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Explicit form of global solution to stochastic logistic differential equation and related topics

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Abstract This paper presents the explicit form of positive global solution to stochastically perturbed nonautonomous logistic equation

$$dN(t) = N(t)\left[(a(t) - b(t)N(t))dt + \alpha(t)dw(t) + \int_{\mathbb{R}} \gamma(t,z)\tilde{\nu}(dt,dz)\right],\ N(0) = N_0,$$

where $w(t)$ is the standard one-dimensional Wiener process, $\tilde{\nu}(t,A) = \nu(t,A) - t\Pi(A)$, $\nu(t,A)$ is the Poisson measure, which is independent on $w(t)$, $E[\nu(t,A)] = t\Pi(A)$, $\Pi(A)$ is a finite measure on the Borel sets in $\mathbb{R}$. If coefficients $a(t), b(t), \alpha(t)$ and $\gamma(t,z)$ are continuous on t, T-periodic on t functions, $a(t) > 0, b(t) > 0$ and

$$\int_0^T \left[a(s) - \alpha^2(s) - \int_{\mathbb{R}} \frac{\gamma^2(s,z)}{1+\gamma(s,z)}\Pi(dz)\right] ds > 0,$$

then there exists unique, positive T-periodic solution to equation for $E[1/N(t)]$.

Keywords Nonautonomous Stochastic Logistic Differential Equation, Explicit Form, Global Solution, Periodical Solution

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1. Introduction

The construction of the logistic model and its properties are presented in M. Iannelli and A. Pugliese [1]. A deterministic nonautonomous logistic equation has a form

$$dN(t) = N(t)\left(a(t) - b(t)N(t)\right)dt,\ N(0) = N_0 > 0,$$

and models the number N of a single species whose members compete among themselves for a limit amount of food and living space. Here $a(t)$ is the rate of growth and $a(t)/b(t)$ is the carrying capacity at time t. In the paper by D. Jiang and N. Shi [2] it is considered the nonautonomous stochastic logistic differential equation

$$dN(t) = N(t)\left[(a(t) - b(t)N(t))dt + \alpha(t)dw(t)\right],\ N(0) = N_0, \tag{1}$$

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where $w(t)$ is the standard one-dimensional Wiener process, $N_0 > 0$ is a random variable independent on $w(t)$, $a(t), b(t)$ and $\alpha(t)$ are bounded, continuous functions. The authors prove, that if $a(t) > 0, b(t) > 0$, then there exists a unique continuous, positive global solution $N(t)$ to equation (1). It is obtained the explicit representation of this global solution. If $b(t) < 0$, then the equation (1) has the only local solution. In the case, when $a(t), b(t)$ and $\alpha(t)$ are continuous T-periodic functions, $a(t) > 0, b(t) > 0$ and $\int_0^T [a(s) - \alpha^2(s)]ds > 0$ the authors prove that equation for function $E[1/N(t)]$ has a unique positive T-periodic solution $E[1/N_p(t)]$ and $\lim_{t\to+\infty}(E[1/N(t)] - E[1/N_p(t)]) = 0$ where $N(t)$ is the solution to the equation (1) for any initial value $N(0) = N_0 > 0$.

In this paper, we consider the stochastic nonautonomous logistic differential equation of the form

$$dN(t) = N(t)\left[(a(t) - b(t)N(t))dt + \alpha(t)dw(t) + \int_{\mathbb{R}} \gamma(t,z)\tilde{\nu}(dt,dz)\right],\ N(0) = N_0, \tag{2}$$

where $w(t)$ is the standard one-dimensional Wiener process, $\tilde{\nu}(t,A) = \nu(t,A) - t\Pi(A)$, $\nu(t,A)$ is the Poisson measure, which is independent on $w(t)$ and $N_0 > 0$, $E[\nu(t,A)] = t\Pi(A)$, $\Pi(A)$ is a finite measure on the Borel sets A in $\mathbb{R}$. The coefficients of equation (2) do not satisfy the linear growth condition, but they are local Lipschitz continuous, so (cf. I.I. Gikhman and A.V. Skorokhod [3]) the local solution to equation (2) exists. We will obtain the explicit form of global solution to equation (2). In the case when coefficients $a(t), b(t), \alpha(t)$ and $\gamma(t,z)$ are continuous on t, T-periodic on t functions, $a(t) > 0, b(t) > 0$ and

$$\int_0^T \left[a(s) - \alpha^2(s) - \int_{\mathbb{R}} \frac{\gamma^2(s,z)}{1+\gamma(s,z)}\Pi(dz)\right] ds > 0,$$

we will show, that the equation for function $E[1/N(t)]$ has a unique positive T-periodic solution.

Mainly we use the notations and approaches proposed in D. Jiang and N. Shi [2]. If coefficient $\gamma(t,z) \equiv 0$, then our results are consistent with the corresponding results in D. Jiang and N. Shi [2].

The rest of this paper is organized as follows. In Section 2, we obtain the explicit representation of the unique global positive solution to equation (2). In Section 3, we derive the explicit form of unique positive T-periodic solution to the equation for $E[1/N(t)]$ in the case of periodical coefficients, and in Section 4, we consider the stochastic nonautonomous logistic differential equation of the form

$$dN(t) = N(t)\left[(a(t) - b(t)N^\theta(t))dt + \alpha(t)dw(t) + \int_{\mathbb{R}} \gamma(t,z)\tilde{\nu}(dt,dz)\right],$$

where $\theta > 0$ is an odd integer.

2. Explicit form of global solution

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $w(t), t \geq 0$ is a standard one-dimensional Wiener process on $(\Omega, \mathcal{F}, P)$, $N_0 > 0$ is a random variable on $(\Omega, \mathcal{F}, P)$, which is independent on $w(t)$, and $\tilde{\nu}(t,A) = \nu(t,A) - t\Pi(A)$ is a centered Poisson measure defined on $(\Omega, \mathcal{F}, P)$ independent on $w(t)$ and on N_0. Here $E[\nu(t,A)] = t\Pi(A)$, $\Pi(\cdot)$ is a finite measure on the Borel sets in $\mathbb{R}$. On the probability space $(\Omega, \mathcal{F}, P)$ we consider an increasing, right continuous family of complete sub-σ-algebras $\{\mathcal{F}_t\}_{t\geq 0}$, where $\mathcal{F}_t = \sigma\{N_0, w(s), \nu(s,A), s \leq t\}$. The main result of this section is following

Theorem 1
Let $a(t) > 0, b(t) > 0$ and $\alpha(t)$ be a bounded continuous functions defined on $[0, +\infty)$. Assume that $\Pi(\mathbb{R}) < \infty$ and $\gamma(t,z)$ is continuous on t function and $|\ln(1+\gamma(t,z))| \leq K$ for some constant $K > 0$. Then there exists a unique positive solution $N(t)$ to equation (2) for any initial value $N(0) = N_0 > 0$, which is global and has a representation

$$N(t) = \frac{\exp\left\{\int_0^t [a(s) - \beta(s)]ds + \int_0^t \alpha(s)dw(s) + \int_0^t \int_{\mathbb{R}} \ln(1+\gamma(s,z))\tilde{\nu}(ds,dz)\right\}}{1/N_0 + \int_0^t b(s)\exp\left\{\int_0^s [a(\tau) - \beta(\tau)]d\tau + \int_0^s \alpha(\tau)dw(\tau) + \int_0^s \int_{\mathbb{R}} \ln(1+\gamma(\tau,z))\tilde{\nu}(d\tau,dz)\right\} ds},$$

where

$$\beta(t) = \frac{\alpha^2(t)}{2} + \int_{\mathbb{R}} [\gamma(t,z) - \ln(1+\gamma(t,z))]\Pi(dz), t \geq 0.$$

Proof. The coefficients of the equation (2) are local Lipschitz continuous. Therefore for any initial random value $N_0 > 0$, which is independent on $w(t)$ and $\nu(t, A)$, there exists a unique local solution $N(t)$ on $[0, \tau_e)$, where $\sup_{t<\tau_e} |N(t)| = +\infty$ (cf. Theorem 6, p.246,[3]). We will derive the explicit form of this solution and we will see, that this solution is global.

Let

$$x(t) = e^{-\eta(t)} \left(1/N_0 + \int_0^t b(s) e^{\eta(s)} ds \right), \tag{3}$$

where

$$\eta(t) = \int_0^t [a(s) - \beta(s)]ds + \int_0^t \alpha(s) dw(s) + \int_0^t \int_{\mathbb{R}} \ln(1 + \gamma(s,z)) \tilde{\nu}(ds, dz).$$

Using the Ito formula, we derive the stochastic differential equation for the process $x(t)$:

$$dx(t) = \left[b(t) - x(t) \left(a(t) - \alpha^2(t) - \int_{\mathbb{R}} \frac{\gamma^2(t,z)}{1+\gamma(t,z)} \Pi(dz) \right) \right] dt$$

$$- x(t) \left[\alpha(t) dw(t) + \int_{\mathbb{R}} \frac{\gamma(t,z)}{1+\gamma(t,z)} \tilde{\nu}(dt, dz) \right], \ x(0) = 1/N_0. \tag{4}$$

Let $N(t) = 1/x(t)$, then $N(t) > 0$ and $N(t)$ is stochastically continuous process with trajectories, which are right continuous and have a left limit at every t almost surely. Under conditions of the theorem, $N(t)$ will not explode in a finite time.

By Ito formula, we obtain from (4):

$$dN(t) = N(t) \left[(a(t) - N(t)b(t))\, dt + \alpha(t) dw(t) + \int_{\mathbb{R}} \gamma(t,z) \tilde{\nu}(dt, dz) \right], \ N(0) = N_0.$$

Thus $N(t)$ is a unique positive solution to equation (2) and $N(t)$ is global on $t \in [0, +\infty)$, i.e. $\tau_e = +\infty$ almost surely.

Remark 1
If $b(t) < 0$, then equation (2) has only the local solution

$$N(t) = \frac{\exp\left\{ \int_0^t [a(s) - \beta(s)]ds + \int_0^t \alpha(s) dw(s) + \int_0^t \int_{\mathbb{R}} \ln(1+\gamma(s,z)) \tilde{\nu}(ds,dz) \right\}}{1/N_0 - \int_0^t |b(s)| \exp\left\{ \int_0^s [a(\tau) - \beta(\tau)]d\tau + \int_0^s \alpha(\tau) dw(\tau) + \int_0^s \int_{\mathbb{R}} \ln(1+\gamma(\tau,z)) \tilde{\nu}(d\tau, dz) \right\} ds},$$

$0 \leq t < \tau_e$, where explosion time defined by

$$\tau_e = \inf \left\{ t : \int_0^t |b(s)| \exp\left\{ \int_0^s [a(\tau) - \beta(\tau)] d\tau \right.\right.$$
$$\left.\left. + \int_0^s \alpha(\tau) dw(\tau) + \int_0^s \int_{\mathbb{R}} \ln(1+\gamma(\tau,z)) \tilde{\nu}(d\tau,dz) \right\} ds = 1/N_0 \right\}.$$

3. Periodic solution to equation for $E[1/N(t)]$

From the Lemma (p.465,[3]) we have the following result.

Lemma 1
Let $\Pi(\mathbb{R}) < \infty$, functions $\alpha(t)$, $\gamma(t,z)$ be continuous on t and bounded, then

$$E\left[\exp\left\{\int_{t_0}^{t}\alpha(s)dw(s)+\int_{t_0}^{t}\int_{\mathbb{R}}\gamma(s,z)\tilde{\nu}(ds,dz)\right\}\right]=\exp\left\{\int_{t_0}^{t}\sigma(s)ds\right\},\ 0\le t_0\le t,$$

where

$$\sigma(t)=\frac{\alpha^2(t)}{2}+\int_{\mathbb{R}}\left[e^{\gamma(t,z)}-1-\gamma(t,z)\right]\Pi(dz).$$

Let us consider equation (2) with T-periodic on t coefficients.

Theorem 2
Let coefficients $a(t), b(t), \alpha(t)$ and $\gamma(t,z)$ be continuous on t and T-periodic on t functions, $a(t) > 0$, $b(t) > 0$ and $\int_0^T [a(s) - \delta^2(s)]ds > 0$, where

$$\delta^2(t)=\alpha^2(t)+\int_{\mathbb{R}}\frac{\gamma^2(t,z)}{1+\gamma(t,z)}\Pi(dz),$$

$\Pi(\mathbb{R}) < \infty$, $|\ln(1+\gamma(t,z))| \le K$ for some constant $K > 0$. Then the equation

$$\frac{d}{dt}E[1/N(t)]=[\delta^2(t)-a(t)]E[1/N(t)]+b(t), \tag{5}$$

where $N(t)$ is a solution to equation (2), has a unique positive T-periodic solution

$$E[1/N_p(t)]=\frac{\int_t^{t+T}\exp\left\{\int_t^s[a(\tau)-\delta^2(\tau)]d\tau\right\}b(s)ds}{\exp\left\{\int_0^T[a(\tau)-\delta^2(\tau)]d\tau\right\}-1}, \tag{6}$$

and for solution $N(t)$ to equation (2) with any initial value $N(0) = N_0 > 0$, $E[1/N_0] < \infty$ we have

$$\lim_{t\to\infty}\left(E[1/N(t)]-E[1/N_p(t)]\right)=0. \tag{7}$$

Proof. From (3) we have

$$x(t)=\exp\left\{\int_0^t[\beta(s)-a(s)]ds\right\}\exp\left\{-\int_0^t\alpha(s)dw(s)-\int_0^t\int_{\mathbb{R}}\ln(1+\gamma(s,z))\tilde{\nu}(ds,dz)\right\}\frac{1}{N_0}$$
$$+\int_0^t b(s)\exp\left\{\int_s^t[\beta(\tau)-a(\tau)]d\tau-\int_s^t\alpha(\tau)dw(\tau)-\int_s^t\int_{\mathbb{R}}\ln(1+\gamma(\tau,z))\tilde{\nu}(d\tau,dz)\right\}ds.$$

By Lemma 1 we obtain

$$E\left[\exp\left\{-\int_{t_0}^t\alpha(s)dw(s)-\int_{t_0}^t\int_{\mathbb{R}}\ln(1+\gamma(s,z))\tilde{\nu}(ds,dz)\right\}\right]$$
$$=\exp\left\{\int_{t_0}^t\frac{\alpha^2(s)}{2}ds+\int_{t_0}^t\int_{\mathbb{R}}\left[\frac{1}{1+\gamma(s,z)}-1+\ln(1+\gamma(s,z))\right]\Pi(dz)ds\right\},\ 0\le t_0<t.$$

Therefore, using the independence of $w(t), \tilde{\nu}(t,A)$ and N_0, we derive

$$E[x(t)]=E[1/N(t)]=\exp\left\{\int_0^t\left[\alpha^2(s)+\int_{\mathbb{R}}\frac{\gamma^2(s,z)}{1+\gamma(s,z)}\Pi(dz)-a(s)\right]ds\right\}E[1/N_0]$$
$$+\int_0^t b(s)\exp\left\{\int_s^t\left[\alpha^2(\tau)+\int_{\mathbb{R}}\frac{\gamma^2(\tau,z)}{1+\gamma(\tau,z)}\Pi(dz)-a(\tau)\right]d\tau\right\}ds.$$

It is easy to see that function $E[x(t)] = E[1/N(t)]$ satisfies the equation (5). By the condition

$$\int_0^T [a(s) - \delta^2(s)]\, ds > 0 \tag{8}$$

the equation (5) has a unique positive T-periodic solution (6)(cf. p.280, [4]).

Let us denote

$$p(t) = a(t) - \delta^2(t),\ \bar{p} = \exp\left\{ \int_0^T p(t)\, dt \right\}.$$

If $N(t)$ is the solution to equation (2) for any initial value $N(0) = N_0 > 0$, then

$$E[1/N(t)] - E[1/N_p(t)] = \exp\left\{ -\int_0^t p(s)\, ds \right\} E[1/N_0]$$

$$+ \int_0^t b(s) \exp\left\{ -\int_s^t p(\tau)\, d\tau \right\} ds - \frac{1}{\bar{p}-1} \int_t^{t+T} b(s) \exp\left\{ -\int_s^t p(\tau)\, d\tau \right\} ds.$$

From (8) we have

$$\lim_{t\to+\infty} E[1/N_0] \exp\left\{ -\int_0^t p(s)\, ds \right\} = 0. \tag{9}$$

Besides

$$\int_0^t b(s) \exp\left\{ -\int_s^t p(\tau)\, d\tau \right\} ds - \frac{1}{\bar{p}-1} \int_t^{t+T} b(s) \exp\left\{ -\int_s^t p(\tau)\, d\tau \right\} ds$$

$$= \exp\left\{ -\int_0^t p(\tau)\, d\tau \right\} \left[\int_0^t b(s) \exp\left\{ \int_0^s p(\tau)\, d\tau \right\} ds - \frac{1}{\bar{p}-1} \int_t^{t+T} b(s) \exp\left\{ \int_0^s p(\tau)\, d\tau \right\} ds \right]$$

$$= \exp\left\{ -\int_0^t p(\tau)\, d\tau \right\} F(t). \tag{10}$$

For $F(t)$ we have

$$F'(t) = b(t) \exp\left\{ \int_0^t p(\tau)\, d\tau \right\} - \frac{1}{\bar{p}-1} \left[b(t+T) \exp\left\{ \int_0^{t+T} p(\tau)\, d\tau \right\} - b(t) \exp\left\{ \int_0^t p(\tau)\, d\tau \right\} \right]$$

$$= b(t) \exp\left\{ \int_0^t p(\tau)\, d\tau \right\} \left[1 - \frac{\exp\left\{ \int_t^{t+T} p(\tau)\, d\tau \right\} - 1}{\bar{p}-1} \right] = 0,$$

because from T-periodicity of $p(t)$ we can derive, that

$$\frac{\exp\left\{ \int_t^{t+T} p(\tau)\, d\tau \right\} - 1}{\exp\left\{ \int_0^T p(\tau)\, d\tau \right\} - 1} = 1.$$

Hence $F(t)$ doesn't depend on t and from (9) and (10) we deduce (7).

4. Some related logistic equations

Let us consider stochastic differential equation of the form

$$dN(t) = N(t)\left[\left(a(t) - b(t)N^{\theta}(t)\right)\,dt + \alpha(t)dw(t) + \int_{\mathbb{R}} \gamma(t,z)\tilde{\nu}(dt,dz)\right],\ t \geq 0 \tag{11}$$

where $\theta > 0$ is an odd integer; $N(0) = N_0 > 0$ is the random variable independent on standard one-dimensional Wiener process and centered Poisson measure $\tilde{\nu}(t,A) = \nu(t,A) - t\Pi(A)$, $\Pi(\mathbb{R}) < \infty$; $w(t)$ and $\tilde{\nu}(t,A)$ are independent. Let non-random functions $a(t), b(t), \alpha(t), \gamma(t,z)$ be bounded, continuous on $t \in [0,\infty)$, $a(t) > 0, b(t) > 0$.

Let $N(t)$ be a solution to equation (11), then by Ito formula applied to stochastic process $N^{\theta}(t)$ we derive

$$dN^{\theta}(t) = N^{\theta}(t)\left[\left(\theta a(t) + \frac{\theta(\theta-1)}{2}\alpha^2(t) - \theta b(t)N^{\theta}(t) + \int_{\mathbb{R}}\left[(1-\gamma(t,z))^{\theta} - 1 - \theta\gamma(t,z)\right]\,\Pi(dz)\right)dt\right.$$
$$\left. + \theta\alpha(t)dw(t) + \int_{\mathbb{R}}\left[(1+\gamma(t,z))^{\theta} - 1\right]\tilde{\nu}(dt,dz)\right],\ N^{\theta}(0) = 1/N_0^{\theta}.$$

For stochastic process $M(t) = N^{\theta}(t)$ we have an (2)-type equation

$$dM(t) = M(t)\left[(\hat{a}(t) - \hat{b}(t)M(t))dt + \hat{\alpha}(t)dw(t) + \int_{\mathbb{R}}\hat{\gamma}(t,z)\tilde{\nu}(dt,dz)\right],\ M(0) = 1/N_0^{\theta},$$

where

$$\hat{a}(t) = \theta a(t) + \frac{\theta(\theta-1)}{2}\alpha^2(t) + \int_{\mathbb{R}}\left[(1+\gamma(t,z))^{\theta} - 1 - \theta\gamma(t,z)\right]\,\Pi(dz),$$
$$\hat{b}(t) = \theta b(t), \quad \hat{\alpha}(t) = \theta\alpha(t), \quad \hat{\gamma}(t,z) = (1+\gamma(t,z))^{\theta} - 1.$$

It is easy to see that under conditions of Theorem 1 on coefficients $a(t), b(t), \alpha(t), \gamma(t,z)$ the same conditions are fulfilled for coefficients $\hat{a}(t), \hat{b}(t), \hat{\alpha}(t), \hat{\gamma}(t,z)$. Therefore we obtain the following results.

Theorem 3
Let conditions of Theorem 1 are fulfilled. Then there exists a unique positive solution $N(t)$ to equation (11) with initial condition $N(0) = N_0 > 0$, which is global and has a representation

$$N(t) = \left[\frac{\exp\left\{\theta\left(\int_0^t[a(s)-\beta(s)]ds + \int_0^t\alpha(s)dw(s) + \int_0^t\int_{\mathbb{R}}\ln(1+\gamma(s,z))\tilde{\nu}(ds,dz)\right)\right\}}{1/N_0^{\theta} + \theta\int_0^t b(s)\exp\left\{\theta\left(\int_0^s[a(\tau)-\beta(\tau)]d\tau + \int_0^s\alpha(\tau)dw(\tau) + \int_0^s\int_{\mathbb{R}}\ln(1+\gamma(\tau,z))\tilde{\nu}(d\tau,dz)\right)\right\}ds}\right]^{1/\theta}$$

where

$$\beta(t) = \frac{\alpha^2(t)}{2} + \int_{\mathbb{R}}[\gamma(t,x) - \ln(1+\gamma(t,z))]\,\Pi(dz).$$

Theorem 4
Let $a(t), b(t), \alpha(t)$ and $\gamma(t,z)$ be continuous on t and T-periodic on t functions, $a(t) > 0, b(t) > 0$ and $\int_0^T[\theta a(t) - \sigma^2(t)]dt > 0$. where

$$\sigma^2(t) = \frac{\theta(\theta+1)}{2}\alpha^2(t) + \int_{\mathbb{R}}\left[(1+\gamma(t,z))^{-\theta} - 1 + \theta\gamma(t,z)\right]\,\Pi(dz),\ \Pi(\mathbb{R}) < \infty,\ |\ln(1+\gamma(t,z))| \leq K$$

for some $K > 0$. Then the equation

$$\frac{d}{dt}E[1/N^{\theta}(t)] = [\sigma^2(t) - \theta a(t)]E[1/N^{\theta}(t)] + \theta b(t)$$

has a unique positive T-periodic solution

$$E[1/N_p^\theta] = \frac{\theta \int_t^{t+T} \exp\left\{\int_t^s [\theta a(\tau) - \sigma^2(\tau)]\, d\tau\right\} b(s)\, ds}{\exp\left\{\int_0^T [\theta a(\tau) - \sigma^2(\tau)]\, d\tau\right\} - 1},$$

and for solution $N(t)$ to equation (11) with any initial value $N(0) = N_0 > 0$, $E[1/N_0] < \infty$, we have

$$\lim_{t\to+\infty} \left(E[1/N^\theta(t)] - E[1/N_p^\theta(t)]\right) = 0.$$

REFERENCES

1. M.Iannelli, and A. Pugliese, *An Introduction to Mathematical Population Dynamics*, Springer, 2014.
2. D. Jiang, and N. Shi, *A note on nonautonomous logistic equation with random perturbation*, J. Math. Anal. Appl., 303, pp. 164 – 172, 2005.
3. I.I Gikhman, and A.V. Skorokhod, *Stochastic Differential Equations and its Applications*, Kiev, Naukova Dumka, 1982.(In Russian)
4. G. Sansone, *Ordinary Differential Equations*, v.1, Moscow, Inostrannaya Literatura, 1953.(In Russian)