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Moment Properties of Generalized Order Statistics From Lindley Pareto Distribution

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Abstract The LindleyPareto distribution is a more flexible model for analyzing the lifetime data. In this paper, the moment properties of generalized order statistics from the Lindley Pareto distribution in terms of exact expression and recurrence relations are studied. The results for order statistics, record values, and progressive type-II right censored order statistics are discussed as particular cases of generalized order statistics. Further, the characterization of the said distribution through recurrence relations between moments of generalized order statistics are presented. Finally, some statistical measures of order statistics and record values for the Lindley Pareto distribution are computed.

Keywords Lindley Pareto distribution, single moment, product moments, recurrence relations, order statistics, record value, generalized order statistics and characterization.

AMS 2010 subject classifications 62G30, 62E10

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1. Introduction

The moment properties of various ordered schemes and recurrence relations between them play crucial roles in statistics, probability theory, and various applied fields. In the characterization of distributions, moments provide essential information about the shape and characteristics of the probability distribution. In statistical inference, higher-order moments can provide insights into the variability, asymmetry, and peakedness of the data. This can enhance the understanding of sample properties and improve decision making in statistical analyses. In the fields like reliability engineering and quality control, moments of order statistics are used to model and assess the performance and failure rates of systems and products. These help in designing and evaluating experiments and in making predictions about future performance. The recurrence relations and identities reduce the amount of direct computations and hence the time and labour. They also express the higher-order moments in terms of the lower-order moments and hence make the evaluation of higher-order moments easy.

1.1. Definition

Kamps [4] proposed the concept of generalized order statistics (GOS). The GOS offers a unified approach to various models of ordered random variables such as upper record values, order statistics, $k-$th upper record values, progressively type-II censoring order statistics, sequential order statistics and Pfeifer records. These models have valuable applications, especially in the field of reliability theory.

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Let $Y_1, Y_2, ..., Y_n$ be a sequence of independent and identically distributed (iid) random variables (rvs) from an absolutely continuous population with cumulative distribution function (CDF) $F(y)$ and probability density function (PDF) $f(y)$, $y \in (\alpha, \beta)$, where α and β may be finite or infinite. Further, assume that $n \in N, \quad n \geq 2, \quad k \geq 1, \quad \tilde{m} = (m_1, m_2, \ldots, m_{n-1}) \in \mathbb{R}^{n-1}$, and $M_i = \sum_{j=i}^{n-1} m_j$ be the parameters such that

$$\gamma_i = k + n - i + M_i \geq 0 \text{ for } 1 \leq i \leq n-1.$$

Then the random variables $Y_{(r,n,\tilde{m},k)}$, $r = 1, 2, \ldots n$ are called GOS from an absolutely continuous population, if their joint PDF is given by [Kamps [4]]

$$k\Big(\prod_{j=1}^{n-1} \gamma_j\Big)\Big(\prod_{i=1}^{n-1} \big[1 - F(y_i)\big]^{m_i} f(y_i)\Big)\big[1 - F(y_n)\big]^{k-1} f(y_n) \tag{1}$$

on the cone $F^{-1}(0) < y_1 \leq y_2 \leq \ldots \leq y_n < F^{-1}(1)$.

Several models of ordered random variables such as order statistics, record values, sequential order statistics, and progressively type II censored order statistics, etc., can be seen as special cases of GOS. If we choose $m_i = m = 0, \; i = 1, 2, \ldots, n$ and $k = 1$ in model (1), then $Y_{(r,n,m,k)}$ reduces to the $r-$th order statistics $Y_{r:n}$. By choosing $m_r = R_r, \; k = R_n + 1$, and $\gamma_r = n - r + 1 + \sum_{i=r}^{m} R_i, \; 1 \leq r \leq m$, where R_i is a set of prefixed integers then the model (1) reduces to the joint density based on progressively type-II censored order statistics. If $m_i = m \to -1, \; i = 1, 2, \ldots, n$ and $k > 0$ be any positive integers, then $Y_{(r,n,m,k)}$ reduces to the $k-$th upper record values $Y_{U_r^{(k)}}$. By setting $m_r = \gamma_r - \gamma_{r+1} - 1$ and $k = \alpha_n, \; \alpha \in R^+$, in this case model (1) reduces to the joint PDF of sequential order statistics.

1.2. Marginal and Joint Distribution

Here, we can discuss two cases of GOS:

Case I. $\gamma_i \neq \gamma_j \; , i, j = 1, 2, \ldots, n-1, i \neq j.$

In view of the model (1), the marginal PDF of the $r-$th GOS is given as (Kamps and Cramer [7])

$$f_{r,n,\tilde{m},k}(y) \;=\; C_{r-1} f(y) \sum_{i=1}^{r} a_i(r) [\bar{F}(y)]^{\gamma_i - 1}, \tag{2}$$

where

$$C_{r-1} \;=\; \prod_{i=1}^{r} \gamma_i, \qquad \gamma_i = k + n - i + \sum_{j=1}^{n-1} m_j > 0,$$

and

$$a_i(r) = \prod_{\substack{j=1 \\ j \neq i}}^{r} \frac{1}{(\gamma_j - \gamma_i)}, \quad 1 \leq i \leq r \leq n.$$

The joint PDF of $r-$th and $s-$th, GOS $1 \leq r < s \leq n$, is given as (Kamps and Cramer [7])

$$\begin{aligned} f_{r,s,n,\tilde{m},k}(y,z) \;=\; & C_{s-1} \sum_{j=r+1}^{s} {a_j}^{(r)}(s) \left(\frac{\bar{F}(z)}{\bar{F}(y)}\right)^{\gamma_j} \\ & \times \left[\sum_{i=1}^{r} a_i(r) [\bar{F}(y)]^{\gamma_i}\right] \frac{f(y)}{\bar{F}(y)} \frac{f(z)}{\bar{F}(z)}, \quad y < z, \end{aligned} \tag{3}$$

where

$$a_j{}^{(r)}(s) = \prod_{\substack{t=r+1 \\ t\neq j}}^{s} \frac{1}{(\gamma_t - \gamma_j)}, \quad r+1 \leq j \leq s \leq n.$$

Case II : $m_i = m, i = 1, 2, \ldots, n-1.$

The marginal PDF of $r-$th GOS of $Y_{r,n,m,k}$ is given as (Kamps [4])

$$f_{r,n,m,k}(y) = \frac{C_{r-1}}{(r-1)!} [\bar{F}(y)]^{\gamma_r - 1} f(y)\, g_m^{r-1}(F(y)), \tag{4}$$

where

$$C_{r-1} = \prod_{i=1}^{r} \gamma_i, \qquad \gamma_i = k + (n-i)(m+1),$$

$$h_m(y) = \begin{cases} -\dfrac{1}{m+1}(1-y)^{m+1}, & m \neq -1 \\ \log\left(\dfrac{1}{1-y}\right), & m = -1 \end{cases}$$

and

$$g_m(y) = h_m(y) - h_m(0) = \int_0^y (1-t)^m dt, \ \ y \in [0,1).$$

The joint PDF of $Y_{r,n,m,k}$ and $Y_{s,n,m,k}$, $1 \leq r < s \leq n$, is given as (Pawlas and Szynal [11])

$$f_{r,s,n,m,k}(y,z) = \frac{C_{s-1}}{(r-1)!\,(s-r-1)!} [\bar{F}(y)]^m g_m^{r-1}(F(y))$$
$$\times [h_m(F(z)) - h_m(F(y))]^{s-r-1} [\bar{F}(z)]^{\gamma_s - 1} f(y)\, f(z), \ \ -\infty \leq y < z \leq \infty. \tag{5}$$

1.3. The Lindley Pareto Distribution

The lifetime distributions have always been useful in different fields such as reliability, engineering, economics, insurance, actuarial sciences, finances, and medicine. Lindley [8] proposed the Lindley distribution as a new distribution to analyze lifetime data. Pareto distribution was established by Pareto [9] to analyze the unequal distribution of wealth. It is commonly used in actuarial science because of its heavy tail properties. In this paper, we consider one of the generalizations of Lindley distribution, which is proposed by Zeghdoudi et al. [1], called Lindley Pareto distribution, to include a wider class of continuous distributions. The Lindley Pareto distribution offers a partical and effective way to model data with heavy tails and it is easy to implement. The Lindley Pareto distribution is a special case of odd-Lindley-G family of distributions seen in Gomes-Silva $et\ al.$[2]. The survival function (SF) of this distribution is given as:

$$\bar{F}(y) = 1 - F(y) = \frac{\alpha^\beta + \theta y^\beta}{(\theta+1)\alpha^\beta} \exp\left\{-\theta\left(\frac{y^\beta}{\alpha^\beta} - 1\right)\right\}, \qquad y > \alpha; \ \ \alpha, \beta, \theta > 0, \tag{6}$$

and the corresponding PDF is given by

$$f(y) = \frac{\beta\,\theta^2\, y^{2\beta-1}}{(\theta+1)\alpha^{2\beta}} \exp\left\{-\theta\left(\frac{y^\beta}{\alpha^\beta} - 1\right)\right\}, \qquad y > \alpha; \ \ \alpha, \beta, \theta > 0. \tag{7}$$

Using (6) and (7), the relationship between SF and PDF of the Lindley Pareto distribution is written as:

$$\bar{F}(y) = \frac{\alpha^\beta}{\beta\,\theta^2}\left(\alpha^\beta\, y^{1-2\beta} + \theta\, y^{1-\beta}\right) f(y). \tag{8}$$

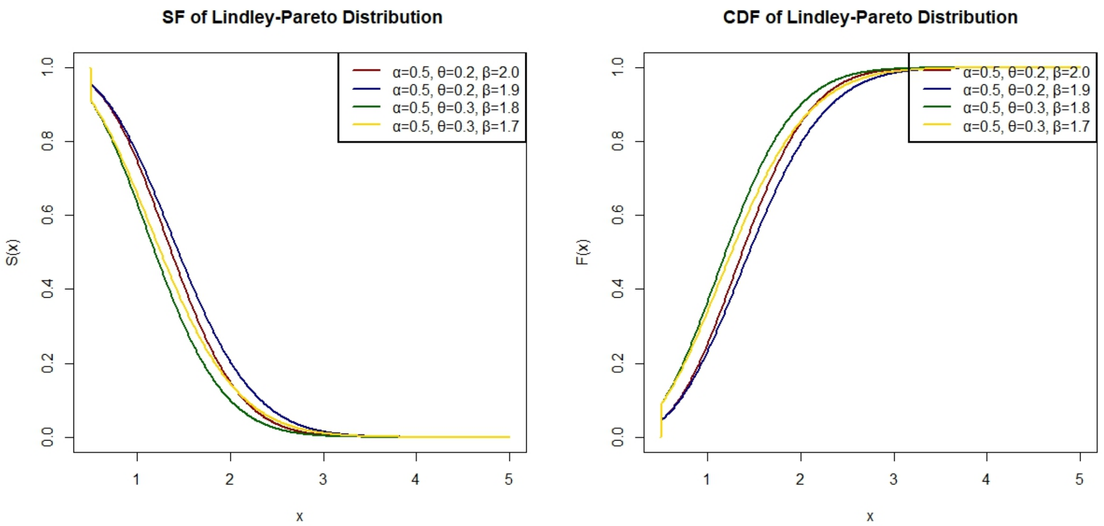

Figure 1. Survival functions and CDFs of the Lindley Pareto distribution for $\alpha = 0.5$ and for selected values of θ and β.

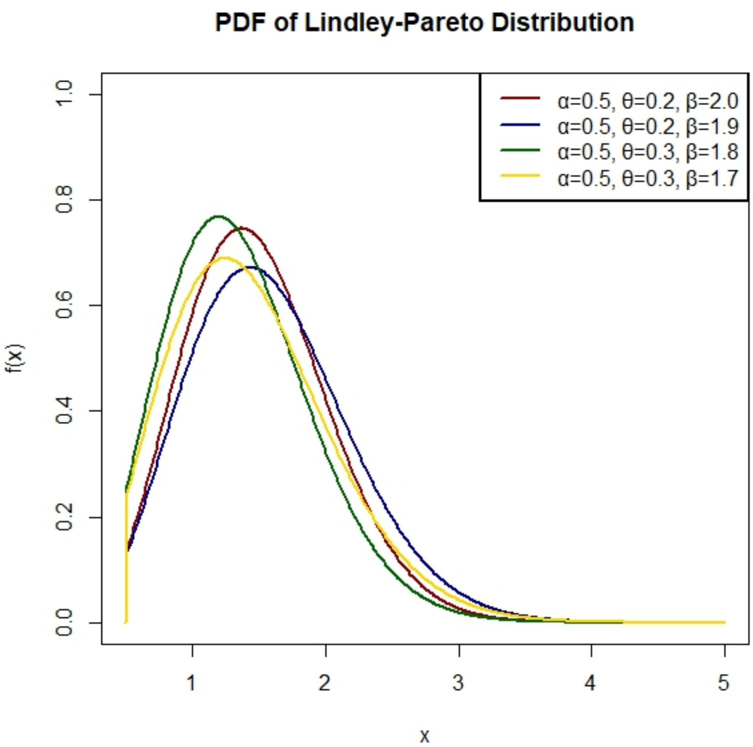

Figure 2. PDFs of the Lindley Pareto distribution for $\alpha = 0.5$ and for selected values of θ and β

1.4. Literature review:

The moment properties of GOS and recurrence relations between them for various distributions have recently attracted much attention. Many authors have studied the moments of GOS for several distributions. For instance, Keseling [13] studied the conditional distribution of GOS and characterized some specific continuous distributions. Pawlas and Szynal [10] presented the recurrence relations for single and product moments of GOS for Pareto, generalized Pareto, and Burr distributions. Athar and Islam [3] gave some general results for the moments of GOS and applied their results to a general class of distributions. Ahmad and Fawzy [15] have proposed recurrence relations for the moments of GOS for a class of doubly truncated distributions. This class includes special cases such as doubly truncated Weibull distribution (Exponential and Rayleigh), Burr type XII (Lomax) and Pareto distributions. Khan and Zia [16] provided the recurrence relations for single and product moments of GOS based on doubly truncated linear exponential distribution; furthermore, the characterization is also discussed using the recurrence relation for single moment of GOS. Athar et al. [17] have presented the recurrence relations for the marginal and joint moment generating functions of GOS based on the Marshall-Olkin extended exponential distribution and characterized the said distribution. Khan et al. [5] derived the exact and explicit expressions for single and product moments of the Lindley distribution based on GOS. Athar et al. [19] have developed the moment properties of GOS and discussed the special cases such as record values and order statistics from Poisson Lomax distribution. Furthermore, they also presented the characterization results using conditional moments and recurrence relations. Alharbi et al. [18] proposed and studied a new exponentiated generalized class of distribution. Further, they studied the moment properties of GOS in terms of recurrence relations and characterization. Athar et al. [6] examined the moment properties of GOS derived from the modified weighted Rayleigh distribution, and also discussed the characterization results based on these moments properties and conditional expectations. Khan [20] has obtained the moment properties of GOS from doubly truncated power linear hazard rate distribution. For more details on moments of GOS and characterization, one may refer to Pawlas and Szynal [11], Cramer and Kamps [12], Saran and Pandey [14], Akhter et al. [39], [40] and references therein.

In the study of probability and statistics, the characterization of probability distributions is an important role in this field. In the literature, a probability distribution can be characterized by various methods, namely conditional expectation, recurrence relations, and truncated moments. For example, Franco and Ruiz [25] studied the characterization of continuous distributions with adjacent order statistics. Ali and Khan [24] investigated the characterization of some types of distribution, such as truncated and non-truncated distributions, by using the recurrence relation between moments of one and two order statistics. Huang and Su [23] characterized the distribution by using the relationships of conditional moments of residual life. Athar and Akhter [29] characterized a general class of continuous probability distribution based on conditional expectation of two order statistics. Using truncated moments and the relationship between the left (right) and failure (reverse) rate function, Ahsanullah et al. [34] produce two characterization results from the Lindley distribution. Athar and Yahia [33] developed the characterization results through the truncated moments for two general classes of continuous probability distribution. For more details one may refer to Khan and Abu- Salih [26], Khan and Aboummaoh [27], Balasubramanian and Beg [28], Ahsanullah [30], Noor et al. [31], [32], Glanzel [35], Kotz and Shanbhag [36], and references given there.

In this paper, moment properties of GOS from the Lindley Pareto distribution in terms of exact expression and recurrence relations are studied. The results for order statistics, record values, and progressive type-II right censored order statistics are discussed as particular cases of GOS. Further, the characterization of the said distribution through recurrence relations between moments of GOS are presented. Finally, some statistical measures of order statistics and record values for the Lindley Pareto distribution are computed.

2. Single moment

In this section, we derived both the exact expression and recurrence relation between single moment of GOS from the Lindley Pareto distribution. For the sake of convenience, we shall consider

$$E\left[Y^p(r,n.\tilde{m},k)\right]=\mu^{(p)}_{r,n,\tilde{m},k}$$

Theorem 1: Suppose case I is true, then for the Lindley Pareto distribution as given in (6) and for $n\in\mathbb{N},\ \tilde{m}\in\mathbb{R},\quad k>0,\ 1\le r\le n$ and $p=1,2,....$

$$\mu^{(p)}_{r,n,\tilde{m},k}=\left(\frac{\alpha^\beta}{\theta}\right)^{[\frac{p}{\beta}]}C_{r-1}\sum_{i=1}^{r}\sum_{j=0}^{\gamma_i-1}a_i(r)\binom{\gamma_i-1}{j}\left(\frac{e^\theta}{\theta+1}\right)^{\gamma_i}\frac{\Gamma\left([\frac{p}{\beta}]+j+2,\,\theta\gamma_i\right)}{\gamma_i^{[\frac{p}{\beta}]+j+2}},\tag{9}$$

where $[\frac{p}{\beta}]$ is a positive integer.

Proof
we have,

$$\begin{aligned}\mu^{(p)}_{r,n,\tilde{m},k}&=\int_\alpha^\infty y^p f_{Y(r,n,\tilde{m},k)}(y)dy\\&=C_{r-1}\int_\alpha^\infty y^p\sum_{i=1}^{r}a_i(r)\left[\bar{F}(y)\right]^{\gamma_i-1}f(y)dy\\&=C_{r-1}\sum_{i=1}^{r}a_i(r)\int_\alpha^\infty y^p\left[\frac{e^\theta(\alpha^\beta+\theta y^\beta)}{(\theta+1)\alpha^\beta}e^{-\theta\frac{y^\beta}{\alpha^\beta}}\right]^{\gamma_i-1}\frac{e^\theta\theta^2\beta y^{2\beta-1}}{(\theta+1)\alpha^{2\beta}}e^{-\theta\frac{y^\beta}{\alpha^\beta}}dy.\end{aligned}$$

Now expanding $(\alpha^\beta+\theta y^\beta)^{\gamma_i-1}$ binomially, we get

$$\mu^{(p)}_{r,n,\tilde{m},k}=C_{r-1}\sum_{i=1}^{r}\sum_{j=0}^{\gamma_i-1}a_i(r)\binom{\gamma_i-1}{j}\left(\frac{\theta}{\alpha^\beta}\right)^{j+2}\left(\frac{e^\theta}{\theta+1}\right)^{\gamma_i}\beta\int_\alpha^\infty y^{p+j\beta+2\beta-1}e^{-\theta\,\gamma_i\frac{y^\beta}{\alpha^\beta}}dy.$$

Let $y^\beta=t$, which implies $\beta y^{\beta-1}dy=dt$. Thus we have,

$$\mu^{(p)}_{r,n,\tilde{m},k}=C_{r-1}\sum_{i=1}^{r}\sum_{j=0}^{\gamma_i-1}a_i(r)\binom{\gamma_i-1}{j}\left(\frac{e^\theta}{\theta+1}\right)^{\gamma_i}\left(\frac{\theta}{\alpha^\beta}\right)^{j+2}\int_{\alpha^\beta}^\infty t^{\frac{p}{\beta}+j+1}e^{-\frac{\theta}{\alpha^\beta}\gamma_i t}dt.\tag{10}$$

Now using the result from Gradshteyn and Ryzhik [19] (pg. 340) of incomplete gamma function in (10), we get (9). Hence the theorem is completed. □

Corollary 1: Suppose case II is true. For the condition as stated in Theorem 1.

$$\begin{aligned}\mu^{(p)}_{r,n,m,k}=&\left(\frac{\alpha^\beta}{\theta}\right)^{[\frac{p}{\beta}]}\frac{C_{r-1}}{(m+1)^{r-1}(r-1)!}\sum_{i=0}^{r-1}\sum_{j=0}^{\gamma_{r-i}-1}(-1)^i\binom{r-1}{i}\binom{\gamma_{r-i}-1}{j}\\&\times\left(\frac{e^\theta}{\theta+1}\right)^{\gamma_{r-i}}\frac{\Gamma\left([\frac{p}{\beta}]+j+2,\,\theta\gamma_{r-i}\right)}{\gamma_{r-i}^{[\frac{p}{\beta}]+j+2}},\qquad m\neq-1,\end{aligned}\tag{11}$$

$$\begin{aligned}=&\frac{k^r}{(r-1)!}\left(\frac{\alpha^\beta}{\theta}\right)\sum_{c=0}^{\infty}\sum_{u=0}^{c+r-1}\sum_{v=0}^{k+u-1}(-1)^u\binom{c+r-1}{u}\binom{k+u-1}{v}\\&\times a_c(r-1)\left(\frac{e^\theta}{\theta+1}\right)^{k+u}\frac{\Gamma\left([\frac{p}{\beta}]+v+2,(k+u)\theta\right)}{(k+u)^{[\frac{p}{\beta}]+v+2}},\qquad m\to-1.\end{aligned}\tag{12}$$

Proof
When, $\gamma_i \neq \gamma_j$, but $m_i = m_j = m, i, j = 1, 2, ..., n-1,$ then

$$a_i(r) = \frac{1}{(m+1)^{r-1}}(-1)^{r-i}\frac{1}{(i-1)!(r-i)!},$$

see Khan et al. [22]

Therefore, in view of (9), we get

$$\mu^{(p)}_{r,n,m,k} = \left(\frac{\alpha^\beta}{\theta}\right)^{[\frac{p}{\beta}]} \frac{C_{r-1}}{(m+1)^{r-1}} \sum_{i=1}^{r}\sum_{j=0}^{\gamma_i-1}(-1)^{r-i}\frac{1}{(i-1)!(r-i)!}\binom{\gamma_i-1}{j}$$
$$\times\left(\frac{e^\theta}{\theta+1}\right)^{\gamma_i}\frac{\Gamma\left([\frac{p}{\beta}]+j+2,\,\theta\gamma_i\right)}{\gamma_i^{[\frac{p}{\beta}]+j+2}}.$$

Hence, the expression (11) holds.

Now, when $m \to -1$, then $h_m(y) = -\log(1-y)$. Thus, the PDF of $r-$th GOS given in ((4)) becomes

$$f_{U_r^{(k)}}(y) = \frac{k^r}{\Gamma r}[-\log \bar{F}(y)]^{r-1}[\bar{F}(y)]^{k-1}f(y).$$

Therefore,

$$E[Y^{(p)}_{U_r^{(k)}}] = \mu^{(p)}_{U_r^{(k)}} = \frac{k^r}{(r-1)!}\int_0^\infty y^p[\bar{F}(y)]^{k-1}[-\ln(1-F(y))]^{r-1}f(y)dy. \tag{13}$$

Since

$$[-\ln(1-t)]^h = \left(\sum_{g=1}^{\infty}\frac{t^g}{g}\right)^h = \sum_{g=0}^{\infty}a_g(h)t^{g+h}, \quad |t|<1,$$

where, $a_g(h)$ is the coefficient of t^{g+h} in the above expansion, [see, Balakrishnan and Cohen [42]]. Thus,

$$\begin{aligned}
\mu^{(p)}_{U_r^{(k)}} =& \frac{k^r}{(r-1)!}\sum_{c=0}^{\infty}a_c(r-1)\int_\alpha^\infty y^p[\bar{F}(y)]^{k-1}[F(y)]^{c+r-1}f(y)dy\\
=& \frac{k^r}{(r-1)!}\sum_{c=0}^{\infty}a_c(r-1)\int_\alpha^\infty y^p[\bar{F}(y)]^{k-1}[1-\bar{F}(y)]^{c+r-1}f(y)dy\\
=& \frac{k^r}{(r-1)!}\sum_{c=0}^{\infty}\sum_{u=0}^{c+r-1}(-1)^u\binom{c+r-1}{u}a_c(r-1)\int_\alpha^\infty y^p[\bar{F}(y)]^{k+u}\frac{f(y)}{\bar{F}(y)}dy\\
=& \frac{k^r}{(r-1)!}\sum_{c=0}^{\infty}\sum_{u=0}^{c+r-1}(-1)^u\binom{c+r-1}{u}a_c(r-1)\left(\frac{e^\theta}{(\theta+1)\alpha^\beta}\right)^{k+u}\\
&\times\int_\alpha^\infty\frac{\beta\theta^2}{\alpha^\beta}y^{p+2\beta-1}\left(\alpha^\beta+\theta y^\beta\right)^{k+u-1}e^{-(k+u)\theta\frac{y^\beta}{\theta^\beta}}dy\\
=& \frac{k^r}{(r-1)!}\sum_{c=0}^{\infty}\sum_{u=0}^{c+r-1}\sum_{v=0}^{k+u-1}(-1)^u\binom{c+r-1}{u}\binom{k+u-1}{v}a_c(r-1)\left(\frac{e^\theta}{\theta+1}\right)^{k+u}\\
&\times\beta\left(\frac{\theta}{\alpha^\beta}\right)^{v+2}\int_\alpha^\infty y^{p+v\beta+2\beta-1}e^{-(k+u)\frac{\theta y^\beta}{\alpha^\beta}}dy.
\end{aligned}$$

Suppose $y^\beta = t$, then we have

$$\mu^{(p)}_{U_r^{(k)}} = \frac{k^r}{(r-1)!} \sum_{c=0}^{\infty} \sum_{u=0}^{c+r-1} \sum_{v=0}^{k+u-1} (-1)^u \binom{c+r-1}{u} \binom{k+u-1}{v} a_c(r-1) \times \left(\frac{e^\theta}{\theta+1}\right)^{k+u} \left(\frac{\alpha^\beta}{\theta}\right)^{[\frac{p}{\beta}]} \int_{\alpha^\beta}^{\infty} t^{[\frac{p}{\beta}]+v+1} e^{-(k+u)\frac{\theta}{\alpha^\beta}t} dt.$$

Now using the results from Gradshteyn and Ryzhik [38] (pg. 340) of incomplete gamma function, we get the result given in ((12)). Hence the theorem is proved. □

Remark 1. By setting $m = 0$ and $k = 1$ in (11), we obtained the expression for single moment of order statistics as:

$$\mu^{(p)}_{r:n} = \left(\frac{\alpha^\beta}{\theta}\right)^{[\frac{p}{\beta}]} C_{r:n} \sum_{i=0}^{r-1} \sum_{j=0}^{n-r+i} (-1)^i \binom{r-1}{i} \binom{n-r+i}{j} \left(\frac{e^\theta}{\theta+1}\right)^{n-r+i+1} \times \frac{\Gamma\left([\frac{p}{\beta}]+j+2, \theta(n-r+i+1)\right)}{(n-r+i+1)^{[\frac{p}{\beta}]+j+2}}, \tag{14}$$

where $\mu^{(p)}_{r:n}$ refers to the $p-$th moment of $r-$th order statistic.

Some particular cases from ((14)) have been stated as follows:

(a) By setting $r = 1$, we get the $p-$th moment of minimum order statistics

$$\mu^{(p)}_{1:n} = \frac{\alpha^p}{\theta^{[\frac{p}{\beta}]}} \left(\frac{e^\theta}{\theta+1}\right)^n \sum_{j=0}^{n-1} \binom{n-1}{j} \frac{\Gamma([\frac{p}{\beta}]+j+2,\ n\theta)}{n^{[\frac{p}{\beta}]+j+1}}.$$

(b) When $r = n$, then we get the $p-$th moment of maximum order statistics

$$\mu^{(p)}_{n:n} = \frac{n\,\alpha^p}{\theta^{[\frac{p}{\beta}]}} \sum_{i=0}^{n-1} \sum_{j=0}^{i} (-1)^i \binom{n-1}{i} \binom{i}{j} \left(\frac{e^\theta}{\theta+1}\right)^{i+1} \frac{\Gamma([\frac{p}{\beta}]+j+2,\ \theta(i+1))}{(i+1)^{[\frac{p}{\beta}]+j+2}}.$$

(c) If $r = n = 1$, then we get the $p-$th moment of the Lindley Pareto distribution

$$\mu^{(p)}_{1:1} = \frac{\alpha^p\, e^\theta\, \Gamma([\frac{p}{\beta}]+2,\ \theta)}{\theta^{[\frac{p}{\beta}]}(\theta+1)}.$$

The above results are also given by Larzi and Zeghdoudi [37].

Remark 2. If $k = 1$ in (12), then we get an expression for single moment of upper records as

$$\mu^{(p)}_{U_r} = \frac{1}{(r-1)!} \left(\frac{\alpha^\beta}{\theta}\right) \sum_{c=0}^{\infty} \sum_{u=0}^{c+r-1} \sum_{v=0}^{u} (-1)^u \binom{c+r-1}{u} \binom{u}{v} \times a_c(r-1) \times \left(\frac{e^\theta}{\theta+1}\right)^{1+u} \frac{\Gamma\left([\frac{p}{\beta}]+v+2, (1+u)\theta\right)}{(1+u)^{[\frac{p}{\beta}]+v+2}}, \tag{15}$$

where $\mu^{p}_{U_r}$, is the $p-$th moment of upper record values.

Remark 3. If $Y^{\tilde{R}}_{1:m:n} < ... < Y^{\tilde{R}}_{r:m:n} < ... < Y^{\tilde{R}}_{m:m:n}$ be the m progressive type-II right censored order statistics and for $1 \le m \le n$, then from (9), we get

$$\mu^{\tilde{R}(p)}_{r:m:n} = \left(\frac{\alpha^{\beta}}{\theta}\right)^{[\frac{p}{\beta}]} C_{r-1} \sum_{i=1}^{r} \sum_{j=0}^{\gamma_i - 1} a_i(r) \binom{\gamma_i - 1}{j} \left(\frac{e^{\theta}}{\theta+1}\right)^{\gamma_i} \frac{\Gamma\left([\frac{p}{\beta}] + j + 2,\, \theta\gamma_i\right)}{\gamma_i^{[\frac{p}{\beta}]+j+2}}, \tag{16}$$

where $E\left[Y^{\tilde{R}}_{r:m:n}\right]^{(p)} = \mu^{\tilde{R}(p)}_{r:m:n}$ refers to the $p-$th moment of progressive type-II right censored order statistics with $\gamma_i = \sum_{j=i}^{m} (R_j + 1)$, $\tilde{R} = (R_1, R_2, ..., R_m)$ and $r = 1, 2, ..., m$.

Theorem 2: Suppose Case I is satisfied. For the Lindley Pareto distributions as given in (6) and $n \in N, \tilde{m} \in \mathbb{R},\ k > 0,\ 1 \le r \le n,\ p = 1,\ 2,\ ...$

$$\mu^{(p)}_{r,n,\tilde{m},k} - \mu^{(p)}_{r-1,n,\tilde{m},k} = \frac{p\,\alpha^{\beta}}{\gamma_r\,\beta\,\theta^2} \left[\alpha^{\beta}\,\mu^{(p-2\beta)}_{r,n,\tilde{m},k} + \theta\,\mu^{(p-\beta)}_{r,n,\tilde{m},k}\right]. \tag{17}$$

Proof
In view of Athar and Islam [3], we have

$$\mu^{(p)}_{r,n,\tilde{m},k} - \mu^{(p)}_{r-1,n,\tilde{m},k}$$
$$= \frac{p\,C_{r-1}}{\gamma_r} \int_{\alpha}^{\infty} y^{p-1} \sum_{i=1}^{r} a_i\,(r)\, \left[\bar{F}(y)\right]^{\gamma_i - 1} \frac{\alpha^{\beta}}{\beta\,\theta^2} \left(\alpha^{\beta}\,y^{1-2\beta} + \theta\,y^{1-\beta}\right) f(y)\,dy.$$
$$= \frac{p\,\alpha^{\beta}}{\beta\,\theta^2\,\gamma_r} C_{r-1} \int_{\alpha}^{\infty} \left(\alpha^{\beta}\,y^{p-2\beta} + \theta\,y^{p-\beta}\right) \sum_{i=1}^{r} a_i\,(r)\, \left[\bar{F}(y)\right]^{\gamma_i - 1} f(y)\,dy,$$

which on simplification gives (17).
Hence the theorem is proved. □

Corollary 2: For case II and the condition as stated in Theorem [2].

$$\mu^{(p)}_{r,n,m,k} - \mu^{(p)}_{r-1,n,m,k} = \frac{p\,\alpha^{\beta}}{\gamma_r\,\beta\,\theta^2} \left[\alpha^{\beta}\,\mu^{(p-2\beta)}_{r,n,m,k} + \theta\,\mu^{(p-\beta)}_{r,n,m,k}\right]. \tag{18}$$

Proof
Since for $\gamma_i \ne \gamma_j;\ i \ne j = 1, 2, \dots, n-1$ but $m_i = m$

$$a_i(r) = \frac{1}{(m+1)^{r-1}} (-1)^{r-i} \frac{1}{(i-1)!(r-i)!}.$$

Therefore, the PDF given in (2) simplified to (4).
The expression given in (18), can be easily obtained by replacing $\tilde{m}$ with m in (17). □

Remark 4: If $m = 0$ and $k = 1$, then the Lindley Pareto distribution has the following relation for a single moment of order statistics

$$\mu^{(p)}_{r:n} - \mu^{(p)}_{r-1:n} = \frac{p\,\alpha^{\beta}}{\beta\,\theta^2\,(n-r+1)} \left[\alpha^{\beta}\mu^{(p-2\beta)}_{r:n} + \theta\,\mu^{(p-\beta)}_{r:n}\right].$$

Remark 5: If $m \to -1$, then the recurrence relation for single moment of $k-$th upper record values is given as

$$\mu^{(p)}_{U_n^{(k)}} - \mu^{(p)}_{U_{n-1}^{(k)}} = \frac{p\,\alpha^{\beta}}{\beta\,\theta^2\,k}\left[\alpha^{\beta}\,\mu^{(p-2\beta)}_{U_n^{(k)}} + \theta\,\mu^{(p-\beta)}_{U_n^{(k)}}\right].$$

Remark 6. If $Y^{\tilde{R}}_{1:m:n} < ... < Y^{\tilde{R}}_{r:m:n} < ... < Y^{\tilde{R}}_{m:m:n}$ be the m progressive type-II right censored order statistics for $1 \leq m \leq n$, then from (17), we get

$$\mu^{\tilde{R}^{(p)}}_{r:m:n} - \mu^{\tilde{R}^{(p)}}_{r-1:m:n} = \frac{p\,\alpha^{\beta}}{\gamma_r\,\beta\,\theta^2}\left[\alpha^{\beta}\,\mu^{\tilde{R}^{(p-2\beta)}}_{r:m:n} + \theta\,\mu^{\tilde{R}^{(p-\beta)}}_{r:m:n}\right].$$

3. Product Moments

In this section, we obtained the recurrence relation between product moments of GOS from the Lindley Pareto distribution. For the sake of convenience, we shall consider

$$E\left[Y^p(r,n.\tilde{m},k).Y^q(s,n.\tilde{m},k)\right] = \mu^{(p,q)}_{r,s,n,\tilde{m},k}$$

Theorem 3: Suppose Case I be satisfied. For the Lindley Pareto distributions as given in (6) and $n \in N, \tilde{m} \in \mathbb{R},\ k > 0,\ 1 \leq r < s \leq n,\ p,q = 1,\ 2,\ ...$

$$\mu^{(p,\,q)}_{r,s,n,\tilde{m},k} - \mu^{(p,\,q)}_{r,s-1,n,\tilde{m},k} = \frac{q\,\alpha^{\beta}}{\beta\,\theta^2\,\gamma_s}\left[\alpha^{\beta}\mu^{(p,\,q-2\beta)}_{r,s,n,\tilde{m},k} + \theta\mu^{(p,\,q-\beta)}_{r,s,n,\tilde{m},k}\right]. \tag{19}$$

Proof
Using the result given by Athar and Islam [3], we get

$$\begin{aligned}
&\mu^{(p,\,q)}_{r,s,n,\tilde{m},k} - \mu^{(p,\,q)}_{r,s-1,n,\tilde{m},k}\\
&= \frac{q\,C_{s-1}}{\gamma_s}\int_{\alpha}^{\infty}\int_{y}^{\infty} y^p\,z^{q-1}\left[\sum_{j=r+1}^{s} a_j^{(r)}(s)\left[\frac{\bar{F}(z)}{\bar{F}(y)}\right]^{\gamma_j}\right]\left[\sum_{i=1}^{r} a_i(r)\,[\bar{F}(y)]^{\gamma_i}\right]\\
&\times \frac{f(y)}{\bar{F}(y)}\frac{f(z)}{\bar{F}(z)}\frac{\alpha^{\beta}}{\beta\,\theta^2}\left(\alpha^{\beta}\,z^{1-2\beta} + \theta\,z^{1-\beta}\right)dz\,dy.\\
&= \frac{q\,\alpha^{\beta}\,C_{s-1}}{\beta\,\theta^2\,\gamma_s}\int_{\alpha}^{\infty}\int_{y}^{\infty}\left(\alpha^{\beta}\,y^p\,z^{q-2\beta} + \theta\,y^p\,z^{q-\beta}\right)\left[\sum_{j=r+1}^{s} a_j^{(r)}(s)\left[\frac{\bar{F}(z)}{\bar{F}(y)}\right]^{\gamma_j}\right]\\
&\times \left[\sum_{i=1}^{r} a_i(r)\,[\bar{F}(y)]^{\gamma_i}\right]\frac{f(y)}{\bar{F}(y)}\frac{f(z)}{\bar{F}(z)}\,dz\,dy.
\end{aligned}$$

This gives (19). □

Corollary 3: For case II and the condition as stated in Theorem [3]

$$\mu^{(p,\,q)}_{r,s,n,m,k} - \mu^{(p,\,q)}_{r,s-1,n,m,k} = \frac{q\,\alpha^{\beta}}{\beta\,\theta^2\,\gamma_s}\left[\alpha^{\beta}\mu^{(p,\,q-2\beta)}_{r,s,n,m,k} + \theta\mu^{(p,\,q-\beta)}_{r,s,n,m,k}\right] \tag{20}$$

Proof

Since, for $\gamma_i \neq \gamma_j;\ \ i \neq j = 1, 2, \dots, n-1$ but $m_i = m$

$$a_i^{(r)}(s) = \frac{1}{(m+1)^{s-r-1}}(-1)^{s-i}\frac{1}{(i-r-1)!(s-i)!}.$$

Therefore, joint PDF of $Y_{r,n,\tilde{m},k}$ and $Y_{s,n,\tilde{m},k}$ given in (3) reduces to (5).

Thus, relation (20) can be obtained by replacing $\tilde{m}$ with m in (19). □

Remark 7: For the Lindley Pareto distribution when $m = 0$ and $k = 1$, the relation for product moments of order statistics is written as

$$\mu_{r,s:n}^{(p,\,q)} - \mu_{r,s-1:n}^{(p,\,q)} = \frac{q\,\alpha^{\beta}}{\beta\,\theta^2\,\gamma_s}\left[\alpha^{\beta}\mu_{r,s:n}^{(p,\,q-2\beta)} + \theta\mu_{r,s:n}^{(p,\,q-\beta)}\right],$$

where $\mu_{r,s:n}^{(p,q)}$ refers to the $(p, q)-$th moment of order statistics.

Remark 8: Let $m \to -1$, then the relation for product moments of $k-$th upper record values is given as

$$\mu_{U_{m,n}^{(k)}}^{(p,q)} - \mu_{U_{m,n-1}^{(k)}}^{(p,q)} = \frac{q\,\alpha^{\beta}}{\beta\,\theta^2\,k}\left\{\alpha^{\beta}\mu_{U_{m,n}^{(k)}}^{(p,q-2\beta)} + \theta\,\mu_{U_{m,n}^{(k)}}^{(p,q-\beta)}\right\},$$

where $E[Y_{U_m^{(k)}}^{(p)}.Y_{U_n^{(k)}}^{(q)}] = \mu_{U_{m,n}^{(k)}}^{(p,q)}$, refer to the $(p, q)-$th moments of the $k-$th record values.

Remark 9. If $Y_{1:m:n}^{\tilde{R}} < \dots < Y_{r:m:n}^{\tilde{R}} < \dots < Y_{m:m:n}^{\tilde{R}}$ be the m progressive type-II right censored order statistics for $1 \leq m \leq n$, then using (19), we get the recurrence relation for product moments of progressive type-II right censored order statistics

$$\mu_{r,s:m:n}^{\tilde{R}^{(p)}} - \mu_{r,s-1:m:n}^{\tilde{R}^{(p)}} = \frac{q\,\alpha^{\beta}}{\beta\,\theta^2\,\gamma_s}\left[\alpha^{\beta}\mu_{r,s:m:n}^{\tilde{R}^{(p,q-2\beta)}} + \theta\mu_{r,s:m:n}^{\tilde{R}^{(p,q-\beta)}}\right],$$

where $E\left[Y_{r:m:n}^{\tilde{R}^{(p)}}.Y_{s:m:n}^{\tilde{R}^{q}}\right] = \mu_{r:m:n}^{\tilde{R}^{(p,q)}}$.

4. Characterization

In this section, using recurrence relations between the moments of GOS, the characterization of Lindley Pareto distributions, whose PDF is given in (7), is studied.

Theorem 4: Fix a positive integer k and assume p to be a non-negative integer. A necessary and sufficient condition for a random variable Y to be distributed with PDF given in (7) is

$$\mu_{r,n,\tilde{m},k}^{(p)} - \mu_{r-1,n,\tilde{m},k}^{(p)} = \frac{p\,\alpha^{\beta}}{\gamma_r\,\beta\,\theta^2}\left[\alpha^{\beta}\,\mu_{r,n,\tilde{m},k}^{(p-2\beta)} + \theta\,\mu_{r,n,\tilde{m},k}^{(p-\beta)}\right]. \tag{21}$$

Proof
Necessary part follows from Theorem [2].

However, suppose the relation in (21) is satisfied. Now, in view of Athar and Islam [3] and using (2) and (17) in (21), we get

$$p\,C_{r-2}\int_{\alpha}^{\infty} y^{p-1}\sum_{i=1}^{r} a_i(r)\,\left[\bar{F}(y)\right]^{\gamma_i}\,dy$$
$$= \frac{p\,\alpha^{\beta}\,C_{r-1}}{\beta\,\theta^2\,\gamma_r}\int_{\alpha}^{\infty}\left(\alpha^{\beta}\,y^{p-2\beta}+\theta\,y^{p-\beta}\right)\sum_{i=1}^{r} a_i\,(r)\,\left[\bar{F}(y)\right]^{\gamma_i-1} f(y)\,dy.$$

This implies

$$\frac{p\,\alpha^{\beta}\,C_{r-1}}{\beta\theta^2\,\gamma_r}\int_{\alpha}^{\infty} y^{p-1}\sum_{i=1}^{r} a_i\,(r)\,\left[\bar{F}(y)\right]^{\gamma_i-1}$$
$$\times\left\{\frac{\beta\,\theta^2}{\alpha^{\beta}}\,\bar{F}(y)-\left(\alpha^{\beta}\,y^{1-2\beta}+\theta\,y^{1-\beta}\right)\,f(y)\right\}dy\;=\;0. \quad (22)$$

Now on applications of *Müntz - Szász* theorem [see, for example, Hwang and Lin [21]] in (22), we get

$$\frac{\bar{F}(y)}{f(y)} = \frac{\alpha^{\beta}}{\beta\,\theta^2}\,\left(\alpha^{\beta}\,y^{1-2\beta}+\theta\,y^{1-\beta}\right).$$

Thus, $f(y)$ has a PDF as given in (7). Therefore, Theorem [4] holds. □

Theorem 5: Fix a positive integer k and assume $p,\,q$ to be non-negative integers. A necessary and sufficient condition for a random variable Y to be distributed with PDF as stated in (7) is that

$$\mu^{(p,\,q)}_{r,s,n,\tilde{m},k}-\mu^{(p,\,q)}_{r,s-1,n,\tilde{m},k} = \frac{q\,\alpha^{\beta}}{\beta\,\theta^2\,\gamma_s}\left[\alpha^{\beta}\mu^{(p,\,q-2\beta)}_{r,s,n,\tilde{m},k}+\theta\mu^{(p,\,q-\beta)}_{r,s,n,\tilde{m},k}\right] \quad (23)$$

Proof
Necessary part follows from Theorem [3].
Now, suppose the relation in (23) is satisfied. So, in view of Athar and Islam [3] and using (3) and (19) in (23), we get

$$q\,C_{s-2}\int_{\alpha}^{\infty}\int_{y}^{\infty} y^{p}\,z^{q-1}\left[\sum_{j=r+1}^{s} a_j^{(r)}\,(s)\,\left(\frac{\bar{F}(z)}{\bar{F}(y)}\right)^{\gamma_j}\right]\left[\sum_{i=1}^{r} a_i\,(r)\,[\bar{F}(y)]^{\gamma_i}\right]\frac{f(y)}{\bar{F}(y)}\,dz\,dy.$$
$$=\frac{q\,\alpha^{\beta}\,C_{s-1}}{\beta\,\theta^2\,\gamma_s}\int_{\alpha}^{\infty}\int_{y}^{\infty}\left(\alpha^{\beta}\,y^{p}\,z^{q-2\beta}+\theta\,y^{p}\,z^{q-\beta}\right)\left[\sum_{j=r+1}^{s} a_j^{(r)}\,(s)\,\left[\frac{\bar{F}(z)}{\bar{F}(y)}\right]^{\gamma_j}\right]$$
$$\times\left[\sum_{i=1}^{r} a_i\,(r)\,[\bar{F}(y)]^{\gamma_i}\right]\frac{f(y)}{\bar{F}(y)}\,\frac{f(z)}{\bar{F}(z)}\,dz\,dy.$$

This implies

$$\frac{q\,\alpha^{\beta}\,C_{s-1}}{\beta\,\theta^2\,\gamma_s}\int_{\alpha}^{\infty}\int_{y}^{\infty} y^{p}\,z^{q-1}\left[\sum_{i=1}^{r} a_i\,(r)\,[\bar{F}(y)]^{\gamma_i}\right]\left[\sum_{j=r+1}^{s} a_j^{(r)}\,(s)\,\left[\frac{\bar{F}(z)}{\bar{F}(y)}\right]^{\gamma_j}\right]\frac{f(y)}{\bar{F}(y)}$$
$$\times\left\{\frac{\beta\,\theta^2}{\alpha^{\beta}}-\left(\alpha^{\beta}\,z^{1-2\beta}+\theta\,z^{1-\beta}\right)\frac{f(z)}{\bar{F}(z)}\right\}dz\,dy\;=0. \quad (24)$$

Applying the extension of *Müntz - Szász* theorem [see, for example, Hwang and Lin [21]] to (24), we have

$$\frac{\bar{F}(z)}{f(z)} = \frac{\alpha^{\beta}}{\beta\,\theta^2}\,\left(\alpha^{\beta}\,z^{1-2\beta}+\theta\,z^{1-\beta}\right).$$

Therefore, $f(z)$ is a PDF as stated in (7). Thus, Theorem [5] holds. □

5. Computations of means and some statistical properties

In this section, we have utilized the results developed in Section 2 to compute the means, variances, skewness, kurtosis and coefficient of variation (CV) of order statistics as well as record values based on the Lindley Pareto distribution.

For $p = 1, 2, 3, 4$ in (14), we systematically computed the first four moments $\mu_{r:n}^{(1)}$, $\mu_{r:n}^{(2)}$, $\mu_{r:n}^{(3)}$ and $\mu_{r:n}^{(4)}$ of the order statistics for sample size $n = 1(1)6$ and then, using these four moments, the statistical measures such as mean, variance, skewness, kurtosis, and coefficient of variation (CV) of order statistics are computed.

It can also be seen that the condition $\sum_{r=1}^{n} \mu_{r:n} = nE(X)$, as shown in David and Nagaraja [41], is satisfied. Further, for the different values of the parameters of the Lindley Pareto distribution and using (12) the first four moments of record values have been computed for $r = 1, 2, \ldots, 10$. Moreover, these first four moments are used to compute mean, variance, skewness, kurtosis and CV of record values. The given tables provide the results rounded to six decimal places.

Table 1: The different statistical measures of order statistics from Lindley Pareto distribution for different values of parameters

		$\theta = 0.2,\ \beta = 0.5,\ \alpha = 1.5$				
n	r	Mean	Variance	Skewness	Kurtosis	CV
1	1	6.106667	84.84892	4.283330	37.11491	1.508409
2	1	2.330278	10.10963	3.752835	28.59220	1.364457
	2	9.883056	131.0659	3.468523	25.74193	1.158387
3	1	1.378866	3.132370	3.472642	24.57440	1.283555
	2	4.233101	18.63304	2.856344	17.94080	1.019726
	3	12.70803	163.3409	3.129529	21.69328	1.005702
4	1	0.967049	1.407326	3.291529	22.15834	1.226730
	2	2.614317	6.272385	2.598078	15.10119	0.957984
	3	5.851885	25.75277	2.479665	14.34289	0.867194
	4	14.99342	188.3117	2.934742	19.54489	0.915247
5	1	0.741975	0.769573	3.161824	20.51614	1.182321
	2	1.867344	2.945178	2.448948	13.60539	0.919034
	3	3.734778	9.170804	2.223734	11.90269	0.810847
	4	7.263291	31.82724	2.264007	12.51549	0.776723
	5	16.92595	208.7594	2.804987	18.18459	0.853631
6	1	0.601660	0.474926	3.062920	19.31349	1.145412
	2	1.443553	1.652149	2.347813	12.65365	0.890414
	3	2.714926	4.453641	2.083443	10.69529	0.777320
	4	4.754629	11.80778	2.007026	10.28530	0.722716
	5	8.517621	37.11695	2.121510	11.40027	0.715267
	6	18.60761	226.1199	2.710806	17.23212	0.808126

Table 2: The different statistical measures of order statistics from Lindley Pareto distribution for different values of parameters

		$\theta = 0.5,\ \beta = 1.0,\ \alpha = 2.5$				
n	r	Mean	Variance	Skewness	Kurtosis	CV
1	1	2.166667	1.888888	1.512286	6.342556	0.634324
2	1	1.444444	0.608026	1.444467	5.941332	0.539834
	2	2.888889	2.126543	1.250180	5.470582	0.504785
3	1	1.176269	0.317238	1.427971	5.809329	0.478835
	2	1.980796	0.758090	1.089533	4.783857	0.439563
	3	3.342936	2.192292	1.165435	5.243208	0.442916
4	1	1.032407	0.200340	1.428066	5.769452	0.433544
	2	1.607853	0.419582	1.034273	4.545319	0.402868
	3	2.353738	0.818429	0.960222	4.456850	0.384355
	4	3.672668	2.215357	1.125735	5.148646	0.405266
5	1	0.941485	0.140200	1.435034	5.770563	0.397705
	2	1.396095	0.275561	1.010501	4.434915	0.376006
	3	1.925490	0.467458	0.884936	4.184263	0.355083
	4	2.639237	0.848634	0.894118	4.309556	0.349046
	5	3.931026	2.223292	1.103565	5.10049	0.379309
6	1	0.878344	0.104627	1.444981	5.792928	0.368262
	2	1.257193	0.198458	0.999934	4.377394	0.354350
	3	1.673900	0.314001	0.848722	4.053974	0.334762
	4	2.177080	0.494318	0.806325	4.019609	0.322946
	5	2.870315	0.865602	0.854299	4.227841	0.324138
	6	4.143168	2.224805	1.089838	5.073163	0.360010

Table 3: The statistical measures of record values from Lindley Pareto distribution

	$\beta = 0.5, \theta = 5.0, \alpha = 0.2, (k = 1)$				
r	Means	Variances	Skewness	Kurtosis	CV
1	0.314667	0.017998	2.969294	18.75761	0.426341
2	0.447613	0.046508	2.284247	12.32454	0.481792
3	0.598546	0.087227	1.977006	9.959527	0.493432
4	0.767239	0.141830	1.787412	8.665135	0.490855
5	0.953512	0.211973	1.652736	7.824828	0.482852
6	1.157217	0.299300	1.549403	7.225164	0.472758
7	1.378231	0.405448	1.466066	6.771094	0.462004
8	1.616451	0.532041	1.396607	6.412882	0.451243
9	1.871790	0.680693	1.337343	6.121720	0.440777
10	2.144171	0.853021	1.285807	5.879600	0.430746

Table 4: The statistical measures of record values from Lindley Pareto distribution

	$\beta = 1.0, \theta = 5.0, \alpha = 0.5$, $(k = 2)$				
r	Means	Variances	Skewness	Kurtosis	CV
1	0.559028	0.003391	1.931688	8.473951	0.104163
2	0.617254	0.006626	1.353308	5.675314	0.131870
3	0.674808	0.009744	1.095115	4.771024	0.146275
4	0.731792	0.012769	0.943648	4.305256	0.154413
5	0.788287	0.015718	0.845134	4.000322	0.159041
6	0.844356	0.018615	0.766371	3.861982	0.161587
7	0.900053	0.021463	0.707878	3.741856	0.162770
8	0.955422	0.024269	0.662086	3.645371	0.163054
9	1.010498	0.027044	0.623771	3.576338	0.162742
10	1.065312	0.029792	0.591635	3.519405	0.162020

Table 5: The statistical measures of record values from Lindley Pareto distribution

	$\beta = 1.0, \theta = 7.5, \alpha = 1.5$, $(k = 3)$				
r	Means	Variances	Skewness	Kurtosis	CV
1	1.575149	0.005593	1.965421	9.096353	0.047478
2	1.649928	0.011082	1.389168	5.857314	0.063803
3	1.724367	0.016485	1.128670	4.881361	0.074458
4	1.798495	0.021809	0.973299	4.418928	0.082112
5	1.872334	0.027069	0.866507	4.127081	0.087872
6	1.945909	0.032263	0.780445	4.341263	0.092305
7	2.019237	0.037404	0.730204	3.774945	0.09578
8	2.092335	0.042502	0.680293	3.693431	0.098531
9	2.165221	0.047549	0.641585	3.599512	0.100709
10	2.237906	0.053991	0.181780	2.097306	0.103829

6. Conclusion

In this paper, we studied the Lindley Pareto distribution, a newly defined three-parameter distribution that provides a more flexible model for lifetime data analysis. This study focused on the moment properties of GOS, which offers a unified approach to several models of order random variables from the Lindley Pareto distribution. The exact and explicit expression for single moment of GOS from the Lindley Pareto distribution is driven. Further, the recurrence relations between them for the single and product moments of GOS are also discussed. The exact expression of moments for order statistics, record values, and progressive type-II right censored order statistics are studied as a particular case of GOS. Moreover, the characterization of probability distribution through recurrence relations are also discussed. The first four moments of order statistics and record values for the different values of parameters are computed. Finally, some statistical measures of order statistics and record values for the Lindley Pareto distribution are computed.

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