Ulf Grenander [9] was the first one where this approach to extrapolation problem for stationary processes was proposed.

Several models of spectral uncertainty and minimax-robust methods of data processing can be found in the survey paper by S. A. Kassam and H. V. Poor [14]. In papers by J. Franke [5], J. Franke and H. V. Poor [6] the minimax extrapolation and filtering problems for stationary sequences were investigated with the help of convex optimization methods. This approach makes it possible to find equations that determine the least favorable spectral densities for different classes of densities.

In papers by M. Moklyachuk [19] – [22] problems of linear optimal estimation of the functionals which depend on the unknown values of stationary sequences and processes were investigated. Methods of solution the interpolation, extrapolation and filtering problems for periodically correlated stochastic sequences and processes were described by M. Moklyachuk and I. Dubovetska [4], M. Moklyachuk and I. Golichenko [23]. The corresponding estimation problems for vector-valued stationary sequences and processes were investigated in papers by M. Moklyachuk and A. Masyutka [24]–[26]. Estimation problems for functionals which depend on the unknown values of stochastic sequences with stationary increments were investigated by M. Luz and M. Moklyachuk [16] - [18]. The problem of interpolation of stationary sequence with missing values was investigated by M. Moklyachuk and M. Sidei [29, 30]. The corresponding problem for harmonizable stable sequences was investigated by M. Moklyachuk and V Ostapenko [27], [28].

Prediction problem for stationary sequences with missing observations was investigated in papers by P. Bondon [1, 2], Y. Kasahara, M. Pourahmadi and A. Inoue [13, 32], R. Cheng, A. G. Miamee, M. Pourahmadi [3]. The problem of interpolation of stationary sequences was considered in the paper of H. Salehi [36].

In this paper we investigate the problem of the mean-square optimal estimation of the functional $A\xi = \sum\limits_{j\in Z^S} a(j)\xi(-j)$ which depends on the unknown values of a stationary sequence $\{\xi(j), j\in\mathbb{Z}\}$ from observations of the sequence $\xi(j)+\eta(j)$ at points $j\in\mathbb{Z}_-\backslash S$, where the stationary sequence $\{\eta(j), j\in\mathbb{Z}\}$ is uncorrelated with the sequence $\xi(j)$, $S=\bigcup\limits_{l=1}^{s}\{-(M_l+N_l),\dots,-M_l\}$, $Z^S=\{1,2,\dots\}\backslash S^+$, $S^+=\bigcup\limits_{l=1}^{s}\{M_l,\dots,M_l+N_l\}$, $M_0=0$, $N_0=0$. The problem is investigated in the case of spectral certainty, where both spectral densities of the sequences $\xi(j)$ and $\eta(j)$ are known. In this case we derive formulas for calculating the spectral characteristic and the mean-square error of the optimal linear estimates using the method of projection in the Hilbert space of random variables with finite second moments proposed by Kolmogorov [15]. In the case of spectral uncertainty, where the spectral densities are not exactly known while a set of admissible spectral densities is given, the minimax method is applied. Formulas for determination the least favorable spectral densities and the minimax-robust spectral characteristics of the optimal estimates of the functional are proposed for some specific classes of admissible spectral densities.

2. Hilbert space projection method of filtering

Consider stationary stochastic sequences $\{\xi(j), j\in\mathbb{Z}\}$ and $\{\eta(j), j\in\mathbb{Z}\}$ with absolutely continuous spectral measures $F(d\lambda)$, $G(d\lambda)$ and correlation functions of the form

$$R_\xi(k)=\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ik\lambda}f(\lambda)d\lambda,\quad R_\eta(k)=\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ik\lambda}g(\lambda)d\lambda,$$

where $f(\lambda)$ and $g(\lambda)$ are the spectral densities of the sequences $\{\xi(j), j\in\mathbb{Z}\}$ and $\{\eta(j), j\in\mathbb{Z}\}$ respectively.

We will suppose that the spectral densities $f(\lambda)$ and $g(\lambda)$ satisfy the minimality condition

$$\int_{-\pi}^{\pi}\frac{1}{f(\lambda)+g(\lambda)}d\lambda<\infty. \tag{1}$$

This condition is necessary and sufficient in order that the error-free filtering of unknown values of the sequences is impossible [35].

The stationary stochastic sequences $\xi(j)$ and $\eta(j)$ admit the following spectral decomposition [8], [12]

$$\xi(j)=\int\limits_{-\pi}^{\pi} e^{ij\lambda}Z_{\xi}(d\lambda), \qquad \eta(j)=\int\limits_{-\pi}^{\pi} e^{ij\lambda}Z_{\eta}(d\lambda),$$

where $Z_{\xi}(d\lambda)$ and $Z_{\eta}(d\lambda)$ are orthogonal measures defined on $(-\pi,\pi]$ that correspond to the spectral measures $F(d\lambda)$ and $G(d\lambda)$

$$EZ_{\xi}(\Delta_1)\overline{Z_{\xi}(\Delta_2)}=F(\Delta_1\cap\Delta_2)=\frac{1}{2\pi}\int_{\Delta_1\cap\Delta_2} f(\lambda)d\lambda,$$

$$EZ_{\eta}(\Delta_1)\overline{Z_{\eta}}(\Delta_2)=G(\Delta_1\cap\Delta_2)=\frac{1}{2\pi}\int_{\Delta_1\cap\Delta_2} g(\lambda)d\lambda.$$

Suppose that we have observations of the sequence $\xi(j)+\eta(j)$ at points $j\in\mathbb{Z}_-\backslash S$, where $S=\bigcup\limits_{l=1}^{s}\{-(M_l+N_l),\dots,-M_l\}$. The problem is to find the mean-square optimal linear estimate of the functional

$$A\xi=\sum_{j\in Z^S} a(j)\xi(-j),$$

which depends on the unknown values of the sequence $\xi(j)$, $Z^S=\{1,2,\dots\}\backslash S^+$, $S^+=\bigcup\limits_{l=1}^{s}\{M_l,\dots,M_l+N_l\}$.

Suppose that the coefficients $\{a(j), j=0,1,\dots\}$ defining the functional $A\xi$ satisfy the following conditions

$$\sum_{k\in Z^S}|a(k)|<\infty, \qquad \sum_{k\in Z^S}(k+1)\,|a(k)|^2<\infty. \tag{2}$$

The first condition ensures that the functional $A\xi$ has finite second moment. The second condition ensures the compactness in ℓ_2 of operators that will be defined below.

It follows from the spectral decomposition of the sequence $\xi(j)$ that the functional $A\xi$ can be represented in the following form

$$A\xi=\int\limits_{-\pi}^{\pi} A(e^{i\lambda})Z_{\xi}(d\lambda), \quad A(e^{i\lambda})=\sum_{j\in Z^S} a(j)e^{-ij\lambda}.$$

Consider values $\xi(j)$ and $\eta(j)$ as elements of the Hilbert space $H=L_2(\Omega,\mathcal{F},P)$ generated by random variables ξ with zero mathematical expectations, $E\xi=0$, finite variations, $E|\xi|^2<\infty$, and inner product $(\xi,\eta)=E\xi\overline{\eta}$. Denote by $H^s(\xi+\eta)$ the closed linear subspace generated by elements $\{\xi(j)+\eta(j) : j\in\mathbb{Z}_-\backslash S\}$ in the Hilbert space $H=L_2(\Omega,\mathcal{F},P)$. Let $L_2(f+g)$ be the Hilbert space of complex-valued functions that are square-integrable with respect to the measure whose density is $f(\lambda)+g(\lambda)$. Denote by $L_2^s(f+g)$ the subspace of $L_2(f+g)$ generated by functions $\{e^{ij\lambda}, j\in\mathbb{Z}_-\backslash S\}$.

The mean-square optimal linear estimate $\hat{A}\xi$ of the functional $A\xi$ from observations of the sequence $\xi(j)+\eta(j)$ can be represented in the form

$$\hat{A}\xi=\int\limits_{-\pi}^{\pi} h(e^{i\lambda})(Z_{\xi}(d\lambda)+Z_{\eta}(d\lambda),$$

where $h(e^{i\lambda})\in L_2^s(f+g)$ is the spectral characteristic of the estimate.

The mean-square error $\Delta(h;f)$ of the estimate $\hat{A}\xi$ is given by the formula

$$\Delta(h;f,g)=E\left|A\xi-\hat{A}\xi\right|^2=\frac{1}{2\pi}\int\limits_{-\pi}^{\pi}\left|A(e^{i\lambda})-h(e^{i\lambda})\right|^2 f(\lambda)d\lambda+\frac{1}{2\pi}\int\limits_{-\pi}^{\pi}\left|h(e^{i\lambda})\right|^2 g(\lambda)d\lambda.$$

The Hilbert space projection method proposed by A. N. Kolmogorov [15] makes it possible to find the spectral characteristic $h(e^{i\lambda})$ and the mean square error $\Delta(h;f)$ of the optimal linear estimate of the functional $A_s\xi$ in the case where spectral densities $f(\lambda)$ and $g(\lambda)$ of the sequences are exactly known and the minimality condition (1) is satisfied. According to this method the optimal estimate of the functional $A\xi$ is a projection of the element $A\xi$ of the space H on the space $H^s(\xi+\eta)$. It can be found from the following conditions:

$$1)\hat{A}\xi \in H^s(\xi+\eta),$$
$$2)A\xi-\hat{A}\xi \perp H^s(\xi+\eta).$$

It follows from the second condition that the spectral characteristic $h(e^{i\lambda})$ for any $j\in\mathbb{Z}_-\backslash S$ satisfies the equations

$$E\left[\left(A\xi-\hat{A}\xi\right)\overline{(\xi(j)+\eta(j))}\right]=$$
$$=\frac{1}{2\pi}\int_{-\pi}^{\pi}\left(A(e^{i\lambda})-h(e^{i\lambda})\right)e^{-ij\lambda}f(\lambda)d\lambda-\frac{1}{2\pi}\int_{-\pi}^{\pi}h(e^{i\lambda})e^{-ij\lambda}g(\lambda)d\lambda=0.$$

The last relation is equivalent to equations

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\left[A(e^{i\lambda})f(\lambda)-h(e^{i\lambda})(f(\lambda)+g(\lambda))\right]e^{-ij\lambda}d\lambda=0,\quad j\in\mathbb{Z}_-\backslash S.$$

Hence the function $\left[A(e^{i\lambda})f(\lambda)-h(e^{i\lambda})(f(\lambda)+g(\lambda))\right]$ is of the form

$$A(e^{i\lambda})f(\lambda)-h(e^{i\lambda})(f(\lambda)+g(\lambda))=C(e^{i\lambda}),$$

$$C(e^{i\lambda})=\sum_{j\in S}c(j)e^{ij\lambda}+\sum_{j=0}^{\infty}c(j)e^{ij\lambda},$$

where $c(j), j\in T=S\cup\{0,1,2,\ldots\}$ are unknown coefficients that we have to find.

From the last relation we deduce that the spectral characteristic of the optimal linear estimate $\hat{A}\xi$ is of the form

$$h(e^{i\lambda})=A(e^{i\lambda})\frac{f(\lambda)}{f(\lambda)+g(\lambda)}-\frac{C(e^{i\lambda})}{f(\lambda)+g(\lambda)}. \tag{3}$$

It follows from the first condition, $\hat{A}\xi\in H^s(\xi+\eta)$, which determine the optimal linear estimate of the functional $A\xi$, that the Fourier coefficients of the function $h(e^{i\lambda})$ are equal to zero for $j\in T$,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}h(e^{i\lambda})e^{-ik\lambda}d\lambda=0,\ k\in T,$$

namely

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\left(A(e^{i\lambda})\frac{f(\lambda)}{f(\lambda)+g(\lambda)}-\frac{C(e^{i\lambda})}{f(\lambda)+g(\lambda)}\right)e^{-ik\lambda}d\lambda=0,\qquad k\in T.$$

We will use the last equality to find equations which determine the unknown coefficients $c(j), j \in T$. After disclosing the brackets we get the relation

$$\sum_{j \in Z^S} a(j) \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{-i(k+j)\lambda} f(\lambda)}{f(\lambda)+g(\lambda)} d\lambda - \left(\sum_{j \in S} c(j) \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{-i(k-j)\lambda}}{f(\lambda)+g(\lambda)} d\lambda + \sum_{j=0}^{\infty} c(j) \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{-i(k-j)\lambda}}{f(\lambda)+g(\lambda)} d\lambda \right) = 0, k \in T. \tag{4}$$

Let us introduce the Fourier coefficients of the functions $\frac{1}{f(\lambda)+g(\lambda)}$, $\frac{f(\lambda)}{f(\lambda)+g(\lambda)}$, $\frac{f(\lambda)g(\lambda)}{f(\lambda)+g(\lambda)}$

$$b_{k,j} = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-i(k-j)\lambda} \frac{1}{f(\lambda)+g(\lambda)} d\lambda, \tag{5}$$

$$r_{k,j} = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-i(k+j)\lambda} \frac{f(\lambda)}{f(\lambda)+g(\lambda)} d\lambda, \tag{6}$$

$$q_{k,j} = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-i(k-j)\lambda} \frac{f(\lambda)g(\lambda)}{f(\lambda)+g(\lambda)} d\lambda. \tag{7}$$

Using the introduced notations we can verify that equality (4) is equivalent to the following system of equations:

$$\sum_{j \in Z^S} r_{k,j} a(j) = \sum_{j \in S} b_{k,j} c(j) + \sum_{j=0}^{\infty} b_{k,j} c(j), \qquad k \in T.$$

Denote by $a(j) = 0, j \in S$, $a(0) = 0$ $a(j) = 0, j \in S^+$. Thus, we can write

$$\sum_{j \in T} r_{k,j} a(j) = \sum_{j \in S} b_{k,j} c(j) + \sum_{j=0}^{\infty} b_{k,j} c(j), \qquad k \in T.$$

The last equations can be rewritten in the following form

$$\mathbf{R}\vec{\mathbf{a}} = \mathbf{B}\vec{\mathbf{c}}, \tag{8}$$

where $\vec{\mathbf{c}}$ is a vector constructed from the unknown coefficients $c(j), j \in T$, vector $\vec{\mathbf{a}}$ has the same with the vector $\vec{\mathbf{c}}$ dimension, it is of the form

$$\vec{\mathbf{a}} = (\vec{0}_0, \vec{a}_1, \vec{0}_1, \vec{a}_2, \vec{0}_2, \ldots \vec{a}_i, \vec{0}_i, \ldots, \vec{a}_s, \vec{0}_s, \vec{a}_{s+1}),$$

where $\vec{0}_0$ is the vector which consists form $|S|+1$ zeros, where $|S| = \sum_{k=1}^{s}(N_k+1)$ is the amount of missing values, vectors $\vec{0}_i$, $i = 1, 2, \ldots, s$, consist from $N_i + 1$ zeros, vectors

$$\vec{a}_1 = (a(1), \ldots, a(M_1 - 1)),$$

$$\vec{a}_i = (a(M_{i-1} + N_{i-1} + 1), \ldots, a(M_i - 1)), \quad i = 2, \ldots, s,$$
$$\vec{a}_{s+1} = (a(M_s + N_s + 1), a(M_s + N_s + 2), \ldots),$$

are constructed from the coefficients that determine the functional $A\xi$.

Operators $\mathbf{B}$, $\mathbf{R}$ are linear operators in the space ℓ_2 defined by matrices with coefficients $(\mathbf{B})_{k,j} = b_{k,j}, k, j \in T$, $(\mathbf{R})_{k,j} = r_{k,j}$, $k, j \in T$,

$$B = \begin{pmatrix} B_{s,s} & B_{s,s-1} & \dots & B_{s,1} & B_{s,n} \\ B_{s-1,s} & B_{s-1,s-1} & \dots & B_{s-1,1} & B_{s-1,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ B_{1,s} & B_{1,s-1} & \dots & B_{1,1} & B_{1,n} \\ B_{n,s} & B_{n,s-1} & \dots & B_{n,1} & B_{n,n} \end{pmatrix},$$

where elements in the last column and the last row are the matrices with the elements

$$\begin{aligned} &B_{l,n}(k,j) = b_{k,j}, \quad l = 1, 2, \dots, s, \\ &k = -M_l - N_l, \dots, -M_l, \quad j = 0, 1, 2, \dots, \\ &B_{n,m}(k,j) = b_{k,j}, \quad m = 1, 2, \dots, s, \\ &k = 0, 1, 2, \dots, \quad j = -M_m - N_m, \dots, -M_m, \\ &B_{n,n}(k,j) = b_{k,j}, \quad k, j = 0, 1, 2, \dots, \end{aligned}$$

and other elements of matrix B are the matrices with elements of the form

$$\begin{aligned} &B_{l,m}(j,k) = b_{k,j}, \quad l, m = 1, 2, \dots, s \\ &k = -M_l - N_l, \dots, -M_l, \quad j = -M_m - N_m, \dots, -M_m. \end{aligned}$$

The unknown coefficients $c(k), k \in T$, which are defined by the equations (8), can be calculated by the formula

$$c(k) = (\mathbf{B}^{-1}\mathbf{R}\vec{\mathbf{a}})_k,$$

where $(\mathbf{B}^{-1}\mathbf{R}\vec{\mathbf{a}}_s)_k$ is the k component of the vector $\mathbf{B}^{-1}\mathbf{R}\vec{\mathbf{a}}$.

The formula for calculating the spectral characteristic $h(e^{i\lambda})$ of the estimate $\hat{A}\xi$ is of the form

$$h(e^{i\lambda}) = A(e^{i\lambda})\frac{f(\lambda)}{f(\lambda)+g(\lambda)} - \frac{\sum_{k\in T}(\mathbf{B}^{-1}\mathbf{R}\vec{\mathbf{a}})_k e^{ik\lambda}}{f(\lambda)+g(\lambda)}. \tag{9}$$

The mean-square error of the estimate $\hat{A}\xi$ can be calculated by the formula

$$\begin{aligned} \Delta(h; f, g) = E\left|A\xi - \hat{A}\xi\right|^2 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\left|A(e^{i\lambda})g(\lambda) + \sum_{k\in T}(\mathbf{B}^{-1}\mathbf{R}\vec{\mathbf{a}})_k e^{ik\lambda}\right|^2}{(f(\lambda)+g(\lambda))^2} f(\lambda)d\lambda \\ &+ \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\left|A(e^{i\lambda})f(\lambda) - \sum_{k\in T}(\mathbf{B}^{-1}\mathbf{R}\vec{\mathbf{a}})_k e^{ik\lambda}\right|^2}{(f(\lambda)+g(\lambda))^2} g(\lambda)d\lambda \\ &= \langle \mathbf{R}\vec{\mathbf{a}}, \mathbf{B}^{-1}\mathbf{R}\vec{\mathbf{a}}\rangle + \langle \mathbf{Q}\vec{\mathbf{a}}, \vec{\mathbf{a}}\rangle, \end{aligned} \tag{10}$$

where $\mathbf{Q}$ is the linear operator in the space ℓ_2 defined by matrix with coefficients $(\mathbf{Q})_{k,j} = q_{k,j}$, $k, j \in T$.

Let us summarize results and present them in the form of a theorem.

Theorem 2.1
Let $\xi(j)$ and $\eta(j)$ be uncorrelated stationary sequences with spectral densities $f(\lambda)$ and $g(\lambda)$ which satisfy the minimality condition (1). The spectral characteristic $h(e^{i\lambda})$ and the mean square error $\Delta(f, g)$ of the optimal linear estimate of the functional $A\xi$ which depends on the unknown values of the sequence $\xi(j)$ based on observations of the sequence $\xi(j) + \eta(j)$, $j \in \mathbb{Z}_-\backslash S$ can be calculated by formulas (9), (10).

Consider the problem of the mean-square optimal linear estimation of the functional

$$A\xi = \sum_{j\in Z^S} a(j)\xi(-j),$$

which depends on the unknown values of the sequence $\xi(j)$ from observations of the sequence $\xi(j)+\eta(j)$ at points $j\in\mathbb{Z}_-\backslash S$, $S=\{-(M+N),\dots,-M\}$, $Z^S=\{1,2,\dots\}\backslash S^+$, $S^+=\{M,\dots,M+N\}$.

From theorem 2.1 the following corollary can be derived.

Corollary 2.1

Let $\xi(j)$ and $\eta(j)$ be uncorrelated stationary sequences with spectral densities $f(\lambda)$ and $g(\lambda)$ which satisfy the minimality condition (1). The spectral characteristic $h(e^{i\lambda})$ and the mean square error $\Delta(f,g)$ of the optimal linear estimate of the functional $A\xi$ which depends on the unknown values of the sequence $\xi(j)$ based on observations of the sequence $\xi(j)+\eta(j)$, $j\in\mathbb{Z}_-\backslash S$ can be calculated by formulas (11), (12)

$$h(e^{i\lambda}) = A(e^{i\lambda})\frac{f(\lambda)}{f(\lambda)+g(\lambda)} - \frac{\sum\limits_{k\in T}(\mathbf{B}^{-1}\mathbf{R}\vec{\mathbf{a}})_k e^{ik\lambda}}{f(\lambda)+g(\lambda)}, \tag{11}$$

$$\Delta(h;f,g) = \langle \mathbf{R}\vec{\mathbf{a}}, \mathbf{B}^{-1}\mathbf{R}\vec{\mathbf{a}}\rangle + \langle \mathbf{Q}\vec{\mathbf{a}},\vec{\mathbf{a}}\rangle, \tag{12}$$

where $\mathbf{B}$, $\mathbf{R}$, $\mathbf{Q}$ are linear operators in the space ℓ_2 defined by matrices with coefficients $(\mathbf{B})_{k,j}=b_{k,j}$, $k,j\in T$, $(\mathbf{R})_{k,j}=r_{k,j}$, $k,j\in T$, $(\mathbf{Q})_{k,j}=q_{k,j}$, $k,j\in T$, $(T=S\cup\{0,1,2,\dots\})$. For example matrix B is of the form

$$B=\begin{pmatrix} B_{s,s} & B_{s,n}\\ B_{n,s} & B_{n,n}\end{pmatrix},$$

where its components are matrices with the elements

$$\begin{aligned}
&B_{s,n}(k,j)=b_{k,j}, && k=-M-N,\dots,-M, && j=0,1,2,\dots,\\
&B_{n,s}(k,j)=b_{k,j}, && k=0,1,2,\dots, && j=-M-N,\dots,-M,\\
&B_{n,n}(k,j)=b_{k,j}, && k,j=0,1,2,\dots, && \\
&B_{s,s}(j,k)=b_{k,j}, && k=-M-N,\dots,-M, && j=-M-N,\dots,-M.
\end{aligned}$$

Consider the problem of the mean-square optimal linear estimation of the functional

$$A\xi = \sum_{j\in Z^S} a(j)\xi(-j),$$

which depends on the unknown values of the sequence $\xi(j)$ from observations of the sequence $\xi(j)+\eta(j)$ at points $j\in\mathbb{Z}_-\backslash\{-s\}$, $Z^S=\{1,2,\dots\}\backslash\{s\}$.

It follows from theorem 2.1 that the following corollary holds true.

Corollary 2.2

Let $\xi(j)$ and $\eta(j)$ be uncorrelated stationary sequences with spectral densities $f(\lambda)$ and $g(\lambda)$ which satisfy the minimality condition (1). The spectral characteristic $h(e^{i\lambda})$ and the mean square error $\Delta(f,g)$ of the optimal linear estimate of the functional $A\xi$ which depends on the unknown values of the sequence $\xi(j)$ based on observations of the sequence $\xi(j)+\eta(j)$, $j\in\mathbb{Z}_-\backslash\{-s\}$ can be calculated by formulas (13), (14)

$$h(e^{i\lambda}) = A(e^{i\lambda})\frac{f(\lambda)}{f(\lambda)+g(\lambda)} - \frac{\sum\limits_{k\in T}(\mathbf{B}^{-1}\mathbf{R}\vec{\mathbf{a}})_k e^{ik\lambda}}{f(\lambda)+g(\lambda)}, \tag{13}$$

$$\Delta(h;f,g) = \langle \mathbf{R}\vec{\mathbf{a}}, \mathbf{B}^{-1}\mathbf{R}\vec{\mathbf{a}}\rangle + \langle \mathbf{Q}\vec{\mathbf{a}},\vec{\mathbf{a}}\rangle, \tag{14}$$

where $\mathbf{B}$, $\mathbf{R}$, $\mathbf{Q}$ are linear operators in the space ℓ_2 defined by matrices with coefficients $(\mathbf{B})_{k,j} = b_{k,j}, k, j \in T$, $(\mathbf{R})_{k,j} = r_{k,j}$, $k, j \in T$, $(\mathbf{Q})_{k,j} = q_{k,j}$, $k, j \in T$, $(T = \{s\} \cup \{0, 1, 2, \ldots\})$,

$$B = \begin{pmatrix} b_{-s,-s} & B_{-s,n} \\ B_{n,-s} & B_{n,n} \end{pmatrix},$$

where elements in the last column and the last row are the matrices with the elements

$$\begin{aligned} B_{-s,n}(k,j) &= b_{k,j}, \quad k = -s, \quad j = 0, 1, 2, \ldots, \\ B_{n,-s}(k,j) &= b_{k,j}, \quad k = 0, 1, 2, \ldots, \quad j = -s, \\ B_{n,n}(k,j) &= b_{k,j}, \quad k, j = 0, 1, 2, \ldots. \end{aligned}$$

Consider the problem of the mean-square optimal linear estimation of the functional

$$A_N\xi = \sum_{j \in Z^S \cap \{0,\ldots,N\}} a(j)\xi(-j),$$

which depends on the unknown values of the sequence $\xi(j)$ from observations of the sequence $\xi(j) + \eta(j)$ at points $j \in \mathbb{Z}_-\backslash S$, where S is defined in the introduction. The linear estimate of the functional $A_N\xi$ has the representation

$$\hat{A}_N\xi = \int_{-\pi}^{\pi} h_N(e^{i\lambda})(Z_\xi(d\lambda) + Z_\eta(d\lambda)).$$

Define the vector $\vec{\mathbf{a}}_N$ as follows: elements with indices from the set $T \cap (S \cup \{0, \ldots, N\})$ coincide with the elements of the vector $\vec{\mathbf{a}}$ with the same indices and elements with indices from the set $T\backslash(S \cup \{0, \ldots, N\})$ are zeros. Let $\mathbf{B}$, $\mathbf{R}$, $\mathbf{Q}$ be linear operators in the space ℓ_2 defined in the theorem 2.1.

The spectral characteristic $h_N(e^{i\lambda})$ and the mean square error $\Delta(h_N; f, g)$ of the optimal linear estimate of the functional $A_N\xi$ can be calculated by formulas (15), (16)

$$h_N(e^{i\lambda}) = A_N(e^{i\lambda})\frac{f(\lambda)}{f(\lambda) + g(\lambda)} - \frac{\sum_{k \in T}(\mathbf{B}^{-1}\mathbf{R}\vec{\mathbf{a}}_N)_k e^{ik\lambda}}{f(\lambda) + g(\lambda)}, \tag{15}$$

$$\Delta(h_N; f, g) = \langle \mathbf{R}\vec{\mathbf{a}}_N, \mathbf{B}^{-1}\mathbf{R}\vec{\mathbf{a}}_N \rangle + \langle \mathbf{Q}\vec{\mathbf{a}}_N, \vec{\mathbf{a}}_N \rangle, \tag{16}$$

where $A_N(e^{i\lambda}) = \sum_{j \in Z^S \cap \{0,\ldots,N\}} a(j)e^{-ij\lambda}$.

The following corollary holds true.

Corollary 2.3

Let $\xi(j)$ and $\eta(j)$ be uncorrelated stationary sequences with spectral densities $f(\lambda)$ and $g(\lambda)$ which satisfy the minimality condition (1). The spectral characteristic $h_N(e^{i\lambda})$ and the mean square error $\Delta(h_N; f, g)$ of the optimal linear estimate of the functional $A_N\xi$ which depends on the unknown values of the sequence $\xi(j)$ from observation of the sequence $\xi(j) + \eta(j)$ at points $j \in \mathbb{Z}_-\backslash S$ can be calculated by formulas (15), (16).

Example 1. Let $\xi(j)$ and $\eta(j)$ be uncorrelated stationary sequences with the spectral densities

$$f(\lambda) = |1 - \alpha e^{-i\lambda}|^2, \quad g(\lambda) = |1 - \beta e^{-i\lambda}|^2$$

respectively, where $|\alpha| < 1$, $|\beta| < 1$. Consider the problem of the mean-square optimal linear estimation of the functional

$$A_2\xi = a(1)\xi(-1) + a(2)\xi(-2)$$

which depends on the unknown values $\xi(-1)$, $\xi(-2)$ based on the observations of the sequence $\xi(j) + \eta(j)$ at points $j \in \mathbb{Z}_-\backslash\{-n, -n+1\}$.

The spectral density of the sequence $\xi(j)+\eta(j)$ has the representation

$$f(\lambda)+g(\lambda)=\left|1-ae^{-i\lambda}\right|^2+\left|1-be^{-i\lambda}\right|^2=\left|x-ye^{-i\lambda}\right|^2,$$

$$x=\frac{1}{2}\left(\pm\sqrt{(1+a)^2+(1+b)^2}\pm\sqrt{(1-a)^2+(1-b)^2}\right),\quad y=\frac{a+b}{x}.$$

Since $|a|<1$, $|b|<1$, then $\left|\frac{y}{x}\right|<1$. Making use of decomposition of the function $\frac{1}{1-t}$ into the power series we can write the canonical factorizations of functions $\frac{1}{f(\lambda)+g(\lambda)}$, $\frac{f(\lambda)}{f(\lambda)+g(\lambda)}$ and $\frac{f(\lambda)g(\lambda)}{f(\lambda)+g(\lambda)}$

$$\frac{1}{f(\lambda)+g(\lambda)}=\frac{1}{|x-ye^{-i\lambda}|^2}=\left|\sum_{k=0}^{\infty}\frac{y^k}{x^{k+1}}e^{-ik\lambda}\right|^2,$$

$$\frac{f(\lambda)}{f(\lambda)+g(\lambda)}=\frac{\left|1-ae^{-i\lambda}\right|^2}{|x-ye^{-i\lambda}|^2}=\left|\sum_{k=0}^{\infty}\frac{y^k}{x^{k+1}}e^{-ik\lambda}-\sum_{k=0}^{\infty}\frac{ay^k}{x^{k+1}}e^{-i(k+1)\lambda}\right|^2=$$
$$=\left|\frac{1}{x}+\sum_{k=0}^{\infty}\left(\frac{y^{k+1}}{x^{k+2}}-\frac{ay^k}{x^{k+1}}\right)e^{-i(k+1)\lambda}\right|^2,$$

$$\frac{f(\lambda)g(\lambda)}{f(\lambda)+g(\lambda)}=\frac{\left|1-ae^{-i\lambda}\right|^2\cdot\left|1-be^{-i\lambda}\right|^2}{|x-ye^{-i\lambda}|^2}=\frac{\left|1-(a+b)e^{-i\lambda}+abe^{-i2\lambda}\right|^2}{|x-ye^{-i\lambda}|^2}=$$
$$=\left|\sum_{k=0}^{\infty}\frac{y^k}{x^{k+1}}e^{-ik\lambda}-\sum_{k=0}^{\infty}\frac{(a+b)y^k}{x^{k+1}}e^{-i(k+1)\lambda}+\sum_{k=0}^{\infty}\frac{aby^k}{x^{k+1}}e^{-i(k+2)\lambda}\right|^2=$$
$$=\left|\frac{1}{x}+\frac{y-ax-bx}{x^2}e^{-i\lambda}+\sum_{k=0}^{\infty}\left(\frac{y^{k+2}}{x^{k+3}}-\frac{(a+b)y^{k+1}}{x^{k+2}}+\frac{aby^k}{x^{k+1}}\right)e^{-i(k+2)\lambda}\right|^2.$$

The spectral characteristic of the optimal linear estimate $\hat{A}_2\xi$ can be calculated by the formula

$$h_2(e^{i\lambda})=(a(1)e^{-i\lambda}+a(2)e^{-2i\lambda})\frac{f(\lambda)}{f(\lambda)+g(\lambda)}-\frac{\sum\limits_{k\in T}(\mathbf{B}^{-1}\mathbf{R}\vec{\mathbf{a}}_2)_k e^{ik\lambda}}{f(\lambda)+g(\lambda)},$$

where unknown components $(\mathbf{B}^{-1}\mathbf{R}\vec{\mathbf{a}}_2)_k$, $k\in T=\{-n,-n+1\}\cup\{0,1,\ldots\}$ are to be found.

Linear operators $\mathbf{B}$, $\mathbf{R}$, $\mathbf{Q}$ are defined by the matrices B, R, Q with elements of the form (5), (6), (7) respectively, and we have vector $\vec{\mathbf{a}}_N=(0,0,0,a(1),a(2),0,\ldots)$,

$$B=\begin{pmatrix} b_{-n,-n} & b_{-n,-n+1} & b_{-n,0} & b_{-n,1} & b_{-n,2} & \ldots\\ b_{-n+1,-n} & b_{-n+1,-n+1} & b_{-n+1,0} & b_{-n+1,1} & b_{-n+1,2} & \ldots\\ b_{0,-n} & b_{1,-n+1} & b_{0,0} & b_{0,1} & b_{0,2} & \ldots\\ b_{1,-n} & b_{1,-n+1} & b_{1,0} & b_{1,1} & b_{1,2} & \ldots\\ b_{2,-n} & b_{2,-n+1} & b_{2,0} & b_{2,1} & b_{2,2} & \ldots\\ \ldots & & & & & \end{pmatrix},$$

$$R=\begin{pmatrix} r_{-n,-n} & r_{-n,-n+1} & r_{-n,0} & r_{-n,1} & r_{-n,2} & \ldots\\ r_{-n+1,-n} & r_{-n+1,-n+1} & r_{-n+1,0} & r_{-n+1,1} & r_{-n+1,2} & \ldots\\ r_{0,-n} & r_{1,-n+1} & r_{0,0} & r_{0,1} & r_{0,2} & \ldots\\ r_{1,-n} & r_{1,-n+1} & r_{1,0} & r_{1,1} & r_{1,2} & \ldots\\ r_{2,-n} & r_{2,-n+1} & r_{2,0} & r_{2,1} & r_{2,2} & \ldots\\ \ldots & & & & & \end{pmatrix},$$

$$Q = \begin{pmatrix} q_{-n,-n} & q_{-n,-n+1} & q_{-n,0} & q_{-n,1} & q_{-n,2} & \dots \\ q_{-n+1,-n} & q_{-n+1,-n+1} & q_{-n+1,0} & q_{-n+1,1} & q_{-n+1,2} & \dots \\ q_{0,-n} & q_{1,-n+1} & q_{0,0} & q_{0,1} & q_{0,2} & \dots \\ q_{1,-n} & q_{1,-n+1} & q_{1,0} & q_{1,1} & q_{1,2} & \dots \\ q_{2,-n} & q_{2,-n+1} & q_{2,0} & q_{2,1} & q_{2,2} & \dots \\ \dots \end{pmatrix}.$$

We need to find the matrix B^{-1} which defines the operator $\mathbf{B}^{-1}$. In order to find it we represent the matrix B in the form

$$B = \begin{pmatrix} K & L \\ M & N \end{pmatrix},$$

where

$$K = \begin{pmatrix} b_{-n,-n} & b_{-n,-n+1} \\ b_{-n+1,-n} & b_{-n+1,-n+1} \end{pmatrix}, \quad L = \begin{pmatrix} b_{-n,0} & b_{-n,1} & b_{-n,2} & \dots \\ b_{-n+1,0} & b_{-n+1,1} & b_{-n+1,2} & \dots \end{pmatrix},$$

$$M = \begin{pmatrix} b_{0,-n} & b_{1,-n+1} \\ b_{1,-n} & b_{1,-n+1} \\ b_{2,-n} & b_{2,-n+1} \\ \vdots & \vdots \end{pmatrix}, \quad N = \begin{pmatrix} b_{0,0} & b_{0,1} & b_{0,2} & \dots \\ b_{1,0} & b_{1,1} & b_{1,2} & \dots \\ b_{2,0} & b_{2,1} & b_{2,2} & \dots \\ \vdots & \vdots & \vdots \end{pmatrix}.$$

By the Frobenius formula [7] the matrix B^{-1} is of the form

$$B^{-1} = \begin{pmatrix} V^{-1} & -V^{-1}LN^{-1} \\ -N^{-1}MV^{-1} & N^{-1} + N^{-1}MV^{-1}LN^{-1} \end{pmatrix},$$

where $V = K - LN^{-1}M$.

The matrix N^{-1} can be found in the following way. Since matrix N is the matrix $B_{n,n}$ defined in theorem 2.1, it is constructed from the Fourier coefficients of the function $\frac{1}{f(\lambda)+g(\lambda)}$. The function $\frac{1}{f(\lambda)+g(\lambda)} = |x - ye^{-i\lambda]}|^{-2}$ admits the following canonical factorization, where z_p are its Fourier coefficients

$$\frac{1}{f(\lambda)+g(\lambda)} = \frac{1}{|x - ye^{-i\lambda}|^2} = \sum_{p=-\infty}^{\infty} z_p e^{ip\lambda} = \left|\sum_{k=0}^{\infty} \psi_k e^{-ik\lambda}\right|^2 = \left|\sum_{j=0}^{\infty} \theta_j e^{-ij\lambda}\right|^{-2},$$

$$\psi_k = \frac{y^k}{x^{k+1}},\ k \geq 0, \quad \theta_0 = x,\ \theta_1 = -y, \quad \theta_j = 0,\ j > 1.$$

Hence $z_p = \sum_{k=0}^{\infty} \psi_k \overline{\psi}_{k+p}$, $p \geq 0$, and $z_{-p} = \overline{z}_p$, $p \geq 0$. In the case $i \geq j$ we have $b_{i,j} = z_{i-j} = \sum_{l=i}^{\infty} \psi_{l-i}\overline{\psi}_{l-j}$, and in the case $i < j$ we have $b_{i,j} = z_{i-j} = \overline{z}_{j-i} = \sum_{l=j}^{\infty} \psi_{l-j}\overline{\psi}_{l-i}$.

Denote $\mathbf{\Psi}$ and $\mathbf{\Theta}$ linear operators in the space ℓ_2 which are defined by matrices with elements $\mathbf{\Psi}_{i,j} = \psi_{i-j}$, $\mathbf{\Theta}_{i,j} = \theta_{i-j}$, when $0 \leq j \leq i$, $\mathbf{\Psi}_{i,j} = 0$, $\mathbf{\Theta}_{i,j} = 0$, when $0 \leq i < j$. Elements of the matrix N can be represented in the form $N(i,j) = (\mathbf{\Psi}'\overline{\mathbf{\Psi}})_{i,j}$. It can be shown that the relation $\mathbf{\Psi\Theta} = \mathbf{\Theta\Psi} = I$ holds true . It follows from this relation that elements of the matrix N^{-1} can be calculated by the formula $N^{-1}(i,j) = (\overline{\mathbf{\Theta}}\mathbf{\Theta}')_{i,j}$.

Denote

$$N^{-1}(i,j) = \gamma_{i,j} = \sum_{l=0}^{\min(i,j)} \overline{\theta}_{i-l}\theta_{j-l}. \tag{17}$$

The matrix N^{-1} is of the form

$$N^{-1} = \begin{pmatrix} \gamma_{0,0} & \gamma_{0,1} & \gamma_{0,2} & \dots \\ \gamma_{1,0} & \gamma_{1,1} & \gamma_{1,2} & \dots \\ \gamma_{2,0} & \gamma_{2,1} & \gamma_{2,2} & \dots \\ \dots \end{pmatrix}.$$

Denote $(LN^{-1}M)_{k,j}=\beta_{k,j}$, $(V^{-1})_{k,j}=\tau_{k,j}$, $(N^{-1}MV^{-1})_{k,j}=-\rho_{k,j}$, $(V^{-1}LN^{-1})_{k,j}=-\sigma_{k,j}$, $(N^{-1}+N^{-1}MV^{-1}LN^{-1})_{k,j}=\omega_{k,j}$.
Thus the matrix B^{-1} is of the form

$$B^{-1}=\left(\begin{array}{cccccc}\tau_{-n,-n} & \tau_{-n,-n+1} & \sigma_{-n,0} & \sigma_{-n,1} & \sigma_{-n,2} & \dots\\ \tau_{-n+1,-n} & \tau_{-n+1,-n+1} & \sigma_{-n+1,0} & \sigma_{-n+1,1} & \sigma_{-n+1,2} & \dots\\ \rho_{0,-n} & \rho_{0,-n+1} & \omega_{0,0} & \omega_{0,1} & \omega_{0,2} & \dots\\ \rho_{1,-n} & \rho_{1,-n+1} & \omega_{1,0} & \omega_{1,1} & \omega_{1,2} & \dots\\ \rho_{2,-n} & \rho_{2,-n+1} & \omega_{2,0} & \omega_{2,1} & \omega_{2,2} & \dots\\ \dots & & & & & \end{array}\right),$$

$$\begin{aligned}
&\tau_{-n,-n}=(b_{-n+1,-n+1}-\beta_{-n+1,-n+1})/d,\\
&\tau_{-n,-n+1}=-(b_{-n,-n+1}-\beta_{-n,-n+1})/d,\\
&\tau_{-n+1,-n}=-(b_{-n+1,-n}-\beta_{-n+1,-n})/d,\\
&\tau_{-n+1,-n+1}=(b_{-n,-n}-\beta_{-n,-n})/d,\\
&d=(b_{-n,-n}-\beta_{-n,-n})(b_{-n+1,-n+1}-\beta_{-n+1,-n+1})-\\
&(b_{-n,-n+1}-\beta_{-n,-n+1})(b_{-n+1,-n}-\beta_{-n+1,-n}),\\
&\beta_{k,j}=\sum_{i=0}^{\infty}\left(\sum_{r=0}^{\infty}b_{k,r}\gamma_{r,i}\right)b_{i,j},\\
&\rho_{k,j}=-\sum_{i=0}^{\infty}\gamma_{k,i}\sum_{r=-n}^{-n+1}b_{i,r}\tau_{r,j},\\
&\sigma_{k,j}=-\sum_{i=-n}^{-n+1}\tau_{k,i}\sum_{r=0}^{\infty}b_{i,r}\gamma_{r,j},
\end{aligned}$$

$$\omega_{k,j}=\gamma_{k,j}+\sum_{i=0}^{\infty}\gamma_{k,i}\sum_{m=-n}^{-n+1}\sum_{r=-n}^{-n+1}b_{i,r}\tau_{r,m}\sum_{l=0}^{\infty}b_{m,l}\gamma_{l,j}=\gamma_{k,j}+\nu_{k,j}. \tag{18}$$

The unknown coefficients $c(k), k\in T$ can be calculated by formulas

$$c(k)=\sum_{j\in T}(B^{-1}R)_{k,j}a(j)=a(1)(B^{-1}R)_{k,1}+a(2)(B^{-1}R)_{k,2}, \tag{19}$$

$$(B^{-1}R)_{k,j}=\sum_{i=-n}^{-n+1}\tau_{k,i}r_{i,j}+\sum_{i=0}^{\infty}\sigma_{k,i}r_{i,j},\quad k=-n,-n+1,\quad j=1,2;$$

$$(B^{-1}R)_{k,j}=\sum_{i=-n}^{-n+1}\rho_{k,i}r_{i,j}+\sum_{i=0}^{\infty}\omega_{k,i}r_{i,j},\quad k=0,1,\dots,\quad j=1,2. \tag{20}$$

The spectral characteristic and the mean-square error of the estimate can be calculated by formulas

$$h_2(e^{i\lambda})=(a(1)e^{-i\lambda}+a(2)e^{-2i\lambda})\frac{f(\lambda)}{f(\lambda)+g(\lambda)}-\frac{\sum_{k\in T}c(k)e^{ik\lambda}}{f(\lambda)+g(\lambda)},$$

$$\begin{aligned}\Delta(h_2;f,g)&=\langle\mathbf{R}\vec{\mathbf{a}}_2,\mathbf{B}^{-1}\mathbf{R}\vec{\mathbf{a}}_2\rangle+\langle\mathbf{Q}\vec{\mathbf{a}}_2,\vec{\mathbf{a}}_2\rangle\\
&=\sum_{k\in T}\left(a(1)r_{k,1}+a(2)r_{k,2}\right)c(k)+a(1)q_{1,1}+a(2)q_{1,2}+a(1)q_{2,1}+a(2)q_{2,2}.\end{aligned}$$

The coefficients $r_{k,j}$ can be found from the canonical factorization of the function $\frac{f(\lambda)}{f(\lambda)+g(\lambda)}$

$$\frac{f(\lambda)}{f(\lambda)+g(\lambda)} = \left|\sum_{k=0}^{\infty}\phi_k e^{-i(k+1)\lambda}\right|^2, \quad \phi_0 = \frac{1}{x}, \quad \phi_k = \frac{y^{k+1}}{x^{k+2}} - \frac{ay^k}{x^{k+1}}, \, k>0.$$

Hence, $r_{i,j} = \sum\limits_{l=max(i,j)}^{\infty} \phi_{l-i}\overline{\phi}_{l-j}$.

Coefficients $q_{k,j}$ we can find from the canonical factorization of the function $\frac{f(\lambda)g(\lambda)}{f(\lambda)+g(\lambda)}$

$$\frac{f(\lambda)g(\lambda)}{f(\lambda)+g(\lambda)} = \left|\sum_{k=0}^{\infty}\varphi_k e^{-i(k+2)\lambda}\right|^2, \quad \varphi_0 = \frac{1}{x}, \quad \varphi_1 = \frac{y-ax-bx}{x^2},$$

$$\varphi_k = \frac{y^{k+2}}{x^{k+3}} - \frac{(a+b)y^{k+1}}{x^{k+2}} + \frac{aby^k}{x^{k+1}}, \, k>1.$$

Hence, $q_{i,j} = \sum\limits_{l=max(i,j)}^{\infty} \varphi_{l-i}\overline{\varphi}_{l-j}$.

Example 2. Consider the problem of the mean-square optimal linear estimation of the functional $A_2\xi = a(1)\xi(-1)+a(2)\xi(-2)$ in the case of observations of the sequence $\xi(j)+\eta(j)$ at all points $j\in\mathbb{Z}_-$ without missing observations.

And compare values of the mean square errors derived in the case of missing observations and in the case of all observations at points $j\in\mathbb{Z}_-$.

The mean-square error of the estimate is determined by the formula [18], [21]

$$\Delta_1(\tilde{h}; f, g) = \langle \tilde{\mathbf{R}}\vec{\mathbf{a}}, \tilde{\mathbf{B}}^{-1}\tilde{\mathbf{R}}\vec{\mathbf{a}}\rangle + \langle \tilde{\mathbf{Q}}\vec{\mathbf{a}}, \vec{\mathbf{a}}\rangle,$$

where vector $\vec{\mathbf{a}} = (0, a(1), a(2), 0, 0, \ldots)$, and $\tilde{\mathbf{B}}, \tilde{\mathbf{R}}, \tilde{\mathbf{Q}}$ are linear operators in the space ℓ_2 defined by matrices $(\tilde{\mathbf{B}})_{k,j} = b_{k,j}, \ \ k,j\geq 0, \ \ (\tilde{\mathbf{R}})_{k,j} = r_{k,j}, \ \ k,j\geq 0, \ \ (\tilde{\mathbf{Q}})_{k,j} = q_{k,j}, \ \ k,j\geq 0$, where elements $b_{k,j}, r_{k,j}, q_{k,j}$ are determined by formulas (5),(6),(7) respectively, $\tilde{h}$ is the spectral characteristic of the estimate of the functional which depends on the unknown values of the sequence without missing values. Hence

$$\Delta_1(\tilde{h}; f, g) = \sum_{k=0}^{\infty}\left(a(1)r_{k,1} + a(2)r_{k,2}\right)\tilde{c}(k) + a(1)q_{1,1} + a(2)q_{1,2} + a(1)q_{2,1} + a(2)q_{2,2},$$

where $\tilde{c}(k) = (\tilde{\mathbf{B}}^{-1}\tilde{\mathbf{R}}\vec{\mathbf{a}})_k, k\geq 0$. Since matrix which determines the operator $\tilde{\mathbf{B}}$ coincides with matrix N from the previous example, elements of the matrix which determines the inverse operator $\tilde{\mathbf{B}}^{-1}$ are calculated by the formula (17). Hence $\tilde{c}(k) = \sum\limits_{j=0}^{\infty}\sum\limits_{i=0}^{\infty}\gamma_{k,i}r_{i,j}a(j), k\geq 0$.

Consider the expression (20). Making use of (18), we can rewrite (20) in the following form

$$(B^{-1}R)_{k,j} = \sum_{i=-n}^{-n+1}\rho_{k,i}r_{i,j} + \sum_{i=0}^{\infty}\gamma_{k,i}r_{i,j} + \sum_{i=0}^{\infty}\nu_{k,i}r_{i,j}, \quad k,j\geq 0.$$

Consider the formula (19) for $k \geq 0$

$$
\begin{aligned}
c(k) = &\sum_{j\in T}(B^{-1}R)_{k,j}a(j) = \sum_{j\in S}(B^{-1}R)_{k,j}a(j) + \sum_{j=0}^{\infty}(B^{-1}R)_{k,j}a(j) \\
= &\sum_{j\in S}(B^{-1}R)_{k,j}a(j) + \sum_{j=0}^{\infty}\left(\sum_{i=-n}^{-n+1}\rho_{k,i}r_{i,j} + \sum_{i=0}^{\infty}\gamma_{k,i}r_{i,j} + \sum_{i=0}^{\infty}\nu_{k,i}r_{i,j}\right)a(j) \\
= &\tilde{c}(k) + \sum_{j\in S}(B^{-1}R)_{k,j}a(j) + \sum_{j=0}^{\infty}\left(\sum_{i=-n}^{-n+1}\rho_{k,i}r_{i,j} + \sum_{i=0}^{\infty}\nu_{k,i}r_{i,j}\right)a(j) \\
= &\tilde{c}(k) + c_T(k).
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\Delta(h_2; f, g) = \Delta_1(\tilde{h}; f, g) + \\
&\sum_{k=0}^{\infty}\left(a(1)r_{k,1} + a(2)r_{k,2}\right)c_T(k) + \sum_{k\in S}\left(a(1)r_{k,1} + a(2)r_{k,2}\right)c(k).
\end{aligned}
$$

3. Minimax-robust method of filtration

Theorem 2.1 and its corollaries can be applied to filtering of the functional in the cases where the spectral densities of the sequences are exactly known. If complete information on the spectral densities is impossible while a class of admissible densities is given, it is reasonable to apply the minimax-robust method of filtering which consists in minimizing the value of the mean-square error for all spectral densities from the given class. For description of the minimax method we propose the following definitions [19].

Definition 3.1. For a given class of spectral densities $D = D_f \times D_g$ the spectral densities $f_0(\lambda) \in D_f$, $g_0(\lambda) \in D_g$ are called least favorable in the class D for the optimal linear filtering of the functional $A\xi$ if the following relation holds true

$$
\Delta\left(f_0, g_0\right) = \Delta\left(h\left(f_0, g_0\right); f_0, g_0\right) = \max_{(f,g)\in D_f\times D_g}\Delta\left(h\left(f, g\right); f, g\right).
$$

Definition 3.2. For a given class of spectral densities $D = D_f \times D_g$ the spectral characteristic $h^0(e^{i\lambda})$ of the optimal linear estimate of the functional $A\xi$ is called minimax-robust if there are satisfied conditions

$$
h^0(e^{i\lambda}) \in H_D = \bigcap_{(f,g)\in D_f\times D_g} L_2^s(f+g),
$$

$$
\min_{h\in H_D}\max_{(f,g)\in D}\Delta\left(h; f, g\right) = \max_{(f,g)\in D}\Delta\left(h^0; f, g\right).
$$

From the introduced definitions and formulas derived above we can obtain the following statement.

Lemma 3.1
Spectral densities $f_0(\lambda) \in D_f$, $g_0(\lambda) \in D_g$ satisfying the minimality condition (1) are the least favorable in the class $D = D_f \times D_g$ for the optimal linear filtering of the functional $A\xi$ if operators B^0, R^0, Q^0 determined by the Fourier coefficients of the functions

$$
(f_0(\lambda) + g_0(\lambda))^{-1}, \quad f_0(\lambda)(f_0(\lambda) + g_0(\lambda))^{-1}, \quad f_0(\lambda)g_0(\lambda)(f_0(\lambda) + g_0(\lambda))^{-1}
$$

determine a solution to the constrain optimization problem

$$
\begin{aligned}
&\max_{(f,g)\in D_f\times D_g}\langle \mathbf{R}\vec{\mathbf{a}}, \mathbf{B}^{-1}\mathbf{R}\vec{\mathbf{a}}\rangle + \langle \mathbf{Q}\vec{\mathbf{a}}, \vec{\mathbf{a}}\rangle = \\
&\qquad\qquad \langle \mathbf{R}^0\vec{\mathbf{a}}, (\mathbf{B}^0)^{-1}\mathbf{R}^0\vec{\mathbf{a}}\rangle + \langle \mathbf{Q}^0\vec{\mathbf{a}}, \vec{\mathbf{a}}\rangle.
\end{aligned} \tag{21}
$$

The minimax spectral characteristic $h^0 = h(f_0, g_0)$ is determined by the formula (9) if $h(f_0, g_0) \in H_D$.

The least favorable spectral densities $f_0(\lambda)$, $g_0(\lambda)$ and the minimax spectral characteristic $h^0 = h(f_0, g_0)$ form a saddle point of the function $\Delta\left(h; f, g\right)$ on the set $H_D \times D$. The saddle point inequalities

$$\Delta\left(h; f_0, g_0\right) \geq \Delta\left(h^0; f_0, g_0\right) \geq \Delta\left(h^0; f, g\right)$$

$$\forall h \in H_D, \forall f \in D_f, \forall g \in D_g$$

hold true if $h^0 = h(f_0, g_0)$ and $h(f_0, g_0) \in H_D$, where (f_0, g_0) is a solution to the constrained optimization problem

$$\sup_{(f,g) \in D_f \times D_g} \Delta\left(h(f_0, g_0); f, g\right) = \Delta\left(h(f_0, g_0); f_0, g_0\right), \tag{22}$$

$$\Delta\left(h(f_0, g_0); f, g\right) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\left|A(e^{i\lambda})g_0(\lambda) + C^0(e^{i\lambda})\right|^2}{(f_0(\lambda) + g_0(\lambda))^2} f(\lambda) d\lambda$$
$$+ \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\left|A(e^{i\lambda})f_0(\lambda) - C^0(e^{i\lambda})\right|^2}{(f_0(\lambda) + g_0(\lambda))^2} g(\lambda) d\lambda,$$

$$C^0(e^{i\lambda}) = \sum_{j \in T} ((\mathbf{B}^0)^{-1} \mathbf{R}^0 \vec{\mathbf{a}})_j e^{ij\lambda},$$

The constrained optimization problem (22) is equivalent to the unconstrained optimization problem [33]:

$$\Delta_D(f, g) = -\Delta(h(f_0, g_0); f, g) + \delta((f, g) \left| D_f \times D_g \right.) \to \inf, \tag{23}$$

where $\delta((f, g) \left| D_f \times D_g \right.)$ is the indicator function of the set $D = D_f \times D_g$. Solution of the problem (23) is characterized by the condition $0 \in \partial\Delta_D(f_0, g_0)$, where $\partial\Delta_D(f_0)$ is the subdifferential of the convex functional $\Delta_D(f, g)$ at point (f_0, g_0) [34].

The form of the functional $\Delta(h(f_0, g_0); f, g)$ admits finding the derivatives and differentials of the functional in the space $L_1 \times L_1$. Therefore the complexity of the optimization problem (23) is determined by the complexity of calculating the subdifferential of the indicator functions $\delta((f, g)|D_f \times D_g)$ of the sets $D_f \times D_g$ [11].

Lemma 3.2
Let (f_0, g_0) be a solution to the optimization problem (23). The spectral densities $f_0(\lambda)$, $g_0(\lambda)$ are the least favorable in the class $D = D_f \times D_g$ and the spectral characteristic $h^0 = h(f_0, g_0)$ is the minimax of the optimal linear estimate of the functional $A\xi$ if $h(f_0, g_0) \in H_D$.

4. Least favorable spectral densities in the class $D_{\varepsilon_1} \times D_{\varepsilon_2}$

Consider the problem of the optimal linear filtering of the functional $A\xi$ in the case where spectral densities $f(\lambda)$, $g(\lambda)$ belong to the set of admissible spectral densities $D_{\varepsilon_1} \times D_{\varepsilon_2}$, where

$$D_{\varepsilon_1} = \left\{ f(\lambda) | f(\lambda) = (1 - \varepsilon_1)u_1(\lambda) + \varepsilon_1 u(\lambda), \; \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\lambda) d\lambda = P_1 \right\},$$

$$D_{\varepsilon_2} = \left\{ g(\lambda) | g(\lambda) = (1 - \varepsilon_2)v_1(\lambda) + \varepsilon_2 v(\lambda), \; \frac{1}{2\pi} \int_{-\pi}^{\pi} g(\lambda) d\lambda = P_2 \right\},$$

where $u_1(\lambda)$, $v_1(\lambda)$ are known and fixed spectral densities and $u(\lambda)$, $v(\lambda)$ are unknown ones. These sets of spectral densities describe "ε-contamination" models of stochastic sequences.

Suppose that the densities $f_0(\lambda)\in D_f^0$, $g_0(\lambda)\in D_g^0$ and functions determined by formulas

$$h_f(f_0,g_0)=\frac{\left|A(e^{i\lambda})g_0(\lambda)+C^0(e^{i\lambda})\right|^2}{(f_0(\lambda)+g_0(\lambda))^2}, \tag{24}$$

$$h_g(f_0,g_0)=\frac{\left|A(e^{i\lambda})f_0(\lambda)-C^0(e^{i\lambda})\right|^2}{(f_0(\lambda)+g_0(\lambda))^2}. \tag{25}$$

are bounded. In this case the functional

$$\Delta(h(f_0,g_0);f,g)=\frac{1}{2\pi}\int_{-\pi}^{\pi}h_f(f_0,g_0)f(\lambda)d\lambda+\frac{1}{2\pi}\int_{-\pi}^{\pi}h_g(f_0,g_0)g(\lambda)d\lambda$$

is continuous and bounded in the space $L_1\times L_1$ and we can use the method of Lagrange multipliers to solve the optimization problem (22) [33].

As a result we obtain the following relations determining the least favorable spectral densities

$$\left|A(e^{i\lambda})g_0(\lambda)+C^0(e^{i\lambda})\right|=(f_0(\lambda)+g_0(\lambda))(\varphi_1(\lambda)+\alpha_1^{-1}), \tag{26}$$

$$\left|A(e^{i\lambda})f_0(\lambda)-C^0(e^{i\lambda})\right|=(f_0(\lambda)+g_0(\lambda))(\varphi_2(\lambda)+\alpha_2^{-1}), \tag{27}$$

where $\varphi_1(\lambda)\leq 0$, and $\varphi_1(\lambda)=0$ when $f_0(\lambda)\geq(1-\varepsilon_1)u_1(\lambda)$, $\varphi_2(\lambda)\leq 0$, and $\varphi_2(\lambda)=0$ when $g_0(\lambda)\geq(1-\varepsilon_2)v_1(\lambda)$.

The following theorem holds true.

Theorem 4.1
Let the spectral densities $f_0(\lambda)\in D_{\varepsilon_1}$, $g_0(\lambda)\in D_{\varepsilon_2}$ and the minimality condition (1) holds true. Suppose that functions determined by formulas (24), (25) are bounded. The functions $f_0(\lambda)$, $g_0(\lambda)$ determined by relations (26), (27) are the least favorable spectral densities in the class $D_{\varepsilon_1}\times D_{\varepsilon_2}$ for the optimal linear filtering of the functional $A\xi$ if they determine a solution to optimization problem (21). The function $h(f_0,g_0)$ determined by formula (9) is the minimax spectral characteristic of the optimal estimate of the functional $A\xi$.

Corollary 4.1
Suppose that the spectral density $f(\lambda)$ is known, the spectral density $g_0(\lambda)\in D_{\varepsilon_2}$. Let the function $f(\lambda)+g_0(\lambda)$ satisfy the minimality condition (1) and the function $h_g(f,g_0)$ determined by (25) is bounded. The spectral density $g_0(\lambda)$ is the least favorable in the class D_{ε_2} for the optimal linear filtering of the functional $A\xi$ if it is of the form

$$g_0(\lambda)=\max\left\{(1-\varepsilon_2)v_1(\lambda),\alpha_2\left|A(e^{i\lambda})f(\lambda)-C^0(e^{i\lambda})\right|-f(\lambda)\right\},$$

and the pair $f(\lambda)$, $g_0(\lambda)$ determine a solution to the optimization problem (21). The function $h(f,g_0)$ determined by formula (9) is the minimax spectral characteristic of the optimal estimate of the functional $A\xi$.

5. Least favorable spectral densities in the class $D=D_\varepsilon^1\times D_v^u$

Consider the problem of the optimal linear filtering of the functional $A\xi$ for the class of admissible spectral densities $D=D_\varepsilon^1\times D_v^u$, where

$$D_\varepsilon^1=\left\{f(\lambda)\left|\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|f(\lambda)-f_1(\lambda)\right|d\lambda\leq\varepsilon\right.\right\},$$

$$D_v^u=\left\{g(\lambda)\left|v(\lambda)\leq g(\lambda)\leq u(\lambda),\frac{1}{2\pi}\int_{-\pi}^{\pi}g(\lambda)d\lambda\leq P\right.\right\},$$

where spectral densities $u(\lambda), v(\lambda), f_1(\lambda)$ are known and fixed and the spectral densities $u(\lambda)$ and $v(\lambda)$ are bounded. The class of admissible spectral densities D_ε^1 describes a model of "ε-neighbourhood" in the space L_1 of the given bounded spectral density $f_1(\lambda)$. The class D_v^u describes the "band" model of stochastic sequences.

Let the sequences $f^0(\lambda) \in D_\varepsilon^1$, $g^0(\lambda) \in D_v^u$ determine the bounded functions $h_f(f_0, g_0)$, $h_g(f_0, g_0)$ defined by the formulas (24), (25). It follows from the condition $0 \in \partial\Delta_D(f_0, g_0)$ that the least favorable densities satisfy the following equations

$$\left|A(e^{i\lambda})g_0(\lambda) + C^0(e^{i\lambda})\right| = (f_0(\lambda) + g_0(\lambda))\Psi(\lambda)\alpha_1, \tag{28}$$

$$\left|A(e^{i\lambda})f_0(\lambda) - C^0(e^{i\lambda})\right| = (f_0(\lambda) + g_0(\lambda))(\gamma_1(\lambda) + \gamma_2(\lambda) + \alpha_2^{-1}), \tag{29}$$

where $|\Psi(\lambda)| \leq 1$, and $\Psi(\lambda) = sign(f_0(\lambda) - f_1(\lambda))$ when $f_0(\lambda) \neq f_1(\lambda)$, α_1, α_2 are fixed values, $\gamma_1 \leq 0$, and $\gamma_1 = 0$ when $g_0(\lambda) \geq v(\lambda)$, $\gamma_2 \geq 0$, and $\gamma_2 = 0$ when $g_0(\lambda) \leq u(\lambda)$.

Equations (28), (29) together with the extremal condition (21) and normalization condition

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(\lambda) - f_1(\lambda)|\, d\lambda = \varepsilon, \tag{30}$$

determine the least favorable spectral densities in the class D.

The following theorem holds true.

Theorem 5.1
Let the spectral densities $f_0(\lambda) \in D_\varepsilon^1$, $g_0(\lambda) \in D_v^u$, and the minimality condition (1) holds true. Suppose the functions defined by the formulas (24), (25) are bounded. Functions determined by equations (28)–(30) are the least favorable spectral densities in the class $D_\varepsilon^1 \times D_v^u$ for the optimal linear filtering of the functional $A\xi$ if they determine a solution to the optimization problem (21). The function $h(f_0, g_0)$ determined by formula (9) is the minimax spectral characteristic of the optimal estimate of the functional $A\xi$.

Corollary 5.1
Suppose that the spectral density $g(\lambda)$ is known, the spectral density $f_0(\lambda) \in D_\varepsilon^1$ and the minimality condition (1) holds true. Let the function $h_f(f_0, g)$ determined by (24) be bounded. The spectral density $f_0(\lambda)$ is the least favorable in the class D_ε^1 for the optimal estimation of the functional $A\xi$ if it is of the form

$$f_0(\lambda) = \max\left\{f_1(\lambda), \alpha_1\left|A(e^{i\lambda})g(\lambda) + C^0(e^{i\lambda})\right| - g(\lambda)\right\},$$

and the pair $f_0(\lambda), g(\lambda)$ determine a solution to the optimization problem (21). The function $h(f_0, g)$ determined by formula (9) is the minimax spectral characteristic of the optimal estimate of the functional $A\xi$.

6. Least favorable spectral densities in the class $D = D_\varepsilon^2 \times D_v^u$

Consider the problem of the optimal linear filtering of the functional $A\xi$ for the class of admissible spectral densities $D = D_\varepsilon^2 \times D_v^u$, where

$$D_\varepsilon^2 = \left\{ f(\lambda) \left| \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(\lambda) - f_1(\lambda)|^2\, d\lambda \leq \varepsilon \right.\right\},$$

$$D_v^u = \left\{ g(\lambda) \left| v(\lambda) \leq g(\lambda) \leq u(\lambda), \frac{1}{2\pi}\int_{-\pi}^{\pi} g(\lambda) d\lambda \leq P \right.\right\},$$

where spectral densities $u(\lambda)$, $v(\lambda)$, $f_1(\lambda)$ are known and fixed, the densities $u(\lambda)$ and $v(\lambda)$ are bounded. The class D_ε^2 describes a model of "ε-district" in the space L_2 of the given bounded spectral density $f_1(\lambda)$.

Suppose that densities $f_0(\lambda) \in D_\varepsilon^2$, $g_0(\lambda) \in D_v^u$ are such that functions $h_f(f_0, g_0)$, $h_g(f_0, g_0)$ determined by the formulas (24), (25) are bounded.

From the condition $0\in\partial\Delta_D(f^0,g^0)$, where $D=D_\varepsilon^2\times D_v^u$, we obtain equations which the least favorable spectral densities should satisfy

$$\left|A(e^{i\lambda})g_0(\lambda)+C^0(e^{i\lambda})\right|^2=(f_0(\lambda)+g_0(\lambda))^2(f_0(\lambda)-f_1(\lambda))\alpha_1, \tag{31}$$

$$\left|A(e^{i\lambda})f_0(\lambda)-C^0(e^{i\lambda})\right|=(f_0(\lambda)+g_0(\lambda))(\gamma_1(\lambda)+\gamma_2(\lambda)+\alpha_2^{-1}), \tag{32}$$

where α_1,α_2 are fixed values, $\gamma_1\leq 0$, and $\gamma_1=0$ when $g_0(\lambda)\geq v(\lambda)$; $\gamma_2\geq 0$ and $\gamma_2=0$ when $g_0(\lambda)\leq u(\lambda)$.

Equations (31), (32) together with the optimization problem (21) and the normalization condition

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(\lambda)-f_1(\lambda)|^2\,d\lambda=\varepsilon, \tag{33}$$

determine the least favorable spectral densities in the class D.

The following theorem holds true.

Theorem 6.1
Let the densities $f_0(\lambda)\in D_\varepsilon^2$, $g_0(\lambda)\in D_v^u$ be such that the minimality condition (1) holds true and the functions determined by (24), (25) are bounded. The spectral densities $f_0(\lambda)$, $g_0(\lambda)$ are the least favorable in the class $D_\varepsilon^2\times D_v^u$ for the optimal linear filtering of the functional $A\xi$ if they satisfy equations (31)–(33) and determine a solution to the optimization problem (21). The function $h(f_0,g_0)$ determined by formula (9) is the minimax spectral characteristic of the optimal estimate of the functional $A\xi$.

Corollary 6.1
Suppose that the spectral density $g(\lambda)$ is known, the spectral density $f_0(\lambda)\in D_\varepsilon^2$ and the minimality condition(1) holds true. Let the function $h_f(f_0,g)$ determined by formula (24) be bounded. The spectral density $f_0(\lambda)$ is the least favorable in the class D_ε^2 for the optimal linear filtering of the functional $A\xi$ if the following relation holds true

$$\left|A(e^{i\lambda})g(\lambda)+C^0(e^{i\lambda})\right|^2=(f_0(\lambda)+g(\lambda))^2(f_0(\lambda)-f_1(\lambda)),$$

and the pair $f_0(\lambda),g(\lambda)$ determine a solution to the optimization problem (21). The function $h(f_0,g)$ determined by formula (9) is the minimax spectral characteristic of the optimal estimate of the functional $A\xi$.

7. Conclusions

In the article we propose methods of the mean-square optimal filtering of functionals which depend on the unknown values of a stationary sequence based on observed data of the sequence with a stationary noise and with missing observations. In the case of spectral certainty, where spectral densities of the stationary sequences are exactly known, we derive formulas for calculating the spectral characteristics and values of the mean-square errors of the optimal estimates of the functionals. In the case of spectral uncertainty, where spectral densities of the stationary sequences are not exactly known while certain sets of admissible densities are given, we derive relations which determine the least spectral densities and the minimax-robust spectral characteristics of estimates.

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