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A stochastic predator-prey system with Watt-type functional response

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Abstract In this paper we consider a stochastic version of predator-prey systems with Watt-type functional response. We first prove the existence and uniqueness of the positive global solution by using the comparison theorem of stochastic equations. Then, we study the boundedness of moments of the solution. Furthermore, the growth rates, persistence and extinction of species are investigated.

Keywords Stochastic prey-predator model, Watt-type functional response, Moment estimates, Growth rates.

AMS 2010 subject classifications 93E03, 92D25

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1. Introduction

It is well known that the predator-prey system is one of the most important models in ecology. The first predator-prey system, described by a system of differential equations, was proposed by Volterra in 1920's. Since then, based on different settings, various types of predator-prey models have been proposed and studied by ecologists and mathematicians, such as considering different functional response types: Holling types [7], Hassel-Varley type [6], Beddington-DeAngelis type [3, 4] and ratio-dependence type [1], adding a prey self-competition term [16], etc. We refer the reader to the book [18] for a detailed presentation.

On the other hand, it is also known that the environmental noise is an important component in an ecosystem. Indeed, for instance, in [15], Mao et al. shown that even a sufficiently small noise can suppress explosions in population dynamics. Since the real world are full of random perturbations, it is of great significance to take into account the effect of noise in the investigation of deterministic predator-prey systems. In fact, a lot of stochastic version of existing deterministic models have been introduced recently by different authors and here we only mention some of them. For example, Khasminskiǐ and Klebaner in [10] gave an analysis of Lotka-Volterra system with small random perturbations, Ji and Jiang in [9] analyzed a stochastic predator-prey system with Beddington-DeAngelis functional response, Bandyopadhyay and Chattopadhyay in [2] studied the effect of environmental fluctuations on a ratio-dependent predator-prey system, Mandal and Banerjee in [13] investigated stochastic persistence and stationary distribution of a Holling-Tanner type prey-predator model, Dung in [5] provided an explicit solution to delayed logistic equations with fractional noise, etc.

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A well-known model of deterministic systems is the predator-prey model with a Watt-type functional response, proposed by Watt [23], which can be described by the following differential equations

$$\begin{cases} \frac{dx(t)}{dt} = x(t)[a_1 - b_1 x(t)] - c_1 y(t)\big[1 - \exp\big(-\frac{cx(t)}{y(t)^m}\big)\big] \\ \frac{dy(t)}{dt} = -a_2 y(t) + c_2 y(t)\big[1 - \exp\big(-\frac{cx(t)}{y(t)^m}\big)\big] \end{cases} \tag{1}$$

where $x(t)$ and $y(t)$ are functions of time representing population density of prey and predator, and all parameters are positive constants, a_1 is the intrinsic growth rate of prey, c_1 the maximum number of prey that can be eaten by a predator per unit of time, c_2 a conversion efficiency and a_2 the mortality rate of the predator.

Although some improvements to the original model (1) have been extensively suggested (see, for instance, [12, 21, 22]), the question of the effect of environmental noise on this model has not yet been addressed. The aim of this paper is to study a stochastic version of (1). More specifically, we assume that the relevant parameters a_1, a_2 are random, then they can be modeled by

$$a_1(\omega) = a_1 + \varepsilon_1 \dot{W}_1(t), \quad a_2(\omega) = a_2 + \varepsilon_2 \dot{W}_2(t), \tag{2}$$

where $a_1 = E[a_1(\omega)], a_2 = E[a_2(\omega)]$ and $\varepsilon_1, \varepsilon_2$ are deterministic and $\dot{\mathrm{W}}_1(t), \dot{\mathrm{W}}_2(t)$ are independent white noises.

Inserting (2) into the system (1), we get a new stochastic version in the following form

$$\begin{cases} dx(t) = \Big(x(t)[a_1 - b_1 x(t)] - c_1 y(t)\big[1 - \exp\big(-\frac{cx(t)}{y(t)^m}\big)\big]\Big)dt + \sigma_1 x(t) dW_1(t) \\ dy(t) = \Big(-a_2 y(t) + c_2 y(t)\big[1 - \exp\big(-\frac{cx(t)}{y(t)^m}\big)\big]\Big)dt + \sigma_2 y(t) dW_2(t), \end{cases} \tag{3}$$

where $W_1(t), W_2(t)$ are independent standard Brownian motions, σ_1, σ_2 represent the intensities of the white noises. The initial conditions $x_0, y_0 > 0$.

In this paper, to investigate the nonlinear stochastic system (3), we mainly use Itô's formula, the theory of stochastic differential equations and the method of Lyapunov functionals to estimate its solution, and to analyze the long time behavior of the system. The structure of the paper is as follows. In Section 2, we establish the existence of unique positive global solution and give an estimate for moments of the solution. In Section 3, we study some features of the system, including the growth rates, persistence and extinction of predator and prey. The conclusion is given in Section 4.

2. The positive solution and its moments

Throughout this paper, we use the following notations. Denote $\mathbb{R}^2_+ = (0,\infty) \times (0,\infty)$, $(\Omega, \mathcal{F}, P)$ the complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions, that is, it is right continuous and increasing while $\mathcal{F}_0$ contains all P-null sets. Let $W_1(t), W_2(t)$ denote the independent standard Brownian motions defined on this probability space.

Since stochastic system (3) describes population dynamics, its solution has to be positive and not to explode at a finite time. By using the Itô formula, first we show that there exists a unique positive local solution of the system (3) and then by using the comparison theorem, we prove that this solution is global. We have the following theorem.

Theorem 2.1
The system (3) has a unique positive global solution on the interval $[0,\infty)$ for any the initial condition $(x_0, y_0) \in \mathbb{R}^2_+$.

Proof
We first consider the following system

$$\begin{cases} du(t) = \Big(a_1 - \frac{\sigma_1^2}{2} - b_1 e^{u(t)} - c_1 e^{v(t)-u(t)}\big[1 - \exp\big(-ce^{u(t)-mv(t)}\big)\big]\Big)dt + \sigma_1 dW_1(t) \\ dv(t) = \Big(-a_2 - \frac{\sigma_2^2}{2} + c_2\big[1 - \exp\big(-ce^{u(t)-mv(t)}\big)\big]\Big)dt + \sigma_2 dW_2(t) \end{cases} \tag{4}$$

with the initial $(u_0, v_0) = (\ln x_0, \ln y_0)$. Since the coefficients of (2.1) are locally Lipschitz continuous, there is a unique local solution to (4) on the interval $[0, \tau)$, where τ is the explosion time. Hence, by Itô's formula, $x(t) = e^{u(t)}, y(t) = e^{v(t)}$ is the unique positive local solution to (3) with the initial $(x_0, y_0) \in \mathbb{R}^2_+$. To show the solution is globally positive, we need to check that $\tau = \infty$ almost surely.

On the interval $[0, \tau)$, $x(t)$ and $y(t)$ are positive and hence,

$$dx(t) \leq x(t)[a_1 - b_1 x(t)]dt + \sigma_1 x(t)dW_1(t),$$

$$dy(t) \leq (c_2 - a_2)y(t)dt + \sigma_2 y(t)dW_2(t).$$

By the comparison theorem for stochastic differential equations (see, [8]) we have, for $0 \leq t < \tau$:

$$0 < x(t) \leq \bar{x}(t) \text{ and } 0 < y(t) \leq \bar{y}(t),\ a.s., \tag{5}$$

where $\bar{x}(t), \bar{y}(t)$ solve the following two equations, respectively

$$d\bar{x}(t) = \bar{x}(t)[a_1 - b_1\bar{x}(t)]dt + \sigma_1\bar{x}(t)dW_1(t), \tag{6}$$

$$d\bar{y}(t) = (c_2 - a_2)\bar{y}(t)dt + \sigma_2\bar{y}(t)dW_2(t). \tag{7}$$

The equation (6) is well known as a stochastic logistic equation in population dynamics, its solution is given by

$$\bar{x}(t) = \frac{x_0 \exp\big((a_1 - \frac{\sigma_1^2}{2})t + \sigma_1 W_1(t)\big)}{1 + x_0 \int\limits_0^t b_1 \exp\big((a_1 - \frac{\sigma_1^2}{2})s + \sigma_1 W_1(s)\big)ds}.$$

Similarly, the solution of (7) is given by

$$\bar{y}(t) = y_0 \exp\big((c_2 - a_2 - \frac{\sigma_2^2}{2})t + \sigma_2 W_2(t)\big).$$

Since both $\bar{x}(t)$ and $\bar{y}(t)$ are defined globally, we conclude that $\tau = \infty$.

The Theorem is proved. □

Remark 2.1. Since $\tau = \infty$, we have the following relations, for all $t \geq 0$

$$0 < x(t) \leq \bar{x}(t) \text{ and } 0 < y(t) \leq \bar{y}(t),\ a.s.$$

Theorem 2.2
For each $n > 0$, the nth-moment of the prey species is bounded uniformly in time, i.e. there exists a finite constant $K_n > 0$ such that

$$\sup_{t \geq 0} Ex(t)^n \leq K_n. \tag{8}$$

Proof
Applying the Itô's formula to $x(t)^n$ shows that

$$dx(t)^n = \Big(nx(t)^n[a_1 - b_1x(t)] - c_1nx(t)^{n-1}y(t)\big[1-\exp\big(-\frac{cx(t)}{y(t)^m}\big)\big] + \frac{1}{2}n(n-1)\sigma_1^2x(t)^n\Big)dt + \sigma_1nx(t)^ndW_1(t).$$

Consequently,

$$Ex(t)^n \le x_0^n + \int_0^t \big((na_1 + \frac{1}{2}n(n-1)\sigma_1^2)Ex(s)^n - nb_1Ex(t)^{n+1}\big)ds.$$

Put $g(t) = Ex(t)^n$ and by Lyapunov's inequality $(Ex(t)^n)^{\frac{1}{n}} \le (Ex(t)^{n+1})^{\frac{1}{n+1}}$ we have

$$g(t) \le x_0^n + \int_0^t \big((na_1 + \frac{1}{2}n(n-1)\sigma_1^2)g(s) - nb_1g(s)^{1+\frac{1}{n}}\big)ds,$$

which can be written as

$$g'(t) \le (na_1 + \frac{1}{2}n(n-1)\sigma_1^2)g(t) - nb_1g(t)^{1+\frac{1}{n}},\ t \ge 0. \tag{9}$$

From (9) and by the differential inequality (see, [20]), $g(t)$ is dominated by the solution of an ordinary Bernouilli differential equation of the form:

$$z'(t) = (na_1 + \frac{1}{2}n(n-1)\sigma_1^2)z(t) - nb_1z(t)^{1+\frac{1}{n}},\ z(0) = g(0).$$

Solving the Bernouilli equation we obtain for all $t \ge 0$

$$\begin{aligned}
[g(t)]^{\frac{1}{n}} &\le \frac{e^{(a_1+\frac{1}{2}(n-1)\sigma_1^2)t}}{(g_0)^{\frac{-1}{p}} + \int_0^t b_1e^{(a_1+\frac{1}{2}(n-1)\sigma_1^2)s}ds} \\
&= \frac{e^{(a_1+\frac{1}{2}(n-1)\sigma_1^2)t}}{(g_0)^{\frac{-1}{n}} + \frac{b_1}{(a_1+\frac{1}{2}(n-1)\sigma_1^2)}[e^{(a_1+\frac{1}{2}(n-1)\sigma_1^2)t} - 1]} \\
&= \frac{e^{(a_1+\frac{1}{2}(n-1)\sigma_1^2)t}}{\frac{1}{x_0} - \frac{b_1}{(a_1+\frac{1}{2}(n-1)\sigma_1^2)} + \frac{b_1}{(a_1+\frac{1}{2}(n-1)\sigma_1^2)}e^{(a_1+\frac{1}{2}(n-1)\sigma_1^2)t}}.
\end{aligned}$$

As a consequence,

$$[g(t)]^{\frac{1}{n}} \le \max\Big\{\frac{a_1+\frac{1}{2}(n-1)\sigma_1^2}{b_1}, x_0\Big\},\ \forall\, t \ge 0.$$

We therefore have

$$\sup_{t\ge0} Ex(t)^n \le \bigg(\max\Big\{\frac{a_1+\frac{1}{2}(n-1)\sigma_1^2}{b_1}, x_0\Big\}\bigg)^n := K_n.$$

The Theorem is proved. □

Remark 2.2. Let $\underline{y}(t)$ be the solution of the following equation

$$d\underline{y}(t) = -a_2\underline{y}(t)dt + \sigma_2\underline{y}(t)dW_2(t).$$

By the comparison theorem we have

$$y(t) \geq \underline{y}(t) = y_0 \exp\big((-a_2 - \frac{\sigma_2^2}{2})t + \sigma_2 W_2(t)\big)\ a.s.,$$

and then

$$Ey(t)^n \geq E\underline{y}(t)^n = y_0^n e^{n(-a_2+\frac{1}{2}(n-1)\sigma_2^2)t}\ \forall\ t \geq 0.$$

We observe that $\lim\limits_{t\to\infty} Ey(t)^n = \infty$ if $n > 1 + \frac{2a_2}{\sigma_2^2}$. Thus it is impossible to establish a similar estimate to (8) for the predator species. When $n = 1$, we have the following.

Theorem 2.3
The first moment of the predator species is bounded uniformly in time:

$$\sup_{t\geq 0} Ey(t) \leq \frac{c_2}{c_1} \max\left(\frac{(a_1+a_2)^2}{4a_2b_1}, x_0 + \frac{c_1}{c_2}y_0\right).$$

Proof
Consider the following stochastic process

$$z(t) = x(t) + \frac{c_1}{c_2}y(t),\ t \geq 0.$$

We have

$$dz(t) = x(t)[a_1 + a_2 - b_1 x(t)] - a_2 z(t) + \sigma_1 x(t)dW_1(t) + \frac{\sigma_2 c_1}{c_2}y(t)dW_2(t),$$

and

$$Ez(t) = z_0 + \int_0^t [(a_1+a_2)Ex(t) - b_1 Ex(t)^2 - a_2 Ez(s)]ds.$$

Obviously, $(a_1+a_2)Ex(t) - b_1 Ex(t)^2 \leq (a_1+a_2)Ex(t) - b_1[Ex(t)]^2 \leq \frac{(a_1+a_2)^2}{4b_1}$. Therefore,

$$Ez(t) \leq z_0 + \int_0^t \left[\frac{(a_1+a_2)^2}{4b_1} - a_2 Ez(s)\right]ds.$$

Once again, by the differential inequality

$$Ez(t) \leq \frac{(a_1+a_2)^2}{4a_2b_1} - \left(\frac{(a_1+a_2)^2}{4a_2b_1} - z_0\right)e^{-a_2 t} \leq \max\left(\frac{(a_1+a_2)^2}{4a_2b_1}, z_0\right)\ \forall\ t \geq 0.$$

Since $Ey(t) \leq \frac{c_2}{c_1}Ez(t)$, we obtain desired estimate.
The Theorem is proved. □

3. The growth rates, persistence and extinction

In this section, we give some estimates for growth rates of prey and predator. We also point out the sufficient conditions for the persistence and extinction of population in terms of its parameters. The obtained results confirm a well-known fact that if the noise is sufficiently large, the population will become extinct with probability one.

We first provide an upper bound for the growth rate of both prey and predator species.

Theorem 3.1
If $0 < m < 1$ then for any $p_1 > 0, p_2 \geq 0$ we have the following estimate for any initial value $(x_0, y_0) \in \mathbb{R}_+$

$$\limsup_{t\to\infty} \frac{\ln[x(t)^{p_1} y(t)^{p_2}]}{\ln t} \leq p_1 + p_2 \quad a.s. \tag{10}$$

Moreover, for all $m > 0$, we have the following estimate for growth rate of prey species

$$\limsup_{t\to\infty} \frac{\ln x(t)}{\ln t} \leq 1 \quad a.s. \tag{11}$$

Proof
We recall that

$$\begin{cases} du(t) = \Big(a_1 - \frac{\sigma_1^2}{2} - b_1 x(t) - c_1 \frac{y(t)}{x(t)}\big[1 - \exp\big(-\frac{cx(t)}{y(t)^m}\big)\big]\Big)dt + \sigma_1 dW_1(t) \\ dv(t) = \Big(-a_2 - \frac{\sigma_2^2}{2} + c_2\big[1 - \exp\big(-\frac{cx(t)}{y(t)^m}\big)\big]\Big)dt + \sigma_2 dW_2(t) \end{cases} \tag{12}$$

Applying Itô's formula to $e^{nt}u(t)$ and $e^{nt}v(t)$ we have, respectively

$$\begin{aligned} e^{nt}u(t) = u_0 &+ \int_0^t ne^{ns}u(s)ds \\ &+ \int_0^t e^{ns}\Big(a_1 - \frac{\sigma_1^2}{2} - b_1 x(s) - c_1\frac{y(s)}{x(s)}\big[1 - \exp\big(-\frac{cx(s)}{y(s)^m}\big)\big]\Big)ds + \int_0^t \sigma_1 e^{ns} dW_1(s), \end{aligned}$$

$$\begin{aligned} e^{nt}v(t) = v_0 &+ \int_0^t ne^{ns}v(s)ds \\ &+ \int_0^t e^{ns}\Big(-a_2 - \frac{\sigma_2^2}{2} + c_2\big[1 - \exp\big(-\frac{cx(s)}{y(s)^m}\big)\big]\Big)ds + \int_0^t \sigma_2 e^{ns} dW_2(s). \end{aligned}$$

Hence,

$$e^{nt}[p_1 u(t) + p_2 v(t)] = p_1 u_0 + p_2 v_0 + \int_0^t e^{ns} g(x(s), y(s))ds + \int_0^t \sigma_1 p_1 e^{ns} dW_1(s) + \int_0^t \sigma_2 p_2 e^{ns} dW_2(s), \tag{13}$$

where

$$\begin{aligned} g(x,y) = n[p_1 \ln x + p_2 \ln y] &+ p_1\Big(a_1 - \frac{\sigma_1^2}{2} - b_1 x(s) - c_1 \frac{y}{x}\big[1 - \exp\big(-\frac{cx}{y^m}\big)\big]\Big) \\ &+ p_2\Big(-a_2 - \frac{\sigma_2^2}{2} + c_2\big[1 - \exp\big(-\frac{cx}{y^m}\big)\big]\Big). \end{aligned}$$

Since $0 < m < 1$, it is easy to check that there exists a positive constant $K = K(n, p_1, p_2)$ such that

$$g(x,y) \leq K \ \forall \ x, y > 0.$$

Inserting the above estimate into (13) we obtain

$$e^{nt}[p_1u(t)+p_2v(t)] \le p_1u_0+p_2v_0+\int_0^t Ke^{ns}ds+\int_0^t \sigma_1p_1e^{ns}dW_1(s)+\int_0^t \sigma_2p_2e^{ns}dW_2(s).$$

Put

$$M_1(t)=\int_0^t \sigma_1e^{ns}dW_1(s),\quad M_2(t)=\int_0^t \sigma_2e^{ns}dW_2(s),$$

then $M_1(t), M_2(t)$ are continuous martingales that have finite quadratic variations:

$$\langle M_i, M_i\rangle_t=\int_0^t \sigma_i^2e^{2ns}ds,\ i=1,2.$$

Fix $\varepsilon\in(0,1)$ and $\theta>1$, by applying the exponential martingale inequality (see [17] or [14, Theorem 7.4]) we have for any $k\ge 1$

$$P\bigg(\sup_{0\le t\le k}\big(M_i(t)-\frac{\varepsilon}{2}e^{-nk}\langle M_i,M_i\rangle_t\big)\ge\frac{e^{nk}\ln k^\theta}{\varepsilon}\bigg)\le\frac{1}{k^\theta},\ i=1,2.$$

Since $\sum\limits_{k=1}^{\infty}\frac{1}{k^\theta}<\infty$, an application of Borel-Cantelli lemma yields there exist $\Omega_i\subset\Omega$ with $P(\Omega_i)=1,\ i=1,2$ such that for any $\omega\in\Omega_i$ there exists an integer $k_i(\omega)$, when $k\ge k_i(\omega)$ and $k-1\le t\le k$,

$$\begin{aligned} M_i(t) &\le \tfrac{\varepsilon}{2}e^{-nk}\langle M_i,M_i\rangle_t+\theta\tfrac{e^{nk}\ln k}{\varepsilon}\\ &=\tfrac{\varepsilon}{2}e^{-nk}\int_0^t\sigma_i^2e^{2ns}ds+\theta\tfrac{e^{nk}\ln k}{\varepsilon},\ i=1,2.\end{aligned}$$

Thus for $\omega\in\Omega_1\cap\Omega_2$, $k\ge k_0(\omega)=k_1(\omega)\vee k_2(\omega)$ and $k-1\le t\le k$

$$\begin{aligned} e^{nt}[p_1u(t)+p_2v(t)] \le p_1u_0+p_2v_0+\int_0^t Ke^{ns}ds+\frac{\varepsilon}{2}e^{-nk}\int_0^t\sigma_1^2p_1e^{2ns}ds\\ +\frac{\varepsilon}{2}e^{-nk}\int_0^t\sigma_2^2p_2e^{2ns}ds+\theta(p_1+p_2)\frac{e^{nk}\ln k}{\varepsilon},\end{aligned}$$

$$\begin{aligned} e^{nt}[p_1u(t)+p_2v(t)] \le p_1u_0+p_2v_0+\frac{K}{n}(e^{nt}-1)+\frac{\varepsilon}{4n}e^{-nk}\sigma_1^2p_1(e^{2nt}-1)\\ +\frac{\varepsilon}{4n}e^{-nk}\sigma_2^2p_2(e^{2nt}-1)+\theta(p_1+p_2)\frac{e^{nk}\ln k}{\varepsilon},\end{aligned}$$

and so

$$p_1u(t)+p_2v(t)\le p_1u_0+p_2v_0+\frac{K}{n}+\left[\frac{\varepsilon}{4n}\sigma_1^2p_1+\frac{\varepsilon}{4n}\sigma_2^2p_2\right]e^{-n(k-t)}+\theta(p_1+p_2)\frac{e^n\ln k}{\varepsilon}.$$

In the above inequality, let $t\to\infty$ we obtain

$$\limsup_{t\to\infty}\frac{p_1u(t)+p_2v(t)}{\ln t}\le\frac{\theta e^n}{\varepsilon}(p_1+p_2).$$

Taking the limits $\theta \to 1^+, \varepsilon \to 1^-$ and $n \to 0^+$ we get

$$\limsup_{t\to\infty} \frac{\ln[x(t)^{p_1}y(t)^{p_2}]}{\ln t} = \limsup_{t\to\infty} \frac{p_1u(t)+p_2v(t)}{\ln t} \le p_1+p_2,$$

which deduces (10) because $P(\Omega_1 \cap \Omega_2) = P(\Omega_1) + P(\Omega_2) - P(\Omega_1 \cup \Omega_2) = 1$.

The inequality (11) follows from (10) by choosing $p_2 = 0$ and using that fact that for any $m > 0$

$$g(x,y) = np_1 \ln x + p_1\Big(a_1 - \frac{\sigma_1^2}{2} - b_1x(s) - c_1\frac{y}{x}\big[1-\exp\big(-\frac{cx}{y^m}\big)\big]\Big) \le K \ \forall\ x,y>0.$$

The Theorem is proved. □

Remark 3.1. In the context of method of Lyapunov functionals, the condition $0 < m < 1$ is inevitable one. Indeed, when $m \ge 1$ we have for fixed x

$$\lim_{y\to\infty} c_1\frac{y}{x}\big[1-\exp\big(-\frac{cx}{y^m}\big)\big] \in \{0, cc_1\},$$

and so, $\lim\limits_{y\to\infty} g(x,y) = \infty$, *i.e.* $g(x,y)$ cannot be bounded by a finite constant K.

Lemma 3.1
(Strong law of large numbers) Let $W(t)$ is a standard Brownian motion. We have

$$\lim_{t\to\infty}\frac{W(t)}{t} = \lim_{t\to\infty}\frac{\min\limits_{0\le s\le t} W(s)}{t} = \lim_{t\to\infty}\frac{\max\limits_{0\le s\le t} W(s)}{t} = 0.$$

Proof
Refer [19]. □

We next study the prey species in the case without the predator species.

Theorem 3.2
I. If $a_1 - \frac{\sigma_1^2}{2} < 0$, then the prey dies with a probability one even if there is no predator.

$$\limsup_{t\to\infty}\frac{\ln x(t)}{t} \le a_1 - \frac{\sigma_1^2}{2} \ a.s.$$

II. In the case of $a_1 - \frac{\sigma_1^2}{2} > 0$, if the predator is absent, i.e., $y(t) = 0$ $a.s.$ then

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t x(s)ds = \frac{a_1 - \frac{\sigma_1^2}{2}}{b_1} \ a.s.,$$

in other words, the prey species is stable in time average.

Proof
I. From the proof of Theorem 2.1 we known that $x(t) = e^{u(t)}$, where

$$du(t) = \Big(a_1 - \frac{\sigma_1^2}{2} - b_1x(t) - c_1\frac{y(t)}{x(t)}\big[1-\exp\big(-\frac{cx(t)}{y(t)^m}\big)\big]\Big)dt + \sigma_1 dW_1(t). \tag{14}$$

Hence,

$$\ln x(t) = u(t) \le u_0 + \int_0^t (a_1 - \frac{\sigma_1^2}{2})ds + \sigma_1 W_1(t).$$

This, together with strong law of large numbers for Brownian motion $W_1(t)$, implies that

$$\limsup_{t\to\infty}\frac{\ln x(t)}{t}\le\lim_{t\to\infty}\frac{\ln x_0}{t}+(a_1-\frac{\sigma_1^2}{2})+\sigma_1\lim_{t\to\infty}\frac{W_1(t)}{t}=a_1-\frac{\sigma_1^2}{2},\ a.s.$$

II. When $y(t)=0\ a.s.$ we have $x(t)=\bar{x}(t)$, where

$$\bar{x}(t)=\frac{x_0\exp\big((a_1-\frac{\sigma_1^2}{2})t+\sigma_1W_1(t)\big)}{1+x_0\int\limits_0^t b_1\exp\big((a_1-\frac{\sigma_1^2}{2})s+\sigma_1W_1(s)\big)ds}.$$

It is clear that

$$\frac{1}{\bar{x}(t)}=e^{-\sigma_1W_1(t)}\left[\frac{1}{x_0}e^{-(a_1-\frac{\sigma_1^2}{2})t}+\int\limits_0^t b_1e^{-(a_1-\frac{\sigma_1^2}{2})(t-s)+\sigma_1W_1(s)}ds\right]$$

$$\ge e^{-\sigma_1W_1(t)}\left[\frac{1}{x_0}e^{-(a_1-\frac{\sigma_1^2}{2})t}+e^{\min\limits_{0\le s\le t}\sigma_1W_1(s)}\int\limits_0^t b_1e^{-(a_1-\frac{\sigma_1^2}{2})(t-s)}ds\right].$$

Since $\min\limits_{0\le s\le t}\sigma_1W_1(s)\le W_1(0)=0$, we can get

$$\frac{1}{\bar{x}(t)}\ge e^{\min\limits_{0\le s\le t}\sigma_1W_1(s)-\sigma_1W_1(t)}\left[\frac{1}{x_0}e^{-(a_1-\frac{\sigma_1^2}{2})t}+\int\limits_0^t b_1e^{-(a_1-\frac{\sigma_1^2}{2})(t-s)}ds\right],\tag{15}$$

$$\ln\bar{x}(t)\le\sigma_1W_1(t)-\min_{0\le s\le t}\sigma_1W_1(s)-\ln\left[\frac{b_1}{a_1-\frac{\sigma_1^2}{2}}+\left(\frac{1}{x_0}-\frac{b_1}{a_1-\frac{\sigma_1^2}{2}}\right)e^{-(a_1-\frac{\sigma_1^2}{2})t}\right].$$

Using the strong law of large numbers and the assumption $a_1-\frac{\sigma_1^2}{2}>0$, the above inequality gives us the following estimate

$$\limsup_{t\to\infty}\frac{\ln\bar{x}(t)}{t}\le 0\ a.s.\tag{16}$$

Similarly, with noting that $\max\limits_{0\le s\le t}\sigma_1W_1(s)\ge W_1(0)=0$, we also have

$$\ln\bar{x}(t)\ge\sigma_1W_1(t)-\max_{0\le s\le t}\sigma_1W_1(s)-\ln\left[\frac{b_1}{a_1-\frac{\sigma_1^2}{2}}+\left(\frac{1}{x_0}-\frac{b_1}{a_1-\frac{\sigma_1^2}{2}}\right)e^{-(a_1-\frac{\sigma_1^2}{2})t}\right]$$

and

$$\liminf_{t\to\infty}\frac{\ln\bar{x}(t)}{t}\ge 0\ a.s.\tag{17}$$

Combining (16) and (17) shows that

$$\lim_{t\to\infty}\frac{\ln\bar{x}(t)}{t}=0\ a.s.\tag{18}$$

An application of Itô's formula to $\bar{u}(t)=\ln\bar{x}(t)$ yields

$$d\bar{u}(t)=\big(a_1-\frac{\sigma_1^2}{2}-b_1\bar{x}(t)\big)dt+\sigma_1dW_1(t),\tag{19}$$

which leads us to

$$\frac{\bar{u}(t)}{t}=\frac{u_0}{t}+\big(a_1-\frac{\sigma_1^2}{2}\big)-b_1\frac{1}{t}\int\limits_0^t\bar{x}(s)ds+\sigma_1\frac{W_1(t)}{t}.\tag{20}$$

This, together with (18), implies that

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t \bar{x}(s)ds = \frac{a_1-\frac{\sigma_1^2}{2}}{b_1}\ a.s.$$

The Theorem is proved. □

Definition 3.1. The population $x(t)$ is said to be strongly persistent in mean if

$$\liminf_{t\to\infty}\frac{1}{t}\int_0^t x(s)ds > 0,\ a.s.$$

Let us now consider the case there is the appearance of the predator species. We have the following properties for the prey species.

Theorem 3.3
If $m = 1$ and $a_1 - \frac{\sigma_1^2}{2} - cc_1 > 0$, then the prey species is strongly persistent in mean. Moreover, we have

$$\frac{a_1-\frac{\sigma_1^2}{2}-cc_1}{b_1} \le \liminf_{t\to\infty}\frac{1}{t}\int_0^t x(s)ds \le \limsup_{t\to\infty}\frac{1}{t}\int_0^t x(s)ds \le \frac{a_1-\frac{\sigma_1^2}{2}}{b_1},\ a.s.$$

Proof
Since $1-\exp\big(-\frac{cx(t)}{y(t)}\big) \le \frac{cx(t)}{y(t)}$, we have

$$dx(t) \ge x(t)[a_1 - cc_1 - b_1x(t)]dt + \sigma_1 x(t)dW_1(t).$$

Consequently,

$$x(t) \ge \underline{x}(t) := \frac{x_0\exp\big(\int_0^t (a_1-\frac{\sigma_1^2}{2}-cc_1)ds + \sigma_1W_1(t)\big)}{1+x_0\int_0^t b_1\exp\big(\int_0^s (a_1-\frac{\sigma_1^2}{2}-cc_1)du+\sigma_1W_1(s)\big)ds}.$$

Since $a_1-\frac{\sigma_1^2}{2}-cc_1>0$, similarly to (18), we also have

$$\lim_{t\to\infty}\frac{\ln \underline{x}(t)}{t} = 0\ a.s. \tag{21}$$

Combining (18) and (21) we obtain

$$\lim_{t\to\infty}\frac{\ln x(t)}{t} = 0\ a.s.$$

Hence, from (14) and the strong law of large numbers we get

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t\bigg(b_1x(s)+c_1\frac{y(s)}{x(s)}\Big[1-\exp\big(-\frac{cx(s)}{y(s)}\big)\Big]\bigg)ds = a_1-\frac{\sigma_1^2}{2},\ a.s.$$

Consequently,

$$\limsup_{t\to\infty}\frac{1}{t}\int_0^t x(s)ds \le \frac{a_1-\frac{\sigma_1^2}{2}}{b_1},\ a.s.$$

On the other hand, it is easy to see that

$$c_1 \frac{y}{x}\Big[1 - \exp\big(-\frac{cx}{y}\big)\Big] < cc_1, \ \forall\ (x, y) \in \mathbb{R}^2_+,$$

and then

$$\liminf_{t\to\infty} \frac{1}{t}\int\limits_0^t x(s)ds \geq \frac{a_1 - \frac{\sigma_1^2}{2} - cc_1}{b_1} > 0, \ a.s.$$

The Theorem is proved. □

We end this paper with the properties of the predator species in the cases where our system contains or does not contain the prey species.

Theorem 3.4
I. If the prey is absent, i.e., $x(t) = 0$ $a.s.$ for all $t \geq 0$, then the predator dies with a probability one. Furthermore, the death's rate of predator is exponential

$$\lim_{t\to\infty} \frac{\ln y(t)}{t} = -a_2 - \frac{\sigma_2^2}{2}, \ a.s. \tag{22}$$

II. If $a_1 - \frac{\sigma_1^2}{2} > 0, i.e.$ the prey species is not extinction. We have

$$\limsup_{t\to\infty} \frac{\ln y(t)}{t} \leq \min\{0, c_2 - a_2 - \frac{\sigma_2^2}{2}\}.$$

Proof
I. When $x(t) = 0$ $a.s.$, the dynamic of predator species is expressed by

$$y(t) = \underline{y}(t) = y_0 \exp\big((-a_2 - \frac{\sigma_2^2}{2})t + \sigma_2 W_2(t)\big).$$

Therefore

$$\lim_{t\to\infty} \frac{\ln y(t)}{t} = \lim_{t\to\infty} \frac{\ln y_0}{t} - a_2 - \frac{\sigma_2^2}{2} + \sigma_2 \lim_{t\to\infty} \frac{W_2(t)}{t}$$

This, together with strong law of large numbers, gives us (22).

II. We recall that

$$y(t) \leq \bar{y}(t) = y_0 \exp\big((c_2 - a_2 - \frac{\sigma_2^2}{2})t + \sigma_2 W_2(t)\big).$$

Then, by using the same arguments as the proof of Part I, we have

$$\limsup_{t\to\infty} \frac{\ln y(t)}{t} \leq c_2 - a_2 - \frac{\sigma_2^2}{2}, \ a.s. \tag{23}$$

From the inequality $1 - \exp\big(-\frac{cx(t)}{y(t)^m}\big) \leq \frac{cx(t)}{y(t)^m}$, and by comparison theorem we have $y(t) \leq Y(t)$, where $Y(t)$ solves the following equation

$$dY(t) = \Big(-a_2 Y(t) + cc_2 x(t) Y(t)^{1-m}\Big)dt + \sigma_2 Y(t) dW_2(t), \ Y(0) = y_0. \tag{24}$$

The solution of (24) can be found explicitly (for instance, see [11, page 125]). Hence,

$$y(t) \leq Y(t) = e^{-(a_2 + \frac{\sigma_2^2}{2})t + \sigma_2 W_2(t)} \Bigg(y_0^m + mcc_2 \int\limits_0^t x(s) e^{-m(-(a_2 + \frac{\sigma_2^2}{2})s + \sigma_2 W_2(s))} ds\Bigg)^{\frac{1}{m}}$$

or equivalently,

$$y(t)^m \le e^{-m(a_2+\frac{\sigma_2^2}{2})t+m\sigma_2 W_2(t)}\left(y_0^m + mcc_2\int_0^t x(s)e^{m(a_2+\frac{\sigma_2^2}{2})s-m\sigma_2 W_2(s)}ds\right),$$

which leads us to the following estimate

$$y(t)^m \le e^{m\sigma_2 W_2(t)-\min\limits_{0\le s\le t} m\sigma_2 W_2(s)}\left(y_0^m e^{-m(a_2+\frac{\sigma_2^2}{2})t} + mcc_2\int_0^t x(s)e^{-m(a_2+\frac{\sigma_2^2}{2})(t-s)}ds\right). \tag{25}$$

We recall that $x(t)\le \bar{x}(t)$. Since $a_1-\frac{\sigma_1^2}{2}>0$, from (15) we have

$$x(t)\le \bar{x}(t)\le \bar{K}e^{\sigma_1 W_1(t)-\min\limits_{0\le s\le t}\sigma_1 W_1(s)},\ \forall\, t\ge 0.$$

where

$$\bar{K}=\frac{1}{\inf\limits_{t\ge 0}\left[\frac{1}{x_0}e^{-(a_1-\frac{\sigma_1^2}{2})t}+\int_0^t b_1 e^{-(a_1-\frac{\sigma_1^2}{2})(t-s)}ds\right]}=\frac{1}{\min\{\frac{1}{x_0},\frac{b_1}{a_1-\frac{\sigma_1^2}{2}}\}}.$$

Hence, it is easy to see that

$$\max_{0\le s\le t} x(s)\le \bar{K}e^{\max\limits_{0\le s\le t}\sigma_1 W_1(s)-\min\limits_{0\le s\le t}\sigma_1 W_1(s)}. \tag{26}$$

Noting that $\max\limits_{0\le s\le t}\sigma_1 W_1(s)-\min\limits_{0\le s\le t}\sigma_1 W_1(s)\ge 0$. Inserting (26) into (25) we find that

$$y(t)^m \le \bar{\bar{K}}e^{\max\limits_{0\le s\le t}\sigma_1 W_1(s)-\min\limits_{0\le s\le t}\sigma_1 W_1(s)+m\sigma_2 W_2(t)-\min\limits_{0\le s\le t} m\sigma_2 W_2(s)} \tag{27}$$

where

$$\bar{\bar{K}}=\sup_{t\ge 0}\left(y_0^m e^{-m(a_2+\frac{\sigma_2^2}{2})t}+mcc_2\bar{K}\int_0^t e^{-m(a_2+\frac{\sigma_2^2}{2})(t-s)}ds\right)=\max\left\{\frac{cc_2\bar{K}}{a_2+\frac{\sigma_2^2}{2}},y_0^m\right\}.$$

By the strong law of large numbers

$$\limsup_{t\to\infty}\frac{\ln y(t)}{t}\le 0 \tag{28}$$

We finish the proof the Theorem by combining (23) and (28).

□

Remark 3.2. Theorem 3.4 gives us some interesting ecological interpretations. Firstly, if the system has no the prey species, then the predator population will go to extinction, which is consistent with our expectation. Secondly, although the prey population will survive, the predators die out because the diffusion coefficient σ_2^2 is too large ($c_2-a_2-\frac{\sigma_2^2}{2}<0$). This means that a relatively large stochastic perturbation can cause the extinction of the population.

4. Conclusion

In this paper, we discuss a stochastic predator-prey system with Watt-type functional response. We show that the model admits a unique global positive solution, and investigate the uniformly finite moments. Moreover, we use the Lyapunov functionals, the strong law of large numbers for Brownian motion and the theory of Itô's stochastic differential equations to study the long-term behaviors of solutions. Our obtained results partly enrich the knowledge of theory of stochastic predator-prey systems.

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