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A Novel Estimator of Kullback-Leibler Information with its Application to Goodness of Fit Tests

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Abstract In this study, our primary focus is on introducing a new estimator of Kullback-Leibler information, which we subsequently utilize for conducting goodness-of-fit tests. We employ this new estimator to propose tests specifically tailored for assessing the fit of data to the normal, exponential, and Weibull distributions. To ensure the reliability and accuracy of our proposed tests, we utilize a Monte Carlo simulation approach. Through this simulation, we obtain percentile points and determine the type I error rates of the tests. This enables us to assess the performance and suitability of the proposed tests under different scenarios. Furthermore, we conduct a comprehensive simulation study to evaluate the power and effectiveness of our proposed tests. We compare their performance with that of other competing tests, allowing us to gauge their relative strengths and weaknesses. To demonstrate the practical applicability of the proposed tests, we provide three real data examples. These examples serve as illustrations of how the tests can be implemented and offer insights into their performance when applied to real-world data. By combining theoretical developments, simulation studies, and real data analysis, we aim to provide a comprehensive evaluation of the proposed goodness-of-fit tests based on the new estimator of Kullback-Leibler information.

Keywords Kullback-Leibler information, Testing normality, Testing exponentiality, Test for Weibull distribution, Critical points, Type I error, Test power, Monte Carlo simulation

AMS 2010 subject classifications: 62G10, 94A17.

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1. Introduction

The normal, exponential, and Weibull distributions find widespread applications in various fields such as environmental sciences, geology, chemistry, physics, medicine, economics, geography, and engineering, attracting the attention of researchers (Krit et al. [20]). Consequently, it becomes crucial to assess the validity of the assumption that a given random sample conforms to these distributions. One common approach for this assessment is through Goodness-of-Fit (GOF) tests.

The GOF test is a statistical concept employed to measure how well an observed dataset aligns with a theoretical or expected distribution. Its purpose is to determine whether the observed data significantly deviates from the expected values based on a given hypothesis (see Cirrone et al. [6]). In this literature review, we highlight recent examples of GOF tests applied to the Weibull distribution across various fields.

For instance, Kreer et al. [19] used GOF tests for non-life insurance, Teamah et al. [24] for earthquakes, Datsiou and Overend [8] for glass strength, Chen et al. [4] for wireless networks, and Tizgui et al. [25] for wind energy. Some researchers employed GOF tests based on the empirical distribution function, while others utilized GOF tests based on Shannon entropy. Wieczorkowski and Grzegorzewski [30] and Ebrahimi et al. [11] developed GOF tests based on Shannon entropy. Their findings indicate that the power of these tests, estimated through simulations,

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outperforms the GOF tests based on the empirical distribution function.
By reviewing these studies and considering the advantages of GOF tests based on Shannon entropy, we aim to contribute to the understanding of GOF testing for the normal, exponential, and Weibull distributions. We will explore the application of Shannon entropy-based GOF tests and assess their power through simulation studies. This research will provide valuable insights into the effectiveness of these tests compared to the traditional empirical distribution function-based GOF tests.
Assume that we have a random variable X with a distribution function $F(x)$ and a probability density function f. Then, Shannon [22] defined entropy $H(f)$ for X as

$$H(f) = -\int_{-\infty}^{\infty} f(x) \log f(x)\, dx. \tag{1}$$

Many researcher have interested to estimating the problem of $H(f)$ based on a set of observations $x_1, x_2, ..., x_n$ from $F(x)$, see for example Van Es [27], Hall and Morton [17], Ebrahimi et al. [12], Correa [7], Wieczorkowski and Grzegorzewski [30] and Alizadeh [2]. A useful display of entropy for the univariate random variable X using the quantile function is presented by Vasicek [28] as follows.

$$H(f) = \int_0^1 \log\left\{\frac{d}{du}Q(u)\right\} du, \tag{2}$$

where $Q(u) = F^{-1}(u) = \inf\{x : F(x) \geq u\}$ is the quantile function. Then, he constructed an estimate by replacing the distribution function F by the empirical distribution function F_n. He used the difference operator instead of differential operator. The derivative of $F^{-1}(p)$ is then estimated by a function of the order statistics obtained from the sample. The estimator with sample $X_1, ..., X_n$ is given by

$$HV_{mn} = \frac{1}{n}\sum_{i=1}^{n} \log\left\{\frac{n}{2m}\left(X_{(i+m)} - X_{(i-m)}\right)\right\}, \tag{3}$$

where m is a positive integer, $m \leq n/2$, and $X_{(1)} \leq X_{(2)} \leq ... \leq X_{(n)}$ are the order statistics and $X_{(i)} = X_{(1)}$ if $i < 1$, $X_{(i)} = X_{(n)}$ if $i > n$. Many researcher have developed entropy based statistical procedures using sample entropy of Vasicek [28], see for example Dudewicz and Van Der Meulen [10], Gokhale [15] and Alizadeh and Arghami [3].
The Kullback-Leibler (KL) divergence is a measure of the difference between two probability distributions, quantifying the extent to which one distribution diverges from another. In our study, we focus on the KL information function between two probability distributions, denoted as f and g. The KL information function is defined as follows:

$$KL(f, g) = \int f(x) \log \frac{f(x)}{g(x)} dx. \tag{4}$$

It is clear, $D(f, g) \geq 0$ and $D(f, g) = 0$ if only if $f(x) = g(x)$ with probability 1.
Section 2 of our study introduces a novel GOF test based on KL information. This test is designed to assess the fit of data to the normal, exponential, and Weibull distributions. We outline the methodology and procedures employed in this test, highlighting its advantages and capabilities. In Section 3, we delve into the specific applications of the suggested GOF test for the normal, exponential, and Weibull distributions. We discuss the results obtained from these tests, including the critical values used and the statistical power achieved. Furthermore, we compare the performance of these tests to gain insights into their effectiveness. Section 4 revolves around the practical application of the GOF test based on KL information in real-life scenarios, with a focus on the Weibull distribution. We provide examples and analyze the performance of the test using real data, showcasing its applicability and usefulness in practical settings. Finally, in Section 5, we present our concluding remarks. We summarize the key findings of our study, discuss the implications of the proposed GOF test based on KL information, and provide suggestions for future research in this area. By following this structure, we aim to present a comprehensive analysis of the suggested GOF test and its applications for the normal, exponential, and Weibull distributions, along with its practical utility in real-life situations.

2. The proposed estimator of KL information

The Kullback-Leibler divergence is a measure used in information theory to quantify the difference between two probability distributions, f and g. It is worth noting that the KL divergence is non-symmetric in nature. In typical scenarios, f represents the true distribution of the observations, while g represents a theoretical model or an assumed distribution.

The KL divergence, denoted as $KL\left(f,g\right)$, captures the dissimilarity between the two distributions and is defined as follows:

$$KL\left(f,g\right)=\int_{0}^{\infty}f(x)\log\frac{f(x)}{g(x)}dx. \tag{5}$$

Here, $f(x)$ represents the probability density function of the true distribution, while $g(x)$ represents the probability density function of the theoretical model. By comparing the logarithmic ratios of the densities at each point, the KL divergence provides a measure of how much information is lost when using the model g to approximate the true distribution f.

In the context of statistical analysis, the KL divergence serves as a valuable tool for assessing the goodness of fit of a theoretical model (g) to the observed data, represented by the true distribution (f). By quantifying the discrepancy between the two distributions, it enables researchers to evaluate the adequacy of the model in capturing the underlying properties of the data.

Suppose $X_1,...,X_n$ be a non-negative random sample of size n from continuous distribution function $F\left(x\right)$ with probability density function $f\left(x\right)$. Let $G\left(x;\theta\right)$ denote a parametric distribution function with a probability density function $g\left(x;\theta\right)$. Then, the hypothesis test is,

$$H_0: f\left(x\right)=g\left(x;\theta\right),$$

The alternative test is

$$H_1: f\left(x\right)\neq g\left(x;\theta\right).$$

In continue, we construct a test statistic for the above hypotheses based on the KL information.

The extension of function $\log$ can be written as follows,

$$\log x=2\left[\left(\frac{x-1}{x+1}\right)+\frac{1}{3}\left(\frac{x-1}{x+1}\right)^3+\frac{1}{5}\left(\frac{x-1}{x+1}\right)^5+\cdots\right], \tag{6}$$

and also the KL information can be written as

$$KL\left(f,g\right)=E_f\left\{\log\frac{f\left(x\right)}{g\left(x\right)}\right\}. \tag{7}$$

Therefore, we can suggest an estimator of KL information, denoted by DS, using extension (6) as follow.

$$\begin{aligned} DS\left(f,g\right)&=\frac{1}{n}\sum_{i=1}^{n}\log\frac{f\left(x_i\right)}{g\left(x_i\right)}\\ &=\frac{1}{n}\sum_{i=1}^{n}2\left[\left(\frac{\frac{f(x_i)}{g(x_i)}-1}{\frac{f(x_i)}{g(x_i)}+1}\right)+\frac{1}{3}\left(\frac{\frac{f(x_i)}{g(x_i)}-1}{\frac{f(x_i)}{g(x_i)}+1}\right)^3+\frac{1}{5}\left(\frac{\frac{f(x_i)}{g(x_i)}-1}{\frac{f(x_i)}{g(x_i)}+1}\right)^5+...\right]\\ &=\frac{2}{n}\sum_{i=1}^{n}\sum_{j=1}^{\infty}\frac{1}{(2j-1)}\left(\frac{\frac{f(x_i)}{g(x_i)}-1}{\frac{f(x_i)}{g(x_i)}+1}\right)^{(2j-1)}, \end{aligned} \tag{8}$$

where f, similar to that in Vasicek [28], is estimated by

$$\hat{f}\left(x_{(i)}\right)=\frac{F_n\left(x_{(i+m)}\right)-F_n\left(x_{(i-m)}\right)}{x_{(i+m)}-x_{(i-m)}}=\frac{2m/n}{x_{(i+m)}-x_{(i-m)}}, \tag{9}$$

where F_n is the empirical distribution function. Moreover, we here use a semi-parametric estimator of $g\left(x_{(i)}\right)$ as

$$g\left(x_{(i)}\right)=\frac{G\left(x_{(i+m)};\hat{\theta}\right)-G\left(x_{(i-m)};\hat{\theta}\right)}{x_{(i+m)}-x_{(i-m)}}, \tag{10}$$

where G is distribution function under the null hypothesis and $\hat{\theta}$ is the maximum likelihood estimation (MLE) of θ. Therefore, by substituting the expressions (9) and (10), we obtain the following estimator of $DS\left(f,g;\theta\right)$ and use it as a test statistic for GOF.

$$\begin{aligned} DS\left(f,g;\hat{\theta}\right) &= \frac{2}{n}\sum_{i=1}^{n}\sum_{j=1}^{\infty}\frac{1}{(2j-1)}\left(\frac{\frac{2m/n}{G\left(x_{(i+m)};\hat{\theta}\right)-G\left(x_{(i-m)};\hat{\theta}\right)}-1}{\frac{2m/n}{G\left(x_{(i+m)};\hat{\theta}\right)-G\left(x_{(i-m)};\hat{\theta}\right)}+1}\right)^{(2j-1)} \\ &= \frac{2}{n}\sum_{i=1}^{n}\sum_{j=1}^{\infty}\frac{1}{(2j-1)}\left(\frac{2m/n\ -\left(G\left(x_{(i+m)};\hat{\theta}\right)-G\left(x_{(i-m)};\hat{\theta}\right)\right)}{2m/n\ +\left(G\left(x_{(i+m)};\hat{\theta}\right)-G\left(x_{(i-m)};\hat{\theta}\right)\right)}\right)^{(2j-1)}, \end{aligned} \tag{11}$$

where m is a positive integer $m\leq n/2$, $X_{(1)}\leq X_{(2)},...,\leq X_{(n)}$ are the order statistics of the random sample and also $X_{(i)}=X_{(1)}$ if $i<1$, $X_{(i)}=X_{(n)}$ if $i>n$. Clearly, for large values of the test statistic the null hypothesis (H_0) is rejected. Let $Q_{1-\alpha}$ be the $1-\alpha$ quantile of the distribution of DS. Then the critical region is $DS>Q_{1-\alpha}$. Moreover, by Vasicek [28], there exists a value $x_i' \in \left(x_{(i-m)},x_{(i+m)}\right)$ such that

$$g\left({x'}_i\,;\hat{\theta}\right)=\frac{G\left(x_{(i+m)};\hat{\theta}\right)-G\left(x_{(i-m)};\hat{\theta}\right)}{x_{(i+m)}-x_{(i-m)}}, \tag{12}$$

and, as $n,m\rightarrow\ \infty$, $m/n\ \rightarrow\ 0$, we have

$$g\left(x,\hat{\theta}\right)\xrightarrow{p}g\left(x,\theta\right). \tag{13}$$

Similarly, the weak law of large number indicates that (Gut [16])

$$F_n\left(x\right)\xrightarrow{p}F\left(x\right),\quad \text{as}\ \ \text{n}\rightarrow\infty.$$

Also, by Vasicek [28], as $n,m\rightarrow\ \infty$, $m/n\ \rightarrow\ 0$, we have

$$\hat{f}\left(x\right)\xrightarrow{p}f\left(x\right). \tag{14}$$

So, by using Theorem 10.3 from Gut [16] and the weak law of large numbers, as $n,m\rightarrow\ \infty$, $m/n\ \rightarrow\ 0$, we have

$$\begin{aligned} DS\left(f,g;\hat{\theta}\right) &= \frac{2}{n}\sum_{i=1}^{n}\sum_{j=1}^{\infty}\frac{1}{(2j-1)}\left(\frac{2m/n\ -\left(G\left(x_{(i+m)};\hat{\theta}\right)-G\left(x_{(i-m)};\hat{\theta}\right)\right)}{2m/n\ +\left(G\left(x_{(i+m)};\hat{\theta}\right)-G\left(x_{(i-m)};\hat{\theta}\right)\right)}\right)^{(2j-1)} \\ &\xrightarrow{P.}\frac{2}{n}\sum_{i=1}^{n}\sum_{j=1}^{\infty}\frac{1}{(2j-1)}\left(\frac{\frac{f(x_i)}{g(x_i)}-1}{\frac{f(x_i)}{g(x_i)}+1}\right)^{(2j-1)}=\frac{1}{n}\sum_{i=1}^{n}\log\frac{f\left(x_i\right)}{g\left(x_i\right)} \\ &\xrightarrow{P.}E_f\left\{\log\frac{f\left(x\right)}{g\left(x\right)}\right\}=KL\left(f,g\right). \end{aligned} \tag{15}$$

3. Tests of fit for some specific distributions

In this section, we focus on three distributions namely normal, exponential and Weibull distributions and we use the proposed test statistic based on KL information for goodness of fit of these distributions. We compare the power of the proposed tests with the general tests based on the empirical distribution function such as Cramer van Mises (W^2), Waston (U^2), Anderson-Darling (A^2), Kolmogorov-Smirnov (D) and Kuiper (V). These tests commonly used in practice and statistical software. The tests can be considered as follows.

- The Cramer-von Mises statistic is

$$W^2 = \frac{1}{12n} + \sum_{i=1}^{n} \left(\frac{2i-1}{2n} - z_{(i)} \right)^2 .$$

- The Watson statistic is computed from

$$U^2 = W^2 - n \left(\bar{z} - \frac{1}{n} \right)^2 ,$$

  where $\bar{z}$ is the mean of the z_i values.
- The Anderson-Darling statistic is

$$A^2 = -n - \frac{1}{n} \sum_{i=1}^{n} (2i-1) \left\{ \log z_{(i)} + \log \left(1 - z_{(n-i+1)} \right) \right\} .$$

- The Kolmogorov statistics are computed from

$$D^+ = \max_{1 \leq i \leq n} \left\{ \frac{i}{n} - z_{(i)} \right\} ; \; D^- = \max_{1 \leq i \leq n} \left\{ z_{(i)} - \frac{i-1}{n} \right\} ,$$

  then, the Kolmogorov-Smirnov statistic is $D = \max (D^+, D^-)$.
- The Kuiper statistic is

$$V = D^+ + D^- ,$$

Here $z_{(i)} = F_0 \left(x_{(i)}, \hat{\theta} \right)$, for $i = 1, 2, ..., n$, and F_0 is the null hypothesis distribution function and $\hat{\theta}$ is the ML estimator of the unknown parameter.

3.1. Testing normality

The normal distribution, also known as the Gaussian distribution, is one of the most important probability distributions in statistics. It is characterized by its bell-shaped curve, symmetric nature, and defined by two parameters: the mean (μ) and the standard deviation (σ).

The normal distribution has numerous applications in statistics due to its remarkable properties. It serves as a fundamental model for many natural phenomena and random variables in various fields. Some key applications of the normal distribution include:

Central Limit Theorem: The central limit theorem states that the sum or average of a large number of independent and identically distributed random variables tends to follow a normal distribution, regardless of the shape of the original distribution. This property allows the normal distribution to be widely used in statistical inference.

Descriptive Statistics: In descriptive statistics, the normal distribution plays a crucial role in summarizing and analyzing data. The mean and standard deviation of a normal distribution provide measures of central tendency and variability, respectively. Moreover, many statistical tests and confidence intervals assume data to be normally distributed.

Statistical Inference: The normal distribution is extensively used in statistical inference techniques such as hypothesis testing and confidence intervals. When sample sizes are large or when certain conditions are met, normality assumptions allow for the application of powerful statistical tests, making the normal distribution a cornerstone of parametric inference.

Quality Control: In quality control and process improvement, the normal distribution is often used to model the distribution of measurements. It helps determine control limits, identify outliers, and assess process capability by comparing observed data to the expected normal distribution.

Regression Analysis: In linear regression, the normal distribution plays a critical role in estimating the parameters of the regression model. The assumption of normal errors allows for valid inference, including confidence intervals and hypothesis tests for regression coefficients.

Risk Management and Finance: The normal distribution is frequently used in financial modeling and risk management. It serves as a foundation for models such as the Black-Scholes option pricing model and the Capital Asset Pricing Model (CAPM), which assume asset returns follow a normal distribution.

These are just a few examples of the wide-ranging applications of the normal distribution in statistics. Its versatility, mathematical tractability, and connection to real-world phenomena make it an indispensable tool for statistical analysis and inference in various disciplines.

The proposed test statistic for the normal distribution is as follows.

$$DS = \frac{2}{n}\sum_{i=1}^{n}\sum_{j=1}^{\infty}\frac{1}{(2j-1)}\left(\frac{2m/n-\left(G\left(x_{(i+m)};\hat{\theta}\right)-G\left(x_{(i-m)};\hat{\theta}\right)\right)}{2m/n+\left(G\left(x_{(i+m)};\hat{\theta}\right)-G\left(x_{(i-m)};\hat{\theta}\right)\right)}\right)^{(2j-1)}, \tag{16}$$

where G is the normal distribution function and $\hat{\theta} = (\hat{\mu}, \hat{\sigma})$, where

$$\hat{\mu} = \bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i \quad ;\hat{\sigma} = s = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x})^2}. \tag{17}$$

The above test statistic is complicated to provide the possibility of extracting its exact distribution under the null hypothesis analytically. Thus, we use the Monte Carlo simulation to obtain the percentiles of the null distribution. To estimate the percentile of null distribution, we generate 50000 random sample with various sample size n from the standard normal distribution.

At significance level α we reject the null hypothesis H_0 if the test statistic exceeds $C(\alpha)$, where the value of $C(\alpha)$ is obtained from the $(1-\alpha)$ quantile of the test statistic under null hypothesis H_0. For each test statistics, we simulated random samples of size n from the standard normal distribution. Since $\alpha = 0.05$, the 47500^{th} order statistic is assessed to determine the critical value $C(\alpha)$. The critical values of the proposed test statistic are presented in Table (1) for different sample sizes $5 \leq n \leq 50$.

Power values of the proposed test DS and the competing tests W^2, U^2, D, V, A^2 are computed by Monte Carlo simulation under 20 alternative distributions. The considered alternatives, based the shape of density function and support are divided into four groups. These alternatives were used earlier by Esteban et al. [13] and Alizadeh and Arghami [3] in their study of several tests for normality. The alternatives are divided into four group as follow.

Group I: Support $(-\infty, \infty)$, symmetric.

- Student t with 1 degree of freedom,
- Student t with 3 degree of freedom,
- Standard logistic,
- Standard Laplace.

Group II: Support $(-\infty, \infty)$, asymmetric.

- Gumbel (α, β) which α is location parameter and β is scale parameter.

Group III: Support$(0, \infty)$.

Table 1. Critical values of the DS statistic for testing normality at $\alpha = 0.05$.

	m									
n	1	2	3	4	5	6	7	8	9	10
5	0.9875	0.7187								
6	0.9297	0.6668	0.6497							
7	0.8772	0.6377	0.5856							
8	0.8284	0.6066	0.5364	0.5679						
9	0.7862	0.5705	0.5083	0.5223						
10	0.7571	0.5449	0.4799	0.4807	0.5215					
15	0.6376	0.4406	0.3920	0.3770	0.3802	0.4022	0.4366			
20	0.5685	0.3792	0.3321	0.3189	0.3205	0.3275	0.3412	0.3637	0.3931	0.4226
25	0.5231	0.3396	0.2916	0.2799	0.2785	0.2839	0.2914	0.3029	0.3200	0.3413
30	0.4963	0.3124	0.2648	0.2517	0.2493	0.2528	0.2590	0.2684	0.2789	0.2918
40	0.4537	0.2788	0.2295	0.2127	0.2074	0.2083	0.2127	0.2197	0.2275	0.2362
50	0.4293	0.2556	0.2069	0.1875	0.1808	0.1798	0.1826	0.1876	0.1929	0.2006

- Exponential with mean 1,
- Gamma(α, β), α is the shape parameter,
- Lognormal(μ, σ), μ is the location parameter and $\sigma$is the scale parameter,
- Weibull(η, β), η is the shape parameter.

Group IV: Support $(0, 1)$.

- Uniform $[0, 1]$,
- Beta$(2, 2)$,
- Beta$(0.5, 0.5)$,
- Beta$(3, 1.5)$,
- Beta$(2, 1)$.

We generated 50,000 random samples of sizes $n = 10$, and 20 from each alternative distribution and evaluated for each sample the test statistics $\left(W^2,\ D,\ V,\ U^2,\ A^2,\ DS\right)$. For the proposed test, we consider $m = 5,\ 9$ for $n = 10,\ 20$, respectively. Through a Monte Carlo simulation the power values of the tests are computed and the results are summarized in Tables (2-5). It can be seen from Tables (3-4) that the proposed test DS is most powerful

Table 2. Power comparisons of the tests under alternatives from group I at $\alpha = 0.05$.

Altern.	n	W^2	U^2	D	V	A^2	DS
$t_{(1)}$	10	**0.6153**	0.6095	0.5796	0.593	0.6145	0.5337
$t_{(1)}$	20	0.8801	0.8792	0.8443	0.8658	**0.8815**	0.8449
$t_{(3)}$	10	0.1783	0.171	0.1592	0.1605	**0.1858**	0.1659
$t_{(3)}$	20	0.307	0.2976	0.2558	0.2758	**0.3260**	0.2998
Logistic	10	0.0739	0.0704	0.0686	0.0689	0.0775	**0.0755**
Logistic	20	0.0987	0.0941	0.0843	0.0891	0.1054	**0.1054**
Laplace	10	0.1558	0.1498	0.1399	0.1409	**0.1593**	0.1337
Laplace	20	0.2632	0.26	0.2177	0.2393	**0.2686**	0.2408

against alternatives in groups III and IV. The difference power of the proposed test with the other tests is substantial and our test has a significantly better performance in compared with the competing tests. In contrast, in the first and fourth groups, for some alternative distributions the proposed test has a high power and for the rest alternatives other tests do. It can be seen that in the first group, the proposed test and Anderson-Darling test have the maximum power and in the fourth group, the tests DS,A^2 and V have the maximum power. It is clear in Tables (2-5).

Table 3. Power comparisons of the tests under alternatives from group II at $\alpha = 0.05$.

Altern.	n	W^2	U^2	D	V	A^2	DS
Gumbel$(0,1)$	10	0.1351	0.1276	0.1167	0.1193	0.1420	**0.1560**
Gumbel$(0,1)$	20	0.2481	0.2207	0.2004	0.1941	0.2726	**0.3204**
Gumbel$(0,2)$	10	0.1350	0.1266	0.1167	0.1173	0.1415	**0.1581**
Gumbel$(0,2)$	20	0.2492	0.2219	0.205	0.1983	0.2748	**0.3228**
Gumbel$(0,3)$	10	0.1355	0.1273	0.1171	0.1186	0.1414	**0.1596**
Gumbel$(0,3)$	20	0.2469	0.2192	0.2006	0.1933	0.2729	**0.3183**

Table 4. Power comparisons of the tests under alternatives from group III at $\alpha = 0.05$.

Altern.	n	W^2	U^2	D	V	A^2	DS
Exp(1)	10	0.3837	0.3659	0.2964	0.359	0.4089	**0.4835**
Exp(1)	20	0.7240	0.6870	0.5857	0.6956	0.7754	**0.8320**
Gamma(2)	10	0.2055	0.1923	0.1704	0.1812	0.2207	**0.2572**
Gamma(2)	20	0.4131	0.3726	0.3207	0.3437	0.4586	**0.5411**
Gamma(0.5)	10	0.6601	0.6460	0.5308	0.6575	0.6912	**0.7677**
Gamma(0.5)	20	0.9515	0.9409	0.8830	0.9540	0.9686	**0.9767**
Lnormal(1)	10	0.5519	0.5330	0.4554	0.5237	0.575	**0.6369**
Lnormal(1)	20	0.8801	0.8584	0.7929	0.8571	0.9042	**0.9293**
Lnormal(2)	10	0.8969	0.8910	0.8233	0.8949	0.9087	**0.9371**
Lnormal(2)	20	0.9975	0.9968	0.9918	0.9976	0.9985	**0.9989**
Lnormal(0.5)	10	0.2152	0.2011	0.1785	0.1881	0.2282	**0.2583**
Lnormal(0.5)	20	0.4230	0.383	0.3395	0.3483	0.4612	**0.5328**
Weibull(0.5)	10	0.8542	0.846	0.7508	0.8555	0.8725	**0.9145**
Weibull(0.5)	20	0.9952	0.9937	0.9828	0.996	0.9974	**0.9977**
Weibull(2)	10	0.0757	0.0728	0.0705	0.0690	0.0795	**0.0872**
Weibull(2)	20	0.1176	0.1066	0.1018	0.0932	0.1299	**0.1629**

Table 5. Power comparisons of the tests under alternatives from group IV at $\alpha = 0.05$.

Altern.	n	W^2	U^2	D	V	A^2	DS
Uniform	10	0.0727	0.0809	0.0628	**0.0811**	0.0776	0.0668
Uniform	20	0.1422	0.1614	0.101	0.1528	**0.172**	0.0664
Beta $(2,2)$	10	0.0443	0.0476	0.0439	**0.0479**	0.0447	0.0417
Beta $(2,2)$	20	0.0571	0.0644	0.0529	**0.067**	0.0592	0.0318
Beta $(0.5,0.5)$	10	0.2247	0.2472	0.1515	0.232	**0.2632**	0.2197
Beta $(0.5,0.5)$	20	0.5048	0.5419	0.3227	0.4921	**0.6155**	0.2505
Beta $(3,0.5)$	10	0.5328	0.521	0.4006	0.5236	0.567	**0.6535**
Beta $(3,0.5)$	20	0.878	0.8579	0.7548	0.8864	0.9155	**0.9314**
Beta $(2,1)$	10	0.1161	0.1166	0.0954	0.1095	0.124	**0.1451**
Beta $(2,1)$	20	0.2299	0.2221	0.1804	0.2075	0.2616	**0.2686**

3.2. Testing Exponentiality

The exponential distribution is a fundamental probability distribution frequently used in statistics to model the time between events in a Poisson process. It is characterized by a constant hazard rate, which means that the probability of an event occurring in a given time interval is independent of how much time has already elapsed.
The exponential distribution finds applications in various areas of statistics due to its unique properties. Some key applications of the exponential distribution include:

Survival Analysis: The exponential distribution is commonly utilized in survival analysis to model the time to failure or time to an event. It is particularly suited for situations where the hazard rate remains constant over time, such as the failure times of electronic components or the lifetimes of certain products.
Reliability Engineering: In reliability engineering, the exponential distribution plays a crucial role in modeling the reliability and failure rates of systems and components. It provides insights into the probability of failure over time and helps estimate mean time between failures (MTBF) or mean time to failure (MTTF).
Queuing Theory: The exponential distribution is essential in queuing theory, where it is used to model the interarrival times or service times in a queuing system. It helps analyze and optimize various aspects of queuing systems, such as waiting times, queue lengths, and system performance.
Financial Modeling: The exponential distribution has applications in financial modeling, particularly in the field of option pricing. It is used to model the time to default or time to maturity of certain financial instruments, allowing for the estimation of probabilities and pricing of options.
Machine Learning and Data Science: In machine learning and data science, the exponential distribution is often used as a building block for certain models and algorithms. For example, it serves as the basis for exponential family distributions, which encompass a wide range of distributions used in statistical modeling.
Actuarial Science: Actuarial science extensively employs the exponential distribution to model various insurance-related quantities, such as claim arrival times, claim amounts, or policy durations. It aids in assessing risk, determining premiums, and evaluating insurance portfolios.
These are just a few examples of the applications of the exponential distribution in statistics. Its simplicity, mathematical tractability, and connection to random time-based events make it a valuable tool for modeling and analyzing a wide range of phenomena in different fields of study.
The probability density of the exponential distribution with parameter λ is as follows.

$$f(x;\lambda) = \lambda e^{-\lambda x} \quad x \geq 0, \ \ \lambda > 0.$$

Based on the new estimator of KL information, we propose the following test statistic for testing exponentiality.

$$DS = \frac{2}{n}\sum_{i=1}^{n}\sum_{j=1}^{\infty}\frac{1}{(2j-1)}\left(\frac{2m/n \ - \Big(\exp\{-\hat{\lambda}x_{(i-m)}\} - \exp\{-\hat{\lambda}x_{(i+m)}\}\Big)}{2m/n \ + \Big(\exp\{-\hat{\lambda}x_{(i-m)}\} - \exp\{-\hat{\lambda}x_{(i+m)}\}\Big)}\right)^{(2j-1)},$$

where $\hat{\lambda} = 1/\bar{x}$ is the maximum likelihood estimator of λ. Clearly, the test statistic is invariant to transformations of scale.
We use the Monte Carlo simulation to calculate the critical values of the proposed test statistic DS for $\alpha = 0.05$. We generate 50,000 random samples with sample sizes $5 \leq n \leq 50$ from the exponential distribution with parameter one. We reject the null hypothesis if the value of the test statistic is greater than the critical point. The critical points of the proposed test are presented in Table (6). Using Monte Carlo simulation with 50000 iterations, the power of the tests can be obtained under 17 different alternative distributions. For this purpose, initially, 50000 random samples are generated at sample sizes of $n = 10, 20, 30$ and 50 under each of the alternative distribution, and the power of each test is estimated by the ratio of the number of times the test statistic exceeds from its critical value to the total number of iterations.
The power comparison of the proposed test with other tests involves the utilization of the following alternatives.

- The Weibull distribution W (η, β);
- The Gamma distribution $G(\alpha, \beta)$;
- The Log normal distribution $LN(\mu, \sigma)$;
- The Modified extreme value $EV(\theta)$ with distribution function $F(x;\theta) = 1 - \exp\left(\theta^{-1}(1-e^{x})\right)$;
- The Half-Normal distribution HN with density function $f(x) = \Gamma\left(\frac{2}{\pi}\right)^{1/2}\exp\left(-x^2/2\right)$;
- The Uniform distribution $U(0,1)$;
- The Chen [5] distribution $C(\lambda, \beta)$, with distribution function

$$F(x) = \left[1 - e^{\lambda\left(1-e^{x^{\beta}}\right)}\right], \quad \lambda, \beta > 0.$$

Table 6. Critical values of the DS statistic for testing exponentiality at $\alpha = 0.05$.

	m									
n	1	2	3	4	5	6	7	8	9	10
5	1.3996	1.3222								
6	1.2485	1.1265	1.1672							
7	1.1335	0.9947	1.0135							
8	1.0396	0.8891	0.8934	0.9433						
9	0.9691	0.8151	0.8043	0.8424						
10	0.9168	0.7473	0.7327	0.7650	0.8125					
15	0.7337	0.5616	0.5295	0.5361	0.5582	0.5878	0.6244			
20	0.6405	0.4613	0.4257	0.4236	0.4358	0.4544	0.4767	0.5050	0.5333	0.5594
25	0.5811	0.4035	0.3640	0.3560	0.3626	0.3741	0.3914	0.4089	0.4323	0.4539
30	0.5390	0.3637	0.3202	0.3097	0.3123	0.3215	0.3339	0.3484	0.3651	0.3817
40	0.4872	0.3134	0.2684	0.2538	0.2510	0.2553	0.2620	0.2721	0.2823	0.2943
50	0.4546	0.2833	0.2361	0.2190	0.2146	0.2148	0.2194	0.2249	0.2338	0.2431

- Dhillon distribution II (Dhillon [9]) $D2\left(\lambda, b\right)$, with distribution function

$$F\left(x\right) = 1 - e^{-\left(\ln(\lambda x+1)\right)^{b+1}}, \;\; \lambda > 0, b \geq 0.$$

- Dhillon distribution I (Dhillon [9]) $D1\left(\lambda, b\right)$, with distribution function

$$F\left(x\right) = 1 - e^{-\left[e^{(\beta x)^{b}-1}\right]}, \;\; b, \beta > 0.$$

Table 7. Power comparison of the exponentiality tests for $n = 10, \;\; m = 5$ and at level $\alpha = 0.05$.

Altern.	W^2	U^2	D	V	A^2	DS
$W\left(0.8\right)$	0.1165	0.0899	0.1058	0.0849	**0.1775**	0.0117
$W\left(1.4\right)$	0.1869	0.1685	0.1605	0.1539	0.1368	**0.2545**
$G\left(0.4\right)$	0.4549	0.3646	0.4151	0.3468	**0.6794**	0.0020
$G\left(1\right)$	0.0500	0.0518	0.0505	**0.0526**	0.0507	0.0466
$G\left(2\right)$	0.2471	0.2228	0.2130	0.2028	0.1869	**0.3518**
$LN\left(0.8\right)$	0.1833	0.1865	0.1634	0.1692	0.1395	**0.2991**
$LN\left(1.5\right)$	0.3475	0.2582	0.3181	0.2440	0.3710	0.0175
HN	0.1244	0.1171	0.1128	0.1094	0.0903	**0.1560**
U	0.3612	0.3367	0.2797	0.3517	0.2900	**0.3970**
$C\left(0.5\right)$	0.3434	0.2722	0.3068	0.2556	**0.5392**	0.0024
$C\left(1\right)$	0.0950	0.0908	0.0875	0.0874	0.0698	**0.1170**
$C\left(1.5\right)$	0.4542	0.4011	0.3665	0.3800	0.3707	**0.5309**
$EV\left(0.5\right)$	0.0944	0.0915	0.0860	0.0869	0.0704	**0.1218**
$EV\left(1.5\right)$	0.2257	0.2062	0.1891	0.1967	0.1708	**0.2658**
$D2\left(1\right)$	0.2306	0.1623	0.2097	0.1557	**0.2582**	0.0185
$D2\left(1.5\right)$	0.1752	0.1210	0.1593	0.1151	**0.1992**	0.0196
$D1\left(0.8\right)$	0.0635	0.0691	0.0625	0.0675	0.0653	**0.0697**

Power of the new test depends on the parameter *m,* and the optimal value of *m*, which maximizes the test power, is not only dependent on the sample size and it is also depended on the alternative distribution. Since the alternative distribution is practically unknown, it is not possible to propose a specific value of *m* that maximizes the test power for all alternative distributions. Therefore, the values of *m* are optimized for different values of *n,* ensuring relatively

Table 8. Power comparison of the exponentiality tests for $n = 20,\ m = 9$ and at level $\alpha = 0.05$.

Altern.	W^2	U^2	D	V	A^2	DS
$W(0.8)$	0.2015	0.1446	0.1729	0.1347	**0.2689**	0.0052
$W(1.4)$	0.3539	0.2943	0.2884	0.2684	0.3106	**0.4704**
$G\,(0.4)$	0.7576	0.6475	0.703	0.6171	**0.8971**	0.0056
$G\,(1)$	0.0503	0.0512	0.0487	0.0511	0.0512	**0.0515**
$G\,(2)$	0.4882	0.4235	0.4072	0.3812	0.4547	**0.6605**
$LN\,(0.8)$	0.3473	0.3693	0.3042	0.3295	0.3395	**0.6253**
$LN\,(1.5)$	0.6174	0.4786	0.5695	0.4551	**0.6253**	0.0702
HN	0.2111	0.1774	0.1767	0.1662	0.1719	**0.2580**
U	0.6758	0.6117	0.5243	0.6705	0.6278	**0.7549**
$C\,(0.5)$	0.6216	0.5135	0.5595	0.4836	**0.7863**	0.0016
$C\,(1)$	0.1519	0.1279	0.1287	0.1223	0.1199	**0.1878**
$C\,(1.5)$	0.7972	0.7112	0.6692	0.6928	0.7629	**0.8727**
$EV\,(0.5)$	0.1491	0.1273	0.1271	0.1202	0.1192	**0.1902**
$EV\,(1.5)$	0.4449	0.3739	0.3550	0.3634	0.3824	**0.4956**
$D2\,(1)$	0.4168	0.2858	0.3755	0.2756	**0.4375**	0.0391
$D2\,(1.5)$	0.3090	0.1999	0.2704	0.1911	**0.3277**	0.0238
$D1\,(0.8)$	0.0831	0.0851	0.0770	0.0832	0.0804	**0.0872**

good power for all alternative distributions. These values are obtained by simulation for all values of $m \leq n/2$ and selecting values that yield good power for all alternative distributions.
The power of tests for the exponential distribution are presented in Tables (7-10). These tables indicate that the proposed test (DS) and Anderson-Darling test (A^2) have the maximum power against various alternatives and different sample sizes.

Table 9. Power comparison of the exponentiality tests for $n = 30,\ m = 13$ and at level $\alpha = 0.05$.

$altern.$	W^2	U^2	D	V	A^2	DS
$W(0.8)$	0.2783	0.1963	0.2375	0.1797	**0.3577**	0.0029
$W(1.4)$	0.5055	0.4208	0.4204	0.3794	0.4814	**0.6413**
$G\,(0.4)$	0.9032	0.8254	0.8693	0.7972	**0.9714**	0.0099
$G\,(1)$	0.0507	0.0508	0.0488	0.0493	0.0504	**0.0521**
$G\,(2)$	0.6795	0.5997	0.5852	0.5394	0.6794	**0.8351**
$LN\,(0.8)$	0.4979	0.5425	0.4511	0.4782	0.5445	**0.8367**
$LN\,(1.5)$	0.7775	0.649	0.7345	0.6216	**0.7798**	0.134
HN	0.2980	0.2440	0.2451	0.2240	0.2593	**0.3613**
U	0.8630	0.8068	0.7251	0.8659	0.8499	**0.9242**
$C\,(0.5)$	0.7929	0.6927	0.7391	0.655	**0.9073**	0.0020
$C\,(1)$	0.2048	0.1668	0.1727	0.1544	0.1751	**0.2605**
$C\,(1.5)$	0.9417	0.8833	0.8536	0.8702	0.9346	**0.9735**
$EV\,(0.5)$	0.2044	0.1660	0.1729	0.1553	0.1738	**0.2619**
$EV\,(1.5)$	0.6249	0.5314	0.5124	0.5187	0.5793	**0.6809**
$D2\,(1)$	0.5558	0.3913	0.5016	0.3755	**0.5726**	0.0634
$D2\,(1.5)$	0.4254	0.2799	0.3762	0.2624	**0.4409**	0.0329
$D1\,(0.8)$	0.0997	0.1030	0.0923	0.0961	0.0984	**0.1029**

Table 10. Power comparison of the exponentiality tests for $n = 50,\ m = 20$ and at level $\alpha = 0.05$.

$altern.$	W^2	U^2	D	V	A^2	DS
$W(0.8)$	0.4259	0.2980	0.3610	0.2778	**0.5109**	0.0010
$W(1.4)$	0.7491	0.6381	0.6429	0.5931	0.7476	**0.8478**
$G(0.4)$	0.9880	0.9654	0.9800	0.9562	**0.9981**	0.0221
$G(1)$	0.0498	0.0491	0.0492	0.0492	0.0498	**0.0513**
$G(2)$	0.8993	0.8347	0.8274	0.7840	0.9151	**0.9690**
$LN(0.8)$	0.7580	0.8031	0.7110	0.7456	0.8448	**0.9793**
$LN(1.5)$	**0.9347**	0.8542	0.9091	0.8383	0.9323	0.2789
HN	0.4834	0.3767	0.3933	0.3578	0.4422	**0.5408**
U	0.9854	0.9648	0.9285	0.9879	0.9867	**0.9967**
$C(0.5)$	0.9512	0.8970	0.9254	0.8760	**0.9862**	0.0032
$C(1)$	0.3297	0.2459	0.2683	0.2373	0.2945	**0.4023**
$C(1.5)$	0.9971	0.9874	0.9806	0.9867	0.9972	**0.9991**
$EV(0.5)$	0.3253	0.2451	0.2628	0.2339	0.2906	**0.3979**
$EV(1.5)$	0.8613	0.7628	0.7484	0.7670	0.8411	**0.8988**
$D2(1)$	0.7536	0.5736	0.6956	0.5641	**0.7612**	0.1182
$D2(1.5)$	0.6032	0.4109	0.5397	0.3993	**0.6125**	0.0537
$D1(0.8)$	**0.1499**	0.1436	0.1304	0.1327	0.1429	0.1383

3.3. Test for Weibull distribution

The Weibull distribution is a versatile probability distribution widely used in statistics to model the lifetimes or failure times of various products and systems. It is characterized by its flexible shape, which can exhibit various types of hazard rates, including increasing, decreasing, and constant.
The Weibull distribution finds applications in several areas of statistics due to its ability to capture different failure patterns. Some key applications of the Weibull distribution include:
Reliability Analysis: The Weibull distribution is extensively used in reliability engineering to model the failure times of components, systems, or structures. It provides a flexible framework for analyzing and predicting the reliability, hazard rates, and failure probabilities of complex systems.
Survival Analysis: In survival analysis, the Weibull distribution is commonly employed to model the time to an event or failure, where the hazard rate can change over time. It allows for the assessment of survival probabilities, estimation of median or mean survival times, and comparison of survival curves between different groups.
Extreme Value Analysis: The Weibull distribution is often utilized in extreme value analysis to model the distribution of extreme events, such as the maximum wind speed in a particular location or the largest flood in a river system. It provides a suitable framework for characterizing tail behavior and estimating extreme quantiles.
Material Science: The Weibull distribution is used in material science to describe the strength and fracture properties of materials. It helps in understanding the variability and reliability of materials under different stress conditions, aiding in design and quality control processes.
Wind Energy: In wind energy studies, the Weibull distribution is employed to model wind speeds and analyze wind power potential. It provides a useful tool for estimating energy production, designing wind turbines, and assessing the performance of wind farms.
Biostatistics and Health Sciences: The Weibull distribution is applied in biostatistics and health sciences to model survival times and analyze the impact of risk factors on disease progression or patient outcomes. It allows for the estimation of survival functions, hazard ratios, and other important measures in clinical research.
These are just a few examples of the applications of the Weibull distribution in statistics. Its flexibility in capturing different failure patterns and its wide-ranging use in modeling lifetime data make it a valuable tool for analyzing reliability, survival, extreme events, and other phenomena in various fields of study.

A random variable X is called Weibull distribution if its density is as follows.

$$f(x;\eta,\beta)=\frac{\beta}{\eta}\left(\frac{x}{\eta}\right)^{\beta-1}e^{-\left(\frac{x}{\eta}\right)^{\beta}},\quad x\geq 0,\ \eta,\beta>0. \tag{18}$$

The proposed test statistic for the Weibull distribution is given by

$$DS\left(f,g;\hat{\theta}\right)=\frac{2}{n}\sum_{i=1}^{n}\sum_{j=1}^{\infty}\frac{1}{(2j-1)}\left(\frac{2m/n\ -\left(G\left(x_{(i+m)};\hat{\theta}\right)-G\left(x_{(i-m)};\hat{\theta}\right)\right)}{2m/n\ +\left(G\left(x_{(i+m)};\hat{\theta}\right)-G\left(x_{(i-m)};\hat{\theta}\right)\right)}\right)^{(2j-1)}, \tag{19}$$

where G is the Weibull distribution and $\hat{\theta}=\left(\hat{\eta},\hat{\beta}\right)$, where the ML estimators can be obtained from solving the following equations.

$$\begin{cases}\hat{\eta}=\left[\frac{1}{n}\sum_{i=1}^{n}X_i^{\hat{\beta}}\right]^{1/\hat{\beta}}\\ \frac{n}{\hat{\beta}}+\sum_{i=1}^{n}\ln X_i-\frac{n}{\sum_{i=1}^{n}X_i^{\hat{\beta}}}\sum_{i=1}^{n}X_i^{\hat{\beta}}\ln X_i=0\,.\end{cases} \tag{20}$$

The proposed test statistic is complicated to provide the possibility of extracting its exact distribution under the null hypothesis analytically. Thus, we use the Monte Carlo simulation to obtain the percentiles of the null distribution. To estimate the percentile of null distribution, we generate 50,000 random samples with various sample sizes from the Weibull distribution. The critical values of the proposed test statistic are presented in Table (11) for sample sizes $5\leq n\leq 50$.

It is worth mentioning that sample size n also depends on the population distribution, which is unknown in practice. In this study, we have chosen the value of $m=[n/3\ +1]$ for our test where $[x]$ represents integer part of x.

In studying test power, we utilized distributions with different hazard rates as a replacement hypothesis. These distributions are presented in Table (12). These distributions include increasing hazard rate (IHR), decreasing hazard rate (DHR), bathtub-shaped hazard rate (BT), and upside-down bathtub-shaped hazard rate (UBT). We calculate the test power against well-known distributions such as gamma (G), log-normal (LN), inverse gamma (IG), inverse Gaussian (IS), as well as the following distributions through simulation (see Krit et al. [20]).

Table 11. Critical values of the DS statistic in testing for Weibull distribution at $\alpha=0.05$.

	m									
n	1	2	3	4	5	6	7	8	9	10
5	1.0041	0.6614								
6	0.9384	0.6220	0.6099							
7	0.8718	0.6064	0.5530							
8	0.8174	0.5766	0.5073	0.5430						
9	0.7765	0.5487	0.4786	0.4985						
10	0.7418	0.5212	0.4562	0.4605	0.5037					
15	0.6192	0.4318	0.3794	0.3654	0.3688	0.3926	0.4302			
20	0.5534	0.3721	0.3252	0.3127	0.3133	0.3197	0.3345	0.3601	0.3908	0.4194
25	0.5154	0.3383	0.2891	0.2748	0.2749	0.2779	0.2886	0.2997	0.3178	0.3403
30	0.4842	0.3104	0.2645	0.2477	0.2454	0.2492	0.2550	0.2651	0.2753	0.2892
40	0.4483	0.2776	0.2282	0.2099	0.2055	0.2066	0.2114	0.2169	0.2246	0.2341
50	0.4228	0.2552	0.2055	0.1869	0.1802	0.1788	0.1813	0.1859	0.19158	0.1990

- Exponentiated Weibull distribution $EW(\theta,\eta,\beta)$ (Mudholkar and Srivastava [21]).

$$F(x) = \left[1 - e^{-(x/\eta)^{\beta}}\right]^{\theta}, \quad \theta,\eta,\beta > 0. \tag{21}$$

- Generalized Gamma distribution $GG(K,\eta,\beta)$ (Stacy [23]).

$$F(x) = \frac{1}{\Gamma(k)}\gamma\left(\kappa,(x/\eta)^{\beta}\right), \quad \kappa,\eta,\beta > 0, \tag{22}$$

where $\gamma(s,x) = \int_0^x v^{s-1}e^{-v}dv$.

- Distribution I of Dhillon [9] $D1(\lambda,b)$ with cdf:

$$F(x) = 1 - e^{-\left[e^{(\beta x)^{b}-1}\right]}, \quad b,\beta > 0. \tag{23}$$

- Distribution II of Dhillon [9] $D2(\lambda,b)$ with cdf:

$$F(x) = 1 - e^{-(\ln(\lambda x+1))^{b+1}}, \quad \lambda > 0, b \geq 0. \tag{24}$$

- Hjorth [18] distribution $H(\beta,\delta,\theta)$ with cdf:

$$F(x) = 1 - \frac{e^{-\frac{\delta x^2}{2}}}{(1+\beta x)^{\theta/\beta}}, \quad \beta,\delta,\theta > 0. \tag{25}$$

- Chen [5] distribution $C(\lambda,\beta)$ with cdf:

$$F(x) = \left[1 - e^{\lambda\left(1-e^{x^{\beta}}\right)}\right], \quad \lambda,\beta > 0. \tag{26}$$

Table 12. Alternative distributions.

IHR	$g(2) \equiv g(2,1)$ $D2(2) = D2(1,2)$	$g(3) \equiv g(3,1)$	$EW1 \equiv EW(6.5,20,6)$
UBT	$LN(0.8) \equiv LN(0,0.8)$ $IS(0.25) \equiv IS(1,0.25)$	$Ig(3) \equiv Ig(3,1)$ $IS(4) \equiv IS(1,4)$	$EW4 \equiv EW(4,12,0.6)$
DHR	$g(0.2) \equiv g(0.2,1)$ $D2(0) = D2(1,0)$	$EW2 \equiv EW(0.1,0.01,0.95)$	$H(0) \equiv H(0,1,1)$
BT	$EW3 \equiv EW(0.1,100,5)$ $C(0,4) \equiv C(2,0.4)$	$gg1 \equiv gg(0.1,1,4)$ $D1(0.8) = D1(1,0.8)$	$gg2 \equiv gg(0.2,1,3)$

Practically, it is useful for researcher to have a general recommendation for choosing the parameter m when the parameter n is fixed. Our simulations indicate that the ideal value of m (based on power) varies depending on the sample size and the alternative hypothesis. However, there is no single value of m that can be considered optimal in all scenarios. Hence, if one aims to protect against all potential alternative scenarios, a compromise needs to be reached.

The power values of the tests against different alternatives are computed using Monte Carlo simulations. For each alternative 50,000 samples are generated with size 10, 20, 30 and 50 with $m = [n/3 + 1]$. Typically, as n increases, the optimal value of m also increases, as the ratio m/n tends to zero.

We use Monte Carlo simulation to calculate the power of the proposed test. In our simulation study, we generate 50,000 sample with different sizes $n = 10,\ 20,\ 30$ and 50 under the alternative hypothesis and calculate the value of the test statistic. Then, by calculating the ration of the number of times the null hypothesis is rejected on the total number of times, the power of the test statistic is estimated. The power values of the test against various alternative

Table 13. Power comparisons of Weibull tests for $n = 10$ at level $\alpha = 0.05$.

$altern.$	W^2	U^2	D	V	A^2	DS
			Increasing Hazard Rate			
$g(2)$	0.0556	0.056	0.053	0.0563	0.0502	**0.0721**
$g(3)$	0.0609	0.0623	0.0573	0.0609	0.0544	**0.0887**
$EW1$	0.0816	0.0819	0.0705	0.0781	0.0714	**0.1297**
$D2(2)$	0.0649	0.0656	0.0584	0.0635	0.0596	**0.0907**
			Upside-down bathtub Hazard			
$LN(0.8)$	0.1113	0.1115	0.0948	0.1028	0.1007	**0.1898**
$Ig(3)$	0.2199	0.2171	0.173	0.199	0.2075	**0.3554**
$EW4$	0.0568	0.0537	0.0558	0.0527	**0.0622**	0.0397
$IS(0.25)$	0.184	0.1821	0.1469	0.1668	0.1696	**0.3211**
$IS(4)$	0.1105	0.111	0.0936	0.1024	0.0995	**0.1903**
			Decreasing Hazard Rate			
$g(0.2)$	0.098	0.0872	0.0918	0.077	**0.1195**	0.0486
$EW2$	0.0516	0.0506	0.0510	0.0506	**0.0520**	0.0488
$H(0)$	**0.0509**	0.0502	0.0498	0.0508	0.0503	0.0503
$D2(0)$	0.1484	0.1477	0.1224	0.1354	0.1384	**0.2248**
			Bathtub Hazard Rate			
$EW3$	0.0814	0.0823	0.0718	0.0781	0.0724	**0.1321**
$gg1$	0.1307	0.1173	0.1179	0.1016	**0.1605**	0.0685
$gg2$	0.1006	0.0889	0.0924	0.0795	**0.1219**	0.0498
$C(0.4)$	0.0592	0.0556	0.0574	0.0538	**0.0657**	0.0372
$D1(0.8)$	0.0692	0.0640	0.0665	0.0604	**0.0813**	0.0385

Table 14. Power comparisons of Weibull tests for $n = 20$ at level $\alpha = 0.05$.

$altern.$	W^2	U^2	D	V	A^2	DS
			Increasing Hazard Rate			
$g(2)$	0.062	0.0633	0.0575	0.0602	0.0593	**0.0989**
$g(3)$	0.0738	0.0746	0.0653	0.0692	0.0713	**0.1314**
$EW1$	0.1282	0.1268	0.1042	0.1138	0.1309	**0.2426**
$D2(2)$	0.0901	0.091	0.0767	0.085	0.0912	**0.1428**
			Upside-down bathtub Hazard			
$LN(0.8)$	0.2056	0.2002	0.1572	0.1756	0.215	**0.3806**
$Ig(3)$	0.458	0.4472	0.3479	0.4096	0.4844	**0.6979**
$EW4$	0.0624	0.0581	0.0613	0.055	**0.0711**	0.0347
$IS(0.25)$	0.3906	0.3764	0.2862	0.3422	0.414	**0.6832**
$IS(4)$	0.2082	0.2026	0.1608	0.178	0.2185	**0.4017**
			Decreasing Hazard Rate			
$g(0.2)$	0.1635	0.1419	0.1425	0.1155	**0.2010**	0.0826
$EW2$	0.0501	0.05	0.0491	0.0495	**0.0506**	0.0482
$H(0)$	0.049	0.0494	0.0483	0.0486	0.0491	0.0494
$D2(0)$	0.2938	0.2917	0.2297	0.2602	0.3069	**0.4283**
			Bathtub Hazard Rate			
$EW3$	0.1282	0.1269	0.1048	0.1129	0.1308	**0.2451**
$gg1$	0.238	0.2081	0.1979	0.166	**0.2972**	0.1519
$gg2$	0.1619	0.1406	0.1408	0.1145	**0.1997**	0.0824
$C(0.4)$	0.0662	0.0619	0.0632	0.0573	**0.0763**	0.0331
$D1(0.8)$	0.0893	0.0792	0.0829	0.0704	**0.1077**	0.0400

Table 15. Power comparisons of Weibull tests for $n = 30$ at level $\alpha = 0.05$.

$altern.$	W^2	U^2	D	V	A^2	DS
			Increasing Hazard Rate			
$g(2)$	0.0708	0.0701	0.0631	0.0653	0.0699	**0.1255**
$g(3)$	0.0928	0.091	0.0793	0.0816	0.0934	**0.1820**
$EW1$	0.1829	0.1756	0.1433	0.1522	0.1964	**0.3630**
$D2(2)$	0.1142	0.1138	0.0918	0.1026	0.121	**0.1978**
			Upside-down bathtub Hazard			
$LN(0.8)$	0.3082	0.2939	0.2306	0.2574	0.3374	**0.5599**
$Ig(3)$	0.6584	0.6388	0.5208	0.6005	0.7016	**0.8784**
$EW4$	0.0699	0.0635	0.0665	0.0581	**0.0804**	0.0273
$IS(0.25)$	0.5857	0.5585	0.4347	0.5226	0.6402	**0.8700**
$IS(4)$	0.3159	0.2993	0.2345	0.2622	0.3486	**0.5872**
			Decreasing Hazard Rate			
$g(0.2)$	0.2393	0.2024	0.1967	0.159	**0.2920**	0.1017
$EW2$	0.0511	0.0502	0.0511	0.0499	**0.0516**	0.0473
$H(0)$	0.0516	0.0509	0.0505	0.0498	**0.0521**	0.0492
$D2(0)$	0.4368	0.4282	0.3387	0.3845	0.4635	**0.6033**
			Bathtub Hazard Rate			
$EW3$	0.1826	0.1752	0.1422	0.1524	0.1946	**0.3602**
$gg1$	0.3607	0.3114	0.289	0.246	**0.4400**	0.2118
$gg2$	0.2386	0.2016	0.1975	0.1587	**0.2917**	0.1023
$C(0.4)$	0.0803	0.0726	0.0748	0.0659	**0.0918**	0.0254
$D1(0.8)$	0.1167	0.1012	0.106	0.0859	**0.1395**	0.0355

Table 16. Power comparisons of Weibull tests for $n = 50$ at level $\alpha = 0.05$.

$altern.$	W^2	U^2	D	V	A^2	DS
			Increasing Hazard Rate			
$g(2)$	0.0834	0.0818	0.0738	0.0757	0.0869	**0.1637**
$g(3)$	0.1218	0.1172	0.1011	0.1031	0.132	**0.2516**
$EW1$	0.2838	0.2672	0.2124	0.2328	0.3226	**0.5223**
$D2(2)$	0.1598	0.1582	0.1227	0.1412	0.1767	**0.2725**
			Upside-down bathtub Hazard			
$LN(0.8)$	0.4947	0.4694	0.3655	0.4216	0.5572	**0.7746**
$Ig(3)$	0.8802	0.8636	0.7635	0.8409	0.9184	**0.9816**
$EW4$	0.0837	0.0745	0.0789	0.0674	**0.0976**	0.0236
$IS(0.25)$	0.8385	0.8101	0.6816	0.7913	0.8942	**0.9843**
$IS(4)$	0.502	0.4717	0.3709	0.4245	0.5720	**0.8080**
			Decreasing Hazard Rate			
$g(0.2)$	0.3821	0.3256	0.3042	0.2562	**0.4611**	0.1855
$EW2$	0.0506	0.05	0.0501	0.05	**0.0513**	0.0446
$H(0)$	0.05	0.0498	0.0495	**0.0506**	0.0502	0.0503
$D2(0)$	0.649	0.6378	0.5275	0.5906	0.6873	**0.7890**
			Bathtub Hazard Rate			
$EW3$	0.2819	0.2665	0.2109	0.2314	0.3203	**0.5215**
$gg1$	0.5786	0.5133	0.4629	0.4176	**0.6810**	0.4252
$gg2$	0.3812	0.3268	0.3071	0.2572	**0.4603**	0.1875
$C(0.4)$	0.1015	0.0909	0.0922	0.08	**0.1206**	0.0267
$D1(0.8)$	0.1654	0.1402	0.1416	0.1163	**0.2036**	0.0486

distributions at the significance level $\alpha = 0.05$ are presented in Tables (13-16).

Tables (13-16) indicate that the proposed test statistic DS is consistently more powerful than the other tests against alternatives with IHR and UBT hazard rates. However, in the DHR and BT groups of hazard rate, the tests A^2 and have the maximum power. Generally, it is evident that the proposed test is powerful for small values of n.

4. Application to real data

This section presents some real data examples of GOF tests applied to social science and industrial data.

Example1: We consider the weights (unit is lb) of a random sample of 40 adult men as given below (Ahsanullah et al. [1], p. 48).
169.1, 144.2, 179.3, 175.8, 152.6, 166.8, 135.0, 201.5, 175.2, 139.0, 156.3, 186.6, 191.1, 151.3, 209.4, 237.1, 176.7, 220.6, 166.1, 137.4, 164.2, 162.4, 151.8, 144.1, 204.6, 193.8, 172.9, 161.9, 174.8, 169.8, 213.3, 198.0, 173.3, 214.5, 137.1, 119.5, 189.1, 164.7, 170.1, 151.0.
The weights of 40 adult men is discussed by Triola [26]. Ahsanullah et al. [1] used Anderson-Darling (A^2) test, Kolmogorov-Smirnov (D) test, Chi squared test and Shapiro-Wilk test for normality and they concluded that these tests do not reject H_0.

Example2: The second data set is the waiting time of 100 bank customers which is used by Ghitany et al. [14]. The date set are as follows.
0.8, 0.8, 1.3, 1.5, 1.8, 1.9, 1.9, 2.1, 2.6, 2.7, 2.9, 3.1, 3.2, 3.3, 3.5, 3.6, 4.0, 4.1, 4.2, 4.2, 4.3, 4.3, 4.4, 4.4, 4.6, 4.7, 4.7, 4.8, 4.9, 4.9, 5.0, 5.3, 5.5, 5.7, 5.7, 6.1, 6.2, 6.2, 6.2, 6.3, 6.7, 6.9, 7.1, 7.1, 7.1, 7.1, 7.4, 7.6, 7.7, 8.0, 8.2, 8.6, 8.6, 8.6, 8.8, 8.8, 8.9, 8.9, 9.5, 9.6, 9.7, 9.8, 10.7, 10.9, 11.0, 11.0, 11.1, 11.2, 11.2, 11.5, 11.9, 12.4, 12.5, 12.9, 13.0, 13.1, 13.3, 13.6, 13.7, 13.9, 14.1, 15.4, 15.4, 17.3, 17.3, 18.1, 18.2, 18.4, 18.9, 19.0, 19.9, 20.6, 21.3, 21.4, 21.9, 23, 27, 31.6 ,33.1, 38.5.

Example3: The third case revolves around reliability analysis, specifically examining the failure times of electronic devices (referenced as Wang [29]). Afterwards, Krit et al. [20] utilized the aforementioned data sets. The number of data is n = 18 and the data sets are:
5, 11, 21, 31, 46, 75, 98, 122, 145, 165, 195, 224, 245, 293, 321, 330, 350, 420. The proposed GOF test is applied to

Table 17. Results of the tests in Examples 1, 2 and 3.

	Null hypothesis	Value of the test statistic	Critical Value	Decision
Example 1	$H_0 : f(x) = N(\mu, \sigma)$	0.2587	0.2935	Not reject H_0
Example 2	$H_0 : f(x) = W(\eta, \beta)$	0.3047	0.3104	Not reject H_0
Example 3	$H_0 : f(x) = W(\eta, \beta)$	0.3405	0.4291	Not reject H_0

evaluate these data examples. The values of each test statistics are computed and then compared with corresponding critical value at significance level 0.05. The findings are presented in Table (17). The results from Table (17) indicate that the values of the test statistics are smaller than the corresponding critical value. Consequently, the null hypothesis is not rejected at the significance level of 0.05. Therefore, it can be inferred that the probability distribution of the data follows a normal distribution in Example (1), and a Weibull distribution in examples (3) and (4).
Figure (1) illustrates the empirical distribution function of the considered data set, which includes the weights of adult men, waiting time of bank customers and failure times of electronic devices.

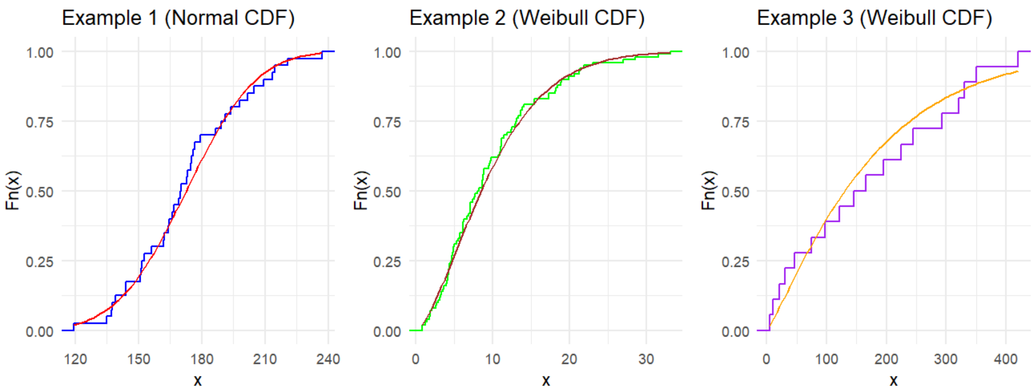

Figure 1. Empirical cumulative distribution and Theoretical Fit of data in Examples 1, 2 and 3.

5. Conclusions

In this paper, our primary focus was on introducing a new estimator of Kullback-Leibler information, which we subsequently utilized for conducting goodness-of-fit tests. We employed this new estimator to propose tests specifically tailored for assessing the fit of data to the normal, exponential, and Weibull distributions. To ensure the reliability and accuracy of our proposed tests, we utilized a Monte Carlo simulation approach. Through this simulation, we obtained percentile points of the proposed test. Furthermore, we conducted a comprehensive simulation study to evaluate the power and effectiveness of our proposed tests. We compared their performance with that of other competing tests, enabling us to gauge their relative strengths and weaknesses. To demonstrate the practical applicability of the proposed test, we provided three real data examples. These examples served as illustrations of how the test can be implemented and offered insights into their performance when applied to real-world data. By combining theoretical developments, simulation studies, and real data analysis, we aimed to provide a comprehensive evaluation of the proposed goodness-of-fit test based on the new estimator of Kullback-Leibler information.

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