Nonlinear Approximation in the Large Deviations Principle

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Abstract   The Markov random processes and their approximations are considered. The main object of study is the exponential generator of random processes with independent increments, which are the solution of the problems of large deviations. These processes satisfy the conditions that make it possible to consider the Poisson and Lévy approximation. Generators of random processes are normalized by nonlinear parameters. Found explicit form of normalization parameter estimation.

Keywords   Markov random processes and their approximations

1. Introduction

The problem of large deviations was originated as a method of solving statistical problems associated with the estimation of probability of rare events. The first work in this direction was the article by Cramér [1], but ultimately the method was developed in the article by Chernoff [2]. Publications [3, 4] are also related to this problem. The purpose of solving the problem of large deviations is finding of action functional

$$I(x) = I(x_0) + \int_0^\infty L(x(s), x'(s))ds,$$

where $x(s)$ is a Markov process, function $L(x, u)$ defined by the exponential generator

$$L(x, u) = \sup_{p \in R}\{pu - H(v, p)\},$$

where $p := \varphi'(u), H(v, \varphi'(u)) := H_\Gamma \varphi(u)$. In the writing [5] in the scheme of Poisson approximation processes with independent increments without diffusion component were considered. Between jumps there were Markov processes with linear normalizing factor.

These processes are defined by the generator

$$\Gamma_\varepsilon^\delta \varphi(u) = \varepsilon^{-1} \int_R \big(\varphi(u + \varepsilon v) - \varphi(u)\big)\Gamma^\delta(dv).$$

in the scale of time t/ε.

In the articles [7]–[9] the generator of Markov process and their evolutions were considered in the scale of time $t/g_1(\varepsilon)$ and $t/g_2(\varepsilon)$ in the Poisson and Lévi approximation.

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The purpose of this work is to find functions that normalize the generator of random process with independent increments in the scale of time $t/g_2(\varepsilon)$ in the Large Deviations Principle.

These processes are defined by the generator

$$\Gamma^\varepsilon \varphi(u) = (g_2(\varepsilon))^{-1} \int_R \left(\varphi(u + g_1(\varepsilon)v) - \varphi(u)\right) \Gamma^\varepsilon(dv).$$

Besides this, in the Poisson approximation the next condition take places

$$(g_2(\varepsilon))^{-1} g_1(\varepsilon) f_1(\varepsilon) \to 1, \quad \varepsilon \to 0,$$

and in the Lévy approximation

$$(g_3(\varepsilon))^{-1} g_1(\varepsilon) f_2(\varepsilon) \to 1, \quad \varepsilon \to 0,$$

where $g_1(\varepsilon)$, $g_2(\varepsilon)$, $g_3(\varepsilon)$, $f_1(\varepsilon)$, $f_2(\varepsilon)$ are normalization function, such that $g_2(\varepsilon) = o(g_1(\varepsilon))$, $g_3(\varepsilon) = o(g_2(\varepsilon))$, $f_2(\varepsilon) = o(f_1(\varepsilon))$, $g_1(\varepsilon), g_2(\varepsilon), g_3(\varepsilon), f_1(\varepsilon), f_2(\varepsilon) \to 0$, $\varepsilon \to 0$.

2. Poisson approximation

We consider Markov processes with locally independent increments $\eta^\varepsilon(\cdot)$ with trajectories in $D^R[0,\infty)$:

$$\eta^\varepsilon = g_1(\varepsilon)\, \eta\left(\frac{t}{g_2(\varepsilon)}\right), \quad t \geq 0.$$

In this normalization $g_1(\varepsilon), g_2(\varepsilon) \to 0$ as $\varepsilon \to 0$.

These processes are defined by the generator

$$\Gamma^\varepsilon \varphi(u) = (g_2(\varepsilon))^{-1} \int_R \left(\varphi(u + g_1(\varepsilon)v) - \varphi(u)\right) \Gamma^\varepsilon(dv),$$

where $\varphi(u)$ is a twice differentiable function on R which tends to 0 at infinity and with sup-norm, $\varphi(u) \in C_0^2(R)$. The kernel of intensity Γ^ε belongs to the class $C^3(R)$. This kernel satisfies the condition $\Gamma^\varepsilon(0) = 0$.

We consider the problem of large deviations in the scheme of Poisson approximation in the case where the following conditions are satisfied:

(P1) Approximation of the mean values:

$$b_\varepsilon = \int_R v\, \Gamma^\varepsilon(dv) = f_1(\varepsilon)\, (b + \theta_b^\varepsilon)$$

and

$$c_\varepsilon = \int_R v^2\, \Gamma^\varepsilon(dv) = f_1(\varepsilon)\, (c + \theta_c^\varepsilon),$$

where $b, c < \infty$, $|\theta_b^\varepsilon| \to 0$, $|\theta_c^\varepsilon| \to 0$, $f_1(\varepsilon) \to 0$, $\varepsilon \to 0$.

(P2) The Poisson approximation condition for the intensity kernel

$$\Gamma_q^\varepsilon = \int_R q(v)\, \Gamma^\varepsilon(dv) = f_1(\varepsilon) \left(\Gamma_q + \theta_q^\varepsilon\right)$$

for all $q(\cdot)$ from $C_3(R)$

The kernel Γ_q has the following representation:

$$\Gamma_q = \int_R q(v)\, \Gamma^0(dv).$$

Items θ_b^ε, θ_c^ε, θ_q^ε satisfied the conditions $|\theta^\varepsilon| \to 0$, $f_1(\varepsilon) \to 0$ as $\varepsilon \to 0$.

(P3) The limiting generator is without diffuse component

$$c = \int_R v^2\, \Gamma^0(dv) = const.$$

(P4) Square-integrability condition

$$\lim_{c \to \infty} \int_{|v|>c} v^2\, \Gamma^\varepsilon(dv) = 0.$$

(P5) The exponential boundedness

$$\int_R e^{p|v|}\, \Gamma_q(dv) < \infty.$$

In the scheme of Poisson approximation the solution of the problem of large deviations for these processes are defined by nonlinear exponential generator:

$$H_\Gamma^\varepsilon \varphi(u) = e^{-\frac{\varphi(u)}{g_1(\varepsilon)}} g_1(\varepsilon) \Gamma^\varepsilon e^{\frac{\varphi(u)}{g_1(\varepsilon)}}.$$

Lemma 1

The exponential generator in the scheme of Poisson approximation has the following asymptotical representation

$$H_\Gamma^\varepsilon \varphi(u) = H_\Gamma \varphi(u) + \theta_\Gamma^\varepsilon \varphi,$$

provided $(g_2(\varepsilon))^{-1} g_1(\varepsilon) f_1(\varepsilon) \to 1$, where $\varphi(u)$ is from $C_0^3(R)$ and $|\theta_\Gamma^\varepsilon \varphi| \to 0$, as $g_1(\varepsilon), f_1(\varepsilon) \to 0$, as $\varepsilon \to 0$,

$$H_\Gamma \varphi(u) = b\varphi'(u) + \int_R \left(e^{v\varphi'(u)} - 1 - v\varphi'(u) \right) \Gamma^0(dv).$$

Proof

The generator of Markov processes has the next form

$$\Gamma^\varepsilon \varphi(u) = (g_2(\varepsilon))^{-1} \int_R \left(\varphi(u + g_1(\varepsilon)v) - \varphi(u) \right) \Gamma^\varepsilon(dv),$$

so, for the exponential generator

$$H_\Gamma^\varepsilon \varphi(u) = (g_2(\varepsilon))^{-1} g_1(\varepsilon) \int_R \left(e^{\Delta_\varepsilon \varphi(u)} - 1 \right) \Gamma^\varepsilon(dv),$$

where $\Delta_\varepsilon \varphi(u) = (g_1(\varepsilon))^{-1} \left(\varphi(u + g_1(\varepsilon)v) - \varphi(u) \right)$.

We can write the generator in the form

$$H_\Gamma^\varepsilon \varphi(u) = (g_2(\varepsilon))^{-1} g_1(\varepsilon) \int_R \left(e^{\Delta_\varepsilon \varphi(u)} - 1 - \Delta_\varepsilon \varphi(u) - \tfrac{1}{2}\left(\Delta_\varepsilon \varphi(u)\right)^2 \right) \Gamma^\varepsilon(dv) \quad +$$

$$(g_2(\varepsilon))^{-1} g_1(\varepsilon) \int_R \left(\Delta_\varepsilon \varphi(u) + \tfrac{1}{2}\left(\Delta_\varepsilon \varphi(u)\right)^2 \right) \Gamma^\varepsilon(dv).$$

where the function $e^{\Delta_\varepsilon \varphi(u)} - 1 - \Delta_\varepsilon \varphi(u) - \tfrac{1}{2}\left(\Delta_\varepsilon \varphi(u)\right)^2$ is from $C_3(R)$, and since

$$\frac{e^{\Delta_\varepsilon \varphi(u)} - 1 - \Delta_\varepsilon \varphi(u) - \tfrac{1}{2}\left(\Delta_\varepsilon \varphi(u)\right)^2}{v^2} \to 0 \quad \text{as } v \to 0.$$

In addition, this function is continuous and bounded, because $\varphi(u)$ is from $C_0^2(R)$. So, from conditions **P1** and **P2** we obtain:

$$H_\Gamma^\varepsilon \varphi(u) = (g_2(\varepsilon))^{-1} g_1(\varepsilon) f_1(\varepsilon) \int_R \left(e^{\Delta_\varepsilon \varphi(u)} - 1 - \Delta_\varepsilon \varphi(u) - \frac{1}{2} \big(\Delta_\varepsilon \varphi(u)\big)^2 \right) \Gamma^0(dv) +$$

$$(g_2(\varepsilon))^{-1} g_1(\varepsilon) \int_R \left(\Delta_\varepsilon \varphi(u) - v\varphi(u) - g_1(\varepsilon) \frac{v^2}{2} \varphi''(u) \right) \Gamma^\varepsilon(dv) +$$

$$(g_2(\varepsilon))^{-1} g_1(\varepsilon) f_1(\varepsilon) b\varphi'(u) + \frac{1}{2} (g_2(\varepsilon))^{-1} (g_1(\varepsilon))^2 f_1(\varepsilon) c\varphi''(u) +$$

$$(g_2(\varepsilon))^{-1} g_1(\varepsilon) \int_R \left(\frac{1}{2} \big(\Delta_\varepsilon \varphi(u)\big)^2 - \frac{v^2}{2} \big(\varphi'(u)\big)^2 \right) \Gamma^\varepsilon(dv) +$$

$$(g_2(\varepsilon))^{-1} g_1(\varepsilon) f_1(\varepsilon) \frac{1}{2} c \big(\varphi'(u)\big)^2.$$

Applying the Taylor's formula for $\varphi(u)$ and making use of the condition **P2**:

$$H_\Gamma^\varepsilon \varphi(u) = (g_2(\varepsilon))^{-1} g_1(\varepsilon) f_1(\varepsilon) \int_R \left(e^{v\varphi'(u)} - 1 - v\varphi'(u) - \frac{v^2}{2} \big(\varphi'(u)\big)^2 \right) \Gamma^0(dv) +$$

$$(g_2(\varepsilon))^{-1} g_1(\varepsilon) f_1(\varepsilon) \int_R \left(e^{v\varphi'(u)} g_1(\varepsilon) \frac{v^2}{2} \varphi''(\tilde{u}) - g_1(\varepsilon) \frac{v^2}{2} \varphi''(\tilde{u}) - \right.$$

$$\left. g_1(\varepsilon) \frac{v^4}{8} \big(\varphi''(\tilde{u})\big)^2 \right) \Gamma^0(dv) +$$

$$(g_2(\varepsilon))^{-1} g_1(\varepsilon) f_1(\varepsilon) \int_R g_1(\varepsilon) \frac{v^3}{3!} \varphi'''(\tilde{u}) \, \Gamma^0(dv) +$$

$$(g_2(\varepsilon))^{-1} g_1(\varepsilon) f_1(\varepsilon) b\varphi'(u) + \frac{1}{2} g_1 (g_2(\varepsilon))^{-1} (g_1(\varepsilon))^2 f_1(\varepsilon) c\varphi''(\tilde{u}) +$$

$$(g_2(\varepsilon))^{-1} g_1(\varepsilon) f_1(\varepsilon) \int_R g_1(\varepsilon) \frac{v^4}{4} \big(\varphi''(\tilde{u})\big)^2 \Gamma^0(dv) +$$

$$(g_2(\varepsilon))^{-1} g_1(\varepsilon) f_1(\varepsilon) c \frac{1}{2} \big(\varphi'(u)\big)^2.$$

From condition **P3** and the condition $(g_2(\varepsilon))^{-1} g_1(\varepsilon) f_1(\varepsilon) \to 1$ we obtain

$$H_\Gamma^\varepsilon \varphi(u) = H_\Gamma \varphi(u) + \theta_\Gamma^\varepsilon \varphi,$$

where $\left| \theta_\Gamma^\varepsilon \varphi \right| \to 0$, as $g_1(\varepsilon), f_1(\varepsilon) \to 0$.
Lemma is proved. □

From this Lemma we have that the following Theorem holds true.

Theorem 1
The solution for the problem of large deviations for the process

$$\eta^\varepsilon = g_1(\varepsilon) \left(\frac{t}{g_2(\varepsilon)} \right), \quad t \geq 0,$$

$$\Gamma^\varepsilon \varphi(u) = (g_2(\varepsilon))^{-1} \int_R \big(\varphi(u + g_1(\varepsilon)v) - \varphi(u) \big) \Gamma^\varepsilon(dv)$$

under conditions **(P1-P5)** and

$$(g_2(\varepsilon))^{-1} g_1(\varepsilon) f_1(\varepsilon) \to 1, \varepsilon \to 0,$$

is determined by the limit generator H_Γ of the form

$$H_\Gamma \varphi(u) = b\varphi'(u) + \int_R \left(e^{v\varphi'(u)} - 1 - v\varphi'(u) \right) \Gamma^0(dv).$$

Remark 1
Condition $(g_2(\varepsilon))^{-1} g_1(\varepsilon) f_1(\varepsilon) \to 1$ can be satisfied not only for trivial functions $g_2(\varepsilon) = \varepsilon^2$, $g_1(\varepsilon) = \varepsilon$, $f_1(\varepsilon) = \varepsilon$, as in the article [6], but for more complex functions, such as $g_2(\varepsilon) = \varepsilon^2$, $g_1(\varepsilon) = \sin \varepsilon$, $f_1(\varepsilon) = \cos \varepsilon$.

3. Lévy approximation

Now consider another normalization for the family of Markov processes with trajectories in $D^R[0, \infty)$

$$\eta^\varepsilon = g_1(\varepsilon) \, \eta \left(\frac{t}{g_3(\varepsilon)} \right), \quad t \geq 0.$$

In this normalization $g_1(\varepsilon), g_3(\varepsilon) \to 0$, as $\varepsilon \to 0$.
These processes are defined by generator

$$\Gamma^\varepsilon \varphi(u) = (g_3(\varepsilon))^{-1} \int_R \left(\varphi(u + g_1(\varepsilon)v) - \varphi(u) \right) \Gamma^\varepsilon(dv),$$

where $\varphi(u)$ is a twice differentiable function on R which tends to 0 at infinity and with sup-norm, $\varphi(u) \in C_0^2(R)$. The kernel of intensity Γ^ε belongs to the class $C^3(R)$. This kernel satisfies the condition $\Gamma^\varepsilon(0) = 0$.
The conditions of Lévy approximation are the following:
(L1) Approximation of the mean values:

$$b_\varepsilon = \int_R v \, \Gamma^\varepsilon(dv) = f_1(\varepsilon)b_1 + f_2(\varepsilon) \left(b + \theta_b^\varepsilon \right)$$

and

$$c_\varepsilon = \int_R v^2 \, \Gamma^\varepsilon(dv) = f_2(\varepsilon) \left(c + \theta_c^\varepsilon \right)$$

where $b < \infty$, $c < \infty$, $|\theta_b^\varepsilon| \to 0$, $|\theta_c^\varepsilon| \to 0$, $f_1(\varepsilon), f_2(\varepsilon) \to 0$. In this normalization $f_2(\varepsilon) = o(f_1(\varepsilon))$.
(L2) Lévy approximation condition for the intensity kernel

$$\Gamma_q^\varepsilon = \int_R q(v) \, \Gamma^\varepsilon(dv) = f_2(\varepsilon) \left(\Gamma_q + \theta_q^\varepsilon \right)$$

for all $q \in C_3(R)$.
This kernel has the following representation:

$$\Gamma_q = \int_R q(v) \, \Gamma^0(dv).$$

Items θ_b^ε, θ_c^ε, θ_q^ε satisfied the conditions $|\theta^\varepsilon| \to 0$, $f_1(\varepsilon) \to 0$ as $\varepsilon \to 0$.
(L3) Square-integrability condition

$$\lim_{c \to \infty} \int_{|v|>c} v^2 \, \Gamma^\varepsilon(dv) = 0.$$

(L5) The exponential boundedness

$$\int_R e^{p|v|} \, \Gamma_q(dv) < \infty.$$

Example 3.1
Consider the example of a family of Markov processes α^ε, for which

$$P\{\alpha^\varepsilon = f_1(\varepsilon)\alpha_1\} = p_0 - f_2(\varepsilon)p_1,$$

$$P\{\alpha^\varepsilon = f_2(\varepsilon)\alpha\} = q_0, p_0 + q_0 = 1,$$

$$P\{\alpha^\varepsilon = d\} = f_2(\varepsilon)p_1.$$

Check the conditions of Lévy approximation.
Firstly, find the first and the second moments for the family of Markov processes

$$b_\varepsilon = E\alpha^\varepsilon = p_0 f_1(\varepsilon)\alpha_1 - f_1(\varepsilon)f_2(\varepsilon)\alpha_1 p_1 + q_0 f_2(\varepsilon)\alpha + f_2(\varepsilon)p_1 d =$$

$$f_1(\varepsilon)(p_0\alpha_1) + f_2(\varepsilon)(\alpha q_0 + p_1 d) + o(f^2(\varepsilon)).$$

$$c_\varepsilon = E(\alpha^\varepsilon)^2 = f_1^2(\varepsilon)\alpha_1(p_0 - f_2(\varepsilon)p_1) + f_2^2(\varepsilon)\alpha^2 q_0 + d^2 f_2(\varepsilon)p_1 =$$

$$f_2(\varepsilon)(p_1 d^2) + o(f_2^2(\varepsilon)).$$

So, we have the next parameters for condition (L1):

$$b_1 = p_0\alpha_1,$$

$$b = \alpha q_0 + p_1 d,$$

$$c = p_1 d^2.$$

Now, find the intensity kernel

$$\Gamma_q^\varepsilon = q(f_1(\varepsilon)\alpha_1)p_0 + q(f_2(\varepsilon)\alpha)q_0 - q(f_1(\varepsilon)\alpha_1)(-f_2 p_1) + q(d)f_2(\varepsilon)p_1 = f_2(\varepsilon)q(d)p_1.$$

These moments satisfied the condition of approximation of the mean values. So, $\Gamma_q = p_1 d^2$.
Thus, we have all parameters for process α^ε in Lévi approximation.

Lemma 2
The exponential generator

$$H_\Gamma^\varepsilon\varphi(u) = e^{-\frac{\varphi(u)}{g_1(\varepsilon)}} g_1(\varepsilon)\Gamma^\varepsilon e^{\frac{\varphi(u)}{g_1(\varepsilon)}}$$

in the scheme of Lévy approximation has the next asymptotically representation

$$H_\Gamma^\varepsilon\varphi(u) = (g_3(\varepsilon))^{-1} g_1(\varepsilon)f_1(\varepsilon)b_1\varphi'(u) + H_\Gamma\varphi(u) + \theta_\Gamma^\varepsilon\varphi$$

with conditions $(g_3(\varepsilon))^{-1}g_1(\varepsilon)f_2(\varepsilon) \to 1, \varepsilon \to 0$, where $\varphi(u)$ is from $C_0^3(R)$ and

$$H_\Gamma\varphi(u) = (b - b_0)\varphi'(u) + \frac{1}{2}(c - c_0)(\varphi'(u))^2 + \int_R \left(e^{v\varphi'(u)} - 1\right)\Gamma^0(dv),$$

$b_0 = \int_R v\,\Gamma^0(dv), c_0 = \int_R v^2\,\Gamma^0(dv)$.
The neglected item $\left|\theta_\Gamma^\varepsilon\varphi\right| \to 0$ as $g_1(\varepsilon) \to 0$.

Proof

We can represent the exponential generator in the form

$$H_\Gamma^\varepsilon \varphi(u) = (g_3(\varepsilon))^{-1} g_1(\varepsilon) \int_R \left(e^{\Delta_\varepsilon \varphi(u)} - 1 \right) \Gamma^\varepsilon(dv),$$

where
$\Delta_\varepsilon \varphi(u) = (g_1(\varepsilon))^{-1}\big(\varphi(u + g_1(\varepsilon)v) - \varphi(u) \big).$
Rewrite the generator:

$$H_\Gamma^\varepsilon \varphi(u) =$$

$$(g_3(\varepsilon))^{-1} g_1(\varepsilon) \int_R \left(e^{\Delta_\varepsilon \varphi(u)} - 1 - \Delta_\varepsilon \varphi(u) - \frac{1}{2}\big(\Delta_\varepsilon \varphi(u)\big)^2 \right) \Gamma^\varepsilon(dv) +$$

$$(g_3(\varepsilon))^{-1} g_1(\varepsilon) \int_R \left(\Delta_\varepsilon \varphi(u) + \frac{1}{2}\big(\Delta_\varepsilon \varphi(u)\big)^2 \right) \Gamma^\varepsilon(dv).$$

Function $e^{\Delta_\varepsilon \varphi(u)} - 1 - \Delta_\varepsilon \varphi(u) - \frac{1}{2}\big(\Delta_\varepsilon \varphi(u)\big)^2$ belongs to $C_3(R)$, since

$$\frac{e^{\Delta_\varepsilon \varphi(u)} - 1 - \Delta_\varepsilon \varphi(u) - \frac{1}{2}\big(\Delta_\varepsilon \varphi(u)\big)^2}{v^2} \to 0 \quad \text{as } v \to 0.$$

In addition, this function is continuous and bounded, since $\varphi(u)$ from $C_0^2(R)$.
So, from conditions **(L1)** and **(L2)** we obtaine

$$H_\Gamma^\varepsilon \varphi(u) =$$

$$(g_3(\varepsilon))^{-1} g_1(\varepsilon) f_2(\varepsilon) \int_R \left(e^{\Delta_\varepsilon \varphi(u)} - 1 - \Delta_\varepsilon \varphi(u) - \frac{1}{2}\big(\Delta_\varepsilon \varphi(u)\big)^2 \right) \Gamma^0(dv) +$$

$$(g_3(\varepsilon))^{-1} g_1(\varepsilon) \int_R \left(\Delta_\varepsilon \varphi(u) - v\varphi'(u) - g_1(\varepsilon)\frac{v^2}{2}\varphi''(u) \right) \Gamma^\varepsilon(dv) +$$

$$(g_3(\varepsilon))^{-1} g_1(\varepsilon) f_1(\varepsilon) b_1 \varphi'(u) + (g_2(\varepsilon))^{-1} g_1(\varepsilon) f_2(\varepsilon) b \varphi'(u) +$$

$$(g_3(\varepsilon))^{-1} g_1^2(\varepsilon) f_2(\varepsilon) \frac{1}{2} c \varphi''(u) +$$

$$(g_3(\varepsilon))^{-1} g_1(\varepsilon) \int_R \left(\frac{1}{2}\big(\Delta_\varepsilon \varphi(u)\big)^2 - \frac{v^2}{2}\big(\varphi'(u)\big)^2 \right) \Gamma^\varepsilon(dv) +$$

$$(g_3(\varepsilon))^{-1} g_1(\varepsilon) f_2(\varepsilon) \frac{1}{2} c \big(\varphi'(u)\big)^2.$$

Applying the Taylor's formula for $\varphi(u)$ and making use the condition **L2** we will have

$$H_\Gamma^\varepsilon \varphi(u) =$$

$$(g_3(\varepsilon))^{-1} g_1(\varepsilon) f_2(\varepsilon) \int_R \left(e^{v\varphi'(u)} - 1 - v\varphi'(u) - \frac{v^2}{2}\big(\varphi'(u)\big)^2 \right) \Gamma^0(dv) +$$

$$(g_3(\varepsilon))^{-1} g_1(\varepsilon) f_2(\varepsilon) \int_R \left(e^{v\varphi'(u)} g_1(\varepsilon)\frac{v^2}{2}\varphi''(\tilde{u}) - g_1(\varepsilon)\frac{v^2}{2}\varphi''(\tilde{u}) - \right.$$

$$g_1(\varepsilon)\frac{v^4}{8}\big(\varphi''(\tilde{u})\big)^2\Big)\,\Gamma^0(dv)+$$

$$(g_3(\varepsilon))^{-1}g_1(\varepsilon)f_2(\varepsilon)\int_R g_1(\varepsilon)\frac{v^3}{3!}\varphi'''(\tilde{u})\,\Gamma^0(dv)+(g_3(\varepsilon))^{-1}g_1(\varepsilon)f_1(\varepsilon)b_1\varphi'(u)+$$

$$(g_3(\varepsilon))^{-1}g_1(\varepsilon)f_2(\varepsilon)b\varphi'(u)+(g_3(\varepsilon))^{-1}g_1^2(\varepsilon)f_2(\varepsilon)\frac{1}{2}c\varphi''(\tilde{u})+$$

$$(g_3(\varepsilon))^{-1}g_1(\varepsilon)f_2(\varepsilon)\int_R g_1(\varepsilon)\frac{v^4}{4}\big(\varphi''(\tilde{u})\big)^2\Gamma^0(dv)+$$

$$(g_3(\varepsilon))^{-1}g_1(\varepsilon)f_2(\varepsilon)c\frac{1}{2}\big(\varphi'(u)\big)^2.$$

Using condition **(L3)** and $(g_3(\varepsilon))^{-1}g_1(\varepsilon)f_2(\varepsilon) \to 1$, finally we obtain

$$H_\Gamma^\varepsilon\varphi(u) = (g_3(\varepsilon))^{-1}g_1(\varepsilon)f_1(\varepsilon)b_1\varphi'(u) + H_\Gamma\varphi(u) + \theta_\Gamma^\varepsilon\varphi,$$

where $\big|\theta_\Gamma^\varepsilon\varphi\big| \to 0$ as $g_1(\varepsilon), f_1(\varepsilon) \to 0$.
Lemma is proved. $\qquad\square$

From this Lemma we have that the following Theorem holds true.

Theorem 2
The solution for the problem of large deviations for the process

$$\eta^\varepsilon = g_1(\varepsilon)\left(\frac{t}{g_3(\varepsilon)}\right), \quad t \geq 0,$$

$$\Gamma^\varepsilon\varphi(u) = (g_3(\varepsilon))^{-1}\int_R \big(\varphi(u + g_1(\varepsilon)v) - \varphi(u)\big)\,\Gamma^\varepsilon(dv)$$

under conditions **(L1-L5)** and

$$(g_3(\varepsilon))^{-1}g_1(\varepsilon)f_2(\varepsilon) \to 1, \varepsilon \to 0$$

is determined by the limit generator H_Γ of the form

$$H_\Gamma\varphi(u) = (b - b_0)\varphi'(u) + \frac{1}{2}(c - c_0)\big(\varphi'(u)\big)^2 + \int_R \Big(e^{v\varphi'(u)} - 1\Big)\Gamma^0(dv).$$

4. Conclusions.

Thus, using normalization parameters as nonlinear functions we can found asymptotic representation for the random process in Poisson and Lévy approximation. This normalization allows to find nonlinear exponential generator that is the solution of the Large Deviations Principle.As normalization factors, we can take not only ε, but other nonlinear functions $f(\varepsilon)$ and $g(\varepsilon)$, that satisfied the conditions

$$(g_2(\varepsilon))^{-1}g_1(\varepsilon)f_1(\varepsilon) \to 1, \quad \varepsilon \to 0,$$

and in the Lévy approximation

$$(g_3(\varepsilon))^{-1}g_1(\varepsilon)f_2(\varepsilon) \to 1, \quad \varepsilon \to 0.$$

This normalization allows us to consider more processes and their small and large deviations.

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