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Mean Variance Complex-Based Portfolio Optimization

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Abstract Mean-Variance (MV) is a method that collects several assets using appropriate weight intending to maximize profits and to reduce risk. Stock market conditions are very volatile, mean variance method does not reach stock market fluctuation well because MV method is only limited to one time period. This study proposes a mean variance complex-based approach that transforms real returns into complex returns by using Hilbert transform to construct an optimal mean-variance portfolio based on complex returns and then find its dynamic asset allocation. The results show that with the same risk tolerance, the mean variance complex-based approach outperforms MV method in profits, losses, and portfolio performance tests.

Keywords Mean Variance (MV), Mean Variance Complex-Based, Hilbert Transform, Portfolio Construction

AMS 2010 subject classifications 62P05

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1. Introduction

Investment means allocating funds to one or more assets over a period of time with the aim of generating profits in the future. Investments can be made in real assets or financial assets. When investors invest in stocks, the investment has benefits and risks that will be borne by investors. The higher the expected rate of return, the higher the level of risk that the investor will bear. The term is "high risk, high return". Therefore, to reduce investment risk, a portfolio is formed. Portfolios collect several investment stocks owned by investors to maximize profits and reduce risks.

When investing, there is a saying, "Don't put all your eggs in one basket". This implies the importance of diversification, as depending on one basket carries the risk of losing all the eggs if the basket falls. Don't invest in just one asset as investors will suffer losses when all stock values fall. So, to reduce the risk, portfolio diversification is practiced. Portfolio diversification involves combining several investment instruments or assets to minimize risk without reducing expected returns.

One of the portfolio diversification strategies, dynamic asset allocation often changes the proportion of asset classes according to current market conditions. These adjustments often require increasing the proportion of the best performing asset classes and decreasing the proportion of the worst performing asset classes.

1.1. Mean Variance Optimization

One method to form a diversified portfolio is the Mean Variance Portfolio (MVP). Mean-Variance Portfolio is a method that collects several assets with the aim of maximizing profits and reducing risk. Research that studies the Mean-Variance Portfolio has been widely researched before, including by Markowitz, an economic scientist who researched investment so that the idea of creating a portfolio emerged.

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1.2. A brief review of existing methods

According to Markowitz in 1952 [15], when constructing a portfolio, it is assumed that the return data follows a normal distribution, and the focus is to reach the optimal return by minimizing risk. The mean-variance of a portfolio includes the expected return with a certain risk, or the return varies with risk. Tobin proposed that investors reduce risk by following the mean-variance principle to maximize returns [23]. Fisher and Statman researched Markowitz's puzzle-like mean-variance optimization framework. Mean Variance Portfolio is a standard portfolio construction model that is rarely used by investors because it is so limited that portfolios reflect constraints rather than optimization [2].

According to research by Gotoh et al. (2018), the mean-variance problem is comparable to an optimization problem for strong distributions, in which the objective's ϕ-divergence penalty controls the departure from the alternative distribution [3]. Despite the fact that semivariance is more in accordance with the preferences of logical investors, Rigamnoti (2020) finds that mean-variance portfolio optimization is more popular than optimization techniques that employ negative risk metrics. The results of the study show how well the mean-variance approach works [18]. Rosadi et al. (2020) investigated how to improve the mean-variance portfolio optimization model while taking into account robust covariance matrix estimator and integer transaction lots. The results show that the robust estimators outperform the standard MLE when there are outliers in the data and the lots are relatively sized—500 shares or less [20].

Iwazaki et al. (2021) applied the Mean-Variance Analysis in Bayesian Optimization (MVA-BO) method to study active learning (AL) problems in an uncertain environment where trade-offs between different risk measures need to be taken into account. Through theoretical study and numerical testing, the results demonstrate the effectiveness of the proposed AL algorithm. [7].

Ma et al. (2022), In the context of a discrete-time situation, investigated the optimistic value iteration for mean-variance optimization within the framework of a Markov decision process (MDP), assuming limited state and action spaces with a bounded reward [13]. The classic Markowitz mean-variance (MV) portfolio is examined by Li et al. (2022) using big dimension matrix theory. The term "large dimensional matrix" refers to a matrix with both large dimensions and huge data sizes. It has been discovered that spectrally-corrected estimates get the best results in terms of portfolio risk and return [12]. Mba et al. (2022) improved the MV formulation with include Behavioral MV (BMV) and Copula Behavioral MV (CBMV) in order to test the robustness of traditional mean-variance (MV) optimization models. Stocks with lower behavioral ratings beat comparable portfolios with higher behavioral scores, according to the research. Conversely, in the Forex market, the BMV strategy exhibits the opposite behavior, but the CBMV strategy stays constant [16].

Uchiyama et al. [24] were the first researchers to incorporate dynamic asset information into portfolio optimization with the CVRD (Complex Valued Risk Diversification) approach. As a result, the CVRD portfolio construction performance outperforms the portfolio construction performance of the Risk Parity (RP) and Maximum Risk Diversification (MRD) methods. Based on several studies that have been conducted, conventional data cannot capture dynamic asset allocation patterns into portfolios. Meanwhile, Hilbert transform, signal processing, and machine learning can capture dynamic asset patterns. The covariance matrix has an important role in constructing an optimal portfolio. Yusuke developed a covariance matrix derived from the return complex [11]. Then it was developed by Sugitomo et al. (2020) [22], Sasmita et al. (2023) for one-dimensional quaternion [1], and Nurwahidah et al. (2024) for two-dimensional quaternion [17] becomes a covariance matrix derived from the return quaternion.

1.3. Contributions

In situations where market conditions tend to be stable or market fluctuations are still within acceptable limits, the conventional or Mean-Variance method can provide reasonable estimates of returns and risks. However, since 2013 the stock market has fluctuated greatly [22], Meanwhile, based on Robbani & Jain [19], mean-variance is not an efficient portfolio for the S&P100 index.This implies that, by investing in the S&P100 index, investors may not achieve the highest return for a given level of risk or the lowest risk for a given level of return. Meanwhile, Uchiyama et al. [24] stated that the method with complex returns is more stable with current fluctuations than the method with conventional returns or real returns. Uchiyama et al. showed that the complex return method provides higher profits and lower losses than the real return method.

This research uses the Mean Variance Complex-Based (MVCB) method developed from the research of Uchiyama et al. which uses Mean Variance with real returns. This research uses Mean Variance with complex returns for portfolio optimization. Complex returns are obtained from the Hilbert transformation of real returns. This research is the first research that examines Mean Variance with complex returns.

The discussion in this paper aims to compare Mean-Variance (MV) portfolio optimization with Mean-Variance Complex-Based (MVCB) portfolio optimization. It is found that MVCB optimization is better than MV in terms of gains, losses, and portfolio performance test. This research contributes to the literature on investment optimization models, specifically in financial mathematics. It is a valuable addition to existing research, provides additional references for future research and suggests alternative perspectives for investors looking to optimize their investment portfolios.

1.4. Organization

The rest of the paper is organized as follows: Section 2 describes the mean variance portfolio optimization procedure with detailed explanations of each step as well as portfolio performance tests. Section 3 presents the experimental results and analysis of the complex-based mean variance portfolio optimization and its comparison with mean variance optimization. Section 4 presents the conclusion of this research.

2. Related Works

2.1. Mean Variance Optimization

Financial theory has grown significantly over the past few decades, and much knowledge is accessible for practical implementation, one of which is portfolio optimization [15]. Portfolio optimization based on mean-variance (MV) has been the main focus in finance since Markowitz (1952) [15] introduced this theory. As the concept goes, portfolio optimizers resolve investment uncertainty by selecting a portfolio that maximizes returns while maintaining a calculated level of risk. Alternatively, minimize variance while achieving the expected rate of return [12].

Modern portfolio theory is based on mean-variance optimization theory, which has many uses in various domains. In supply chain management, for example, the theory is used to analyze and design option contracts using mean-variance models [27]. Similarly, in asset pricing, the theory provides a mean-variance framework to test models through likelihood ratios [8]. The Mean Variance method is widely accepted in both academia and business, and because of its simplicity, it can be used to represent various risk minimization rules, making it a useful and common tool [13].

The calculation results will show the optimal portfolio return composition for individual stocks. The construction of this portfolio optimization model is expected to contribute to the existing literature, offering an alternative approach for investors who want to optimize their investment portfolio [6].

Investors are constantly making rational choices to increase their utility, according to Markowitz [15]. As expected, the main target of an investor is to minimize the portfolio's standard deviation (risk) or maximize the portfolio's average (return) [16]. This suggests that certain investors may seek to minimize risk, while others choose to maximize returns. Such choices depend on the investor's overall strategy, which is shaped by the investor's risk tolerance and investment goals. Investors carefully consider their asset allocation in hopes to maximize efficiency [19].

One aspect that encourages investors to invest is return because investors can see the investment results from the return. Realized return is the percentage change in the closing price of stock i on day t minus the closing price of stock i on day $t-1$ then the result is divided by the closing price of stock i on $t-1$.

$$R_{t(i)} = \frac{P_{t(i)} - P_{t-1(i)}}{P_{t-1(i)}}, \tag{1}$$

with $R_{t(i)}$: return of stock i, $i = 1, \dots, N.N$ is the number of stocks in the portfolio, $P_{t(i)}$: closing price stock i on day $t, i = 1, \dots, N, t = 1, \dots, n.n$ is the number of elements of a stock, and $P_{t-1(i)}$: the closing price of stock i on $t-1$.

Furthermore, the portfolio return is as follows:

$$R_t = \sum_{i=1}^{n} w_i R_{t(i)}, \tag{2}$$

with w_i: portfolio coefficient weight, and $R_{t(i)}$: Stock realized return i,$i = 1, \dots, N$.

Expected return, which is not the exact return received but is the average of all possible returns, recognizing that some returns are more likely than others from various investment scenarios, is calculated using the formula:

$$E[R_p] = \mathbf{w^T}\mu = \sum_{i=1}^{N} w_i.E[R_i], \tag{3}$$

in which $E[R_p]$: portfolio Expected Return, $\mathbf{w^T}$: transpose vector of weight coefficients, w_i: proportion of stock fund i,$i = 1, \dots, N$, and $E[R_i] = \mu$: expected Return of stock i, $i = 1, \dots, N$.

In the financial sector, risk is always significant as it affects trading, which depends on the pricing of market risk and the effectiveness of different investment techniques. In the financial sector, particularly in portfolio optimization, risk is represented by the covariance matrix in this context. Greater market volatility is indicated by high covariance values, while decreasing values indicate the opposite.

The covariance matrix is used by almost all portfolio design techniques to estimate portfolio risk and it serves as the optimization objective function. For random vectors, it is possible to extract stationary information from the corresponding covariance matrix. However, it is important to remember that asset prices usually exhibit erratic fluctuations and are not stationary. Therefore, accurate estimation of portfolio risk requires utilizing dynamic information about asset movements [22].

Calculate the stock covariance variance to determine the tendency of stocks to move together with the formula:

$$\sum = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1n} \\ \sigma_{21} & \sigma_{22} & \cdots & \sigma_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{n1} & \sigma_{n2} & \cdots. & \sigma_{nn} \end{bmatrix} = E[(R_{t(i)} - E[R_i])(R_{t(j)} - E[R_j]^T)], \tag{4}$$

with $\sum = \sigma_{ij} : cov(R_i, R_j)$ if $i \neq j$ and $Var(R_i, R_j)$ if $i = j, i, j = 1, \dots, N$, $R_{t(i)}$: stock return i in period $t, i = 1, \dots, N$, $R_{t(j)}$: stock return j in period $t, j = 1, \dots, N$, $E[R_i]$: the expected return value of stock i in period $t, i = 1, \dots, N$, and $(E[R_j])^T$: transpose of the expected return value with stock j in period $t, j = 1, \dots, N$ [24].

The squared average difference between actual returns and average returns is measured using variance. The greater the variance value, the further the difference between the actual return and the average return. Determine the optimal portfolio variance using the formula (see [20]):

$$\sigma^2 = \mathbf{w^T} \sum \mathbf{w} = \sum_{i=1}^{N}\sum_{j=1}^{N} w_i w_j \sigma_{ij}, \tag{5}$$

with σ^2: variance of portfolio, $\mathbf{w}$: weight coefficient vector, w_i: weight of the stock $i, i = 1, \dots, N$, w_j: weight of the stock $j, j = 1, \dots, N$, and $\sum = \sigma_{ij} : cov(R_i, R_j), i, j = 1, ..., N$.

The risk or standard revision of the portfolio (σ_p) is a weighted average of the individual standard deviation of each stock forming the portfolio, calculated using the formula:

$$\sigma_p = \sqrt{\sigma}. \tag{6}$$

Here, σ_p: standard deviation of the portfolio, and σ^2: variance of the portfolio.

To obtain an efficient portfolio, the portfolio optimization problem is solved using the Lagrange function as follows:

$$max\left(2\tau\mu^{\mathbf{T}}\mathbf{w}-\mathbf{w}^{\mathbf{T}}\sum\mathbf{w}\right). \tag{7}$$

Subject to $\mathbf{e}^{\mathbf{T}}\mathbf{w}=\mathbf{1}$ with μ: vector of expected returns for each stock $i, i=1,\ldots,N$, $\mathbf{w}$: weight vector of each stock $i, i=1,\ldots,N$, $\sum$: covariance matrix.

In $\mathbf{e}^{\mathbf{T}}=(1,1,...,N), \mu^{\mathbf{T}}\mathbf{w}=\mathbf{w}^{\mathbf{T}}\mu$, and $\mathbf{w}^{\mathbf{T}}\mathbf{e}=\mathbf{e}^{\mathbf{T}}\mathbf{w}$ and the parameter τ is the risk tolerance. The Lagrange function in (7), where the multiper is λ, can be expressed in the following:

$$L(w,\lambda)=\left(2\tau\mathbf{w}^{\mathbf{T}}\mu-\mathbf{w}^{\mathbf{T}}\sum\mathbf{w}\right)+\lambda\left(\mathbf{w}^{\mathbf{T}}\mathbf{e}-1\right). \tag{8}$$

Equation (8) by using the necessary condition theorem Kuhn-Tucker $\frac{\partial L}{\partial \mathbf{w}}=0$ and $\frac{\partial L}{\partial \lambda}=0$, is obtained:

$$\frac{\partial L}{\partial \mathbf{w}}=2\tau\mu-2\sum\mathbf{w}+\lambda\mathbf{e}=0. \tag{9}$$

$$\frac{\partial L}{\partial \lambda}=\mathbf{w}^{\mathbf{T}}\mathbf{e}-1=0. \tag{10}$$

Multiply equation (9) by inverse of the covariance matrix ($\sum^{-\mathbf{1}}$) and express it to $\mathbf{w}$, then multiply the result by $\mathbf{e}^{\mathbf{T}}$ to get

$$\mathbf{w}=\frac{1}{\mathbf{e}^{\mathbf{T}}\sum^{-\mathbf{1}}\mathbf{e}}\sum^{-\mathbf{1}}\mathbf{e}+\tau\left\{\sum^{-\mathbf{1}}\mu-\frac{\mathbf{e}^{\mathbf{T}}\sum^{-\mathbf{1}}\mu}{\mathbf{e}^{\mathbf{T}}\sum^{-\mathbf{1}}\mathbf{e}}\sum^{-\mathbf{1}}\mathbf{e}\right\};\tau\geq 0. \tag{11}$$

When $\tau=0$, gives a minimum variance portfolio with weights

$$\mathbf{w_0}=\frac{1}{\mathbf{e}^{\mathbf{T}}\sum^{-\mathbf{1}}\mathbf{e}}\sum^{-\mathbf{1}}\mathbf{e}. \tag{12}$$

Using the Markowitz model approach, followings are several benefits in solving optimization problems, namely:

- Risk tolerance τ is determined.
- Profit (μ_i) and risk (σ_{ij}) are required from asset returns [21].

2.2. *Hilbert Transform*

Like the quaternion Fourier transform [1], the Hilbert transform is important in various fields of science and technology, including optics, waves in stratified fluids, signal processing, and more [26].

The Hilbert transform of a time series function $x(t)$, where $x(t)$ is a real signal, at $t\in[0,\infty)$ is defined as the integral value of the Cauchy principle.

$$\mathcal{H}[x(t)]=\frac{1}{\pi}\int_0^{\infty}\frac{x(\tau)}{t-\tau}. \tag{13}$$

In practice, empirical time series is re-encoded at a specific sampling rate Δx, which results in a discrete-time $t_n=n\Delta t$ where n is an integer. The Hilbert transform for discrete time series is given by:

$$\mathcal{H}_D[X_k]=-isgn(k-\frac{N}{2})\sum_{n=0}^{N-1}x_n e^{i\frac{2\pi n}{N}}, \tag{14}$$

where $sgn(.)$ is the signum function.

2.3. Portfolio Performance Test

2.3.1. Risk Adjusted Return

The risk-adjusted return is the estimate of an investment's profit or projected profit that accounts for the degree of risk necessary to achieve that profit. The most basic method of evaluating portfolio performance is risk-adjusted return. The Risk Adjusted Return formula is as follows:

$$RiskAdjustedReturn = \frac{\mu_p}{\sigma_p}, \tag{15}$$

with μ_p is expected return portfolio and σ_p is standard deviation portfolio.

2.3.2. Sharpe Ratio

The Sharpe ratio test was first presented by Sharpe in 1966. The Sharpe ratio involves determining the optimal portfolio positioned at the efficient frontier, which represents the highest risk-adjusted portfolio return.

This ratio, which is determined by subtracting the risk-free rate return or the so-called excess return from the portfolio return and then divided by portfolio risk, which is characterized by its volatility [4]. The formula of Sharpe Ratio Test is:

$$SharpeRatio = \frac{R_p - R_f}{\sigma_p}. \tag{16}$$

In this case, R_p represent expected return portfolio, R_f is risk free rate, and σ_p is standard deviation portfolio.

2.3.3. Omega Ratio

Keating and Shadwick argue that their Omega ratio offers a comprehensive representation of the risk-return attributes in the distribution of investment returns. They suggest a metric that considers probability-weighted gains and losses [14].

$$\omega = \frac{E[r] - T}{E[T - r]^+} + 1. \tag{17}$$

Here, $E[r]$: expected return portfolio, T: a threshold for returns that divides desired returns (gains) and undesirable returns (losses). In general, it can be determined by the investor according to his beliefs. r: return of asset, and $E[.]^+$: the value if positive or otherwise [25].

3. Experimental results

3.1. Mean-Variance Complex-Based (MVCB)

As discussed in the previous section, the portfolio construction method uses the covariance matrix to estimate the portfolio risk as an optimization objective function. The covariance matrix of a random vector includes the autocorrelation of pairs of vector components. Therefore, stationary information can be obtained from the covariance matrix of the corresponding random vector. Usually, however, asset prices exhibit nonstationary random fluctuations. Therefore, dynamic information regarding asset fluctuations should be used to accurately estimate portfolio risk.

To include asset price dynamics into the portfolio construction, we find weights at each time on the time series of asset returns. This portfolio construction involves using time series with complex values obtained through the Hilbert transform, along with their covariance matrices, which we refer to as Mean Variance Complex-Based (MVCB) portfolio construction.

The Mean Variance Complex-Based procedure is as follows:

- Calculate the real return using equation (1).
- Transform the real return into complex return using Hilbert transform in (14), then based on equation (1) the return complex is as follows [24]:

$$z_t = r_t + i\mathcal{H}_D[r_t], \tag{18}$$

with z_t: complex return on t time, $t = 1, \ldots, 119$, r_t: return real of asset, and $\mathcal{H}_D[r_t]$: Hilbert transform operator for discrete time series of assets.
- Calculate the descriptive statistics.
- Calculate the covariance matrix using equation (4).
- Calculate the optimal weight of each asset. In calculating the weight, the inverse of the covariance matrix is required with (11). In portfolio optimization, $\tau \geq 0$ is simulated by taking several values that satisfy the condition $e^T w = 1$. Risk tolerance, which is denoted by τ. Taking the tolerance value is stopped if the risk tolerance value after being substituted into equation (11) generates a weight $w_i (i = 1, 2, \cdots, 6)$ which is continuous negative number and does not meet the requirement $e^T w = 1$
- Calculate the expected portfolio return using equation (3).
- Calculate the portfolio variance using equation (5).
- Calculate the standard deviation using equation (6).
- Calculate the dynamic assets.
- Calculate portfolio return of Mean-Variance Complex-Based using equation (2).
- Calculate portfolio performance tests, namely the Risk Adjusted Return (RAR) Test using equation (15), Sharpe Ratio Test using equation (16), and Omega Ratio Test using equation (17).
- Calculate the annualized return and annualized risk of Mean-Variance Complex-Based.

3.2. Data

The data in this study consists of 8 assets, which are divided into three types of assets, namely two foreign currencies or foreign exchange (forex), four stock indices, and two commodities. The data used is monthly data obtained through the investing.com website. The data period starts from October 22, 2013, to October 23, 2023.

Table 1. Asset List

Name	Type	Definition
REURUSD	Foreign currency or foreign exchange (forex)	Euro/USD
RCHFUSD	Foreign currency or foreign exchange (forex)	Franc Swiss (CHF)/USD
RWMT	Stock index	Walmart
RIBM	Stock index	International Business Machines Corporation
RMRK	Stock index	Merck & Co.
RNKE	Stock index	Nike Incorporation
RRICE	Commodities	Rough Rice Futures
RALUM	Commodities	Aluminium

In Table 2, n indicates the amount of return data from each asset; in this case, there are 119 data for each asset. The Shapiro-Wilk test is a statistical test that tests whether the data is normally distributed or not. The Shapiro-Wilk test begins by formulating a hypothesis, namely:

Null hypothesis (H_0): The data sample is derived from a normal distribution.

Hypothesis one (H_1): The data sample is not derived from a normal distribution.

In this test, if the $p - value \leq \alpha$ or in this study, the value of $\alpha = (0.05)$ is determined, then reject H_0, which means that the data is is not distributed normally. Meanwhile, if $p - value > \alpha$, then accept H_0, which means the data is derived from a normal distribution. Based on Table 2, the $p - value$ of each asset is $0.8417, 0.238, 0.2047, 0.0599, 0.05903, 0.05329 > 0.05$, so accept H_0, which means the data is normally distributed. Average or mean is the most common measure of center used in statistics to measure the middle value of data.

Table 2. Descriptive Statistics of Each Asset

Asset Name	n	Shapiro Wilk Test (P-Value)	Distribution	Mean	Standard Deviation
REURUSD	119	0.8417	Normal	-0.001878112	0.021343443
RCHFUSD	119	0.238	Normal	0.000202361	0.022122081
RWMT	119	0.2047	Normal	0.007288183	0.052508543
RIBM	119	0.0599	Normal	0.000633781	0.064300283
RMRK	119	0.3003	Normal	0.007964969	0.054464237
RNKE	119	0.7418	Normal	0.010669324	0.072431587
RRICE	119	0.05903	Normal	0.00222043	0.0657499
RALUM	119	0.05329	Normal	-0.000662755	0.053781273

From the mean, a vector mean μ_i will be formed, where i consists of the number of assets included in the portfolio $(i = 1, 2, 3, 4, 5, 6)$ which in vector form, the transpose mean is :

$\mu_i = (-0.001878, 0.000202, 0.007288, 0.000634, 0.007964969, 0.010669324, 0.00222, -0.000662)$

Vector mean μ_i will be used to calculate weights and expected returns.

Standard deviation is calculated by taking the square root of the variance. Standard deviation provides a measure that is easier to interpret than variance because it has the same units as the original data. The REURUSD return has the lowest standard deviation of 0.021343443, which means the lowest level of variation. Meanwhile, the RNKE return has the highest standard deviation of 0.072431587, which means the highest level of variation.

3.3. Covariance Matrix

Covariance matrices have been used to construct portfolios in the financial industry due to their ease of estimation for practitioners. The covariance matrix is used in determining the portfolio variance. Real return covariance matrix is used for Mean-Variance portfolio construction. Real return data will be transformed using Hilbert transformation. Equation (14) is the first step to transform the real return. Then, Equation (18) is an equation to form a complex return. The complex return will be used to form a complex covariance matrix, in this case the covariance matrix derived from the complex return is used for Mean-Variance Complex-Based portfolio optimization. Then, the inverse of the covariance matrix is used to determine each asset's weight. The covariance matrix with real returns and its inverse are expressed as follows:

$$\sum_R =$$
$$\begin{bmatrix} 0.0004555 & 0.0003073 & 0.0001661 & 0.0002219 & 0.0001519 & 0.0003690 & 0.0000417 & 0.0000526 \\ 0.0003073 & 0.0004894 & 0.0000666 & 0.0001198 & 0.0002198 & 0.0002186 & -0.0001058 & 0.0000902 \\ 0.0001661 & 0.0000666 & 0.0027571 & 0.0006841 & 0.0008592 & 0.0012567 & 0.0005755 & -0.0002196 \\ 0.0002219 & 0.0001198 & 0.0006841 & 0.0041345 & 0.0008115 & 0.0012769 & -0.0000191 & 0.0001807 \\ 0.0001519 & 0.0002198 & 0.0008592 & 0.0008115 & 0.0029664 & 0.0009050 & 0.0001001 & -0.0000399 \\ 0.0003690 & 0.0002186 & 0.0012567 & 0.0012769 & 0.0009050 & 0.0052463 & 0.0000730 & -0.0001128 \\ 0.0000417 & -0.0001058 & 0.0005755 & -0.0000191 & 0.0001001 & 0.0000730 & 0.0043230 & 0.0000783 \\ 0.0000526 & 0.0000902 & -0.0002196 & 0.0001807 & -0.0000399 & -0.0001128 & 0.0000783 & 0.0028924 \end{bmatrix}$$

$$\sum_R^{-1} =$$
$$\begin{bmatrix} 4073.31030 & -2510.07992 & -100.28849 & -100.59238 & 81.59270 & -146.21816 & -87.29872 & 0.66960 \\ -2510.07992 & 3711.48754 & 69.46159 & 49.93655 & -192.11222 & 23.12018 & 111.44604 & -72.72425 \\ -100.28849 & 69.46159 & 455.05328 & -30.08517 & -97.09428 & -79.29648 & -55.05113 & 33.13796 \\ -100.59238 & 49.93655 & -30.08517 & 276.41520 & -51.81657 & -46.75021 & 9.80996 & -22.08700 \\ 81.59270 & -192.11222 & -97.09428 & -51.81657 & 398.91076 & -30.55102 & -1.60561 & 4.72967 \\ -146.21816 & 23.12018 & -79.29648 & -46.75021 & -30.55102 & 235.60682 & 8.92213 & 7.36856 \\ -87.29872 & 111.44604 & -55.05113 & 9.80996 & -1.60561 & 8.92213 & 242.38123 & -12.91318 \\ 0.66960 & -72.72425 & 33.13796 & -22.08700 & 4.72967 & 7.36856 & -12.91318 & 352.58446 \end{bmatrix}$$

The covariance matrix with complex returns and its inverse are expressed as follows.

$$\sum_Z = \begin{bmatrix} 0.0004517 & 0.0003047 & 0.0001647 & 0.0002201 & 0.0001506 & 0.0003659 & 0.0000414 & 0.0000522 \\ 0.0001647 & 0.0000660 & 0.0027340 & 0.0006784 & 0.0008520 & 0.0012462 & 0.0005706 & -0.0002177 \\ 0.0002201 & 0.0001188 & 0.0006784 & 0.0040998 & 0.0008047 & 0.0012662 & -0.0000190 & 0.0001792 \\ 0.0001506 & 0.0002180 & 0.0008520 & 0.0008047 & 0.0029414 & 0.0008974 & 0.0000993 & -0.0000396 \\ 0.0003659 & 0.0002168 & 0.0012462 & 0.0012662 & 0.0008974 & 0.0052022 & 0.0000724 & -0.0001119 \\ 0.0000414 & -1.0497470 & 0.0005706 & -0.0000190 & 0.0000993 & 0.0000724 & 0.0042867 & 0.0000776 \\ 0.0000522 & 0.0000894 & -0.0002177 & 0.0001792 & -0.0000396 & -0.0001119 & 0.0000776 & 0.0028681 \end{bmatrix}$$

$$\sum_Z^{-1} = \begin{bmatrix} 4107.91581 & -2531.47570 & -101.15679 & -101.45571 & 82.31333 & -147.45175 & -88.05543 & 0.65365 \\ -2531.47570 & 3743.12093 & 70.07774 & 50.37363 & -193.78230 & 23.30807 & 112.41497 & -73.30840 \\ -101.15679 & 70.07774 & 458.91070 & -30.33989 & -97.91890 & -79.96849 & -55.51660 & 33.41893 \\ -101.45571 & 50.37363 & -30.33989 & 278.75815 & -52.25849 & -47.14592 & 9.89472 & -22.27384 \\ 82.31333 & -193.78230 & -97.91890 & -52.25849 & 402.29581 & -30.80953 & -1.62168 & 4.76882 \\ -147.45175 & 23.30807 & -79.96849 & -47.14592 & -30.80953 & 237.60320 & 8.99747 & 7.43133 \\ -88.05543 & 112.41497 & -55.51660 & 9.89472 & -1.62168 & 8.99747 & 244.43667 & -13.02209 \\ 0.65365 & -73.30840 & 33.41893 & -22.27384 & 4.76882 & 7.43133 & -13.02209 & 355.57111 \end{bmatrix}$$

The covariance matrix with Mean-Variance produces matrix elements that tend to be higher than the covariance matrix with Mean-Variance Complex-Based. This means that the volatility or risk of Mean-Variance optimization is higher than Mean-Variance Complex-Based optimization. High volatility indicates significant price fluctuations, while low volatility indicates price stability. Thus, Mean-Variance Complex-Based volatility tends to be more stable than Mean-Variance volatility.

3.4. *Mean-variance Diversified Portfolio Optimization Process*

The mean-variance diversification optimization problem refers to equation (7). Using vectors vector $\mu^{\mathbf{T}}$ and $\mathbf{e}^{\mathbf{T}}$(where $\mathbf{e}^{\mathbf{T}} = (1, 1, 1, 1, 1, 1)$ and matrix $\sum^{-1}$. Then, the weight vector $\mathbf{w}$ is calculated using equation (11). Risk tolerance τ with the condition $\tau \geq 0$ in diversification portfolio optimization is simulated by taking several values that meet the conditions $\mathbf{e}^{\mathbf{T}}\mathbf{w} = 1$. Taking the tolerance value is stopped if a risk tolerance value after being substituted in equation (11) produces a weight $w_i (i = 1, 2, \ldots, 6)$,which is continuously not a positive real number and does not satisfy the condition $\mathbf{e}^{\mathbf{T}}\mathbf{w} = 1$. The risk tolerance values and the results of the calculation of the weight of each asset are as follows:

Table 3. Optimization process of Mean Variance portfolio

τ	REURUSD	RCHFUSD	RWMT	RIBM	RMRK	RNKE	RRICE	RALUM	$e^T w$	μ_p	σ_p^2	$\frac{\mu_p}{\sigma_p^2}$
0	0.3700248	0.3637433	0.0598340	0.0259182	0.0342356	-0.0084932	0.0658998	0.0888374	1	0.0001007	0.0003055	0.3295424
0.01	0.2682334	0.4126980	0.0769994	0.0186313	0.0532433	0.0111057	0.0698892	0.0891995	1	0.0007914	0.0003124	2.5329174
0.02	0.1664421	0.4616526	0.0941649	0.0113445	0.0722511	0.0307046	0.0738786	0.0895616	1	0.0014821	0.0003332	4.4485505
0.03	0.0646507	0.5106073	0.1113303	0.0040577	0.0912588	0.0503035	0.0778680	0.0899236	1	0.0021728	0.0003677	5.9091821
0.04	-0.0371407	0.5595620	0.1284957	-0.0032291	0.1102665	0.0699024	0.0818574	0.0902857	1	0.0028634	0.0004160	6.8826257
0.05	-0.1389321	0.6085166	0.1456612	-0.0105159	0.1292743	0.0895013	0.0858468	0.0906478	1	0.0035541	0.0004782	7.4322924

From Table 3, it can be seen that τ or risk tolerance for the Mean-Variance method stops at $\tau = 0.04$. This is because the REURUSD and RIBM weights do not fulfill the optimal portfolio requirement, which is continuously not a positive number so the iteration stops. To get the optimal portfolio, it is seen from the value of $\frac{\mu_p}{\sigma_p^2}$. The optimal portfolio will produce the largest $\frac{\mu_p}{\sigma_p^2}$ value because it provides the highest profit value with a certain volatility or loss which when the highest profit is divided by a certain loss will provide the largest ratio. In Table 3, the highest $\frac{\mu_p}{\sigma_p^2}$ value is 5.909182088 with a risk tolerance of $\tau = 0.03$.

Based on the expected return portfolio using equation (3) and the portfolio variance using equation (5), with a risk tolerance of $\tau = 0.03$, the optimal expected return with the mean-variance method is 0.002172755 and the optimal variance is 0.000367691.

Table 4. Optimization process of Mean-Variance Complex-Based

τ	REURUSD	RCHFUSD	RWMT	RIBM	RMRK	RNKE	RRICE	RALUM	$e^T w$	μ_p	σ_p^2	$\frac{\mu_p}{\sigma_p^2}$
0.00	0.3699980	0.3637696	0.0598361	0.0259188	0.0342302	-0.0084936	0.0659017	0.0888391	1	0.0001008	0.0003030	0.3328277
0.01	0.2673332	0.4131496	0.0771471	0.0185716	0.0533977	0.0112712	0.0699252	0.0892039	1	0.0007974	0.0003099	2.5728916
0.02	0.1646692	0.4625296	0.0944582	0.0112244	0.0725651	0.0310360	0.0739487	0.0895687	1	0.0014940	0.0003308	4.5159424
0.03	0.0620049	0.5119096	0.1117692	0.0038772	0.0917325	0.0508007	0.0779723	0.0899336	1	0.0021905	0.0003656	5.9908111
0.04	-0.040659	0.5612896	0.1290803	-0.0034701	0.1109000	0.0705655	0.0819958	0.0902984	1	0.0028871	0.0004144	6.9667968
0.05	-0.14332387	0.61066962	0.1463913	-0.01081726	0.13006739	0.09033027	0.08601929	0.09066325	1	0.00358366	0.0004771	7.51136083

From Table 4, it can be seen that τ or risk tolerance for the Mean-Variance Complex-Based method stops at $\tau = 0.04$. This is because the REURUSD and RIBM weight does not meet the optimal portfolio requirements, which is continuously not a positive number so the iteration stops. To get the optimal portfolio, it can be seen from the $\frac{\mu_p}{\sigma_p^2}$ value. In Table 4, the highest $\frac{\mu_p}{\sigma_p^2}$ value is 5.9908111 with a risk tolerance of $\tau = 0.03$. Based on the expected return portfolio using equation (3) and the portfolio variance using equation (5), with a risk tolerance of $\tau = 0.03$, optimal expected return with the mean-variance method is 0.0021905 and the optimal variance is 0.0003656. Optimal expected return of Mean Variance Complex-Based is 0.000017775 higher than mean variance method. This indicates that the Mean Variance Complex-Based is more profitable than Mean-Variance method. The optimal variance of the Mean Variance Complex-Based is -0.000002041 lower than the mean variance method. This indicates that the volatility of the Mean Variance Complex-Based is lower than the Mean Variance method. The weights of the entire portfolio are rebalanced monthly without transaction costs or dynamic assets with the same risk tolerance (τ) between Mean-Variance and Mean-Variance Based-Complex.

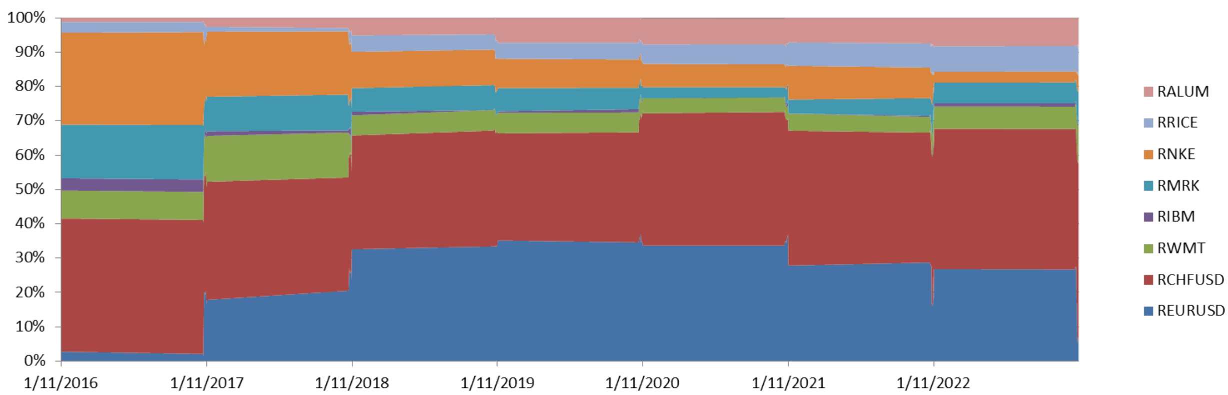

Figure 1. Dynamic Assets Using Mean-Variance Method

The order of Mean-Variance-Based and Mean-Variance Complex-Based portfolio dynamic assets is shown in Figures 1 and 2. The Mean Variance and Mean Variance Based Complex graphs can be said to be almost similar, but based on the dynamic asset data that has been obtained, it certainly has a different value. The difference in dynamic asset data tends to be small so that it is not readable on the dynamic asset graph. It can be seen that both Mean-Variance and Mean-Variance Complex-Based mostly consist of forex. Forex investment is quite volatile but always consistent at 1 USD, so the weight proportion is always consistent. As for stock assets, both Mean-Variance and Mean-Variance Complex-Based, RMRK and RNKE stocks have a fairly good weight proportion at the beginning, but every year the stock is very volatile, so the proportion of its weight is decreasing every year. In RWMT stocks, the proportion of weight is quite consistent every year and for RIBM stocks which have the

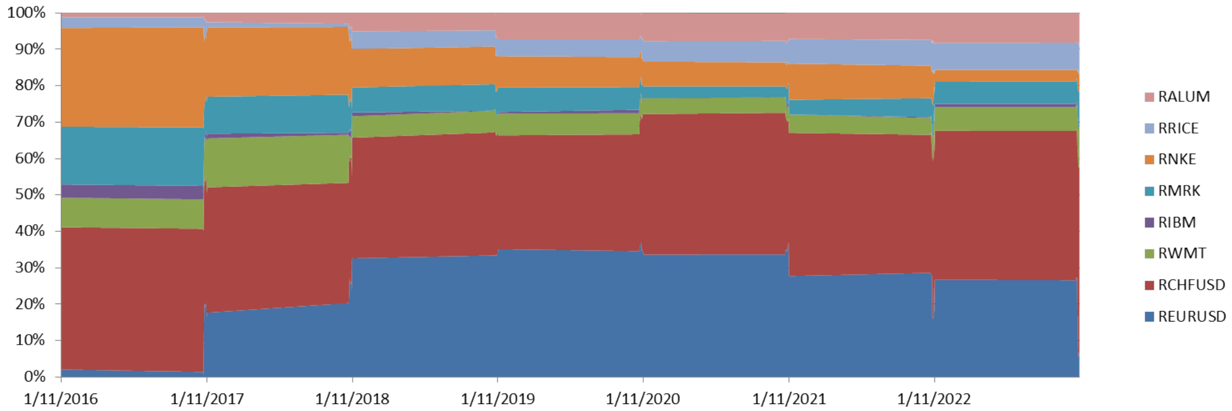

Figure 2. Dynamic Assets Using Mean-Variance Complex-Based

smallest proportion of weight is very volatile every year. As for commodities with the lowest risk level, both Mean-Variance and Mean-Variance Complex-Based, RRICE, and RALUM commodities every year the proportion of weight continues to increase.

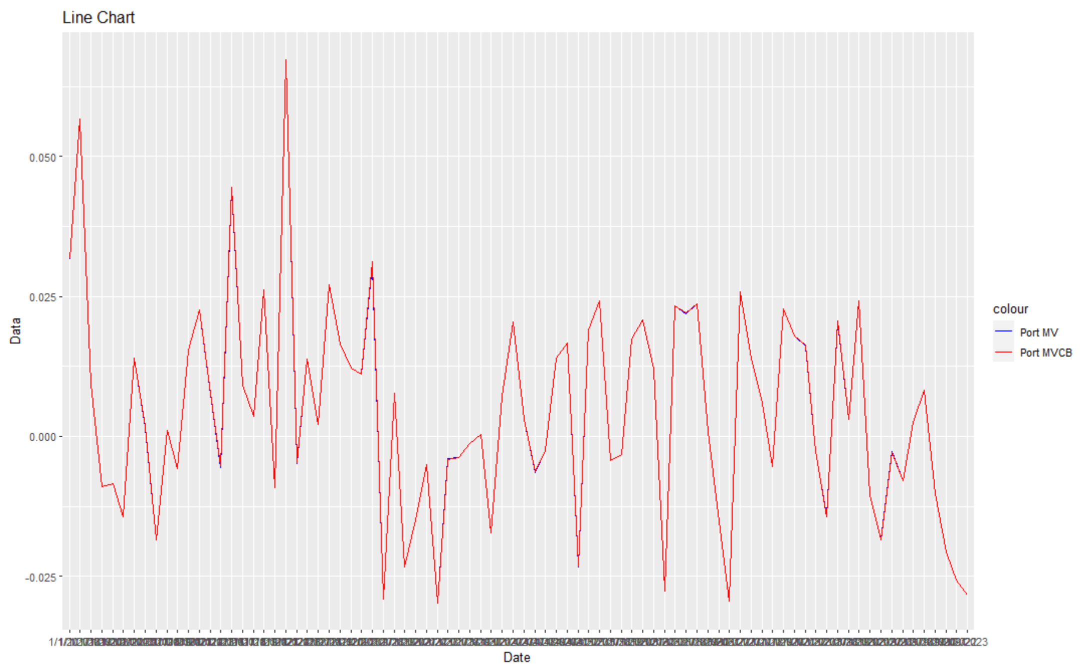

Figure 3. Portfolio Mean-Variance and Mean-Variance Complex-Based

Using equation (2), Mean-Variance Based and Mean-Variance Complex-Based portfolios are obtained as shown in Figure 3. It is seen that the Mean-Variance Complex-Based portfolio is more favorable than the Mean-Variance method. Although Mean-Variance outperforms Mean-Variance Complex-Based several times, Mean-Variance Complex-Based has more advantages compared to Mean-Variance. When the portfolios are summed up, the Mean-Variance Complex-Based portfolio sum of 0.355979 is higher than the Mean-Variance portfolio sum of 0.354231875.

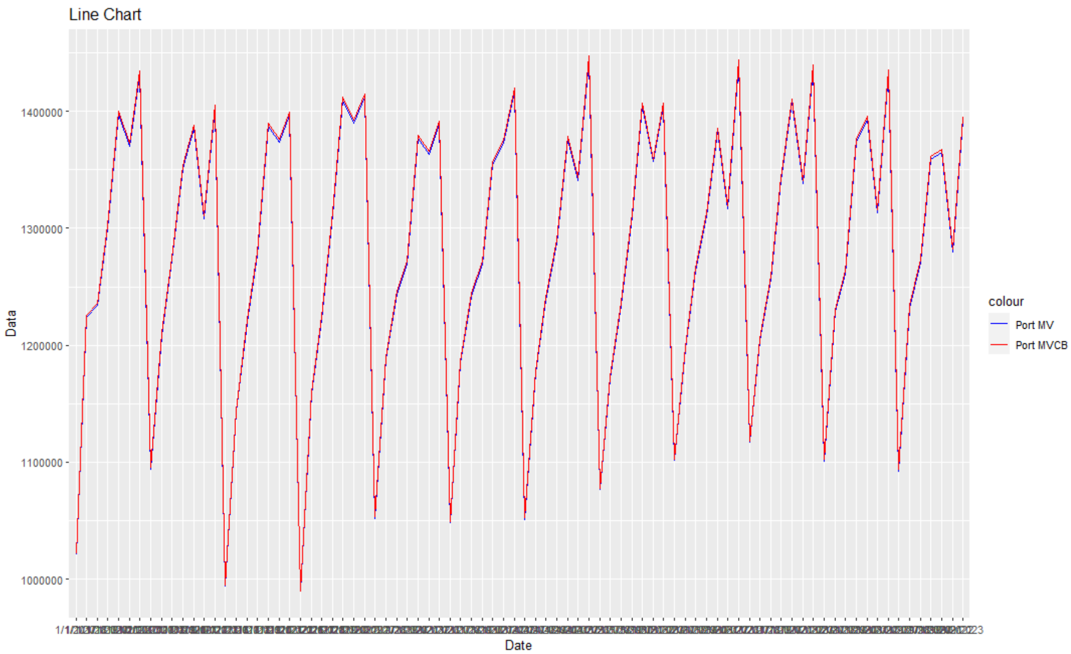

Figure 4. Portfolio Mean-Variance and Mean-Variance Complex-Based with an initial capital of 1000000 USD

Based on Figure 4, it can be seen that the Mean-Variance and Mean-Variance Complex-Based portfolios with an initial capital of 1000000 USD are more favourable to the Mean-Variance Complex-Based portfolio because they outperform the Mean-Variance portfolio. Investor funds that were originally 1000000 USD on 1 October 2016, when using the Mean-Variance diversification portfolio, then 7 years later on 1 October 2023 the investor's funds became 1403153.069 USD. While the original investor's funds were 1000000 USD on 1 October 2016, when using the Mean-Variance Complex-Based diversified portfolio, then 7 years later the investor's funds became 1405521.577 USD. The difference between the Mean-Variance and Mean-Variance Complex-Based portfolios is 2368.508231 USD.

Table 5. Performance Test Portfolio

Method	Mean-Variance	Mean-Variance Complex-Based
Expected Return Portfolio	0.001218	0.001229
Standard Deviation Portfolio	0.000346	0.000343
Risk-Adjusted Return	0.065550786	0.0663467
Sharpe Ratio	0.199525314	0.202217892
Omega Ratio	1.065434353	1.069484687

The Risk-Adjusted Return (RAR), Sharpe ratio, and omega ratio are assessed as indicators of portfolio performance. Risk-adjusted return measures the return of an investment instrument by first taking into account the risk contained in the instrument. Sharpe ratio measures the relationship between return versus volatility [5]. However, the Sharpe ratio has some drawbacks; if the standard deviation performance is poor, it will cause poor performance of the Sharpe ratio [9], and the Sharpe ratio only considers the first two moments of the return distribution in the construction.

In addition, the omega ratio is used because it can cover all the statistical information of the portfolio [5]. Omega ratio is a probability-weighted ratio of gains versus losses to a prospect or the ratio of upside return (good) compared to downside return (bad)[10].

Table 5 shows the Expected Return Portfolio, Standard Deviation Portfolio, Risk Adjusted Return Portfolio, Sharpe ratio, and omega ratio of the Mean-Variance and Mean-Variance Complex-Based portfolios.

The Risk-Adjusted Return test value of the Mean Variance Complex-Based method of 0.114555624 is higher than the Mean-Variance method of 0.106522995. So, for every 1% risk borne by investors, 0.114555624 or 0.114555624% return is obtained for the Mean Variance Complex-Based method.

The Sharpe ratio test value obtained for the Mean Variance Complex-Based is 0.373382995, and the Mean-Variance method is 0.346949787. A positive Sharpe ratio value indicates that the rate of return on the investment or portfolio is higher than the risk-free risk rate (usually measured by government bond interest rates). Sharpe ratio test value Mean Variance Complex-Based is higher than Mean-Variance method. So at 1% risk, if investors use Mean Variance Complex-Based optimization, then investors can get an excess return (return or return from a stock or bond, etc., after deducting interest rate risk) is 0.373382995%. Furthermore, if the investor uses Mean-Variance optimization, then the investor's excess return is 0.346949787% on 1% risk.

The omega ratio test value obtained Mean Variance Complex-Based of 1.246936104 is greater than the mean-variance of 1.243838544. If the Omega Ratio test value is more than 1, it indicates that the investment return tends to exceed the set positive limit more often. Intuitively, this is an indication that the investment has a tendency to deliver the desired results or even exceed expectations at a predetermined return level.

The Mean Variance Complex-Based portfolio outperforms the Mean-Variance portfolio in all measures, starting from the higher annualized return, lowest risk, higher risk-adjusted return, higher Sharpe ratio, and higher omega ratio.

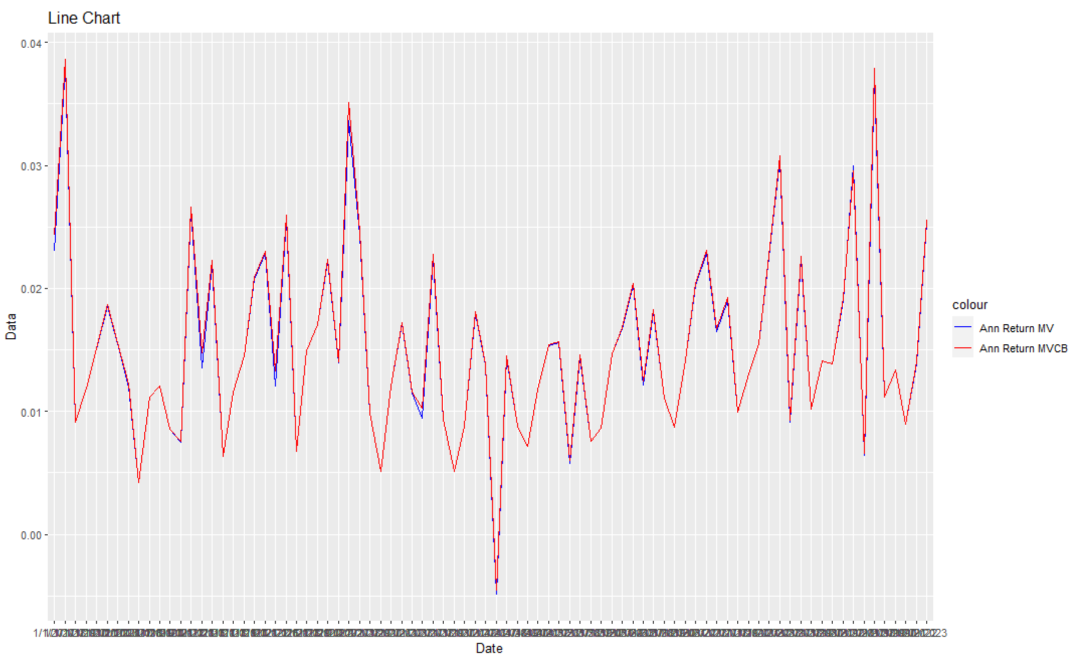

Figure 5. Annual Return Mean-Variance and Mean-Variance Based Complex

Based on equation (3), the expected return of the Mean-Variance and Mean Variance Complex-based portfolios is obtained which is still in the form of monthly data. Furthermore, the expected return for each research period is sought, or in this study is the expected return for each month of the research time. Therefore, to get an annualized return, the monthly expected return is converted into an annualized expected return or annualized return. Then the annualized return for each period is sought. Figure 5 shows the annualized returns of the Mean-Variance and Mean Variance Complex-Based methods. It can be seen that the annualized returns of Mean-Variance and Mean Variance Complex-Based fluctuate. However, the annualized return of Mean Variance Complex-Based is always superior to the Mean-Variance method. One of them is on 1 October 2023, the annual return of Mean Variance Complex-Based

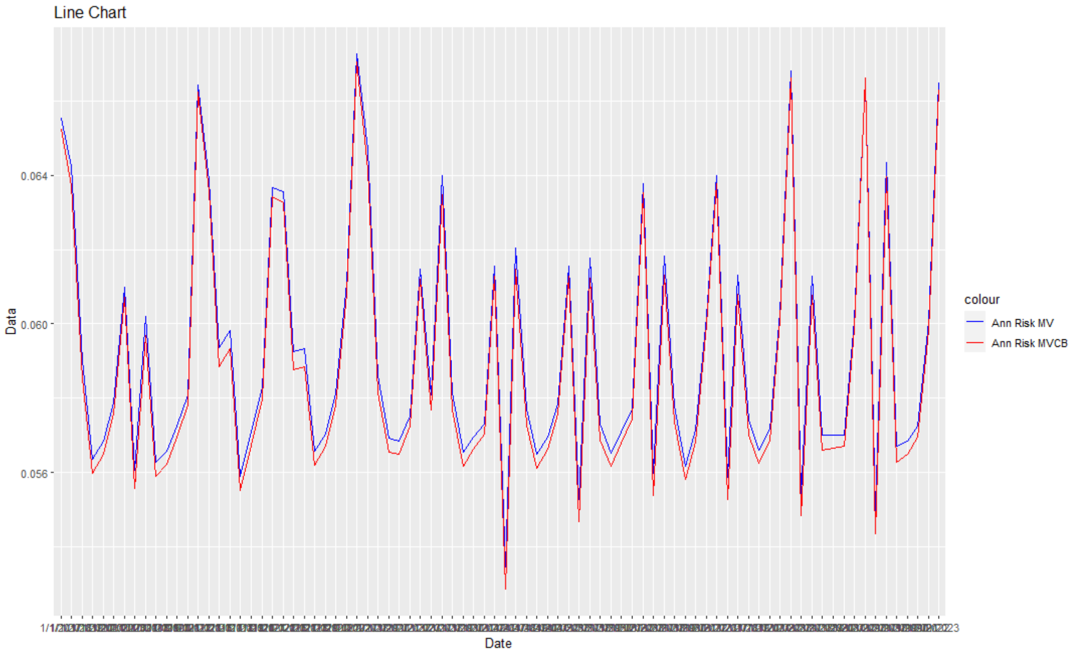

Figure 6. Annual Risk Mean-Variance and Mean-Variance Based Complex

which is 0.02660538 is superior to the Mean-Variance annualized return of 0.0263869. So the annualized return of Mean Variance Complex-Based is more profitable.

Based on equation (6), the portfolio standard deviation or volatility of Mean-Variance and Mean Variance Complex-Based is obtained which is still in the form of monthly data. Then look for volatility for each month of the study time. Therefore, to get the annualized risk, monthly volatility is converted into annual volatility or annualized risk. Then look for annualized risk for each period. With an asset risk tolerance of $0 \leq \tau \leq 0.03$ in the portfolio optimization process, Figure 5 shows the annualized risk of each method. Based on Figure 6, it can be seen that the annualized risk of Mean-Variance is higher than Mean Variance Complex-Based. One of them is on October 1, 2023, the annualized risk of Mean-Variance which is 0.066425109 is higher than the annualized risk of Mean Variance Complex-Based which is 0.066240471. Therefore, the annualized risk of Mean Variance Complex-Based is lower than that of Mean-Variance. Thus, Mean Variance Complex-Based is more profitable.

4. Conclusion

This study compared two portfolio construction performances: Mean-Variance and Mean Variance Complex-Based. In the study, the annual return of Mean Variance Complex-Based portfolio construction outperforms the return of Mean-Variance. In addition, with the asset risk level $0 \leq \tau \leq 0.03$ in the portfolio optimization process, the annual risk of Mean-Variance is higher than that of Mean Variance Complex-Based, so the asset risk in Mean Variance Complex-Based portfolio construction is well allocated compared to Mean-Variance. The Risk-Adjusted Return (RAR) test, Sharpe ratio test, and omega ratio test of the Mean-Variance and Mean Variance Complex-Based portfolios were conducted to measure portfolio performance. The Mean Variance Complex-Based portfolio outperformed the Mean-Variance in all measures. These results proved that the Mean Variance Complex-Based portfolio construction successfully constructs the optimal portfolio.

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