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Relations for Single and Product Moments of Odds Generalized Exponential-Pareto Distribution Based on Generalized Order Statistics and Characterization

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Abstract This paper deals with explicit expressions and recurrence relations for single, inverse, product and ratio moments of Generalized Order Statistics from Odds Generalized Exponential-Pareto Distribution (OGEPD). Characterization results have also been carried out.

Keywords Generalized order statistics, order statistics, record values, recurrence relations.

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1. Introduction

The concept of generalized order statistics (*gos*) was first introduced by [8] which envelops variety of models of ordered random variables that acts as a flexible model in various directions such as order statistics, upper record values, progressive type II censoring order statistics, sequential order statistics and Pfeiffer's records.
Let $X_1, X_2, ...$ be a sequence of independent identically distributed (*iid*) random variables with distribution function (*df*) $F(x)$ and probability density function (*pdf*) $f(x)$. Let $k \geq 1, n \geq 2, n \in N, \tilde{m} = (m_1, m_2, ..., m_{n-1}) \in \Re^{n-1}, M_r = \sum_{j=r}^{n-1} m_j$ such that $\gamma_r = k + n - r + M_r > 0 \ \forall \ \ r \in \{1, 2, ..., n-1\}$. Then $X(r, n, \tilde{m}, k)$, $r = 1, 2, \dots, n$, is called *gos* based on $F(x)$, if their joint *pdf* is of the form

$$k \left(\prod_{j=1}^{n-1} \gamma_j \right) \left(\prod_{i=1}^{n-1} [1 - F(x_i)]^{m_i} f(x_i) \right) [1 - F(x_n)]^{k-1} f(x_n), \qquad F^{-1}(0) < x_1 \leq .... \leq x_n < F^{-1}(1).$$

For different values of m_i's, k and γ_i's the model of *gos* reduces to various models *e.g.*, when $(m_1 = m_2 = ... = m_{n-1} = 0, \ k = 1, \ \gamma_i = 1 + n - i)$, this model reduces to order statistics and for k^{th} upper record values $(m_1 = m_2 = ... = m_{n-1} = -1, \ \ i.e. \ \gamma_i = k, \ \ k \in N)$. Here we take the case $m_i = m_j = m$. Then the density function of r^{th} *gos* $X(r, n, m, k)$ is given by

$$f_{X(r,n,m,k)} = \frac{C_{r-1}}{(r-1)!} \left[\bar{F}(x) \right]^{\gamma_r - 1} f(x) \, g_m^{r-1}(F(x)). \tag{1}$$

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The joint *pdf* of $X(r,n,m,k)$ and $X(s,n,m,k)$, $1 \le r < s \le n$ is

$$f_{X(r,n,m,k)X(s,n,m,k)} = \frac{C_{s-1}}{(r-1)!\,(s-r-1)!}\left[\bar{F}(x)\right]^{m} f(x)\, g_m^{r-1}(F(x))\left[h_m F(y) - h_m F(x)\right]^{s-r-1} \times \left[\bar{F}(y)\right]^{\gamma_s - 1} f(y), \tag{2}$$

where

$$\bar{F}(x) = 1 - F(x), C_{r-1} = \prod_{i=1}^{r}\gamma_i,$$

$$h_m(x) = \begin{cases} -\frac{1}{m+1}(1-x)^{m+1}, & m \neq -1 \\ -\ln(1-x), & m = -1 \end{cases}$$

and

$$g_m(x) = h_m(x) - h_m(0), x \in [0,1).$$

A lot of work has been carried out by several researchers in the field of *gos*. Recurrence relations for moments of *gos* for specific as well as for general class of distribution have been well investigated hitherto. [2] and [11] derived the recurrence relations for single and product moments of *gos* for some general class of distributions, whereas explicit expressions for exact moments of *gos* were probed by [4] and the generalized record values from additive Weibull distribution derived by [13]. For more details of *gos* we have [9, 12, 3, 1, 14, 10] and [6].
A random variable X is said to have OGEPD [15] if its *df* is given by

$$F(x) = 1 - e^{-\lambda\left[\left(\frac{x}{a}\right)^{\theta} - 1\right]}, \qquad x > a,\ \theta, \lambda > 0. \tag{3}$$

The corresponding *pdf* is of the form

$$f(x) = \frac{\lambda\theta}{a^{\theta}} x^{\theta-1} e^{-\lambda\left[\left(\frac{x}{a}\right)^{\theta} - 1\right]}, \qquad x > a,\ \theta, \lambda > 0. \tag{4}$$

The relation between $\bar{F}(x)$ and $f(x)$ can be easily derived as

$$\bar{F}(x) = \frac{a^{\theta}}{\lambda\theta} x^{1-\theta} f(x), \tag{5}$$

where $\bar{F}(x) = 1 - F(x)$.

2. Relation for single moments

2.1. Explicit expressions for single moments

We first derive explicit expression for single moments of r^{th} *gos*, $X(r,n,m,k)$. The following theorem shows the explicit expression for $E\left[X^{j}(r,n,m,k)\right] = \mu_{r:n}^{(j)}$.

Theorem 1
For $1 \le r \le n$, $k \ge 1$ and $j = 0, 1, 2, \ldots,$

$$\mu_{r:n,m,k}^{(j)} = \frac{C_{r-1}}{(r-1)!(m+1)^{r-1}} \frac{a^{j}}{\lambda^{j/\theta}} \sum_{u=0}^{r-1} (-1)^{u} \binom{r-1}{u} e^{\lambda\gamma_{r-u}} (\gamma_{r-u})^{-j/\theta-1}\, \Gamma\left((j/\theta+1), \lambda\gamma_{r-u}\right), \quad m \neq -1 \tag{6}$$

and for $m = -1$, we have

$$\mu_{r:n,-1,k}^{(j)} = \frac{e^{\lambda k} a^{j}}{(r-1)!} \sum_{u=0}^{r-1} (-1)^{r-u-1} \binom{r-1}{u} \frac{\Gamma\left((j/\theta+u+1), \lambda k\right)}{(\lambda k)^{j/\theta+u-r+1}}. \tag{7}$$

Proof
Using (1), we get

$$\mu_{r:n,m,k}^{(j)} = \frac{C_{r-1}}{(r-1)!} \int_a^\infty x^j \left[\bar{F}(x)\right]^{\gamma_r - 1} g_m^{r-1}(F(x)) f(x)\, dx.$$

On expanding $g_m^{r-1}(F(x)) = \left[\frac{1}{m+1}\left(1 - (\bar{F}(x))^{m+1}\right)\right]^{r-1}$ binomially in above expression, we get

$$\mu_{r:n,m,k}^{(j)} = \frac{C_{r-1}}{(r-1)!(m+1)^{r-1}} \sum_{u=0}^{r-1} (-1)^u \binom{r-1}{u} \int_a^\infty x^j \left[\bar{F}(x)\right]^{\gamma_{r-u}-1} f(x)\, dx. \quad (8)$$

On using (3) and (4) in (8), we obtain

$$\mu_{r:n,m,k}^{(j)} = \frac{\lambda\theta}{a^\theta} \frac{C_{r-1}}{(r-1)!(m+1)^{r-1}} \sum_{u=0}^{r-1} (-1)^u \binom{r-1}{u} e^{\lambda\gamma_{r-u}} \int_a^\infty x^{j+\theta-1} e^{-\frac{\lambda}{a^\theta}\gamma_{r-u} x^\theta}\, dx$$

$$\mu_{r:n,m,k}^{(j)} = \frac{C_{r-1}}{(r-1)!(m+1)^{r-1}} \frac{\lambda\theta}{a^\theta} \sum_{u=0}^{r-1} (-1)^u \binom{r-1}{u} e^{\lambda\gamma_{r-u}} \int_a^\infty x^{j+\theta-1} e^{-\frac{\lambda}{a^\theta}\gamma_{r-u} x^\theta}\, dx. \quad (9)$$

From [5] p-346, we have

$$\int_u^\infty x^m e^{-\beta x^n}\, dx = \frac{\Gamma((m+1)/n, \beta u^n)}{n\beta^{(m+1)/n}}, \qquad \beta, m, n > 0. \quad (10)$$

By substituting (10) in (9) and simplifying the resulting expression, we get the required result for $m \neq -1$.
When $m = -1$, we have

$$\mu_{r:n,-1,k}^{(j)} = \frac{k^r}{(r-1)!} \int_a^\infty x^j \left[-\ln \bar{F}(x)\right]^{r-1} \left[\bar{F}(x)\right]^{k-1} f(x)\, dx.$$

From (3) and (4), we have

$$\mu_{r:n,-1,k}^{(j)} = \frac{k^r}{(r-1)!} \frac{\lambda^r\theta}{a^\theta} e^{\lambda k} \int_a^\infty x^{j+\theta-1} \left(\left(\frac{x}{a}\right)^\theta - 1\right)^{r-1} e^{-\lambda k\left(\frac{x}{a}\right)^\theta}\, dx$$

$$\mu_{r:n,-1,k}^{j} = \frac{(\lambda k)^r e^{\lambda k}}{(r-1)!\theta} \sum_{u=0}^{r-1} (-1)^{r-u-1} \binom{r-1}{u} \frac{1}{a^{\theta(u+1)}}$$
$$\times \int_a^\infty x^{j+\theta(u+1)-1} e^{-\frac{\lambda k}{a^\theta} x^\theta}\, dx. \quad (11)$$

On using (10) in (11), we get the required result for $m = -1$. □

Special Cases

(i) For $m = 0$, $k = 1$ in (6), we get the explicit formula for the single moments of ordinary order statistics for the OGEPD as

$$\mu_{r:n}^{(j)} = C_{r:n} a^j \lambda^{-j/\theta} \sum_{u=0}^{r-1} (-1)^u \binom{r-1}{u} e^{\lambda(n-r+u+1)} \Gamma\left((j/\theta+1), \lambda(n-r+u+1)\right)$$
$$\times (n-r+u+1)^{-(j/\theta+1)}.$$

(ii) For $k = 1$ in (7), we get the explicit expression for the single moments of upper record values for the OGEPD as

$$\mu_{r:n,-1,1}^{(j)} = \frac{e^\lambda a^j}{(r-1)!} \sum_{u=0}^{r-1} (-1)^{r-u-1} \binom{r-1}{u} \frac{\Gamma\left((j/\theta+u+1), \lambda\right)}{\lambda^{j/\theta+u-r+1}}.$$

2.2. Recurrence relation for single moments

Theorem 2
If X is a *rv* with *df* (3) with $2 \leq r \leq n$, $j \geq 0$, $k \geq 1$ the following recurrence relation

$$\mu_{r:n,m,k}^{(j)} = \mu_{r-1:n,m,k}^{(j)} + \frac{a^{\theta} j}{\lambda\theta\gamma_r} \mu_{r:n,m,k}^{(j-\theta)}, \qquad m \neq -1 \tag{12}$$

is satisfied.

Proof
To obtain the recurrence relation for single moments of *gos*, we use the result of [2] for the OGEPD

$$\mu_{r:n,m,k}^{(j)} - \mu_{r-1:n,m,k}^{(j)} = \frac{jC_{r-1}}{\gamma_r\,(r-1)\,!} \int_a^{\infty} x^{j-1} \left[\bar{F}\,(x)\right]^{\gamma_r} g_m^{r-1}\,(F\,(x))\,dx. \tag{13}$$

Using (5) in (13), we get

$$\mu_{r:n,m,k}^{(j)} = \mu_{r-1:n,m,k}^{(j)} + \frac{a^{\theta} jC_{r-1}}{\lambda\theta\gamma_r\,(r-1)\,!} \int_a^{\infty} x^{j-\theta} \left[\bar{F}\,(x)\right]^{\gamma_r - 1} g_m^{r-1}\,(F\,(x))\,f\,(x)\,dx.$$

On simplifying the above expression, we get the required result of (12). □

Special Cases

(i) For $m = 0$, $k = 1$ in (12), we get the recurrence relations for single moments of ordinary order statistics for the OGEPD as

$$\mu_{r:n}^{(j)} = \mu_{r-1:n}^{(j)} + \frac{a^{\theta} j}{\lambda\theta\,(n-r+1)} \mu_{r:n}^{(j-\theta)}.$$

(i) For $m = -1$, $k = 1$ in (12), we get the recurrence relations for single moments of upper record values for the OGEPD as

$$\mu_{r:n,-1,1}^{(j)} = \mu_{r-1:n,-1,1}^{(j)} + \frac{a^{\theta} j}{\lambda\theta} \mu_{r:n,-1,1}^{(j-\theta)}.$$

Table 1(a): First Four Moments of Order Statistics

		$a=0.5,\lambda=1,\theta=0.5$				$a=0.5,\lambda=1.5,\theta=0.5$			
n	r	$\mu^{(1)}$	$\mu^{(2)}$	$\mu^{(3)}$	$\mu^{(4)}$	$\mu^{(1)}$	$\mu^{(2)}$	$\mu^{(3)}$	$\mu^{(4)}$
1	1	2.50000	16.2500	244.620	6850.11	1.61110	5.21290	35.3780	440.660
2	1	1.25000	2.62500	10.3438	72.7188	0.94444	1.21296	2.39660	7.68527
	2	3.75000	29.8750	478.906	13627.4	2.27778	9.21296	68.3596	873.628
3	1	0.94444	1.21296	2.39660	7.68527	0.77161	0.70085	0.81081	1.29473
	2	1.86111	5.44907	26.2380	202.786	1.29012	2.23720	5.56819	20.4663
	3	4.69444	42.0880	705.240	20339.7	2.77160	12.7008	99.7553	1300.21
4	1	0.81250	0.80467	1.06689	2.05457	0.69444	0.53241	0.47184	0.512817
	2	1.34028	2.43778	6.38574	24.5774	1.00309	1.20616	1.82774	3.64048
	3	2.38194	8.46036	46.0903	380.994	1.57716	3.26823	9.30864	37.2922
	4	5.46528	53.2972	924.957	26992.6	3.16975	15.8451	129.904	1721.18
5	1	0.74000	0.62760	0.65156	0.89225	0.65111	0.45279	0.34574	0.30127
	2	1.10250	1.51304	2.72823	6.70384	0.86778	0.85089	0.97621	1.35901
	3	1.69694	3.82492	11.872	51.3877	1.20605	1.73906	3.10503	7.0627
	4	2.83861	11.5507	68.9026	600.732	1.82457	4.28768	13.4444	57.4452
	5	6.12194	63.7338	1138.97	33590.6	3.50605	18.7344	159.019	2137.11
		$a=1,\lambda=2,\theta=1$				$a=1.5,\lambda=2,\theta=2$			
1	1	1.50000	2.50000	4.75000	10.5000	1.81600	3.37500	6.43945	12.6556
2	1	1.25000	1.62500	2.21875	3.21875	1.66975	2.81250	4.78385	8.22656
	2	1.75000	3.37500	7.28125	17.7813	1.96230	3.93750	8.09524	17.0859
3	1	1.16667	1.38889	1.69444	2.12963	1.61648	2.62500	4.28427	7.03125
	2	1.41667	2.09722	3.26736	5.39699	1.77631	3.18750	5.78303	10.6172
	3	1.91667	4.01389	9.28819	23.9734	1.58874	2.53125	4.04525	6.48633
4	1	1.12500	1.28125	1.48047	1.74023	1.69968	2.90625	5.00132	8.66602
	2	1.29167	1.71181	2.33637	3.29782	1.69968	2.90625	5.00132	8.66602
	3	1.54167	2.48264	4.19835	7.49617	1.85293	3.46875	6.56473	12.5684
	4	2.04167	4.52431	10.9848	29.4658	2.12275	4.59375	10.1469	22.9043
5	1	1.10000	1.22000	1.36600	1.54640	1.57171	2.47500	3.90545	6.17625
	2	1.22500	1.52625	1.93834	2.51557	1.65690	2.75625	4.60445	7.72664
	3	1.39167	1.99014	2.93341	4.47118	1.76385	3.13125	5.59662	10.0751
	4	1.64167	2.81097	5.04164	9.51282	1.91232	3.69375	7.21014	14.2305
	5	2.14167	4.95264	12.4706	34.4540	2.17536	4.81875	10.8811	25.0727

Table 1(b): First Four Moments of Upper Record Values

	$a=1,\lambda=1.5,\theta=1.5$				$a=1,\lambda=1.5,\theta=2$			
r	$\mu^{(1)}$	$\mu^{(2)}$	$\mu^{(3)}$	$\mu^{(4)}$	$\mu^{(1)}$	$\mu^{(2)}$	$\mu^{(3)}$	$\mu^{(4)}$
2	2.1815	2.3019	1.7407	0.9008	2.0469	2.3330	2.2169	1.7407
3	5.3586	7.6724	8.4676	7.2166	4.7019	6.6249	8.0576	8.4676
4	6.5712	10.015	11.276	9.1401	5.6509	8.4167	10.622	11.276
5	8.4753	15.086	20.191	20.352	7.0222	11.716	16.667	20.191
6	9.8931	19.427	28.089	29.546	7.9757	14.395	21.931	28.089
	$a=1,\lambda=0.5,\theta=2.5$				$a=1,\lambda=0.5,\theta=3$			
2	1.2316	3.0133	8.7309	31.545	1.0698	2.2041	5.0000	13.055
3	2.2616	5.1485	15.429	59.819	2.0309	3.7902	8.6253	23.565
4	2.5384	6.6505	22.047	93.163	2.2088	4.703	11.746	34.691
5	2.9423	8.2108	29.176	131.96	2.5389	5.6769	15.018	46.973
6	3.2511	9.6893	36.614	175.62	2.7729	6.5583	18.283	60.130

3. Relations for inverse moments

In this section, we derive recurrence relation for inverse moments of *gos*. The inverse moments of *gos* are defined as

$$\mu_{r:n,m,k}^{(-j)} = E\left(X_{r:n,m,k}^{-j}\right) = \int_{-\infty}^{\infty} x^{-j} f_{r:n,m,k}(x)\, dx. \tag{14}$$

Theorem 3
Fix a positive integer k and for *df* (3) with $2 \leq r \leq n$, $j = 0, 1, 2, ...$ the following recurrence relation

$$\mu_{r:n,m,k}^{(-j)} = \mu_{r-1:n,m,k}^{(-j)} - \frac{ja^{\theta}}{\gamma_r \lambda\theta} \mu_{r:n,m,k}^{(-j-\theta)}, \qquad m \neq -1 \tag{15}$$

is satisfied.

Proof
From (14) and (1), we have

$$\mu_{r:n,m,k}^{(-j)} = \frac{C_{r-1}}{(r-1)\,!} \int_a^{\infty} x^{-j} \left[\bar{F}(x)\right]^{\gamma_{r-1}} f(x)\, g_m^{r-1}(F(x))\, dx.$$

Proceeding in similar manner as we have done for Theorem 2, we get

$$\mu_{r:n,m,k}^{(-j)} = \mu_{r-1:n,m,k}^{(-j)} - \frac{jC_{r-1}}{\gamma_r (r-1)\,!} \int_a^{\infty} x^{-j-1} \left[\bar{F}(x)\right]^{\gamma_r} g_m^{r-1}(F(x))\, dx.$$

By using (5) in above expression, we have

$$\mu_{r:n,m,k}^{(-j)} = \mu_{r-1:n,m,k}^{(-j)} - \frac{jC_{r-1}}{\gamma_r (r-1)\,!} \frac{a^{\theta}}{\lambda\theta} \int_a^{\infty} x^{-j-\theta} \left[\bar{F}(x)\right]^{\gamma_r - 1} g_m^{r-1}(F(x))\, f(x)\, dx. \tag{16}$$

On simplifying (16), we get the required result of (15). □

Special Cases

(i) For $m = 0$, $k = 1$ in (15), we get the recurrence relation for inverse moments of ordinary order statistics for the OGEPD as

$$\mu_{r:n}^{(-j)} = \mu_{r-1:n}^{(-j)} - \frac{a^{\theta} j}{\lambda\theta(n-r+1)} \mu_{r:n}^{(-j-\theta)}.$$

(ii) For $m = -1$, $k = 1$ in (15), we get the recurrence relation for inverse moments of upper record values for the OGEPD as

$$\mu_{r:n,-1,1}^{(-j)} = \mu_{r-1:n,-1,1}^{(-j)} - \frac{a^{\theta} j}{\lambda\theta} \mu_{r:n,-1,1}^{(-j-\theta)}.$$

4. Relations for product moments

Theorem 4
For the OGEPD given in (3), $1 \leq r < s \leq n-1$, $k \geq 1$, $j \geq 0$ the following recurrence relation

$$\mu_{r,s:n,m,k}^{(i,j)} = \mu_{r,s-1:n,m,k}^{(i,j)} + \frac{a^{\theta} j}{\lambda\theta\gamma_s} \mu_{r,s:n,m,k}^{(i,j-\theta)}, m \neq -1 \tag{17}$$

is satisfied.

Proof
From (2), we have

$$\mu_{r,s:n,m,k}^{(i,j)} = \frac{C_{s-1}}{(r-1)!\,(s-r-1)!} \int_{-\infty}^{\infty} \int_{x}^{\infty} x^i y^j \left[\bar{F}(x)\right]^m f(x)\, g_m^{r-1}(F(x))\, [h_m F(y) - h_m F(x)]^{s-r-1} \times \left[\bar{F}(y)\right]^{\gamma_s - 1} f(y)\, dy\, dx.$$

By considering the result of [2] for product moments of any distribution, we have

$$\mu_{r,s:n,m,k}^{(i,j)} = \mu_{r,s-1:n,m,k}^{(i,j)} + \frac{C_{s-1} j}{\gamma_s (r-1)!\,(s-r-1)!} \int_{-\infty}^{\infty} \int_{x}^{\infty} x^i y^{j-1} \left[\bar{F}(x)\right]^m f(x)\, g_m^{r-1}(F(x)) [h_m F(y) - h_m F(x)]^{s-r-1} \left[\bar{F}(y)\right]^{\gamma_s} dy dx. \quad (18)$$

Using the relation (5) in (18) for the OGEPD, we have

$$\mu_{r,s:n,m,k}^{(i,j)} = \mu_{r,s-1:n,m,k}^{(i,j)} + \frac{C_{s-1} j}{\gamma_s (r-1)!\,(s-r-1)!} \frac{a^\theta}{\lambda\theta} \int_{a}^{\infty} \int_{x}^{\infty} x^i y^{j-\theta} \left[\bar{F}(x)\right]^m f(x)\, g_m^{r-1}(F(x)) [h_m F(y) - h_m F(x)]^{s-r-1} f(y) \left[\bar{F}(y)\right]^{\gamma_s - 1} dy dx. \quad (19)$$

On simplifying (19), we get the result of (17). □

Special Cases

(i) For $m = 0$, $k = 1$ in (17), we get the recurrence relations for product moments of ordinary order statistics for the OGEPD as

$$\mu_{r,s:n}^{(i,j)} = \mu_{r,s-1:n}^{(i,j)} + \frac{a^\theta j}{\lambda\theta\,(n-s+1)} \mu_{r,s:n}^{(i,j-\theta)}.$$

(ii) For $m = -1$, $k = 1$ in (17), we get the recurrence relations for product moments of upper record values for the OGEPD as

$$\mu_{r,s:n,-1,1}^{(i,j)} = \mu_{r,s-1:n,-1,1}^{(i,j)} + \frac{a^\theta j}{\lambda\theta} \mu_{r:n,-1,1}^{(i,j-\theta)}.$$

5. Relations for ratio moments

The ratio moments of *gos* are defined as

$$\mu_{r,s:n,m,k}^{(i,-j)} = \int_{-\infty}^{\infty} \int_{x}^{\infty} \frac{x^i}{y^j} f_{r,s:n,m,k}(x,y)\, dy\, dx = \int_{-\infty}^{\infty} \int_{x}^{\infty} x^i y^{-j} f_{r,s:n,m,k}(x,y)\, dy\, dx. \quad (20)$$

Theorem 5
For the OGEPD given in (3), $1 \le r < s \le n-1$, $k \ge 1$ and $j \ge 0$ the following recurrence relation for ratio moments

$$\mu_{r,s:n,m,k}^{(i,-j)} = \mu_{r,s-1:n,m,k}^{(i,-j)} - \frac{a^\theta j}{\lambda\theta\gamma_s} \mu_{r,s:n,m,k}^{(i,-j-\theta)}, \qquad m \neq -1 \quad (21)$$

is satisfied.

Proof
We derive the ratio moments for the OGEPD by using (20) and proceeding in similar way as Theorem 4, we have

$$\mu_{r,s:n,m,k}^{(i,-j)} = \mu_{r,s-1:n,m,k}^{(i,-j)} - \frac{C_{s-1} j}{\gamma_s (r-1)!\,(s-r-1)!} \int_{a}^{\infty} \int_{x}^{\infty} x^i y^{-j-1} \left[\bar{F}(x)\right]^m f(x)\, g_m^{r-1}(F(x)) \times [h_m F(y) - h_m F(x)]^{s-r-1} \left[\bar{F}(y)\right]^{\gamma_s} dy dx.$$

On using (5), we get the result of (21) . □

Special cases

(i) For $m=0$, $k=1$ in (21), we get the recurrence relations for ratio moments of ordinary order statistics for the OGEPD as

$$\mu_{r,s:n}^{(i,-j)} = \mu_{r,s-1:n}^{(i,-j)} - \frac{a^{\theta} j}{\lambda\theta\,(n-s+1)} \mu_{r,s:n}^{(i,-j-\theta)}.$$

(ii) For $m=-1$, $k=1$ in (21), we get the recurrence relations for ratio moments of upper record values for the OGEPD as

$$\mu_{r,s:n,-1,1}^{(i,-j)} = \mu_{r,s-1:n,-1,1}^{(i,-j)} - \frac{a^{\theta} j}{\lambda\theta} \mu_{r:n,-1,1}^{(i,-j-\theta)}.$$

6. Characterization

6.1. Characterization of OGEPD based on a recurrence relation for single moments

Theorem 6
For the positive integers k and j, a necessary and sufficient condition for a random variable X to be distributed with *cdf* given in (3) is that

$$\mu_{r:n,m,k}^{(j)} = \mu_{r-1:n,m,k}^{(j)} + \frac{a^{\theta} j}{\lambda\theta\gamma_r} \mu_{r:n,m,k}^{(j-\theta)}. \tag{22}$$

Proof
The necessary part follows immediately from (12). On the other hand if the recurrence relation in (22) is satisfied, then from (1), we have

$$\frac{C_{r-1}}{\gamma_r\,(r-1)\,!} \int_a^{\infty} x^j \left[\bar{F}(x)\right]^{\gamma_r} g_m^{r-2}(F(x))\, f(x) \left(\frac{\gamma_r g_m(F(x))}{\bar{F}(x)} - (r-1)\left[\bar{F}(x)\right]^m \right) dx = \frac{C_{r-1} j}{(r-1)\,!\gamma_r} \frac{a^{\theta}}{\lambda\theta} \times \int_a^{\infty} x^{j-\theta} \left[\bar{F}(x)\right]^{\gamma_r - 1} g_m^{r-1}(F(x))\, f(x)\, dx. \tag{23}$$

Consider $\xi(x) = -g_m^{r-1}(F(x)) \left[\bar{F}(x)\right]^{\gamma_r}$.
Differentiating $\xi(x)$ with respect to x, we have

$$\xi'(x) = g_m^{r-2}(F(x))\, f(x) \left[\bar{F}(x)\right]^{\gamma_r} \left[\frac{g_m(F(x))\,\gamma_r}{\bar{F}(x)} - (r-1)\left[\bar{F}(x)\right]^m \right].$$

Thus, we have

$$\frac{C_{r-1}}{(r-1)\,!\gamma_r} \int_a^{\infty} x^j \xi'(x)\, dx = -\frac{C_{r-1}\, j}{(r-1)\,!\gamma_r} \frac{a^{\theta}}{\lambda\theta} \int_a^{\infty} x^{j-\theta} \xi(x) \left[\bar{F}(x)\right]^{-1} f(x) dx. \tag{24}$$

Integrating left hand side of (24) and using the expression of $\xi(x)$, we have

$$\frac{C_{r-1}}{(r-1)\,!\gamma_r} \int_a^{\infty} x^{j-1} g_m^{r-1}(F(x)) \left[\bar{F}(x)\right]^{\gamma_r} \left(1 - \frac{a^{\theta}}{\lambda\theta} x^{1-\theta} \frac{f(x)}{\bar{F}(x)} \right) = 0.$$

Using generalization result of Müntz-Szász Theorem of [7], we get $\frac{\bar{F}(x)}{f(x)} = \frac{a^{\theta}}{\lambda\theta} x^{1-\theta}$ which proves that,

$$\bar{F}(x) = \frac{a^{\theta}}{\lambda\theta} x^{1-\theta} f(x),\, x > a,\ \theta, \lambda > 0.$$

By simplifying above expression, we can also get the following result,

$$f(x) = \frac{\lambda\theta}{a^{\theta}} x^{\theta-1} e^{-\lambda\left[\left(\frac{x}{a}\right)^{\theta} - 1\right]},\, x > a,\ \theta, \lambda > 0.$$

□

6.2. *Characterization of OGEPD based on conditional moments*

Theorem 7
If X is an absolutely continuous positive random variable with *df* $G\left(x\right)$and $g\left(x\right)$ such that $E\left(X\right)$ exist, then $E\left(X|X\leq x\right)=h\left(x\right)\varphi\left(x\right)$ where $\varphi\left(x\right)=g\left(x\right)/G\left(x\right)$, then

$$h\left(x\right)=\frac{a^{\theta+1}}{\lambda\theta}x^{1-\theta}e^{\lambda\left[\left(\frac{x}{a}\right)^{\theta}-1\right]}-\frac{a^{\theta}}{\lambda\theta}x^{2-\theta}+\frac{a^{\theta}}{\lambda\theta}x^{1-\theta}e^{\lambda\left[\left(\frac{x}{a}\right)^{\theta}-1\right]}\int_{a}^{x}e^{-\lambda\left[\left(\frac{u}{a}\right)^{\theta}-1\right]}du \tag{25}$$

holds if and only if

$$g\left(x\right)=\frac{\lambda\theta}{a^{\theta}}x^{\theta-1}e^{-\lambda\left[\left(\frac{x}{a}\right)^{\theta}-1\right]},\qquad x\geq a,\ \theta,\lambda>0.$$

Proof
We first prove the necessity part. For this, we consider

$$g\left(x\right)=\frac{\lambda\theta}{a^{\theta}}x^{\theta-1}e^{-\lambda\left[\left(\frac{x}{a}\right)^{\theta}-1\right]},$$

then, we take

$$h\left(x\right)=\frac{1}{g\left(x\right)}\int_{a}^{x}uf\left(u\right)du. \tag{26}$$

On simplifying (26) and using $g(x)$, we get

$$h\left(x\right)=\frac{a^{\theta+1}}{\lambda\theta}x^{1-\theta}e^{\lambda\left[\left(\frac{x}{a}\right)^{\theta}-1\right]}-\frac{a^{\theta}}{\lambda\theta}x^{2-\theta}+\frac{a^{\theta}}{\lambda\theta}x^{1-\theta}e^{\lambda\left[\left(\frac{x}{a}\right)^{\theta}-1\right]}\int_{a}^{x}e^{-\lambda\left[\left(\frac{u}{a}\right)^{\theta}-1\right]}du.$$

which is (25).
Now for sufficient condition we need to prove (25) which implies (4). We consider (25) as

$$h\left(x\right)=\frac{a^{\theta+1}}{\lambda\theta}x^{1-\theta}e^{\lambda\left[\left(\frac{x}{a}\right)^{\theta}-1\right]}-\frac{a^{\theta}}{\lambda\theta}x^{2-\theta}+\frac{a^{\theta}}{\lambda\theta}x^{1-\theta}e^{\lambda\left[\left(\frac{x}{a}\right)^{\theta}-1\right]}\int_{a}^{x}e^{-\lambda\left[\left(\frac{u}{a}\right)^{\theta}-1\right]}du.$$

Differentiating above equation with respect to x, we have

$$h'\left(x\right)=\left(\frac{(1-\theta)}{x}+\frac{\lambda\theta}{a^{\theta}}x^{\theta-1}\right)\left(\frac{a^{\theta+1}}{\lambda\theta}x^{1-\theta}e^{\lambda\left[\left(\frac{x}{a}\right)^{\theta}-1\right]}+\frac{a^{\theta}}{\lambda\theta}x^{1-\theta}e^{\lambda\left[\left(\frac{x}{a}\right)^{\theta}-1\right]}\int_{a}^{x}e^{-\lambda\left[\left(\frac{u}{a}\right)^{\theta}-1\right]}du\right)-\frac{(1-\theta)}{\lambda\theta}a^{\theta}x^{1-\theta}.$$

Thus we have,

$$\frac{x-h'\left(x\right)}{h\left(x\right)}=-\left(\frac{(1-\theta)}{x}+\frac{\lambda\theta}{a^{\theta}}x^{\theta-1}\right).$$

Hence, using the result of Lemma 6.1 from [16], we have

$$\frac{g'\left(x\right)}{g\left(x\right)}=-\left(\frac{(1-\theta)}{x}+\frac{\lambda\theta}{a^{\theta}}x^{\theta-1}\right).$$

On integrating, we get

$$g\left(x\right)=c\frac{e^{-\lambda\left(\frac{x}{a}\right)^{\theta}}}{x^{1-\theta}},$$

where c is determined by using $\int_{a}^{\infty}g\left(x\right)dx=1$, hence, we get

$$g\left(x\right)=\frac{\lambda\theta}{a^{\theta}}x^{\theta-1}e^{-\lambda\left[\left(\frac{x}{a}\right)^{\theta}-1\right]},x>a,\ \ \theta,\lambda>0,$$

which implies (4). □

Theorem 8
Suppose that X is an absolutely continuous random variable with *cdf* $G(x)$ such that $G(a) = 0$ and $G(x) > 0$ for all $x > a$. We assume that the *pdf* of X and $g(x)$ and $g'(x)$ exists for all $x > 0$ and $E(X)$ also exists. Then $E(X|X \geq x) = h_0(x)\varsigma(x)$, where $\varsigma(x) = g(x)/[1 - G(x)]$,

$$h_0(x) = x^{1-\theta} e^{\lambda\left[\left(\frac{x}{a}\right)^{\theta}-1\right]} \int_x^{\infty} u^{\theta} e^{-\lambda\left[\left(\frac{u}{a}\right)^{\theta}-1\right]} du,$$

holds if and only if

$$g(x) = \frac{\lambda\theta}{a^{\theta}} x^{\theta-1} e^{-\lambda\left[\left(\frac{x}{a}\right)^{\theta}-1\right]}, \qquad x \geq a,\ \theta, \lambda > 0.$$

Proof
We have

$$g(x)\, h_0(x) = \frac{\lambda\theta}{a^{\theta}} \int_x^{\infty} u^{\theta} e^{-\lambda\left[\left(\frac{u}{a}\right)^{\theta}-1\right]} du$$

$$h_0(x) = x^{1-\theta} e^{\lambda\left[\left(\frac{x}{a}\right)^{\theta}-1\right]} \int_x^{\infty} u^{\theta} e^{-\lambda\left[\left(\frac{u}{a}\right)^{\theta}-1\right]} du.$$

By differentiating the above expression, we have

$$h_0'(x) = x + \left(\frac{(1-\theta)}{x} + \frac{\lambda\theta}{a^{\theta}} x^{\theta-1}\right) x^{1-\theta} e^{\lambda\left[\left(\frac{x}{a}\right)^{\theta}-1\right]} \int_x^{\infty} u^{\theta} e^{-\lambda\left[\left(\frac{u}{a}\right)^{\theta}-1\right]} du$$

$$h_0'(x) = x + \left(\frac{(1-\theta)}{x} + \frac{\lambda\theta}{a^{\theta}} x^{\theta-1}\right) h_0(x).$$

Using result of Lemma 6.2 from [16], we have

$$-\frac{x - h_0'(x)}{h_0(x)} = -\left(\frac{(1-\theta)}{x} + \frac{\lambda\theta}{a^{\theta}} x^{\theta-1}\right)$$

$$\frac{g'(x)}{g(x)} = -\frac{(1-\theta)}{x} - \frac{\lambda\theta}{a^{\theta}} x^{\theta-1}.$$

Integrating both sides with respect to x, we have

$$g(x) = c x^{\theta-1} e^{-\lambda\left(\frac{x}{a}\right)^{\theta}},$$

where c is a constant determined by using the condition $\int_a^{\infty} g(x)\, dx = 1$, hence, we get

$$g(x) = \frac{\lambda\theta}{a^{\theta}} x^{\theta-1} e^{-\lambda\left[\left(\frac{x}{a}\right)^{\theta}-1\right]}, x \geq a,\ \theta, \lambda > 0,$$

which implies (4). □

7. Conclusion

In this paper, exact form for single moments from OGEPD has been established in conjunction with derivations for some recurrence relations for single, inverse, product and ratio moments. Further, first four moments of *gos* and upper record values from the said distribution have been calculated. A characterization by two methods is also given.

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