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Results on relative mean residual Life and relative cumulative residual entropy

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Abstract Wei [21] has proposed the relative mean residual life function for comparing two lifetime distributions and studied its properties and relationship with other stochastic orders. In this paper, we obtain some new results on the relative mean residual life function and give a characterization result for a relative ordering based on this function. Motivated by this notion, we also introduce two notions of the dynamic relative cumulative residual entropy functions. Their properties and relationship with other relative orderings are investigated.

Keywords Ageing class, Cumulative residual entropy, Residual lifetime, Relative ageing, Reliability measures, Partial ordering

AMS 2010 subject classifications 62G10, 62G09, 62N05

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1. Introduction

In reliability and survival analysis and other life sciences, it is of interest to compare between lifetime distributions. This comparison is usually done based on partial orderings between their corresponding reliability measures and ageing functions. The relative ageing is another comparison method between lifetime distributions which has been studied in the literature. For example, Kalashnikov and Rachev [9] have studied the comparison of life distributions through a partial ordering which is equivalent to a monotone hazard ratio. Wei [21] has introduced the relative mean residual life (RMRL) function and investigated its relationship with other orderings. Sengupta and Deshpande [18] have considered two other models of relative ageing which include an increasing cumulative hazard ratio. Finkelstein [6] has studied the relative characteristics of the mean residual life functions. Hazra and Nanda [8] have introduced some new generalized stochastic orderings in the spirit of relative ageing. Li and Li [12] have studied the relative ageing order on series and parallel systems with mutually statistically independent and heterogeneous component lifetimes. Misra et al. [13] have introduced a new notion of relative aging based on the ratio of the mean inactivity time functions.

Let X be a non-negative random variable with distribution function F, survival function $\bar{F} = 1 - F$ and cumulative hazard functions $\Lambda = -\log \bar{F}$. The uncertainty contained in random variable X, since its first mathematical formulation by Shannon in 1948, has been defined in several formula. Recently, Rao et al. [16] defined a new uncertainty measure, the cumulative residual entropy (CRE), through

$$\mathcal{E}(X) = -\int_0^\infty \bar{F}(x) \log \bar{F}(x) dx = \int_0^\infty \bar{F}(x)\Lambda(x) dx. \quad (1)$$

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Properties of the CRE can be found in [15], [5], and [14]. Asadi and Zohrevand [1] have considered the corresponding dynamic measure, the CRE corresponding to the residual lifetime variable:

$$\begin{aligned} \mathcal{E}(X;t) &= -\int_t^\infty \frac{\bar{F}(x)}{\bar{F}(t)} \log \frac{\bar{F}(x)}{\bar{F}(t)} dx \\ &= -\frac{1}{\bar{F}(t)} \int_t^\infty \bar{F}(x) \log \bar{F}(x) dx - m(t)\Lambda(t), \end{aligned} \tag{2}$$

where, $m(t) = \frac{1}{\bar{F}(t)} \int_t^\infty \bar{F}(x)dx$ is the mean residual life (MRL) function. They have also obtained results about comparing between lifetime variables based on $\mathcal{E}(X;t)$. In this paper, we introduce a definition of dynamic relative cumulative residual entropy (RCRE) using $\mathcal{E}(X;t)$.

Wei [21] has defined the relative mean residual of F_1 with respect to F_2 through

$$m_{1.2}(t) = \frac{1}{\bar{F}_2(t)} \int_t^\infty \bar{F}_2(x) d\Lambda_1(x), \tag{3}$$

provided that $\bar{F}_2(t) > 0$. Motivated by this, we define the dynamic relative cumulative residual entropy of F_1 with respect to F_2 by

$$\mathcal{E}_{1.2}(t) = -\int_t^\infty \frac{\bar{F}_2(x)}{\bar{F}_2(t)} \log \frac{\bar{F}_2(x)}{\bar{F}_2(t)} d\Lambda_1(x), \tag{4}$$

provided that $\bar{F}_2(t) > 0$. For $\bar{F}_2(t) = 0$, we set $\mathcal{E}_{1.2}(t) = 0$. Note that if F_1 is an exponential distribution with parameter 1, then $\mathcal{E}_{1.2}(t)$ is reduced to $\mathcal{E}(X_2;t)$. That is, $\mathcal{E}_{1.2}(t)$ measures the dynamic entropy of F_2 with respect to a general distribution F_1.

The aim of this paper is twofold. In the first part, we give some new results on RMRL function and related orders. In the Second part, we study properties of $\mathcal{E}_{1.2}(t)$ along with introducing and investigating properties of another dynamic RCRE function. The rest of the paper is organized as follows. In Section 2, we provide various standard definitions and notations that are used later in the paper. In Section 3 we obtain some new results about the RMRL function. In Section 4 we investigate properties of the dynamic RCRE. Section 5 is devoted to the definition and properties of the other dynamic relative cumulative residual entropy and finally, some concluding remarks are given in section 6.

2. Some Preliminaries

Before proceeding to give the main results of the paper, we overview some preliminary concepts of ageing and standard partial orderings describing relative ageing. (For more details of these concepts see, for example, [18], [6], [19], [10], [13]).

Let $X_1(X_2)$ be a non-negative random variable with distribution function $F_1(F_2)$, survival function $\bar{F}_1(\bar{F}_2) = 1 - F_1(F_2)$, hazard function $\lambda_1(\lambda_2)$, cumulative hazard functions $\Lambda_1(\Lambda_2)$ and mean residual life functions $m_1(m_2)$, respectively. Throughout this paper we assume that these functions all exist and increasing (decreasing) means non-decreasing (non-increasing).

The following definitions pertaining to ageing classes and stochastic orders have been taken from [19].

*Definition 1***(i)** The random variable X is said to have increasing (decreasing) failure rate, IFR (DFR), if $X_t = X - t \mid X > t$ is stochastically decreasing (increasing) in $t \geq 0$. If X has a density f this is equivalent to say that the failure rate $\lambda(t) = \frac{f(t)}{\bar{F}(t)}$ is increasing (decreasing) in t.

(ii) The random variable X is said to be new better (worse) than used, NBU(NWU), if $\bar{F}(t+x) \leq (\geq)\bar{F}(t)\bar{F}(x)$ for all $x, t \geq 0$.

(iii) The random variable X is said to be new better (worse) than used (of second order), NBU(2)[NWU(2)], if $\int_0^x F(y)dy \leq (\geq) \frac{1}{\bar{F}(t)} \int_0^x [F(y+t) - F(t)]dy$, for all $x, t > 0$.

(iv) The random variable X is said to be new better (worse) than used in expectation, NBUE(NWUE), if $m(t) \leq (\geq) m(0)$, for all $t > 0$.

(v) X is said to be a new better (worse) than used in failure rate, NBUFR (NWUFR), if $\lambda(0) \leq (\geq) \lambda(t)$ for all $t > 0$ ([4]).

(vi) The random variable X is said to be increasing (decreasing) mean residual life (IMRL(DMRL)) if $m(t)$ is increasing (decreasing) in t.

(vii) The random variable X_1 is said to be smaller than X_2 in the usual stochastic order (denoted by $X_1 \leq_{st} X_2$) if $\bar{F}_1(x) \leq \bar{F}_2(x)$ for all x.

(viii) The random variable X_1 is said to be smaller than X_2 in the hazard rate ordering (denoted by $X_1 \leq_{hr} X_2$) if $\lambda_1(t) \geq \lambda_2(t)$ for all t.

(ix) The random variable X_1 is said to be smaller than X_2 in the mean residual life ordering (denoted by $X_1 \leq_{mrl} X_2$) if $m_1(t) \leq m_2(t)$, for all $t > 0$.

(x) The random variable X_1 is said to be smaller than X_2 in the convex order (denoted by $X_1 \leq_{cx} X_2$) if $E[\phi(X_1)] \leq E[\phi(X_2)]$, for all convex functions ϕ.

(xi) The random variable X_1 is said to be smaller than X_2 in the dilation order (denoted by $X_1 \leq_{dil} X_2$) if $[X_1 - E(X_1)] \leq_{cx} [X_2 - E(X_2)]$.

Definition 2
The random variable X_1 is said to be aging faster than X_2 in the

(i) Failure rate (written as $X_1 \prec_c X_2$) if $\Lambda_1(X_2)$ has a decreasing failure rate (DFR) distribution (equivalently, $\lambda_1(t)/\lambda_2(t)$ is increasing in t, [18]),

(ii) Failure rate average (written as $X_1 \prec_* X_2$) if $\Lambda_1(X_2)$ has a decreasing failure rate average (DFRA) distribution ([18]),

(iii) quantile (written as $X_1 \prec_{su} X_2$) if $\Lambda_1(X_2)$ has a new better than used (NBU) distribution ([18]),

(iv) the mean residual life (written as $X_1 \prec_a X_2$) if $m^*_{1.2}(t) = m_1(t)/m_2(t)$ is a decreasing function of $t \in (0, \infty)$ ([10]).

In contrast to Definition 2 (iv), Finkelstein [6] defined the random variable X_1 to be aging faster than X_2 in mean residual life ($X_1 \prec_a X_2$) if $m_1(t)/m_2(t)$ is increasing on $[0, \infty)$. As Misra et al. [13] have mentioned, since increasing mean residual life describes negative aging, we believe that the correct way of defining relative aging in terms of mean residual function is as given in Definition 2 (iv).

Remark 1
As a special case of Proposition 2.1 (iii) in [8] (for $s = 1$), one can see easily that $X_1 \prec_a X_2$ if and only if $X_1^e \prec_c X_2^e$, where X^e is the equilibrium random variable (see, for example, [7], and references therein) corresponding to X with survival function

$$\bar{F}_e(t) = \frac{1}{\mu} \int_t^\infty \bar{F}(x) dx, \quad t \geq 0.$$

3. Results on the RMRL and relative orders

Sengupta and Deshpande [18] have generalized the $\prec_c$ ordering to the $\prec_*$ and $\prec_{su}$ orderings. Hazra and Nanda [8] have generalized these relative orderings to s-IFR(R), s-IFRA(R), s-NBU(R), s-NBUFR(R) and s-NBAFR(R) orderings.

As it is clear from the complete chain of implications among the positive ageing criteria in [4], one can see that the NBU class has two sperate ways of generalization. The one is toward the classes NBUFR and NBUFRA and the other one is toward the classes NBU(2) and NBUE. Here, we follow [8] and define two generalized relative ordering classes. Let X_{1s} and X_{2s} be random variables as defined in [8].

Definition 3
For any positive integer s, X_1 is said to be more

(i) s-NBU(2)(R) than X_2 (written as $X_1 \leq_{s-NBU(2)(R)} X_2$) if the random variable $\Lambda_{X_{2s}}(X_{1s})$ has an NBU(2) distribution,

(ii) s-NBUE(R) than X_2 (written as $X_1 \leq_{s-NBUE(R)} X_2$) if the random variable $\Lambda_{X_{2s}}(X_{1s})$ has an NBUE distribution.

The properties of these classes can be obtained in similar way as those of the other classes in [8]. For $s = 1$, we denote the s-NBUE(R) ordering between the random variables X_1 and X_2 by $X_1 \prec_{rm} X_2$ which is equivalent to that $Z = \Lambda_1(X_2)$ has a NWUE distribution. It follows from Proposition 6.3 in [4] that Z is NWUE if and only if Z^e is NWUFR, where Z^e is the equilibrium random variable corresponding to Z. The following theorem gives another characterization of this ordering based on the RMRL function.

Theorem 1
Suppose that F_1 is continuous and strictly increasing. Then $X_1 \prec_{rm} X_2$ if and only if $m_{1.2}(t) \geq E[\Lambda_1(X_2)]$ for all $t > 0$.

Proof: Using the integration by part, it follows from the equation (3) that

$$m_{1.2}(t) = E[\Lambda_1(X_2)|X_2 > t] - \Lambda_1(t) = m_Z(\Lambda_1(t)).$$

The result now is clear.

Remark 2
It follows form the proof of the above theorem that $m_{1.2}(t)$ is an increasing function if and only if $Z = \Lambda_1(X_2)$ has an IMRL distribution.

The following theorem gives a bound for $m_{1.2}(t)$ and shows that the behavior of $m_{1.2}(t)$ is related to the behavior of the hazard rate function of X_2.

Theorem 2 **(a)** For all $t > 0$, $m_{1.2}(t) \leq \Lambda_1(t)$.

(b) If X_2 is IFR(DFR), then $m_{1.2}(t)$ is increasing (decreasing) in t.

Proof: (a) is clear form equation (3). To prove (b), first note that if X_2 is IFR(DFR), then so is $Z = \Lambda_1(X_2)$. Now the assertion follows on $m_{1.2}(t) = m_Z(\Lambda_1(t))$, Remark 2 and the fact that IFR(DFR) implies DMRL(IMRL).

It is worth to mention that if X_1 is distributed as exponential with $E(X_1) = E(X_2)$, then $m_{1.2}(t) = \frac{1}{E(X_2)} m_{X_2}(t)$. Hence, $m_{1.2}(t) \geq 1$ implies that X_2 is NWUE. The next result gives a bound for $m_{1.2}(t)$, in terms of the survival functions, when X_1 and X_2 are stochastically ordered and gives conditions under which $m_{1.2}(t) \geq 1$ and $X_1 \leq_{mrl} X_2$ follow each other.

Theorem 3 **(a)** If $X_1 \leq_{st} X_2$, then $m_{1.2}(t) \geq \bar{F}_1(t)/\bar{F}_2(t)$.

(b) If X_1 is IFR and $X_1 \leq_{mrl} X_2$, then $m_{1.2}(t) \geq 1$.

(c) If X_1 is DFR and $m_{1.2}(t) \geq 1$, then $X_1 \leq_{mrl} X_2$.

Proof: (a) If $X_1 \leq_{st} X_2$, then $\bar{F}_1(t) \leq \bar{F}_2(t)$. Now, the result follows from the fact that

$$m_{1.2}(t) = E[\frac{\bar{F}_1(t)\bar{F}_2(X_1)}{\bar{F}_2(t)\bar{F}_1(X_1)}|X_1 > t].$$

To prove part (b), first note that the assumption that X_1 is IFR implies that $\Lambda_1(t)$ is a convex function. On the other hand, it is easy to see that the random variable $U = \Lambda_1(X_1)$ is distributed according as standard exponential distribution. It follows now from the proof of Theorem 1 and the Theorem 2.A.19 in [19], p. 92 that

$$m_{1.2}(t) = m_Z(\Lambda_1(t)) \geq m_U(\Lambda_1(t)) = 1.$$

This completes the proof of (b). The proof of part (c) is similar.

Remark 3
Let X_1, X_2 and X be three independent random variables. Assume that X_1 and X are IFR and that $X_1 \leq_{mrl} X_2$. Lemma 2.A.8. in [19], p. 86, follows that $X_1 + X \leq_{mrl} X_2 + X$. The assumption also implies that $X_1 + X$ is IFR (cf. [2], p. 100). Then, it follows from part (b) of Theorem 3 that $m^X_{1.2}(t) \geq 1$, where $m^X_{1.2}(t)$ represents the RMRL of $X_1 + X$ with respect to $X_2 + X$.

Let $X_{k:n}$ be the kth smallest order statistic of random variables $X_1, \ldots, X_n$, $k = 1, \ldots, n$. In reliability theory, $X_{k:n}$ indicates the lifetime of a $n-k+1$-out-of-n system, which works if at least $n-k+1$ of the n components function normally. In particular, $X_{1:n}$ and $X_{n:n}$ represent lifetimes of the series and parallel systems, respectively. Let also $X_{11}, \ldots, X_{1n}$ be i.i.d. random lifetimes of type X_1 and $X_{21}, \ldots, X_{2n}$ be i.i.d. ones of type X_2 with $X^1_{k:n}$ and $X^2_{k:n}$ as their corresponding kth smallest order statistics. The following examples give some applications of the above theorem.

Example 1
Let X_1 and X_2 have Pareto distributions with survival functions

$$\bar{F}_1(t) = (\frac{a}{a+t})^\beta, \;\; t > 0, \;\; a, \beta > 0,$$

and

$$\bar{F}_2(t) = (\frac{b}{b+t})^\beta, \;\; t > 0, \;\; b, \beta > 0.$$

For $a < b$, we have $\bar{F}_1(t) \leq \bar{F}_2(t)$, i.e. $X_1 \leq_{st} X_2$. Thus from part (a) of Theorem 3 it follows that

$$m_{1.2}(t) \geq (\frac{at+ab}{bt+ab})^\beta.$$

On the other hand, it follows form Theorem 1.A.23. (a) in [19], p. 13 that $X^1_{k:n} \leq_{st} X^2_{k:n}$ for $k = 1, \ldots, n$. Then, using again the part (a) of Theorem 3 we get that

$$m^k_{1.2}(t) \geq \frac{1 - I_{F_1(t)}(k, n-k+1)}{1 - I_{F_2(t)}(k, n-k+1)},$$

where $m^k_{1.2}(t)$ stands for the RMRL of $X^1_{k:n}$ with respect to $X^2_{k:n}$, and $I_p(a,b) = \int_0^p t^{a-1}(1-t)^{b-1}dt/B(a,b)$ is the incomplete beta function.

Example 2
Unloaded redundancy. Consider the unloaded redundancy when one of the identical components starts operating and the other $n-1$ are in stand by. As the operating one fails, it is immediately replaced by the stand by one etc. The system fails when the last component fails. We compare the RMRL for two objects with exponentially distributed components lifetimes and different levels of redundancy: $m < n$. For a component with exponential distribution, $\bar{F}(t) = e^{-\lambda t}$, the lifetime distribution of the object having n components is the Gamma (or the Erlangian) distribution with parameters n and λ. Let X_1, X_2 be the lifetimes of the objects with m and n components, respectively. For $1 < m < n$, X_1 is IFR and $X_1 \leq_{mrl} X_2$. Then, if $m_{1.2}(t)$ is the RMRL of X_1 with respect to X_2, it follows from part (b) of Theorem 3 that $m_{1.2}(t) \geq 1$.

4. Properties of $\mathcal{E}_{1.2}(t)$

In this section we investigate some properties of the dynamic RCRE. First, note that $\mathcal{E}_{1.2}(t)$ can also be given by

$$\mathcal{E}_{1.2}(t) = \frac{1}{\bar{F}_2(t)} \int_t^{\infty} \bar{F}_2(x)\Lambda_2(x) d\Lambda_1(x) - \Lambda_2(t) m_{1.2}(t).$$

Asadi and Zohrevand [1] have shown that

$$\mathcal{E}(X) = E[m(X)], \quad \mathcal{E}(X;t) = E[m(X)|X > t],$$

where $m(t)$ is the MRL function. The following theorem gives the direct relation between the RMRL and dynamic RCRE functions.

Theorem 4
Let X_1 and X_2 be two non-negative random variables with distribution functions F_1 and F_2, respectively, and finite dynamic RCRE function $\mathcal{E}_{1.2}(t)$. Then,

$$\mathcal{E}_{1.2}(t) = E[m_{1.2}(X_2)|X_2 > t].$$

Proof: From (3) we have

$$\begin{aligned}
E[m_{1.2}(X_2)|X_2 > t] &= \frac{1}{\bar{F}_2(t)} \int_t^{\infty} \int_y^{\infty} \frac{\bar{F}_2(x)}{\bar{F}_2(y)} d\Lambda_1(x) dF_2(y) \\
&= \frac{1}{\bar{F}_2(t)} \int_t^{\infty} \int_y^{\infty} \bar{F}_2(x) d\Lambda_1(x) d\Lambda_2(y) \\
&= \frac{1}{\bar{F}_2(t)} \int_t^{\infty} \int_t^{x} d\Lambda_2(y) \bar{F}_2(x) d\Lambda_1(x) \\
&= \frac{1}{\bar{F}_2(t)} \int_t^{\infty} \Lambda_2(x) \bar{F}_2(x) d\Lambda_1(x) - \Lambda_2(t) m_{1.2}(t) \\
&= \mathcal{E}_{1.2}(t).
\end{aligned}$$

This completes the proof.

The following theorem shows that the behavior of $\mathcal{E}_{1.2}(t)$ is closely related to the behavior of $m_{1.2}(t)$ and the ratio of the hazard rates of X_1 and X_2. In particular, the theorem proves that the constancy of $\mathcal{E}_{1.2}(t)$ gives a characterization of proportional hazards model. Proportional hazards model, introduced by Cox [3], plays an important role in reliability and survival analysis. Two random variables X_1 and X_2 with survival functions $\bar{F}_1$ and $\bar{F}_2$ are said to have proportional hazards if there exists $\theta > 0$ such that $\bar{F}_1(t) = \bar{F}_2^{\theta}(t)$ for all $t \geq 0$. If the hazard rates of X_1 and X_2 exist, i.e. the survival functions are absolutely continuous, this is equivalent to say that for all $t > 0$, $\lambda_1(t) = \theta\lambda_2(t)$.

Theorem 5
Let $\lambda_{1.2}(t) = \lambda_1(t)/\lambda_2(t)$, (for all t for which the ratio is well defined) and $S_{\mathcal{E}} = \{t : \bar{F}_1(t) > 0, \bar{F}_2(t) > 0\}$. Then

(a) $\mathcal{E}_{1.2}(t)$ is constant iff $\lambda_{1.2}(t)$ is constant $\forall t \in S_{\mathcal{E}}$,

(b) $\mathcal{E}_{1.2}(t)$ is increasing (decreasing) iff $m_{1.2}(t)$ is increasing (decreasing) $\forall t \in S_{\mathcal{E}}$.

Proof: Proposition 2.1 in [21] gives that if $\lambda_{1.2}(t) = \theta$, then $m_{1.2}(t) = \theta$. This along with Theorem 4 imply that $\mathcal{E}_{1.2}(t) = \theta$. Inversely, if $\mathcal{E}_{1.2}(t) = \theta$, then using Theorem 4 again follows that

$$\theta \bar{F}_2(t) = \int_t^{\infty} m_{1.2}(y) dF_2(y).$$

Derivation from both side of the above equation with respect to t implies that $m_{1.2}(t) = \theta$. On the other hand $m_{1.2}(t)$ can also be written as

$$m_{1.2}(t) = \frac{1}{\bar{F}_2(t)} \int_t^\infty \lambda_{1.2}(y) dF_2(y).$$

Derivation again follows that $\lambda_{1.2}(t) = \theta$. This proves part (a) of the theorem. To prove part (b), it is easy to see that the derivative of $\mathcal{E}_{1.2}(t)$ is equal to

$$\mathcal{E}'_{1.2}(t) = \lambda_2(t)[\mathcal{E}_{1.2}(t) - m_{1.2}(t)].$$

The if part now follows from the definition of $\mathcal{E}_{1.2}(t)$. For the only if part, note that if $\mathcal{E}_{1.2}(t)$ is increasing (decreasing) then $\mathcal{E}_{1.2}(t) \geq (\leq) m_{1.2}(t)$ which is equivalent to that

$$\int_t^\infty m_{1.2}(x) dF_2(x) \geq (\leq) \int_t^\infty \lambda_{1.2}(x) dF_2(x).$$

The result now follows from the integral comparison lemma (cf. [17], p. 336) and the fact that $m'_{1.2}(t) = \lambda_2(t)[m_{1.2}(t) - \lambda_{1.2}(t)]$.

Proposition 2.2 in [21] gives that if $\lambda_{1.2}(t)$ is increasing (decreasing) function then $m_{1.2}(t)$ is also increasing (decreasing) function. However, the reverse is not true. For a counterexample, let F_1 be the standard exponential distribution and F_2 the distribution given in Example 4.9 in [11], p. 114.

Wei [21] has defined another relative mean residual life of F_1 with respect to F_2 as $m^*_{1.2}(t) = m_1(t)/m_2(t)$. It is well-known that if $\lambda_{1.2}(t) \geq 1$, then $m^*_{1.2}(t) \leq 1$. The reverse is not necessarily true(cf. [19], p. 83). The following example shows that the same is true for the relation between $\lambda_{1.2}(t)$, $m_{1.2}(t)$ and $\mathcal{E}_{1.2}(t)$. That is, $\mathcal{E}_{1.2}(t) \geq (\leq) 1$, for all t, does not necessarily imply that $m_{1.2}(t) \geq (\leq) 1$ or $\lambda_{1.2}(t) \geq (\leq) 1$. Though, it follows from Theorem 3(b) that if X_1 is IFR and $X_1 \leq_{mrl} (\geq_{mrl}) X_2$, then $m_{1.2}(t) \geq (\leq) 1$ and hence, $\mathcal{E}_{1.2}(t) \geq (\leq) 1$.

Example 3
Let X_1 and X_2 be distributed as Weibull with survival functions

$$\bar{F}_1(t) = e^{-(t/\beta_1)^{\alpha_1}}, \quad t > 0, \quad \alpha_1 > 0, \quad \beta_1 > 0,$$
$$\bar{F}_2(t) = e^{-(t/\beta_2)^{\alpha_2}}, \quad t > 0, \quad \alpha_2 > 0, \quad \beta_2 > 0,$$

respectively. Then

$$\lambda_{1.2}(t) = \frac{\alpha_1 \beta_2^{\alpha_2}}{\alpha_2 \beta_1^{\alpha_1}} t^{\alpha_1 - \alpha_2},$$

$$m_{1.2}(t) = \frac{\alpha_1 \beta_2^{\alpha_1} e^{(t/\beta_2)^{\alpha_2}}}{\alpha_2 \beta_1^{\alpha_1}} \Gamma((t/\beta_2)^{\alpha_2}, \frac{\alpha_1}{\alpha_2}),$$

$$\mathcal{E}_{1.2}(t) = \frac{\alpha_1 \beta_2^{\frac{\alpha_1}{\alpha_2} - \alpha_2 + 1} e^{(t/\beta_2)^{\alpha_2}}}{\alpha_2 \beta_1^{\alpha_1}} \Gamma((t/\beta_2)^{\alpha_2}, \frac{\alpha_1}{\alpha_2} + 1) - (t/\beta_2)^{\alpha_2} m_{1.2}(t),$$

where $\Gamma(t, a) = \int_t^\infty x^{a-1} e^{-x} dx$ is the incomplete gamma function. Figure 1 depicts the graphs of $\lambda_{1.2}(t)$, $m_{1.2}(t)$ and $\mathcal{E}_{1.2}(t)$ for different values of (α_1, β_1) and (α_2, β_2).

Example 4
Let X be a continuous non-negative random variable with survival function $\bar{F}$ and a finite mean μ. It can be easily seen that the RMRL and RCRE of F with respect to F_e are given as follows

$$m_{1.2}(t) = \frac{\mathcal{E}(X;t)}{m(t)}, \quad \mathcal{E}_{1.2}(t) = \frac{1}{m(t)\bar{F}(t)} \int_t^\infty \frac{\mathcal{E}(X;x)}{m(x)} \bar{F}(x) dx,$$

respectively, which are both less (greater) than one if X is DMRL (IMRL).

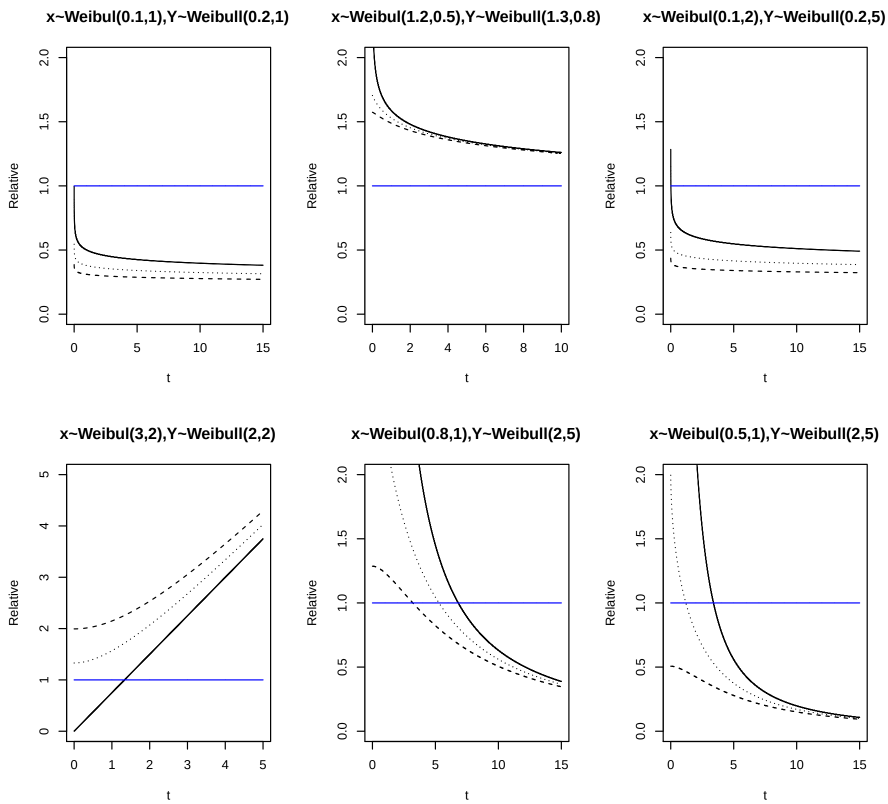

Figure 1. Graph of $\lambda_{1.2}(t)$(solid line), $m_{1.2}(t)$(dotted line), and $\mathcal{E}_{1.2}(t)$(dashed line)

5. Another relative dynamic cumulative residual entropy

It is easy to show that the cumulative residual entropy (1) and the dynamic cumulative residual entropy (2) can also be given by

$$\mathcal{E}(X) = Cov(X, \Lambda(X)), \;\; \mathcal{E}(X;t) = Cov(X_t, \Lambda(X_t)). \tag{5}$$

Motivated by this, we define the relative cumulative residual entropy of F_1 with respect to F_2 by

$$\mathcal{E}^*(X_1, X_2) = Cov(X_1, F_2^{-1}(F_1(X_1))) = \int_0^1 F_1^{-1}(u)F_2^{-1}(u)du - E(X_1)E(X_2), \tag{6}$$

where $F_i^{-1}(u) = \inf\{x : F_i(x) \geq u\}$, $0 \leq u \leq 1$, is the quantile function associated with F_i, $i = 1, 2$. Equivalently, we define the another dynamic relative cumulative residual entropy of F_1 with respect to F_2 by

$$\begin{aligned} \mathcal{E}^*_{1.2}(t) &= Cov(X_{1t}, F_2^{-1}(F_{1t}(X_{1t}))) \\ &= \int_0^1 F_1^{-1}(u\bar{F}_1(t) + F_1(t))F_2^{-1}(u)du - (t + m_{X_1}(t))E(X_2) \\ &= \frac{1}{\bar{F}_1(t)} \int_t^\infty x F_2^{-1}\left(1 - \frac{\bar{F}_1(x)}{\bar{F}_1(t)}\right) dF_1(x) - (t + m_{X_1}(t))E(X_2) \\ &= E[X_1\{F_2^{-1}\left(1 - \frac{\bar{F}_1(X_1)}{\bar{F}_1(t)}\right) - E(X_2)\}|X_1 > t], \end{aligned} \tag{7}$$

where $F_{it}(x) = \frac{F_i(x+t)-F_i(t)}{\bar{F}_i(t)}$ and F_{it}^{-1} are the distribution and quantile functions corresponding to residual lifetime variable $X_{it} = X_i - t | X_i > t$, $i = 1, 2$, respectively. Note that if F_2 is an exponential distribution with parameter 1, then

$$F_2^{-1}(F_1(X_1)) = -\ln(1 - F_1(X_1)) = \Lambda_1(X_1),$$
$$F_2^{-1}(F_{1t}(X_{1t})) = -\ln(1 - F_{1t}(X_{1t})) = \Lambda_1(X_{1t}),$$

and $\mathcal{E}^*(X_1, X_2)$ and $\mathcal{E}^*_{1.2}(t)$ are reduced to $\mathcal{E}(X_1)$ and $\mathcal{E}(X_1; t)$, respectively.

Example 5
Let X_1 and X_2 be uniformly distributed on intervals $(0, b_1)$ and $(0, b_2)$, respectively. Then

$$\mathcal{E}^*_{1.2}(t) = \frac{b_2(b_1 - t)}{12}.$$

The following theorem gives the relation between the dilation order and the above RCRE.

Theorem 6
If $X_1 \leq_{dil} X_2$, then $\sigma_1 \leq \mathcal{E}^*(X_1, X_2) \leq \sigma_2$, where σ_i is the standard deviation of X_i, $i = 1, 2$.

Proof: First note that $X_1 \leq_{dil} X_2$ is equivalent to (see equation 3.A.37 in [19], p. 118)

$$\nu_1(F_1^{-1}(p)) \leq \nu_2(F_2^{-1}(p)) + E(X_1) - E(X_2), \quad p \in [0, 1),$$

where $\nu_i(t) = m_i(t) + t = E(X_i | X_i > t)$, $i = 1, 2$. On the other hand

$$\begin{aligned} \mathcal{E}^*(X_1, X_2) &= \int_0^\infty x F_2^{-1}(F_1(x)) dF(x) - E(X_1)E(X_2) \\ &= \int_0^\infty \int_0^x F_2^{-1}(F_1(x)) dt dF(x) - E(X_1)E(X_2) \\ &= \int_0^\infty \bar{F}_1(t) \nu_2(F_2^{-1}(F_1(t))) dt - E(X_1)E(X_2). \end{aligned}$$

Thus,

$$\begin{aligned} \mathcal{E}^*(X_1, X_2) &\geq \int_0^\infty \bar{F}_1(t) \nu_2(t) dt - E^2(X_1) \\ &= \int_0^\infty \int_t^\infty \bar{F}_1(y) dy dt + \frac{1}{2} E(X_1^2) - E^2(X_1) \\ &= Var(X_1). \end{aligned}$$

The result now follows from the Cauchy-Schwarz inequality which implies that $\mathcal{E}^*(X_1, X_2) \leq \sigma_1 \sigma_2$.

The above proof implies that (7) can also given by

$$\mathcal{E}^*_{1.2}(t) = \frac{1}{\bar{F}_1(t)} \int_t^\infty \nu_2(F_2^{-1}\left(1 - \frac{\bar{F}_1(x)}{\bar{F}_1(t)}\right)) \bar{F}_1(x) dx - m_{X_1}(t) E(X_2). \tag{8}$$

6. Conclusion

Models of relative aging are tools for comparing two lifetime distributions. In this paper, we obtained some new properties of the relative mean residual life (RMRL) and obtained a characterization result for a special case of s-NBUE(R) ordering based on the RMRL. We also introduced two definitions of the dynamic relative cumulative residual entropy (RCRE) for comparing the uncertainty within two lifetime random variables. It was shown that the behavior of the dynamic RCRE has close relation with the behavior of the RMRL and the ratio of the hazard functions. Some results obtained to explore the relation between these new notions and some well-known stochastic orderings such as dilation order.

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