Mid-Truncated Burr XII distribution and its applications in order statistics

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Abstract   In this paper we define the Mid-Truncated Burr XII distribution and derive some of its statistical properties such as moments, moment generating function, characteristic function, likelihood function, etc. We have also obtained the recurrence relations for single and product moments of order statistics in a random sample of size n drawn from Mid-Truncated Burr XII distribution. Further, the characterization of Mid-Truncated distributions using conditional moments has been given.

1. Introduction

Many authors like Malik [10], Balakrishnan and Joshi [1], Balakrishnan et al. [2], Saran and Pushkarna [17, 18, 19], etc. have obtained several results for the single and product moments of order statistics from the un-truncated, left truncated, right truncated and doubly truncated distributions. Mohie El-Din and Sultan [12] have obtained recurrence relations for moments of order statistics from doubly truncated continuous distributions. They have also presented some applications of their results. Mohie El-Din et al. [11] have obtained moment generating functions of order statistics from doubly truncated exponential distribution in terms of hypergeometric function. They have also derived some recurrence relations between these moment generating functions.The truncated distributions are quite effectively used where a random variable is restricted to be observed on some range and these situations are common in various fields. For instance, in survival analysis, failures during the warranty period may not be counted. Therefore many researchers were being attracted to the problem of analysing such truncated data encountered in various disciplines, proposed the truncated versions of the usual statistical distributions, to improve a forecasting actuarial model and particularly for modelling data from insurance policies. But there exist life models which do not obey the complete or truncated distribution, for example, in Microbiology (cf. Tortora et al. [21]), when a bacterial strain is inoculated into a liquid growth medium, the population is counted at intervals, it is possible to plot a bacterial growth curve. There are three basic phases of growth: the log, stationary, and death phases. During log phase, the bacterial cells are most metabolically active and are preferred for industrial purposes. During stationary phase we find that the number of cells will still remain constant due to the number of bacterial death balances with the number of new cells. Consequently during this phase no investigation is required. After the stationary phase the most bacterial cells will die because of the exclusion of nutrients and lead to accumulation of waste products. In such live models mid-truncated distributions are quite effectively used where the random variable is restricted

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to be observed on some sub intervals of the given specified range (cf. Okasha et al. [14] and Mohie El-Din et al. [13]).

Burr system of distributions was constructed in 1942 by Irving W. Burr [3]. Since the corresponding density functions have a wide variety of shapes, this system is useful for approximating histograms, particularly when a simple mathematical structure for the fitted cumulative distribution function (cdf) is required. Other applications include simulation, quantal response, approximation of distributions, and development of non-normal control charts. Pathak and Chaturvedi [15] evaluated reliability function for Burr (X) distribution for its wide variety of application in life time data. Recently work on optimal sample size determination for Burr (XII) population has been done by Hashempour [5]. According to Soliman [20], this distribution covers the curve shape characteristic for a large number of distributions. Due to versatility and flexibility of the Burr (XII) distribution, this distribution is quite attractive as a tentative model for data whose underlying distribution is unknown.

In this paper, we propose to consider the mid-truncated Burr distribution (MTBD) (Type XII) with three parameters, which contains, as special sub models, mid-truncated Weibull distribution, mid-truncated exponential distribution, mid-truncated log-logistic distribution, among others. Due to the flexibility of the considered distribution in accommodating different forms of risk function, this MTBD is appropriate for a variety of problems modelling lifetime data.

2. Mid-truncated distribution

We define the Mid-truncated distribution as follows:

Let Y be a continuous random variable with baseline probability density function (pdf) $g(y)$, cumulative distribution function (cdf) $G(y)$, define X as a corresponding mid-truncated variable, of the random variable Y with pdf $f(x)$. We define

$$f(x) = \begin{cases} p\frac{g(x)}{G(Q_1)}, & \text{if } x \leq Q_1 \\ q\frac{g(x)}{1-G(P_1)}, & \text{if } x \geq P_1 \end{cases}, \quad 0 \leq p \leq 1,\ 0 \leq q \leq 1,\ \text{s.t. } p+q=1,\ Q_1 \leq P_1, \tag{2.1}$$

which is called the Mid-truncated density function and, Q_1 and P_1 are the points of mid truncation of the baseline distribution under consideration. Also we assume that

$$Q = \int_{-\infty}^{Q_1} g(x)dx, \tag{2.2}$$

and

$$1 - P = \int_{P_1}^{\infty} g(x)dx. \tag{2.3}$$

Then equation (2.1) can be written as

$$f(x) = \begin{cases} p\frac{g(x)}{Q}, & \text{if } x \leq Q_1 \\ q\frac{g(x)}{1-P}, & \text{if } x \geq P_1 \end{cases}, \quad 0 \leq p \leq 1,\ 0 \leq q \leq 1,\ \text{s.t. } p+q=1,\ Q_1 \leq P_1. \tag{2.4}$$

2.1. k-th moment

The k-th moment of a mid-truncated random variable X (denoted by $\mu_*^{(k)}$) is given by:

$$\mu_*^{(k)} = \frac{p}{G(Q_1)} \int_{-\infty}^{Q_1} x^k g(x)dx + \frac{q}{1-G(P_1)} \int_{P_1}^{\infty} x^k g(x)dx.$$

Integrating by parts, we get

$$\mu_*^{(k)} = \frac{p}{G(Q_1)} \left[Q_1^k G(Q_1) - \int_{-\infty}^{Q_1} kx^{k-1}G(x)dx \right] + \frac{q}{(1-G(P_1))} \left[\mu^{(k)} - P_1^k G(P_1) + \int_{-\infty}^{P_1} kx^{k-1}G(x)dx \right],$$
(2.5)

where $\mu^{(k)}$ is the k-th moment of the un-truncated distribution.

2.2. *Moment generating function and characteristic function*

The moment generating function of a mid-truncated random variable X (denoted by $M^*(t)$) is given by:

$$M^*(t) = \frac{p}{G(Q_1)} \int_{-\infty}^{Q_1} e^{xt}g(x)dx + \frac{q}{1-G(P_1)} \int_{P_1}^{\infty} e^{tx}g(x)dx.$$

Integrating by parts, we get

$$M^*(t) = pe^{tQ_1} - \frac{p}{G(Q_1)} \int_{-\infty}^{Q_1} te^{xt}G(x)dx + \frac{q}{1-G(P_1)} \left[M(t) - e^{tP_1}G(P_1) + \int_{-\infty}^{P_1} te^{tx}G(x)dx \right], \quad (2.6)$$

where $M(t)$ is the moment generating function of the un-truncated random variable.

Similarly, the characteristic function of the mid-truncated random variable is given by:

$$\Phi^*(t) = pe^{itQ_1} - \frac{p}{G(Q_1)} \int_{-\infty}^{Q_1} ite^{ixt}G(x)dx + \frac{q}{1-G(P_1)} \left[\Phi(t) - e^{itP_1}G(P_1) + \int_{-\infty}^{P_1} ite^{itx}G(x)dx \right],$$

where $\Phi(t)$ is the characteristic function of the un-truncated random variable.

2.3. *Distribution function*

The distribution function of a mid-truncated random variable X is given by:

$$F(x) = \begin{cases} p\frac{G(x)}{G(Q_1)}, & -\infty < x \le Q_1 \\ 1 - q\frac{1-G(x)}{1-G(P_1)}, & P_1 \le x < \infty. \end{cases}$$
(2.7)

2.4. *Generating data*

If the distribution function of the mid-truncated random variable exists we can use the inverse transform method to generate the data. To generate a random variable X with distribution function $F(x)$ as given in (2.7), draw U_1, U_2 from $U(0,1)$ and then solve each of the following equations

$$u_1 = \frac{G(x)}{G(Q_1)}, \quad u_2 = \left(1 - \frac{1-G(x)}{1-G(P_1)} \right)$$
(2.8)

with respect to x.

2.5. *Likelihood function*

For n observations $x_1, x_2, \ldots, x_n$ from a mid-truncated distribution, the likelihood function is given by:

$$L(\theta) = \prod_{i=1}^{r} \frac{pg(x_i)}{G(Q_1)} \prod_{i=r+1}^{n} \frac{qg(x_i)}{(1-G(P_1))}, \text{ where } r \text{ is the number of } x_i\text{'s} \le Q_1.$$

$$L(\theta) = \frac{p^r q^{n-r}}{(G(Q_1))^r (1-G(P_1))^{n-r}} \prod_{i=1}^{n} g(x_i).$$
(2.9)

3. Mid-truncated Burr XII distribution

The probability density function (pdf) of Burr distribution (type XII) is given by

$$g(x) = \nu\beta\lambda x^{\beta-1}(1+\lambda x^\beta)^{-(\nu+1)}, \quad 0 \le x < \infty, \ \nu, \beta, \lambda > 0, \tag{3.1}$$

and the cumulative distribution function (cdf) is given by

$$G(x) = 1 - (1+\lambda x^\beta)^{-\nu}, \quad 0 \le x < \infty, \ \nu, \beta, \lambda > 0. \tag{3.2}$$

Then the probability density function (pdf) of mid-truncated Burr distribution (type XII) is given by

$$f(x) = \begin{cases} \dfrac{p\nu\beta\lambda x^{\beta-1}(1+\lambda x^\beta)^{-(\nu+1)}}{Q}, & 0 < x \le Q_1 \\[3mm] \dfrac{q\nu\beta\lambda x^{\beta-1}(1+\lambda x^\beta)^{-(\nu+1)}}{1-P}, & P_1 \le x < \infty \end{cases} \tag{3.3}$$

and the cumulative distribution function (cdf) is given by

$$F(x) = \begin{cases} \dfrac{p(1-(1+\lambda x^\beta)^{-\nu})}{Q}, & 0 < x \le Q_1 \\[3mm] 1 - \dfrac{q(1+\lambda x^\beta)^{-\nu}}{1-P}, & P_1 \le x < \infty. \end{cases} \tag{3.4}$$

Using (3.3) and (3.4), we get the relation between pdf and cdf as

$$1 - F(x) = \begin{cases} 1 - \dfrac{p}{Q} + f(x)\dfrac{x^{1-\beta}(1+\lambda x^\beta)}{\nu\beta\lambda}, & 0 < x \le Q_1 \\[3mm] f(x)\dfrac{x^{1-\beta}(1+\lambda x^\beta)}{\nu\beta\lambda}, & P_1 \le x < \infty. \end{cases} \tag{3.5}$$

Using (2.2) and (2.3), we get

$$Q_1 = \left\{ \frac{(1-Q)^{\frac{-1}{\nu}} - 1}{\lambda} \right\}^{\frac{1}{\beta}} \tag{3.6}$$

$$P_1 = \left\{ \frac{(1-P)^{\frac{-1}{\nu}} - 1}{\lambda} \right\}^{\frac{1}{\beta}}. \tag{3.7}$$

Using (3.6) and (3.7), equations (3.3) and (3.4) can be rewritten as

$$f(x) = \begin{cases} \dfrac{p\nu\beta\lambda x^{\beta-1}(1+\lambda x^\beta)^{-(\nu+1)}}{1-(1+\lambda Q_1{}^\beta)^{-\nu}}, & 0 < x \le Q_1 \\[3mm] \dfrac{q\nu\beta\lambda x^{\beta-1}(1+\lambda x^\beta)^{-(\nu+1)}}{(1+\lambda P_1{}^\beta)^{-\nu}}, & P_1 \le x < \infty, \end{cases} \tag{3.8}$$

and

$$F(x) = \begin{cases} \dfrac{p(1-(1+\lambda x^\beta)^{-\nu})}{1-(1+\lambda Q_1{}^\beta)^{-\nu}}, & 0 < x \le Q_1 \\[3mm] 1 - \dfrac{q(1+\lambda x^\beta)^{-\nu}}{(1+\lambda P_1{}^\beta)^{-\nu}}, & P_1 \le x < \infty. \end{cases} \tag{3.9}$$

The mid-truncated Burr XII density function for $\beta = 2$, $\lambda = 4$, $\nu = 0.5$, $p = q = 0.5$ truncated at $Q_1 = 1.5$ and $P_1 = 3.25$ is provided in Figure 1.
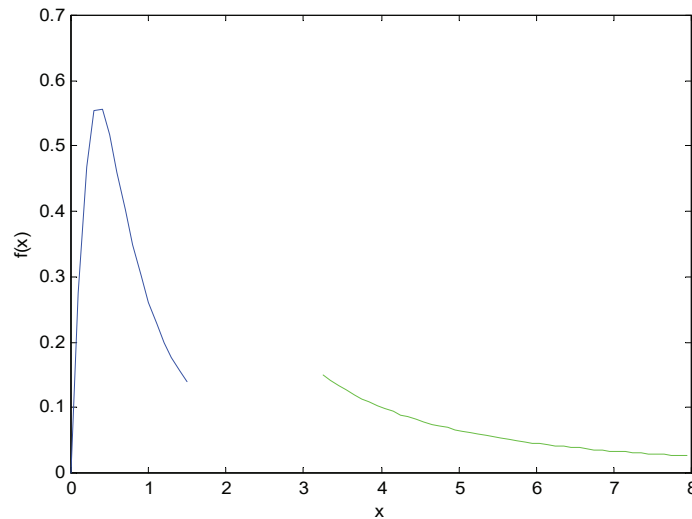

Figure 1. Probability density function for mid-truncated Burr XII distribution.

The mid-truncated Burr XII density function for $\beta = 2.6$, $\lambda = 0.5$, $\nu = 2$, $p = 0.8$ and $q = 0.2$ truncated at $Q_1 = 1.9$ and $P_1 = 5.9$ is provided in Figure 2.
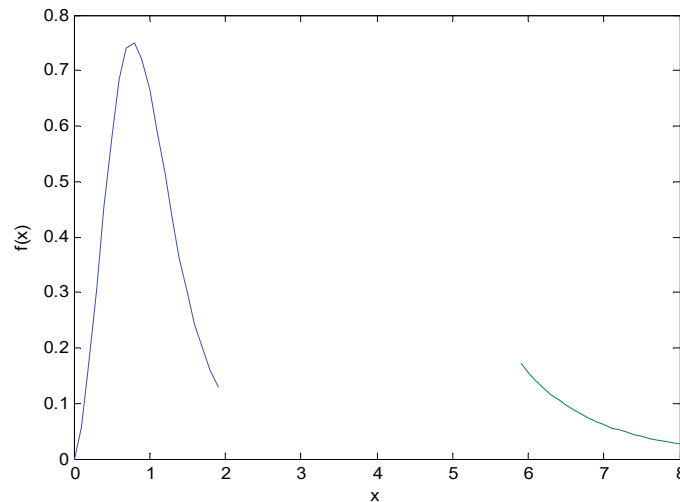

Figure 2. Probability density function for mid-truncated Burr XII distribution.

Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from the mid-truncated Burr XII distribution defined in (3.3) and let $X_{1:n} \leq X_{2:n} \leq \ldots \leq X_{n:n}$ be the corresponding order statistics. Thus the probability density function (pdf) of $X_{r:n}(1 \leq r \leq n)$ is given by:

$$f_{r:n}(x) = c_{r:n}[F(x)]^{r-1}[1 - F(x)]^{n-r}f(x), \quad 0 \leq x < \infty, \tag{3.10}$$

where $c_{r:n} = \frac{n!}{(r-1)!(n-r)!}$.

The joint density function of order statistics $X_{r:n}$ and $X_{s:n}$ $(1 \leq r < s \leq n)$ is given by

$$f_{r,s:n}(x,y) = c_{r,s:n}[F(x)]^{r-1}[F(y) - F(x)]^{s-r-1}[1 - F(y)]^{n-s}f(y)f(x), \;\; 0 \leq x < y < \infty, \qquad (3.11)$$

where

$$c_{r,s:n} = \frac{n!}{(r-1)!(s-r-1)!(n-s)!}$$

(cf. David and Nagaraja [4]).

The single moments of order statistics $X_{r:n}$ $(1 \leq r \leq n)$ are given by

$$\mu_{r:n}^{(k)} = \; E(X_{r:n}^k) = \int_0^{Q_1} x^k f_{r:n}(x)dx + \int_{P_1}^{\infty} x^k f_{r:n}(x)dx, \;\; k = 0, 1, 2, \ldots \qquad (3.12)$$

Similarly, the product moments of $X_{r:n}$ and $X_{s:n}$ $(1 \leq r < s \leq n)$ are given by

$$\mu_{r,s:n}^{(j,k)} = E(X_{r:n}^j X_{s:n}^k) = \int_0^{Q_1} \int_x^{Q_1} x^j y^k f_{r,s:n}(x,y)dydx + \int_{P_1}^{\infty} \int_x^{\infty} x^j y^k f_{r,s:n}(x,y)dydx. \qquad (3.13)$$

3.1. k-th moment

The k-th moment of the mid-truncated Burr XII distribution can be obtained using (2.5) and is given by:

$$\mu_*^{(k)} = \frac{p}{G(Q_1)}\left[Q_1^k G(Q_1) - k\int_0^{Q_1} x^{k-1}G(x)dx\right] + \frac{q}{(1-G(P_1))}\left[\mu^{(k)} - P_1^k G(P_1) + k\int_0^{P_1} x^{k-1}G(x)dx\right].$$

Substituting the values of $G(Q_1)$, $G(P_1)$ and $G(x)$ from equation (3.2) and solving, we get

$$\mu_*^{(k)} = \frac{p}{G(Q_1)}\left[-Q_1^k(1+\lambda Q_1^\beta)^{-\nu}\right] + \frac{q}{(1-G(P_1))}\left[\mu^{(k)} + P_1^k(1+\lambda P_1^\beta)^{-\nu}\right]$$

$$+ \frac{pk}{G(Q_1)}\int_0^{Q_1} x^{k-1}(1+\lambda x^\beta)^{-\nu}dx - \frac{qk}{1-G(P_1)}\int_0^{P_1} x^{k-1}(1+\lambda x^\beta)^{-\nu}dx$$

$$= \frac{p}{G(Q_1)}\left[-Q_1^k(1+\lambda Q_1^\beta)^{-\nu}\right] + \frac{q}{(1-G(P_1))}\left[\mu^{(k)} + P_1^k(1+\lambda P_1^\beta)^{-\nu}\right]$$

$$+ \frac{pk}{G(Q_1)}\frac{\beta_{m(Q_1)}(\frac{k}{\beta}, \nu - \frac{k}{\beta})}{\beta\lambda^\beta} - \frac{qk}{1-G(P_1)}\frac{\beta_{m(P_1)}(\frac{k}{\beta}, \nu - \frac{k}{\beta})}{\beta\lambda^\beta},$$

where $m(Q_1) = \frac{\lambda Q_1^\beta}{1+\lambda Q_1^\beta}$, $m(P_1) = \frac{\lambda P_1^\beta}{1+\lambda P_1^\beta}$ and $\beta_x(a,b) = \int_0^x t^{a-1}(1-t)^{b-1}dt, 0 < x < 1$, is the incomplete beta function and can be obtained using tables given by Pearson et al. [16].

To investigate the effect of the shape parameters β, ν and scale parameter λ on the mid-truncated Burr XII density function we have computed mean, variance, skewness and kurtosis for different values of the parameters, which are presented in following Tables 1 to 3.

Table 1. $\nu = 6$ and $\lambda = 1.5$

β	Mean	Variance	Skewness (β_1)	Kurtosis (β_2)
0.8	2.2156	5.5589	1.1701	10.6781
0.9	2.1373	4.7813	0.487	4.7715
0.95	2.1076	4.5052	0.3348	3.6722
1	2.0826	4.2775	0.2375	2.9912
1.5	1.9648	3.1544	0.0195	1.3656
2	1.939	2.7109	0.0033	1.1638

Table 2. $\nu = 5$ and $\beta = 1$

λ	Mean	Variance	Skewness (β_1)	Kurtosis (β_2)
1	2.2716	5.1371	0.9897	8.3788
1.23	2.2294	5.0299	0.8434	7.5447
2	2.1553	4.8732	0.6276	6.2366
2.5	2.1309	4.829	0.5663	5.846
3	2.1144	4.8009	0.5274	5.5938
4	2.0937	4.7674	0.481	5.2883

Table 3. $\beta = 1$ and $\lambda = 1.5$

ν	Mean	Variance	Skewness (β_1)	Kurtosis (β_2)
5	2.1953	4.9532	0.7379	6.9181
6	2.0826	4.2775	0.2375	2.9912
6.5	2.0412	4.0658	0.15	2.3661
7	2.0067	3.9016	0.0996	1.9957
8	1.9523	3.6639	0.049	1.5964
10	1.8796	3.3818	0.0159	1.283

3.2. *Moment generating function and characteristic function*

The moment generating function of the mid-truncated Burr XII distribution is given by:

$$M^*(t) = \frac{p}{G(Q_1)} \int_0^{Q_1} e^{xt} g(x) dx + \frac{q}{1 - G(P_1)} \int_{P_1}^{\infty} e^{tx} g(x) dx.$$

Integrating by parts, we get

$$M^*(t) = pe^{tQ_1} - \frac{p}{G(Q_1)} \int_0^{Q_1} te^{xt} G(x) dx + \frac{q}{1 - G(P_1)} \left[M(t) - e^{tP_1} G(P_1) + \int_0^{P_1} te^{tx} G(x) dx \right]$$

$$= pe^{tQ_1} + \frac{q}{1 - G(P_1)} [M(t) - e^{tP_1} G(P_1)] - \frac{pt}{G(Q_1)} \int_0^{Q_1} e^{xt} G(x) dx + \frac{qt}{1 - G(P_1)} \int_0^{P_1} e^{tx} G(x) dx.$$

Substituting the value of $G(x) = 1 - (1 + \lambda x^\beta)^{-\nu}$ from (3.2), and solving we get

$$M^*(t) = pe^{tQ_1} + \frac{q}{1 - G(P_1)} [M(t) - e^{tP_1} G(P_1)] - \frac{p}{G(Q_1)} (e^{tQ_1} - 1) + \frac{q}{1 - G(P_1)} (e^{tP_1} - 1)$$

$$+ \frac{pt}{G(Q_1)} \sum_{m=0}^{\infty} \frac{t^m}{m!} \int_0^{Q_1} x^m (1 + \lambda x^\beta)^{-\nu} dx - \frac{qt}{1 - G(P_1)} \sum_{m=0}^{\infty} \frac{t^m}{m!} \int_0^{P_1} x^m (1 + \lambda x^\beta)^{-\nu} dx$$

$$= pe^{tQ_1} + \frac{q}{1 - G(P_1)} [M(t) - e^{tP_1} G(P_1)] - \frac{p}{G(Q_1)} (e^{tQ_1} - 1) + \frac{q}{1 - G(P_1)} (e^{tP_1} - 1)$$

$$+ \frac{pt}{G(Q_1)} \sum_{d=0}^{\infty} \frac{t^d}{d!} \frac{\beta_{m(Q_1)}(\frac{d+1}{\beta}, \nu - \frac{d+1}{\beta})}{\beta \lambda^{(d+1)/\beta}} - \frac{qt}{1 - G(P_1)} \sum_{d=0}^{\infty} \frac{t^d}{d!} \frac{\beta_{m(P_1)}(\frac{d+1}{\beta}, \nu - \frac{d+1}{\beta})}{\beta \lambda^{(d+1)/\beta}},$$

where $m(Q_1) = \frac{\lambda Q_1^\beta}{1 + \lambda Q_1^\beta}$, $m(P_1) = \frac{\lambda P_1^\beta}{1 + \lambda P_1^\beta}$ and $\beta_x(a, b) = \int_0^x t^{a-1} (1 - t)^{b-1} dt, 0 < x < 1$, is the incomplete beta function and can be obtained using tables given by Pearson et al. [16].

Similarly the characteristic function of the mid-truncated Burr XII distribution can be evaluated.

3.3. Distribution function

The cumulative distribution function of the mid-truncated Burr XII distribution (as in equation (3.9)) is given by

$$F(x) = \begin{cases} p\dfrac{1 - (1 + \lambda x^\beta)^{-\nu}}{1 - (1 + \lambda Q_1^\beta)^{-\nu}}, & 0 < x \le Q_1 \\[4mm] 1 - q\dfrac{(1 + \lambda x^\beta)^{-\nu}}{(1 + \lambda P_1^\beta)^{-\nu}}, & P_1 \le x < \infty. \end{cases} \tag{3.14}$$

3.4. Generating data

To generate random variables from mid-truncated Burr XII distribution, we use equations (3.14) and (2.8), and we get

$$u_1 = \frac{1 - (1 + \lambda x^\beta)^{-\nu}}{1 - (1 + \lambda Q_1^\beta)^{-\nu}},$$

$$u_2 = 1 - \frac{(1 + \lambda x^\beta)^{-\nu}}{(1 + \lambda P_1^\beta)^{-\nu}}.$$

Solving the above two equations for x, we get

$$x = \begin{cases} \left[\dfrac{[1 - u_1(1 - (1 + \lambda Q_1^\beta)^{-\nu})]^{-1/\nu} - 1}{\lambda}\right]^{1/\beta}, & 0 < x \le Q_1 \\[6mm] \left[\dfrac{[(1 - u_2)(1 + \lambda P_1^\beta)^{-\nu}]^{-1/\nu} - 1}{\lambda}\right]^{1/\beta}, & P_1 \le x < \infty. \end{cases}$$

3.5. Likelihood function

For n observations $x_1, x_2, \ldots, x_n$ from mid-truncated Burr XII distribution the likelihood function is obtained from equation (2.9) and is given by:

$$L(\nu, \beta, \lambda) = \frac{p^r q^{n-r}}{\left(1 - (1 + \lambda Q_1^\beta)^{-\nu}\right)^r \left((1 + \lambda P_1^\beta)^{-\nu}\right)^{(n-r)}} \prod_{i=1}^{n} \nu \beta \lambda x_i^{\beta-1} (1 + \lambda x_i^\beta)^{-(\nu+1)}$$

$$\log L(\nu, \beta, \lambda) = r \log(p) + (n - r)\log(q) + n(\log \nu + \log \beta + \log \lambda) + (\beta - 1)\sum_{i=1}^{n} \log(x_i)$$

$$- (\nu + 1)\sum_{i=1}^{n} \log(1 + \lambda x_i^\beta) - r\log(1 - (1 + \lambda Q_1^\beta)^{-\nu}) + \nu(n - r)\log(1 + \lambda P_1^\beta). \tag{3.15}$$

4. Recurrence relations for single moments of order statistics from mid-truncated Burr XII distribution

In this section we will derive recurrence relations for single moments of order statistics from mid-truncated Burr XII distribution as defined in Section 3.

Theorem 1
For $k = \beta$, $\nu > 1$ and $\lambda, \beta > 0$, we have

$$\mu_{1:n}^{(\beta)}\left(1 - \frac{1}{n\nu}\right) = \frac{(Q - p)\beta}{Q} I_{1:n}^{(\beta)} - q^n\left[\left\{\frac{(1 - Q)^{\frac{-1}{\nu}} - 1}{\lambda}\right\} - \left\{\frac{(1 - P)^{\frac{-1}{\nu}} - 1}{\lambda}\right\}\right], \tag{4.1}$$

and, for $k > \beta$, $\beta \in Z^+$, $\nu > 1$ and $\lambda, \beta > 0$, we have

$$\mu_{1:n}^{(k)} = \frac{k}{n\nu\beta\lambda}\mu_{1:n}^{(k-\beta)} + \frac{k}{n\nu\beta}\mu_{1:n}^{(k)} + \frac{(Q-p)k}{Q}I_{1:n}^{(k)} - q^n\left[\left[\left\{\frac{(1-Q)^{\frac{-1}{\nu}}-1}{\lambda}\right\}^{\frac{1}{\beta}}\right]^k - \left[\left\{\frac{(1-P)^{\frac{-1}{\nu}}-1}{\lambda}\right\}^{\frac{1}{\beta}}\right]^k\right],$$
(4.2)

where

$$I_{1:n}^{(k)} = \begin{cases} \dfrac{Q_1^k}{k}, & n = 1 \\[2ex] \left\{1 + \displaystyle\sum_{j=1}^{n-1}(-1)^j\left(\dfrac{p}{Q}\right)^j\binom{n-1}{j}\right\}\dfrac{Q_1^k}{k} + \displaystyle\sum_{j=1}^{n-1}\sum_{i=1}^{j}(-1)^{j+i}\left(\dfrac{p}{Q}\right)^j\binom{n-1}{j}\binom{j}{i}\dfrac{\beta_{m(Q_1)}(\frac{k}{\beta},\nu i - \frac{k}{\beta})}{\beta\lambda^{k/\beta}}, & n > 1, \end{cases}$$

$I_{1:n}^{(\beta)}$ is the same as $I_{1:n}^{(k)}$ with k replaced by β, $m(Q_1) = \frac{\lambda Q_1^\beta}{1+\lambda Q_1^\beta}$ and $\beta_x(a,b) = \int_0^x t^{a-1}(1-t)^{b-1}dt, 0 < x < 1$, is the incomplete beta function of first kind and can be obtained by using tables given by Pearson et al. [16].

Proof
Relations in (4.1) and (4.2) may be proved by following exactly the same steps as those in proving Theorem 2 which is presented below. □

Theorem 2
For $2 \leq r \leq n$, $k = \beta$, $\nu > 1$, $\lambda, \beta > 0$, we have

$$\mu_{r:n}^{(\beta)}\left(1 - \frac{1}{n\nu}\right) - \mu_{r-1:n-1}^{(\beta)} = \frac{(Q-p)\beta}{Q}I_{r:n}^{(\beta)} - \binom{n-1}{r-1}p^{r-1}q^{n-r+1}\left[\left\{\frac{(1-Q)^{\frac{-1}{\nu}}-1}{\lambda}\right\} - \left\{\frac{(1-P)^{\frac{-1}{\nu}}-1}{\lambda}\right\}\right],$$
(4.3)

and, for $k > \beta$, $\beta \in Z^+$, $\nu > 1$, $\lambda, \beta > 0$, we have

$$\mu_{r:n}^{(k)} - \mu_{r-1:n-1}^{(k)} = \frac{k}{n\nu\beta\lambda}\mu_{r:n}^{(k-\beta)} + \frac{k}{n\nu\beta}\mu_{r:n}^{(k)} + \frac{(Q-p)k}{Q}I_{r:n}^{(k)}$$
$$- \binom{n-1}{r-1}p^{r-1}q^{n-r+1}\left[\left[\left\{\frac{(1-Q)^{\frac{-1}{\nu}}-1}{\lambda}\right\}^{\frac{1}{\beta}}\right]^k - \left[\left\{\frac{(1-P)^{\frac{-1}{\nu}}-1}{\lambda}\right\}^{\frac{1}{\beta}}\right]^k\right],$$
(4.4)

where

$$I_{r:n}^{(k)} = \begin{cases} \displaystyle\sum_{j=r}^{n}\sum_{i=1}^{j-1}(-1)^{j-r+i}\left(\dfrac{p}{Q}\right)^{j-1}\binom{n-1}{j-1}\binom{j-1}{r-1}\binom{j-1}{i}\dfrac{\beta_{m(Q_1)}(\frac{k}{\beta},\nu i - \frac{k}{\beta})}{\beta\lambda^{k/\beta}}, & 2 \leq r < n \\[2ex] \left(\dfrac{p}{Q}\right)^{n-1}\left\{\dfrac{Q_1^k}{k} + \displaystyle\sum_{i=1}^{n-1}(-1)^i\binom{n-1}{i}\dfrac{\beta_{m(Q_1)}(\frac{k}{\beta},\nu i - \frac{k}{\beta})}{\beta\lambda^{k/\beta}}\right\}, & r = n, \end{cases}$$

$I_{r:n}^{(\beta)}$ is the same as $I_{r:n}^{(k)}$ with k replaced by β, $m(Q_1) = \frac{\lambda Q_1^\beta}{1+\lambda Q_1^\beta}$ and

$$\beta_x(a,b) = \int_0^x t^{a-1}(1-t)^{b-1}dt, \quad 0 < x < 1,$$

is the incomplete beta function of first kind and can be obtained by using tables by Pearson et al. [16].

Proof
To prove the above theorem, first we prove the following two lemmas. □

Lemma 1

In usual notations, for an arbitrary mid-truncated distribution discussed in Section 2, we have

$$\mu_{r:n}^{(k)} - \mu_{r-1:n-1}^{(k)} = \binom{n-1}{r-1}\left[k\int_0^{Q_1} x^{k-1}(F(x))^{r-1}(1-F(x))^{n-r+1}dx \right.$$

$$\left. + k\int_{P_1}^{\infty} x^{k-1}(F(x))^{r-1}(1-F(x))^{n-r+1}dx - (Q_1{}^k - P_1{}^k)\, p^{r-1}q^{n-r+1}\right]. \quad (4.5)$$

Proof

Using (3.12), we get

$$\mu_{r:n}^{(k)} - \mu_{r-1:n-1}^{(k)} = C_{r:n}\int_0^{Q_1} x^k(F(x))^{r-1}(1-F(x))^{n-r}f(x)dx$$

$$+ C_{r:n}\int_{P_1}^{\infty} x^k(F(x))^{r-1}(1-F(x))^{n-r}f(x)dx$$

$$- C_{r-1:n-1}\int_0^{Q_1} x^k(F(x))^{r-2}(1-F(x))^{n-r}f(x)dx$$

$$- C_{r-1:n-1}\int_{P_1}^{\infty} x^k(F(x))^{r-2}(1-F(x))^{n-r}f(x)dx$$

$$= \binom{n-1}{r-1}\left[\int_0^{Q_1} x^k(F(x))^{r-2}(1-F(x))^{n-r}f(x)\{nF(x)-(r-1)\}dx \right.$$

$$\left. + \int_{P_1}^{\infty} x^k(F(x))^{r-2}(1-F(x))^{n-r}f(x)\{nF(x)-(r-1)\}dx\right]. \quad (4.6)$$

Let

$$\varphi(x) = -(F(x))^{r-1}(1-F(x))^{n-r+1}. \quad (4.7)$$

Differentiating both sides with respect to x, we get

$$\frac{d\varphi(x)}{dx} = (F(x))^{r-2}(1-F(x))^{n-r}f(x)\{nF(x)-(r-1)\}. \quad (4.8)$$

On using (4.8), equation (4.6) becomes

$$\mu_{r:n}^{(k)} - \mu_{r-1:n-1}^{(k)} = \binom{n-1}{r-1}\left[\int_0^{Q_1} x^k\left(\frac{d\varphi(x)}{dx}\right)dx + \int_{P_1}^{\infty} x^k\left(\frac{d\varphi(x)}{dx}\right)dx \right]$$

$$= \binom{n-1}{r-1}\left[[x^k\varphi(x)]_0^{Q_1} - k\int_0^{Q_1} x^{k-1}\varphi(x)dx + [x^k\varphi(x)]_{P_1}^{\infty} - k\int_{P_1}^{\infty} x^{k-1}\varphi(x)dx \right]$$

$$= \binom{n-1}{r-1}\left[Q_1^k\varphi(Q_1) - P_1^k\varphi(P_1) - k\int_0^{Q_1} x^{k-1}\varphi(x)dx - k\int_{P_1}^{\infty} x^{k-1}\varphi(x)dx \right]. \quad (4.9)$$

Now, since $F(Q_1) = p$ and $F(P_1) = 1 - q$, therefore, from (4.7), we have

$$\varphi(Q_1) = \varphi(P_1) = -p^{r-1}q^{n-r+1}.$$

Substituting the values of $\varphi(x)$, $\varphi(Q_1)$ and $\varphi(P_1)$ in (4.9), it leads to Lemma 1.   □

Lemma 2

If for $1 \le r \le n$ and $k = 1, 2, \ldots,$

$$I_{r:n}^{(k)} = \binom{n-1}{r-1}\int_0^{Q_1} x^{k-1}(F(x))^{r-1}(1-F(x))^{n-r}dx, \quad (4.10)$$

then, it may be written as

$$
I_{1:n}^{(k)} = \begin{cases} \dfrac{Q_1^k}{k}, & n = 1 \\ \left\{ 1 + \displaystyle\sum_{j=1}^{n-1} (-1)^j \left(\dfrac{p}{Q} \right)^j \binom{n-1}{j} \right\} \dfrac{Q_1^k}{k} + \displaystyle\sum_{j=1}^{n-1} \sum_{i=1}^{j} (-1)^{j+i} \left(\dfrac{p}{Q} \right)^j \binom{n-1}{j} \binom{j}{i} \dfrac{\beta_{m(Q_1)}(\frac{k}{\beta}, \nu i - \frac{k}{\beta})}{\beta \lambda^{k/\beta}}, & n > 1, \end{cases}
$$

$$(4.11)$$

and

$$
I_{r:n}^{(k)} = \begin{cases} \displaystyle\sum_{j=r}^{n} \sum_{i=1}^{j-1} (-1)^{j-r+i} \left(\dfrac{p}{Q} \right)^{j-1} \binom{n-1}{j-1} \binom{j-1}{r-1} \binom{j-1}{i} \dfrac{\beta_{m(Q_1)}(\frac{k}{\beta}, \nu i - \frac{k}{\beta})}{\beta \lambda^{k/\beta}}, & 2 \le r < n \\ \left(\dfrac{p}{Q} \right)^{n-1} \left\{ \dfrac{Q_1^k}{k} + \displaystyle\sum_{i=1}^{n-1} (-1)^i \binom{n-1}{i} \dfrac{\beta_{m(Q_1)}(\frac{k}{\beta}, \nu i - \frac{k}{\beta})}{\beta \lambda^{k/\beta}} \right\}, & r = n, \end{cases}
$$

$$(4.12)$$

where $m(Q_1) = \frac{\lambda Q_1^\beta}{1 + \lambda Q_1^\beta}$ and $\beta_x(a,b) = \int_0^x t^{a-1}(1-t)^{b-1} dt$, $0 < x < 1$, is the incomplete beta function of first kind, which can be found out using tables given by Pearson et al. [16].

Proof
Relation in (4.11) may be proved by following exactly the same steps as those in proving (4.12) which is presented below.

Expanding $(1 - F(x))^{n-r}$ binomially in powers of $F(x)$ and then substituting the same in (4.10), we get for $2 \le r < n$,

$$
I_{r:n}^{(k)} = \binom{n-1}{r-1} \sum_{l=0}^{n-r} (-1)^l \binom{n-r}{l} \int_0^{Q_1} x^{k-1} (F(x))^{l+r-1} dx.
$$

On putting $j = l + r$, we get

$$
I_{r:n}^{(k)} = \binom{n-1}{r-1} \sum_{j=r}^{n} (-1)^{j-r} \binom{n-r}{j-r} \int_0^{Q_1} x^{k-1} (F(x))^{j-1} dx.
$$

Substituting the value of $F(x)$ from (3.4), we get

$$
I_{r:n}^{(k)} = \sum_{j=r}^{n} (-1)^{j-r} \binom{n-1}{j-1} \binom{j-1}{r-1} \int_0^{Q_1} x^{k-1} \left(\frac{p(1 - (1 + \lambda x^\beta)^{-\nu})}{Q} \right)^{j-1} dx
$$

$$
= \sum_{j=r}^{n} (-1)^{j-r} \binom{n-1}{j-1} \binom{j-1}{r-1} \frac{p^{j-1}}{Q^{j-1}} \int_0^{Q_1} x^{k-1} ((1 - (1 + \lambda x^\beta)^{-\nu}))^{j-1} dx.
$$

Expanding $((1 - (1 + \lambda x^\beta)^{-\nu}))^{j-1}$ binomially, we get

$$
I_{r:n}^{(k)} = \sum_{j=r}^{n} (-1)^{j-r} \binom{n-1}{j-1} \binom{j-1}{r-1} \frac{p^{j-1}}{Q^{j-1}} \sum_{i=1}^{j-1} \binom{j-1}{i} \int_0^{Q_1} (-1)^i x^{k-1} (1 + \lambda x^\beta)^{-\nu i} dx
$$

$$
= \sum_{j=r}^{n} \sum_{i=1}^{j-1} (-1)^{j-r+i} \left(\frac{p}{Q} \right)^{j-1} \binom{n-1}{j-1} \binom{j-1}{r-1} \binom{j-1}{i} \int_0^{Q_1} x^{k-1} (1 + \lambda x^\beta)^{-\nu i} dx. \qquad (4.13)
$$

Now consider

$$
\begin{aligned}
J &= \int_0^{Q_1} x^{k-1}(1 + \lambda x^\beta)^{-\nu i} dx \\
&= \frac{\int_1^{1+\lambda Q_1^\beta} y^{-\nu i}(y-1)^{\frac{k}{\beta}-1} dy}{\beta \lambda^{k/\beta}}, \text{ by taking } y = 1 + \lambda x^\beta \\
&= \frac{1}{\beta \lambda^{k/\beta}} \int_0^{\frac{\lambda Q_1^\beta}{1+\lambda Q_1^\beta}} z^{\frac{k}{\beta}-1}(1-z)^{\nu i - \frac{k}{\beta}-1} dz, \text{ by taking } z = (y-1)/y \\
&= \frac{1}{\beta \lambda^{k/\beta}} \beta_{m(Q_1)}\left(\frac{k}{\beta}, \nu i - \frac{k}{\beta}\right),
\end{aligned}
\tag{4.14}
$$

where $m(Q_1) = \frac{\lambda Q_1^\beta}{1+\lambda Q_1^\beta}$ and $\beta_x(a,b) = \int_0^x t^{a-1}(1-t)^{b-1} dt, 0 < x < 1$, is the incomplete beta function and can be obtained using tables given by Pearson et al. [16].

Substituting the value of J from (4.14) in (4.13), it leads to the result (4.12) for the case $2 \le r < n$. The result (4.12) for the case $r = n$ can similarly be established. □

Proof of the main Theorem 2
Using (3.5) in (4.5), we get

$$
\begin{aligned}
\mu_{r:n}^{(k)} - \mu_{r-1:n-1}^{(k)} &= \binom{n-1}{r-1}\left[k \int_0^{Q_1} x^{k-1}(F(x))^{r-1}(1-F(x))^{n-r}\left(1 - \frac{p}{Q} + f(x)\frac{x^{1-\beta}(1+\lambda x^\beta)}{\nu\beta\lambda}\right) dx \right. \\
&\quad \left. + k \int_{P_1}^\infty x^{k-1}(F(x))^{r-1}(1-F(x))^{n-r}f(x)\frac{x^{1-\beta}(1+\lambda x^\beta)}{\nu\beta\lambda} dx - (Q_1^k - P_1^k)p^{r-1}q^{n-r+1}\right] \\
&= \binom{n-1}{r-1}\left[k\frac{(Q-p)}{Q} \int_0^{Q_1} x^{k-1}(F(x))^{r-1}(1-F(x))^{n-r} dx \right. \\
&\quad + \frac{k}{\nu\beta\lambda}\left\{ \int_0^{Q_1} x^{k-\beta}(F(x))^{r-1}(1-F(x))^{n-r}f(x)dx \right. \\
&\quad \left. + \int_{P_1}^\infty x^{k-\beta}(F(x))^{r-1}(1-F(x))^{n-r}f(x)dx \right\} \\
&\quad + \frac{k}{\nu\beta}\left\{ \int_0^{Q_1} x^k(F(x))^{r-1}(1-F(x))^{n-r}f(x)dx \right. \\
&\quad \left. \left. + \int_{P_1}^\infty x^k(F(x))^{r-1}(1-F(x))^{n-r}f(x)dx \right\} - (Q_1^k - P_1^k)p^{r-1}q^{n-r+1}\right],
\end{aligned}
$$

which on simplification and using (3.12) gives

$$
\begin{aligned}
\mu_{r:n}^{(k)} - \mu_{r-1:n-1}^{(k)} &= \frac{k}{n\nu\beta\lambda}\mu_{r:n}^{(k-\beta)} + \frac{k}{n\nu\beta}\mu_{r:n}^{(k)} + \binom{n-1}{r-1} \\
&\quad \cdot \left[\frac{(Q-p)k}{Q} \int_0^{Q_1} x^{k-1}(F(x))^{r-1}(1-F(x))^{n-r} dx - (Q_1^k - P_1^k) p^{r-1}q^{n-r+1}\right].
\end{aligned}
\tag{4.15}
$$

Now, on using the definition of $I_{r:n}^{(k)}$ as given in Lemma 2 and putting the values of Q_1 and P_1 in (4.15), we get

$$
\begin{aligned}
\mu_{r:n}^{(k)} - \mu_{r-1:n-1}^{(k)} &= \frac{k}{n\nu\beta\lambda}\mu_{r:n}^{(k-\beta)} + \frac{k}{n\nu\beta}\mu_{r:n}^{(k)} + \frac{(Q-p)k}{Q}I_{r:n}^{(k)} \\
&\quad - \binom{n-1}{r-1}p^{r-1}q^{n-r+1}\left[\left[\left\{\frac{(1-Q)^{\frac{-1}{\nu}}-1}{\lambda}\right\}^{\frac{1}{\beta}}\right]^k - \left[\left\{\frac{(1-P)^{\frac{-1}{\nu}}-1}{\lambda}\right\}^{\frac{1}{\beta}}\right]^k\right],
\end{aligned}
$$

where the values of $I_{r:n}^{(k)}$, for different values of r, are given in (4.11) and (4.12).This completes the proof of (4.4). Further, on putting $k = \beta$ in (4.4), it reduces to (4.3). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

Recursive Algorithm

Utilizing the knowledge of recurrence relations obtained above one can evaluate the moments of order statistics from mid-truncated Burr XII distribution for $\beta \in Z^+$, $k \geq \beta$, $\nu > 1$ and $\lambda > 0$. Algorithm is as following:

Case 1: For $k = m\beta$, $m \in Z^+$

From equation (4.2), we observe that $\mu_{1:n}^{(m\beta)}$ can be evaluated by the knowledge of $\mu_{1:n}^{((m-1)\beta)}$ which in turn can be evaluated with the knowledge of $\mu_{1:n}^{((m-2)\beta)}$ and so on, and after the recursive use of (4.2), $(m-1)$ times, we conclude that $\mu_{1:n}^{(m\beta)}$ comes out to be a function of parameter values only. Thus, if the parameters are known then one can evaluate $\mu_{1:n}^{(k)}$ for all n and k in a simple recursive manner for this case, i.e., for $k = m\beta$.

From equation (4.4), we observe $\mu_{r:n}^{(m\beta)}$ can be evaluated by the knowledge of $\mu_{r-1:n-1}^{(m\beta)}$ and $\mu_{r:n}^{((m-1)\beta)}$ and after $(r-2)$ iterations of equation (4.4) we conclude that to evaluate $\mu_{r:n}^{(m\beta)}$ the required values are $\mu_{1:n-r+1}^{(m\beta)}$, $\mu_{2:n-r+2}^{((m-1)\beta)}, \ldots, \mu_{r:n}^{((m-r+1)\beta)}$.

In particular, to calculate $\mu_{2:n}^{(m\beta)}$ we require the values of $\mu_{1:n-1}^{(m\beta)}$ and $\mu_{2:n}^{((m-1)\beta)}$. As $\mu_{1:n-1}^{(m\beta)}$ is already known and after the recursive use of (4.2), $(m-1)$ times, we conclude that to evaluate $\mu_{2:n}^{(m\beta)}$, the required values are $\mu_{1:n-1}^{(t\beta)}$ for all $t = 1, 2, 3, \ldots, m$, which are already known.

Now for $r = 3$ to evaluate $\mu_{3:n}^{(m\beta)}$, for odd m, we require the values of $\mu_{1:n-2}^{(m\beta)}$, $\mu_{1:n-2}^{((m-2)\beta)}, \ldots, \mu_{1:n-2}^{(3\beta)}$ and $\mu_{2:n-1}^{((m-1)\beta)}, \mu_{2:n-1}^{((m-3)\beta)}, \ldots, \mu_{2:n-1}^{(2\beta)}$ (which have already been calculated) and $\mu_{3:n}^{(\beta)}$. Using equation (4.3) we can evaluate $\mu_{3:n}^{(\beta)}$, thus by using the values calculated above and utilizing Theorem 2 we can calculate $\mu_{3:n}^{(m\beta)}$ for all odd values of m.

Similarly for even values of m, one can evaluate $\mu_{3:n}^{(m\beta)}$, by utilizing the above calculated values. In this way one can calculate $\mu_{4:n}^{(m\beta)}, \mu_{5:n}^{(m\beta)}, \ldots$ in a recursive manner.

Case 2: when $k \neq m\beta$, $m \in Z^+$

From equation (4.2) we observe that $\mu_{1:n}^{(k)}$ can be evaluated by the knowledge of $\mu_{1:n}^{(k-\beta)}$ and after the recursive use of (4.2) $(\alpha - 1)$ times (where α is the greatest integer satisfying the condition $(k - \alpha\beta) > 0$), we conclude that $\mu_{1:n}^{(k)}$ is a function of $\mu_{1:n}^{(k-\alpha\beta)}$. Thus in this case, by the knowledge of $\mu_{1:n}^{(k-\alpha\beta)}$ one can evaluate values of $\mu_{1:n}^{(k)}$ for all n and k in a simple recursive manner.

From equation (4.3) we observe that to evaluate $\mu_{r:n}^{(k)}$ we should have the knowledge of only one value with sample size n i.e. $\mu_{r:n}^{(k-\alpha\beta)}$ (where α is the greatest integer satisfying the condition $(k - \alpha\beta) > 0$) and some moments of lower orders. Using the above information one can calculate $\mu_{2:n}^{(k)}, \mu_{3:n}^{(k)}, \ldots$ in a recursive manner.

5. Recurrence relations for product moments of order statistics from mid-truncated Burr XII distribution

Theorem 3
For $1 \leq r < s \leq n$, $x < y$ and $j, k > 0$, we have

$$\mu_{r,s:n}^{(j,k)} - \mu_{r,s-1:n}^{(j,k)} = \frac{k}{(n-s+1)\nu\beta\lambda}\mu_{r,s:n}^{(j,k-\beta)} + \frac{k}{(n-s+1)\nu\beta}\mu_{r,s:n}^{(j,k)}$$

$$+ C_{r,s:n}^* \left[k\left(1 - \frac{p}{Q}\right) \int_0^{Q_1} \int_x^{Q_1} x^j y^{k-1} [F(x)]^{r-1} [F(y) - F(x)]^{s-r-1}\right.$$

$$\times [1 - F(y)]^{n-s} f(x) dy dx - (1-p)^{n-r} Q_1{}^k \sum_{i=0}^{s-r-1} \binom{s-r-1}{i} (-1)^i \left(\frac{p}{q}\right)^{s-r-i-1}$$

$$\times \int_0^{Q_1} x^j [F(x)]^{r+i-1}[1 - F(x)]^{s-r-i-1} f(x)dx \Bigg], \tag{5.1}$$

where $C_{r,s:n}^* = \dfrac{n!}{(r-1)!(s-r-1)!(n-s+1)!}.$

Proof

To prove the above theorem, first we evaluate the following term,

$$E(j, r+i, s) = \int_0^{Q_1} x^j [F(x)]^{r+i-1}[1 - F(x)]^{s-r-i-1} f(x)dx \quad \text{(say)}$$

$$= \int_0^{Q_1} x^j [F(x)]^{\alpha-1}[1 - F(x)]^{s-\alpha-1} f(x)dx, \text{ by taking } r + i = \alpha.$$

From equation (3.5) we know for $x \le Q_1$

$$F(x) = \frac{p}{Q} - f(x)\frac{x^{1-\beta}(1 + \lambda x^\beta)}{\nu\beta\lambda}$$

$$\Rightarrow \quad \frac{p}{Q} - F(x) = f(x)\frac{x^{1-\beta}(1 + \lambda x^\beta)}{\nu\beta\lambda}$$

$$\Rightarrow \quad \nu\beta\lambda\left[\frac{p}{Q} - F(x)\right]x^{\beta-1} = f(x)(1 + \lambda x^\beta). \tag{5.2}$$

Now consider,

$$E(j, \alpha, s) + \lambda E(j + \beta, \alpha, s) = \int_0^{Q_1} x^j [F(x)]^{\alpha-1}[1 - F(x)]^{s-\alpha-1} f(x)dx$$

$$+ \lambda \int_0^{Q_1} x^{j+\beta} [F(x)]^{\alpha-1}[1 - F(x)]^{s-\alpha-1} f(x)dx$$

$$= \int_0^{Q_1} x^j [F(x)]^{\alpha-1}[1 - F(x)]^{s-\alpha-1} f(x)(1 + \lambda x^\beta)dx$$

$$= \nu\beta\lambda \int_0^{Q_1} x^j [F(x)]^{\alpha-1}[1 - F(x)]^{s-\alpha-1}\left[\frac{p}{Q} - F(x)\right]x^{\beta-1}dx \quad \text{(using (5.2))}$$

$$= \nu\beta\lambda\left[\frac{p}{Q} \int_0^{Q_1} x^{j+\beta-1} [F(x)]^{\alpha-1}[1 - F(x)]^{s-\alpha-1}dx\right.$$

$$\left. - \int_0^{Q_1} x^{j+\beta-1} [F(x)]^{\alpha}[1 - F(x)]^{s-\alpha-1}dx\right]$$

$$= \nu\beta\lambda\left[\frac{p}{Q} \frac{I_{\alpha:s-1}^{(j+\beta)}}{\binom{s-2}{\alpha-1}} - \frac{I_{\alpha+1:s}^{(j+\beta)}}{\binom{s-1}{\alpha}}\right],$$

where $\alpha = r + i$, and

$$I_{r:n}^{(k)} = \binom{n-1}{r-1} \int_0^{Q_1} x^{k-1}(F(x))^{r-1}(1 - F(x))^{n-r}dx,$$

which is evaluated in Lemma 2. $\qquad \square$

Proof of the main Theorem 3

Using equation (3.13), and letting $C_{r,s:n}^* = \dfrac{n!}{(r-1)!(s-r-1)!(n-s+1)!}$, consider

$$\mu_{r,s:n}^{(j,k)} - \mu_{r,s-1:n}^{(j,k)} = \frac{n!}{(r-1)!(s-r-1)!(n-s)!}$$

$$\times \left[\int_0^{Q_1} \int_x^{Q_1} x^j y^k [F(x)]^{r-1} [F(y) - F(x)]^{s-r-1} [1 - F(y)]^{n-s} f(x) f(y) dy dx \right.$$

$$\left. + \int_{P_1}^{\infty} \int_x^{\infty} x^j y^k [F(x)]^{r-1} [F(y) - F(x)]^{s-r-1} [1 - F(y)]^{n-s} f(x) f(y) dy dx \right]$$

$$- \frac{n!}{(r-1)!(s-r-2)!(n-s+1)!}$$

$$\times \left[\int_0^{Q_1} \int_x^{Q_1} x^j y^k [F(x)]^{r-1} [F(y) - F(x)]^{s-r-2} [1 - F(y)]^{n-s+1} f(x) f(y) dy dx \right.$$

$$\left. + \int_{P_1}^{\infty} \int_x^{\infty} x^j y^k [F(x)]^{r-1} [F(y) - F(x)]^{s-r-2} [1 - F(y)]^{n-s+1} f(x) f(y) dy dx \right]$$

$$= C_{r,s:n}^* \left[\int_0^{Q_1} \int_x^{Q_1} x^j y^k [F(x)]^{r-1} [F(y) - F(x)]^{s-r-2} \left[1 - F(y) \right]^{n-s} f(x) f(y) \right.$$

$$\times \left\{ (n-s+1)(F(y) - F(x)) - (s-r-1)(1 - F(y)) \right\} dy dx$$

$$+ \int_{P_1}^{\infty} \int_x^{\infty} x^j y^k [F(x)]^{r-1} [F(y) - F(x)]^{s-r-2} [1 - F(y)]^{n-s} f(x) f(y)$$

$$\left. \times \left\{ (n-s+1)(F(y) - F(x)) - (s-r-1)(1 - F(y)) \right\} dy dx \right]$$

$$= C_{r,s:n}^* \left[\int_0^{Q_1} \int_x^{Q_1} x^j y^k [F(x)]^{r-1} [F(y) - F(x)]^{s-r-2} [1 - F(y)]^{n-s} f(x) f(y) \right.$$

$$\times \left\{ (n-r)F(y) - (n-s+1)F(x) - (s-r-1) \right\} dy dx$$

$$+ \int_{P_1}^{\infty} \int_x^{\infty} x^j y^k [F(x)]^{r-1} [F(y) - F(x)]^{s-r-2} [1 - F(y)]^{n-s} f(x) f(y)$$

$$\left. \times \left\{ (n-r)F(y) - (n-s+1)F(x) - (s-r-1) \right\} dy dx \right]. \tag{5.3}$$

Let

$$\varphi(x, y) = -[F(y) - F(x)]^{s-r-1} [1 - F(y)]^{n-s+1}. \tag{5.4}$$

Then,

$$\frac{\partial \varphi(x, y)}{\partial y} = -(s-r-1)[F(y) - F(x)]^{s-r-2} [1 - F(y)]^{n-s+1} f(y)$$

$$+ (n-s+1)[F(y) - F(x)]^{s-r-1} [1 - F(y)]^{n-s} f(y)$$

$$= [F(y) - F(x)]^{s-r-2} [1 - F(y)]^{n-s} f(y)[(n-r)F(y)$$

$$- (n-s+1)F(x) - (s-r-1)]. \tag{5.5}$$

Putting the above value in (5.3), we get

$$\mu_{r,s:n}^{(j,k)} - \mu_{r,s-1:n}^{(j,k)} = C_{r,s:n}^* \left[\int_0^{Q_1} \int_x^{Q_1} x^j y^k [F(x)]^{r-1} \frac{\partial \varphi(x, y)}{\partial y} f(x) dy dx \right.$$

$$\left. + \int_{P_1}^{\infty} \int_x^{\infty} x^j y^k [F(x)]^{r-1} \frac{\partial \varphi(x, y)}{\partial y} f(x) dy dx \right]$$

$$= C_{r,s:n}^* \left[\int_0^{Q_1} x^j [F(x)]^{r-1} \left(\int_x^{Q_1} y^k \frac{\partial \varphi(x, y)}{\partial y} dy \right) f(x) dx \right.$$

$$+ \int_{P_1}^{\infty} x^j [F(x)]^{r-1} \left(\int_x^{\infty} y^k \frac{\partial \varphi(x,y)}{\partial y} dy \right) f(x) dx \Bigg]. \tag{5.6}$$

Now, consider

$$\int_x^{Q_1} y^k \frac{\partial \varphi(x,y)}{\partial y} dy = Q_1{}^k \varphi(x, Q_1) + k \int_x^{Q_1} y^{k-1} [F(y) - F(x)]^{s-r-1} [1 - F(y)]^{n-s+1} dy. \tag{5.7}$$

From equations (5.4) and (3.9), we have

$$\varphi(x, Q_1) = -[p - F(x)]^{s-r-1} [1 - p]^{n-s+1}. \tag{5.8}$$

Putting (5.8) in equation (5.7), we get

$$\int_x^{Q_1} y^k \frac{\partial \varphi(x,y)}{\partial y} dy = k \int_x^{Q_1} y^{k-1} [F(y) - F(x)]^{s-r-1} [1 - F(y)]^{n-s+1} dy$$
$$- Q_1{}^k [p - F(x)]^{s-r-1} [1 - p]^{n-s+1}, \tag{5.9}$$

also,

$$\int_x^{\infty} y^k \frac{\partial \varphi(x,y)}{\partial y} dy = k \int_x^{\infty} y^{k-1} [F(y) - F(x)]^{s-r-1} [1 - F(y)]^{n-s+1} dy. \tag{5.10}$$

Putting (5.9) and (5.10) in equation (5.6), we get

$$\mu_{r,s:n}^{(j,k)} - \mu_{r,s-1:n}^{(j,k)} = C_{r,s:n}^* \Bigg[\int_0^{Q_1} x^j [F(x)]^{r-1} \bigg\{ k \int_x^{Q_1} y^{k-1} [F(y) - F(x)]^{s-r-1} [1 - F(y)]^{n-s+1} dy$$

$$- Q_1{}^k [p - F(x)]^{s-r-1} [1 - p]^{n-s+1} \bigg\} f(x) dx$$

$$+ \int_{P_1}^{\infty} x^j [F(x)]^{r-1} \bigg\{ k \int_x^{\infty} y^{k-1} [F(y) - F(x)]^{s-r-1} [1 - F(y)]^{n-s+1} dy \bigg\} f(x) dx \Bigg]$$

$$= C_{r,s:n}^* \Bigg[k \int_0^{Q_1} \int_x^{Q_1} x^j y^{k-1} [F(x)]^{r-1} [F(y) - F(x)]^{s-r-1} [1 - F(y)]^{n-s+1} f(x) dy dx$$

$$+ k \int_{P_1}^{\infty} \int_x^{\infty} x^j y^{k-1} [F(x)]^{r-1} [F(y) - F(x)]^{s-r-1} [1 - F(y)]^{n-s+1} f(x) dy dx$$

$$- \int_0^{Q_1} x^j Q_1{}^k [p - F(x)]^{s-r-1} [1 - p]^{n-s+1} [F(x)]^{r-1} f(x) dx \Bigg]. \tag{5.11}$$

Considering the last term of the above equation and writing $(n - s + 1) = (n - r) - (s - r - 1)$, we get

$$Q_1{}^k [1 - p]^{n-s+1} \int_0^{Q_1} x^j [p - F(x)]^{s-r-1} [F(x)]^{r-1} f(x) dx$$

$$= Q_1{}^k [1 - p]^{n-r} \int_0^{Q_1} x^j \frac{[p - F(x)]^{s-r-1}}{(1-p)^{s-r-1}} [F(x)]^{r-1} f(x) dx$$

$$= Q_1{}^k [1 - p]^{n-r} \int_0^{Q_1} x^j \left(\frac{p - F(x)}{q} \right)^{s-r-1} [F(x)]^{r-1} f(x) dx$$

$$= Q_1{}^k [1 - p]^{n-r} \int_0^{Q_1} x^j [F(x)]^{r-1} \left(\frac{p(1 - F(x))}{q} - F(x) \right)^{s-r-1} f(x) dx.$$

Expanding $\left(\dfrac{p}{q}(1 - F(x)) - F(x)\right)^{s-r-1}$ binomially we get the above expression equal to

$$Q_1{}^k[1-p]^{n-r}\int_0^{Q_1} x^j[F(x)]^{r-1}\sum_{i=0}^{s-r-1}\binom{s-r-1}{i}(-F(x))^i\left(\frac{p}{q}(1-F(x))\right)^{s-r-i-1}f(x)dx$$

$$= Q_1{}^k[1-p]^{n-r}\sum_{i=0}^{s-r-1}\binom{s-r-1}{i}\left(\frac{p}{q}\right)^{s-r-i-1}(-1)^i\int_0^{Q_1} x^j[F(x)]^{r+i-1}[1-F(x)]^{s-r-i-1}f(x)dx.$$

$$(5.12)$$

Putting (5.12) in (5.11), we get

$$\mu_{r,s:n}^{(j,k)} - \mu_{r,s-1:n}^{(j,k)}$$

$$= C_{r,s:n}^*\bigg[k\int_0^{Q_1}\int_x^{Q_1} x^j y^{k-1}[F(x)]^{r-1}[F(y)-F(x)]^{s-r-1}[1-F(y)]^{n-s+1}f(x)dydx$$

$$+ k\int_{P_1}^{\infty}\int_x^{\infty} x^j y^{k-1}[F(x)]^{r-1}[F(y)-F(x)]^{s-r-1}[1-F(y)]^{n-s+1}f(x)dydx$$

$$- Q_1{}^k[1-p]^{n-r}\sum_{i=0}^{s-r-1}\binom{s-r-1}{i}\left(\frac{p}{q}\right)^{s-r-i-1}(-1)^i\int_0^{Q_1} x^j[F(x)]^{r+i-1}[1-F(x)]^{s-r-i-1}f(x)dx\bigg].$$

$$(5.13)$$

Using (3.5) in (5.13), we get

$$\mu_{r,s:n}^{(j,k)} - \mu_{r,s-1:n}^{(j,k)} = C_{r,s:n}^*\bigg[k\left(1-\frac{p}{Q}\right)\int_0^{Q_1}\int_x^{Q_1} x^j y^{k-1}[F(x)]^{r-1}[F(y)-F(x)]^{s-r-1}[1-F(y)]^{n-s}f(x)dydx$$

$$+ \frac{k}{\nu\beta\lambda}\int_0^{Q_1}\int_x^{Q_1} x^j y^{k-\beta}[F(x)]^{r-1}[F(y)-F(x)]^{s-r-1}[1-F(y)]^{n-s}f(x)f(y)dydx$$

$$+ \frac{k}{\nu\beta}\int_0^{Q_1}\int_x^{Q_1} x^j y^{k}[F(x)]^{r-1}[F(y)-F(x)]^{s-r-1}[1-F(y)]^{n-s}f(x)f(y)dydx$$

$$+ \frac{k}{\nu\beta\lambda}\int_{P_1}^{\infty}\int_x^{\infty} x^j y^{k-\beta}[F(x)]^{r-1}[F(y)-F(x)]^{s-r-1}[1-F(y)]^{n-s}f(x)f(y)dydx$$

$$+ \frac{k}{\nu\beta}\int_{P_1}^{\infty}\int_x^{\infty} x^j y^{k}[F(x)]^{r-1}[F(y)-F(x)]^{s-r-1}[1-F(y)]^{n-s}f(x)f(y)dydx$$

$$- (1-p)^{n-r}Q_1{}^k\sum_{i=0}^{s-r-1}\binom{s-r-1}{i}(-1)^i\left(\frac{p}{q}\right)^{s-r-i-1}\int_0^{Q_1} x^j[F(x)]^{r+i-1}$$

$$\times [1-F(x)]^{s-r-i-1}f(x)dx\bigg]$$

$$= \frac{k}{(n-s+1)\nu\beta\lambda}\mu_{r,s:n}^{(j,k-\beta)} + \frac{k}{(n-s+1)\nu\beta}\mu_{r,s:n}^{(j,k)}$$

$$+ C_{r,s:n}^*\bigg[k\left(1-\frac{p}{Q}\right)\int_0^{Q_1}\int_x^{Q_1} x^j y^{k-1}[F(x)]^{r-1}[F(y)-F(x)]^{s-r-1}$$

$$\times [1-F(y)]^{n-s}f(x)dydx - (1-p)^{n-r}Q_1{}^k\sum_{i=0}^{s-r-1}\binom{s-r-1}{i}(-1)^i\left(\frac{p}{q}\right)^{s-r-i-1}$$

$$\times \int_0^{Q_1} x^j[F(x)]^{r+i-1}[1-F(x)]^{s-r-i-1}f(x)dx\bigg],$$

which completes the proof of Theorem 3.   □

Remark 1

Proceeding in a similar manner as in Section 3, and considering the following baseline distributions, which are particular cases of Burr XII distribution, namely

 (i) Lomax distribution:

$$F(x) = 1 - \left[1 + \frac{x}{a}\right]^{-\nu}$$

 (ii) Weibull distribution:

$$F(x) = 1 - e^{-x^{\beta}}$$

 (iii) Compound Weibull or Weibull–Gamma distribution:

$$F(x) = 1 - \left[1 + \frac{x^{\beta}}{\delta}\right]^{-\nu}$$

 (iv) Weibull-Exponential distribution:

$$F(x) = \frac{x^{\beta}/\delta}{(1 + x^{\beta}/\delta)}$$

 (v) Log logistic distribution:

$$F(x) = \frac{(x/a)^{\beta}}{1 + (x/a)^{\beta}}$$

 (vi) Exponential distribution:

$$F(x) = 1 - e^{-x}$$

 (vii) Rayleigh distribution:

$$F(x) = 1 - e^{-x^{2}/2}$$

(viii) Generalized Rayleigh distribution:

$$F(x) = 1 - [1 + \alpha x^{2}/2]^{-(1/\alpha)}$$

 (ix) Generalized Pareto distribution:

$$F(x) = 1 - [1 + px]^{-(1/p)},$$

we can construct the corresponding mid-truncated distributions and find corresponding recurrence relations for moments of order statistics arising from these distributions.

Thus, the recurrence relations obtained in Sections 4 and 5 are also true for mid-truncated
Lomax ($\beta = 1$, $\lambda = 1/a$), Weibull ($\nu = 1/\lambda$, $\lambda \to 0$), Weibull-Gamma ($\lambda = 1/\delta$), Weibull-Exponential ($\nu = 1, \lambda = 1/\delta$), Log logistic ($\nu = 1, \lambda = a^{-\beta}$), Exponential ($\nu = 1/\lambda, \beta = 1, \lambda \to 0$), Rayleigh ($\beta = 2, \nu = 1/(2\lambda)$, $\lambda \to 0$), Generalized Rayleigh ($\beta = 2$, $\nu = 1/\alpha$, $\lambda = \alpha/2$) and Generalized Pareto ($\beta = 1, \nu = 1/p$, $\lambda = p$) distributions.

Remark 2

The results deduced for the mid-truncated Weibull distribution, so obtained, are in agreement with the results of Okasha et al. [14].

6. Characterization of mid-truncated distributions using order statistics

In this section we will use the conditional moments of order statistics to characterize mid-truncated distributions.

Let $X_1, X_2, X_3, \ldots, X_n$ be a random samples of size n from the distribution defined in (3.3) and $X_{1:n} \leq X_{2:n} \leq \ldots \leq X_{n:n}$ be the corresponding order statistics. Then the probability density function (pdf) of $X_{r:n}$ $(1 \leq r \leq n)$ is given by $f_{r:n}(x)$ as defined by equation (3.10) and joint probability density function of $X_{r:n}$ and $X_{s:n}$ $(1 \leq r < s \leq n)$ is given by $f_{r,s:n}(x,y)$ as defined in (3.11).

Many authors have studied characterizations depending on order statistics. Among those Khan and Khan [8] characterized Burr XII distribution through moments of order statistics. Lin [9] used product moments to characterize power function, exponential and normal distributions. For extensive survey refer to Huang and Lin [7] and Huang [6].

Next, we establish a theorem to characterize mid-truncated distributions based on order statistics.

Theorem 4
Let X be a random variable having probability density function as (2.1), then X is having mid-truncated distribution iff

$$E(X_{r+1:n}^k | X_{r:n} = x) = x^k + \frac{k\Delta(x)}{(1 - F(x))^{n-r}},$$

where

$$\Delta(x) = \int_x^\infty y^{k-1} \left[1 - \left[\frac{p}{G(Q_1)} \int_{-\infty}^y g(t)dt + \frac{p}{G(Q_1)} \int_{-\infty}^{Q_1} g(t)dt + \frac{q}{(1 - G(P_1))} \int_{P_1}^y g(t)dt \right] \right]^{n-r} dy,$$

and $F(x)$ is the cumulative distribution function of the mid-truncated random variable.

Proof
Necessary condition:
We have

$$E(X_{r+1:n}^k | X_{r:n} = x) = \frac{C_{r,r+1:n}}{C_{r:n}} \left[\int_x^\infty \frac{y^k (F(x))^{r-1} (1 - F(y))^{n-r-1} f(x) f(y)}{(F(x))^{r-1} (1 - F(x))^{n-r} f(x)} dy \right]$$
$$= \frac{(n-r)}{(1 - F(x))^{n-r}} \int_x^\infty y^k (1 - F(y))^{n-r-1} f(y) dy,$$

where $X_{r+1:n} = y$. Integrating by parts, we get

$$E(X_{r+1:n}^k | X_{r:n} = x) = \frac{(n-r)}{(1 - F(x))^{n-r}} \left(\frac{-y^k (1 - F(y))^{n-r}}{n-r} \bigg|_x^\infty + k \int_x^\infty \frac{y^{k-1} (1 - F(y))^{n-r}}{n-r} dy \right)$$
$$= x^k + \frac{k}{(1 - F(x))^{n-r}} \int_x^\infty y^{k-1} \left[1 - \left[\frac{p}{G(Q_1)} \int_{-\infty}^y g(t)dt + \frac{p}{G(Q_1)} \int_{-\infty}^{Q_1} g(t)dt \right. \right.$$
$$\left. \left. + \frac{q}{(1 - G(P_1))} \int_{P_1}^y g(t)dt \right] \right]^{n-r} dy$$
$$= x^k + \frac{k\Delta(x)}{(1 - F(x))^{n-r}}.$$

Hence necessary part is proved.

Sufficient condition:
Let

$$E(X_{r+1:n}^k | X_{r:n} = x) = x^k + \frac{k\Delta(x)}{(1 - F(x))^{n-r}},$$

then

$$\frac{(n-r)}{(1-F(x))^{n-r}} \int_x^\infty y^k (1-F(y))^{n-r-1} f(y) dy = x^k + \frac{k\Delta(x)}{(1-F(x))^{n-r}}.$$

Multiplying both sides by $(1-F(x))^{n-r}$, and differentiating both sides with respect to x, we get

$$-(n-r)x^k(1-F(x))^{n-r-1}f(x)$$
$$= kx^{k-1}(1-F(x))^{n-r} - (n-r)x^k(1-F(x))^{n-r-1}f(x)$$
$$- kx^{k-1}\left[1 - \left[\frac{p}{G(Q_1)} \int_{-\infty}^y g(t)dt + \frac{p}{G(Q_1)} \int_{-\infty}^{Q_1} g(t)dt + \frac{q}{(1-G(P_1))} \int_{P_1}^y g(t)dt\right]\right]^{n-r}.$$

On simplification, we get

$$F(x) = \frac{p}{G(Q_1)} \int_{-\infty}^y g(t)dt + \frac{p}{G(Q_1)} \int_{-\infty}^{Q_1} g(t)dt + \frac{q}{(1-G(P_1))} \int_{P_1}^y g(t)dt.$$

This is the cumulative distribution function of the mid-truncated random variable.

Hence sufficient part is proved.

By applying the above theorem, X has mid-truncated Burr XII distribution iff

$$E(X_{r+1:n}^k | X_{r:n} = x) = x^k + \frac{k\Delta(x)}{(1-F(x))^{n-r}},$$

where

$$F(x) = p\frac{1-(1+\lambda x^\beta)^{-\nu}}{1-(1+\lambda Q_1^\beta)^{-\nu}} I_{0<x\le Q_1} + \left(1 - q\frac{(1+\lambda x^\beta)^{-\nu}}{(1+\lambda P_1^\beta)^{-\nu}}\right) I_{P_1 \le x < \infty},$$

$$\Delta(x) = \int_x^\infty y^{k-1}\left[1 - \left[p\left\{1 + \frac{1-(1+\lambda x^\beta)^{-\nu}}{1-(1+\lambda Q_1^\beta)^{-\nu}}\right\} + q\left\{\frac{(1+\lambda P_1^\beta)^{-\nu} - (1+\lambda y^\beta)^{-\nu}}{(1+\lambda P_1^\beta)^{-\nu}}\right\}\right]\right]^{n-r} dy,$$

$$I_{0<x\le Q_1} = \begin{cases} 1, & 0 < x \le Q_1 \\ 0, & \text{elsewhere}, \end{cases}$$

and

$$I_{P_1 \le x < \infty} = \begin{cases} 1, & P_1 \le x < \infty \\ 0, & \text{elsewhere}. \end{cases}$$

7. Conclusion

In live models as discussed earlier in the paper, mid-truncated Burr distribution is quite effectively used when the random variable is restricted to be observed on some sub intervals of the given specified range. Figures 1 and 2 depict the probability density function of Mid-truncated Burr distribution for different values of parameters and for different points of truncation.

We have derived the recurrence relations for single moments and product moments of order statistics from mid-truncated Burr XII distribution, and utilizing the knowledge of recurrence relations we can evaluate the moments of order statistics using the recursive algorithm as discussed in Section 4. Using these moments, to investigate the effect of the shape parameters β, ν and scale parameter λ on the mid-truncated Burr XII density function, we have computed mean, variance, skewness and kurtosis for different values of the parameters, which are presented in

Tables 1 to 3. From these tables we observe that by keeping any two parameters fixed, the increase in the value of the third parameter results in the decrease of the values of the measures calculated.

Also, due to the flexibility of the considered distribution, i.e., mid-truncated Burr XII distribution, which contains, as special sub models, mid-truncated Weibull distribution, mid-truncated exponential distribution, mid-truncated log-logistic distribution, among others, in accommodating different forms of risk functions, this mid-truncated Burr XII distribution is appropriate for a variety of problems modelling lifetime data.

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