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Estimating Stress-Strength Reliability in the Beta-Pareto Distribution Using Ranked Set Sampling

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Abstract This paper introduces a novel approach for estimating the stress-strength reliability in the beta-pareto (BP) distribution by employing ranked set sampling (RSS). Stress-strength reliability (SSR) is a crucial measure that quantifies the probability of an item or system operating without failure under random stress and strength conditions. The study focuses on estimating the reliability function ($R(t)$) and the probability (P) of stress being lower than strength when both stress and strength variables follow independent random variables from the BP distribution. The maximum likelihood ML estimator of $R(t)$ and P is obtained, and its performance is compared with the estimator based on simple random sampling (SRS). The proposed methodology is evaluated using real data from the Wheaton River experiment, showcasing its practical applicability and effectiveness. The findings highlight the superiority of our approach in accurately estimating stress-strength reliability in the BP distribution, providing valuable insights for various fields such as engineering, finance, and risk analysis.

Keywords Beta-Pareto distribution; Maximum likelihood estimator; Ranked set sampling; Stress-strength reliability

AMS 2010 subject classifications 62N05, 90B25, 60E05

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1. Introduction

The family of the Pareto distribution is well known in the literature for its capability in modelling the heavy-tailed distributions that are mostly common in data on income distribution, city population size, and size of firms. Various forms of the Pareto distribution and its generalization exist in the literature. The name generalized Pareto distribution (GPD) was first used by Pickands [31] when making statistical inferences about the upper tail of a distribution function. The GPD is found useful in modelling extreme value data because of its long tail feature. The distribution is often called the 'peaks over thresholds'model since it is used to model exceedances over threshold level in flood control. The Pareto distribution is a special case of the GPD. The Pareto distribution is also obtained as a special case of another generalized Pareto distribution, which is generated by compounding a heavy-tailed skewed conditional gamma density function with parameters α and β^{-1}, where the weighting function for β has a gamma distribution with parameters k and θ [32]. The beta-Pareto distribution is a compound distribution that combines the beta distribution and the Pareto distribution. It was introduced as a flexible model for heavy-tailed data with additional shape parameters to capture various data characteristics. Among the important features of the beta-pareto distribution, three items can be mentioned: A) Flexibility: The beta-Pareto(BP) distribution (PD) is very flexible due to its fore parameters $(\alpha, \beta, \theta, k)$, allowing it to model a wide range of data shapes and tail behaviors. B) Heavy tail behavior: It can capture heavy tail phenomena, which is critical in many real-world

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applications, including reliability analysis. C) Special cases: when $\alpha = \beta = 1$, it reduces to the Pareto distribution white parameters (θ, k).

Let $F(x)$ denote the cumulative distribution function (CDF) of a random variable X. The CDF for a generalized class of distribution for the random variable X, as defined by [10], is generated by applying the inverse CDF to a beta distributed random variable to obtain

$$F(x) = \frac{1}{B(\alpha, \beta)} \int_0^{G(x)} t^{\alpha-1}(1-t)^{\beta-1} dt \qquad \alpha > 0,\ \beta > 0,$$

The corresponding probability density function (PDF) for $F(x)$ is given by

$$f(x) = \frac{1}{B(\alpha, \beta)} [G(X)]^{\alpha-1} [1 - G(X)]^{\beta-1} \acute{G}(X) \tag{1}$$

In the present study, we let $G(x)$ be the CDF of the Pareto random variable with density function $g(x) = \frac{k\theta^k}{x^{k+1}}$ and CDF $G(x) = 1 - (\frac{x}{\theta})^{-k}$ for $x \geq \theta$. From Equation (1), the probability density function for the BP distribution random variable is given by

$$f(x) = \frac{k}{\theta B(\alpha, \beta)} \left[1 - \left(\frac{x}{\theta}\right)^{-k}\right]^{\alpha-1} \left(\frac{x}{\theta}\right)^{-k\beta-1}, \quad \alpha, \beta, \theta, k > 0,\ x \geq \theta, \tag{2}$$

let $F^*(X) = 1 - F(X)$, then, $F^*(X)$ for the BPD with density function given in (1) is

$$F^*(x) = \int_x^{\infty} f(t) dt = \int_x^{\infty} \frac{k}{\theta B(\alpha, \beta)} \left[1 - \left(\frac{t}{\theta}\right)^{-k}\right]^{\alpha-1} \left(\frac{t}{\theta}\right)^{-k\beta-1} dt$$

By setting $y = (\frac{t}{\theta})^{-k}$, the above integration becomes

$$\begin{aligned} F^*(x) &= \int_0^{z} \frac{1}{B(\alpha, \beta)} y^{\beta-1} (1-y)^{\alpha-1} dy \\ &= \frac{B(z; \beta, \alpha)}{B(\alpha, \beta)} \qquad 0 < z < 1 \end{aligned} \tag{3}$$

where $B(z; \beta, \alpha)$ is an incomplete beta function with $z = (\frac{x}{\theta})^{-k}$, and $x \sim PD$ white parameters (θ, k). Hence,

$$B(\alpha, \beta) F(x) = B(\alpha, \beta)[1 - F^*(x)] = B(\alpha, \beta) - B(z; \beta, \alpha) \tag{4}$$

$$= B(\alpha, \beta) - \frac{z^{\beta}(1-z)^{\alpha}}{\beta} {}_2F_1(\alpha + \beta, 1; \beta + 1; z). \tag{5}$$

by [19] and the ${}_2F_1$ is a generalised hypergeometric function, using the infinite series expansion for incomplete beta function.

Stress-strength reliability estimation is a fundamental concept in reliability engineering and is a fundamental concept in reliability engineering, involving the determination of the probability that a component's strength exceeds the stress applied to it. It involves determining the probability that a component's strength exceeds the stress applied to it. The beta-Pareto distribution is particularly useful in this context for several reasons: A) Modeling material properties: The strength of materials often exhibits heavy-tailed behavior, which the beta-Pareto distribution can effectively capture. B) Stress modeling: Environmental stresses or loads can also follow heavy-tailed distributions, making the beta-Pareto suitable for modeling both stress and strength. C) Flexibility in reliability analysis: The additional shape parameters of the beta-Pareto distribution allow for more accurate modeling of complex stress-strength scenarios. The stress-strength reliability estimation, which calculates the probability of Y being less than X $(P = P(Y < X))$, has become a popular problem in statistical literature. Y represents stress and X represents strength, making P a measure of system reliability since it's the probability of Y being less than X. If stress exceeds strength, then the system fails; otherwise, it continues to operate. This concept was first proposed by [6], and since then, numerous studies have been conducted on estimating P under different distributions of stress Y and strength X, including Weibull [16], Frechet [2] and Pareto [12] distributions. Traditional methods for stress-strength reliability estimation often rely on simple random sampling (SRS) and make assumptions about the independence of stress and strength variables. However, these approaches have several limitations: A) Dependency between stress and strength: In many real-world scenarios, stress and strength variables are not independent. This dependency can lead to biased estimates when using traditional methods that assume independence. B) Efficiency: Simple random sampling may not always provide the most efficient estimates, especially when dealing with complex distributions like the beta-Pareto. C) Sample size requirements: Accurate estimation using traditional methods often requires larger sample sizes, which can be costly or impractical in some situations. To address these limitations, researchers have explored alternative sampling methods, such as ranked set sampling (RSS) and its variations. The motivation for using RSS in stress-strength reliability estimation includes: A) Improved efficiency: RSS has been shown to provide more efficient estimates compared to SRS, often requiring smaller sample sizes to achieve the same level of precision. B) Flexibility: RSS and its variations (e.g., median ranked set sampling, extreme ranked set sampling) can be adapted to different distribution types and estimation scenarios. C) Handling complex distributions: RSS methods have demonstrated effectiveness in estimating parameters and reliability for various distributions, including the exponentiated Pareto distribution. D) Practical applicability: In some situations, it may be easier or more cost-effective to rank a small set of units rather than obtain precise measurements for a large sample. Recent research has focused on applying RSS and its variations to stress-strength reliability estimation for various distributions. Al-Omari [26] investigated the use of RSS and median RSS for estimating stress-strength reliability in the exponentiated Pareto distribution. Their results showed that RSS-based estimators were more efficient than those based on SRS. Furthermore, researchers have explored the use of copula functions to model the dependency between stress and strength variables, addressing one of the key limitations of traditional methods [28]. This approach, combined with advanced sampling techniques like RSS, offers promising avenues for improving the accuracy and efficiency of stress-strength reliability estimation. In conclusion, the beta-Pareto distribution and ranked set sampling methods represent important advancements in stress-strength reliability estimation. By addressing the limitations of traditional approaches and offering greater flexibility and efficiency, these techniques are helping to improve the accuracy and applicability of reliability analysis across various engineering domains. The RSS procedure, initially suggested by [20], is an economical method that ranks experimental units with significantly less effort than direct measurement. [24] established the mathematical theory of RSS, while [8] demonstrated that RSS is more efficient than SRS, even if the ranking isn't perfect. Over time, various authors applied RSS in several scientific fields such as environment and ecology [5], [25] and [11] quality control [4], [21] and medicine [22], and recently, for making inferences about stress-strength reliability. For example [23], considered unbiased estimation of stress-strength reliability using RSS. In addition, and [1], [9], [18], [17], [3] and [14] used parametric or non-parametric methods to estimate stress-strength reliability. In [27] the author advances the field by using ranked set sampling techniques in the inverse Kumaraswamy distribution in multi-stress resistance reliability estimation.

The RSS consists of two stages. The first stage involves identifying and ranking units, and the second stage entails taking measurements from a fraction of the ranked elements. This approach yields more efficient estimators of population parameters of interest (e.g., mean, median, variance, and quantiles) compared to SRS of the same size. To obtain an RSS sample of size n, we draw a random sample of size n from the population and order them without measuring. Then, the smallest observation is measured, and the remaining are not measured. Next, we draw another sample of size n, order them, and measure only the second smallest observation. This procedure repeats until the largest observation of the n^{th} sample of size n is measured. We call this process a one-cycle ranked set sample of size n, and the data thus observed are denoted by $X_{RSS} = \{X_{(11)}, X_{(22)}, ..., X_{(nn)}\}$. The observational process is illustrated in the following figure:

$$\begin{array}{cccccc}
\underline{X_{1:1}} & X_{2:1} & \ldots & X_{n:1} & -> & X_{(11)} \\
X_{1:2} & \underline{X_{2:2}} & \ldots & X_{n:2} & -> & X_{(22)} \\
\ldots & \ldots & \ldots & \ldots & -> & \ldots \\
X_{1:n} & X_{2:n} & \ldots & \underline{X_{n:n}} & -> & X_{(nn)}
\end{array}$$

To obtain an RSS sample of size $m{=}nr$, the process is repeated r times. It's important to note that the set size m plays a critical role in the RSS procedure. We aim to choose m as large as possible to gather more information about the variable of interest. However, it's essential to consider imperfect ranking errors which increase as m increases. Imperfect ranking refers to errors made during the ranking process. Therefore, selecting the optimal value of m is crucial to minimize the effects of imperfect ranking.
The organization of this paper is as follows: Parameters estimation of the models are discussed in section 2. To compare the performance of the estimators, a simulation study has been conducted in section 2.2. Two real-life data are analyzed to illustrate the findings of the paper in section 2.4. Finally, some concluding remarks are presented in section 3.

2. Parameters Estimation

2.1. MLE *of the parameters with* SRS

Maximum Likelihood Estimation (MLE) is a method of estimating the parameters of a probability distribution by maximizing a likelihood function. This function expresses the likelihood of observing the given data as a function of the parameters. The core idea is to choose parameter values that make the observed data "most likely" to have occurred. As sample size increases, the estimate converges to the true parameter value and For large samples, the distribution of the MLE is approximately normal. In general, MLEs achieve the Kramer-Rao lower bound asymptotically, which makes them asymptotically efficient. MLE can be applied to a wide range of statistical models and distributions. It's particularly useful for complex distributions often encountered in reliability analysis, such as the inverted Kumaraswamy or beta-Pareto distributions.
The log-likelihood function of BPD may be expressed as,

$$\begin{aligned}\ln L(x;\alpha,\beta,\theta,k) &= n\ln k - n\ln\theta + n\left(\ln\Gamma(\alpha+\beta) - \ln\Gamma(\alpha) - \ln\Gamma(\beta)\right)\\ &\quad + (\alpha-1)\sum_{i=1}^{n}\ln\left[1-\left(\frac{x_i}{\theta}\right)^{-k}\right] - (k\beta+1)\sum_{i=1}^{n}\left(\frac{x_i}{\theta}\right). \end{aligned} \tag{6}$$

Differentiating Equation (6) with respect to k, α, and β, respectively, and setting the results equal to zero, we have

$$\frac{\partial\ln L(x)}{\partial k} = \frac{n}{k} - \sum_{i=1}^{n}\left\{\beta + (\alpha-1)\left[1-\left(\frac{x_i}{\theta}\right)^{k}\right]^{-1}\right\}\ln\left(\frac{x_i}{\theta}\right) = 0 \tag{7}$$

$$\frac{\partial\ln L(x)}{\partial\alpha} = n\left\{\Psi(\alpha+\beta) - \Psi(\alpha)\right\} + \sum_{i=1}^{n}\ln\left[1-\left(\frac{x_i}{\theta}\right)^{-k}\right] = 0 \tag{8}$$

$$\frac{\partial\ln L(x)}{\partial\beta} = n\left\{\Psi(\alpha+\beta) - \Psi(\beta)\right\} - k\sum_{i=1}^{n}\ln\left(\frac{x_i}{\theta}\right) = 0. \tag{9}$$

Since $x \geq \theta$, the maximum likelihood estimate of θ is the first-order statistic $x_{(1)}$. The maximum likelihood estimates $\hat{\alpha}$, $\hat{\beta}$, and $\hat{k}$ for the parameters α, β, and k, respectively, are obtained by solving alternatively Equations (7) - (9). The initial estimates of α, β, and k can be obtained as follows: fit the Pareto density to the data. The maximum likelihood of $\hat{\theta} = x_{(1)}$, the first-order statistic, and the maximum likelihood of k is $\widehat{k} = n\left[\sum\ln\left(\frac{x_i}{\widehat{\theta}}\right)\right]^{-1}$. By using $\hat{\theta}$ and $\hat{k}$, we transform the data to beta density data and then find the maximum likelihood estimates of α and β or the moment estimates of α and β from the beta density. The initial estimates for the BPD are the moment or maximum likelihood estimates of α and β, and the estimate $\hat{k}$. By using Equations (7) - (9), the second partial derivatives may be expressed as

$$\frac{\partial^2 \ln L(x)}{\partial k \partial \beta} = -\sum_{i=1}^{n} \ln\left(\frac{x_j}{\theta}\right),$$
$$\frac{\partial^2 \ln L(x)}{\partial k \partial \alpha} = \sum_{i=1}^{n} \left[\left(\frac{x_i}{\theta}\right)^k - 1\right]^{-1} \ln\left(\frac{x_i}{\theta}\right),$$
$$\frac{\partial^2 \ln L(x)}{\partial k^2} = -\frac{n}{k^2} - (\alpha - 1)\sum_{i=1}^{n} \left\{\frac{\ln\left(\frac{x_i}{\theta}\right)}{1-\left(\frac{x_i}{\theta}\right)^k}\right\}^2 \left(\frac{x_i}{\theta}\right)^k,$$
$$\frac{\partial^2 \ln L(x)}{\partial \alpha \partial \beta} = n\Psi''(\alpha+\beta),$$
$$\frac{\partial^2 \ln L(x)}{\partial \alpha^2} = n\left\{\Psi'(\alpha+\beta) - \Psi'(\alpha)\right\},$$
$$\frac{\partial^2 \ln L(x)}{\partial \beta^2} = n\left\{\Psi'(\alpha+\beta) - \Psi'(\beta)\right\}.$$

These second partial derivatives can be used to compute Fisher's information matrix. However, the expectations of these second partial derivatives cannot be expressed in a closed form. By following a similar procedure for the second sample Y_j, $(j = 1, ...z)$, we can obtain results that are comparable to those of the higher process. To obtain the expectations, a numerical method may be employed.

2.2. MLE *of the parameters with* RSS

RSS is an effective technique for acquiring data when measuring units in a population is costly, but ranking them according to the variable of interest is relatively easy. RSS-based estimators, including MLE, have been shown to outperform their Simple Random Sampling (SRS) counterparts significantly, providing more accurate parameter estimates. Studies have demonstrated that RSS-based estimators are more efficient than SRS-based methods, especially when using the same number of measured units. MLE with RSS has been successfully applied to estimate parameters of various complex distributions used in reliability studies and life testing, such as the Inverted Kumaraswamy distribution [33]. The effectiveness of RSS-based MLE has been demonstrated through applications to real-world datasets, such as waiting times between consecutive eruptions of natural phenomena [33]. RSS can be used not only with MLE but also with other estimation techniques like maximum product of spacings, least squares, and various goodness-of-fit based methods, allowing for comprehensive comparisons. These motivations highlight that MLE with RSS offers a powerful and flexible approach to parameter estimation, particularly valuable in scenarios where data collection is challenging or expensive, and when dealing with complex or bounded distributions common in reliability and life testing applications.

We can represent the i th ordered statistics from the ith set of size n_x in the sth cycle of size r_x as $X_{(i)is}$, where $(i = 1, ...n_x, s = 1, ..., r_x)$. For X with a $BP(\alpha, \beta, \theta, k)$ density, we simplify the notation by using X_{is} instead of $X_{(i)is}$.

Similarly, for Y with a $BP(c, d, \sigma, l)$ density, we denote the jth ordered statistic from the jth set of size n_y in the lth cycle of size r_y as $Y_{(j)jl}$, where $(j = 1, ...n_y, l = 1, ..., r_y)$. To simplify the notation, we use Y_{jl} instead of $Y_{(j)jl}$. It is worth mentioning that if the judgment ranking is perfect, we can express the PDF of the ith ordered statistic X_{is} using the following expression:

$$f_i(x_{is}) = \frac{1}{B(i, n_x - i + 1)}[F(x_{is})]^{i-1}[1 - F(x_{is})]^{n_x - i} f(x_{is}), \tag{10}$$

In addition, the PDF of Y_{jl} has a form similar to (10). First, we need to calculate the relationship (10) for the beta distribution with $BP(\alpha, \beta, \theta, k)$.

$$f_{i:n}(x) = \frac{n!}{i!(n-i)!}[F(x)]^{i-1}[1-F(x)]^{n-i}f(x)$$
$$= \frac{1}{B(i,n-i+1)}[F(x)]^{i-1}\sum_{j=0}^{n-i}(-1)^j\binom{n-i}{j}[F(x)]^j f(x)$$
$$= \frac{1}{B(i,n-i+1)}\sum_{j=0}^{n-i}(-1)^j\binom{n}{j}[F(x)]^{i+j-1}f(x).$$

Then we can write:

$$F_{i:n}(x) = \int f_{i:n}(x)dx$$
$$= \int \frac{1}{B(i,n-i+1)}\sum_{j=0}^{n-i}(-1)^j\binom{n-i}{j}[F(x)]^{i+j-1}f(x)dx$$
$$= \sum_{j=0}^{n-i}\frac{(-1)^j\binom{n-i}{j}}{B(i,n-i+1)}\int [F(x)]^{i+j-1}f(x)dx$$
$$= \sum_{j=0}^{n-i}\frac{(-1)^j\binom{n-i}{j}}{B(i,n-i+1)(i+j)}[F(x)]^{i+j}.$$

In this part, we use the relationships found in the [13] to calculate

$$(I) \quad F(x) = \textstyle\sum_{t=0}^{\infty} b_t G(x)^t$$
$$(II) \quad F(x)^n = \textstyle\sum_{t=0}^{\infty} d_{n,t} G(x)^t$$

where

$$b_t = \sum_{j=0}^{\infty}\sum_{l=t}^{\infty} p_j(-1)^{l+t}\binom{\alpha+j}{l}\binom{l}{t} \quad \& \quad p_j = \frac{(-1)^j\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta-j)\Gamma(j+1)(\alpha+j)}$$

and $d_{n,t};\ t = 1, 2, ...$ are easily determined from the recurrence equation

$$d_{n,t} = (tb_0)^{-1}\sum_{m=1}^{t}[m(n+1)-t]b_m d_{n,t-m}$$

and $d_{n,0} = b_0^n$. Hence, $d_{n,t}$ comes directly from $d_{n,0}, ..., d_{n,t-1}$ and, therefore, from $b_0, ...b_t$. With this description, proposition (II) is confirmed. And finally, according to the presented relations, we have

$$F_{i:n}(x) = \sum_{j=0}^{n-i}\frac{(-1)^j\binom{n-i}{j}}{B(i,n-i+1)(i+j)}[G(x)]^{i+j}$$
$$= \sum_{j=0}^{n-i}\frac{(-1)^j\binom{n-i}{j}}{B(i,n-i+1)(i+j)}\sum_{t=0}^{\infty} d_{i+j,t}G(x)^t$$

And we have,

$$F_{i:n}(x) = \sum_{t=0}^{\infty}\sum_{j=0}^{n-i} \frac{(-1)^j \binom{n-i}{j} d_{i+j,t}}{B(i, n-i+1)(i+j)} G(x)^t \tag{11}$$

Were $G(X)$ denot the CDF of the Pareto random variable white parameters (θ, k), And similarly for the density function we have:

$$\begin{aligned} f_{i:n}(x) &= \frac{\partial F_{i:n}}{\partial x} &(12)\\ &= \frac{\partial}{\partial x}\left[\sum_{j=0}^{n-i} \frac{(-1)^j \binom{n-i}{j}}{B(i, n-i+1)(i+j)} \sum_{t=0}^{\infty} d_{i+j,t} G(x)^t\right] &(13)\\ &= \sum_{t=0}^{\infty}\sum_{j=0}^{n-i} \frac{t(-1)^j \binom{n-i}{j} d_{i+j,t}}{B(i, n-i+1)(i+j)} G(x)^{t-1} g(x). &(14) \end{aligned}$$

And finally we have:

$$f_{i:n}(x) = \sum_{t=0}^{\infty}\sum_{j=0}^{n-i} \frac{t(-1)^j \binom{n-i}{j} d_{i+j,t}}{B(i, n-i+1)(i+j)} G(x)^{t-1} g(x). \tag{15}$$

Were $G(X)$ denot the CDF of the Pareto random variable, and $g(X)$ denot the PDF of the Pareto random variable white parameters (θ, k) . To obtain the ML estimator of R we first derive the ML estimators of parameters. Therefore, the likelihood function based on RSS is written as shown below

$$\begin{aligned} L(\alpha, \beta, \theta, k) &= \prod_{i=1}^{n_x}\prod_{s=1}^{r_x} f_i(x_{is}) &(16)\\ &= \prod_{i=1}^{n_x}\prod_{s=1}^{r_x}\sum_{l=0}^{\infty}\sum_{t=0}^{\infty}\sum_{j=0}^{m-i} \frac{kt(-1)^{j+l}\binom{t-1}{l}\binom{m-i}{j} d_{i+j,t}}{B(i, m-i+1)(i+j)} \theta^{k(l+1)} {x_{is}}^{-k(l+1)-1} \\ &= \left(\sum_{l=0}^{\infty}\sum_{t=0}^{\infty}\sum_{j=0}^{m-i} \frac{kt(-1)^{j+l}\binom{t-1}{l}\binom{m-i}{j} d_{i+j,t}}{B(i, m-i+1)(i+j)}\right)^{n_x r_x} \left(\sum_{l=0}^{\infty} \theta^{k(l+1)}\right)^{n_x r_x} \\ &\times \left(\prod_{i=1}^{n_x}\prod_{s=1}^{r_x}\sum_{l=0}^{\infty} {x_{is}}^{-k(l+1)-1}\right). &(17) \end{aligned}$$

Then, the log-likelihood function is

$$\begin{aligned} l = \ln(L) &= Q + n_x r_x \ln(k) + n_x r_x \ln(\sum_{t=0}^{\infty}\sum_{j=0}^{m-i} d_{i+j,t}) + n_x r_x \ln(\sum_{l=0}^{\infty} \theta^{k(l+1)}) \\ &+ \sum_{i=1}^{n_x}\sum_{s=1}^{r_x} \ln(\sum_{l=0}^{\infty} {x_{is}}^{-k(l+1)-1}) &(18) \end{aligned}$$

Where Q is a fixed number and indicates the sum of the logarithms of sentences without parameters. Since $x \geq \theta$, the MLE of θ is the first-order statistic $x_{(1)}$. Differentiating Equation (18) with respect to k, α, and β, respectively, and setting the results equal to zero, we have

$$\frac{\partial l}{\partial k} = \frac{n_x r_x}{k} + n_x r_x \sum_{l=0}^{\infty} \frac{\theta^{k(l+1)} \ln \theta}{\theta^{k(l+1)}} + \sum_{i=1}^{n_x} \sum_{s=1}^{r_x} \sum_{l=0}^{\infty} \frac{{x_{is}}^{-k(l+1)-1} \ln x_{is}}{{x_{is}}^{-k(l+1)-1}} \tag{19}$$

$$= \frac{n_x r_x}{k} + n_x r_x \ln \theta + \sum_{i=1}^{n_x} \sum_{s=1}^{r_x} \ln x_{is} = 0, \tag{20}$$

$$\frac{\partial l}{\partial \alpha} = n_x r_x \sum_{t=0}^{\infty} \sum_{j=0}^{m-i} \frac{\frac{\partial}{\partial \alpha} d_{i+j,t}}{d_{i+j,t}} = 0, \tag{21}$$

$$\frac{\partial l}{\partial \beta} = n_x r_x \sum_{t=0}^{\infty} \sum_{j=0}^{m-i} \frac{\frac{\partial}{\partial \beta} d_{i+j,t}}{d_{i+j,t}} = 0. \tag{22}$$

The MLEs $\hat{\alpha}$, $\hat{\beta}$, and $\hat{k}$ for the parameters α, β, and k, respectively, are obtained by solving alternatively Equations (19) - (22). By applying the same procedure for the second sample $Y_{(j)jl}$, $(j = 1, ...n_y, l = 1, ..., r_y)$, our results are similar to those of the higher process. The expectations are not in close form. we resort to iterative methods for the ML estimators.

2.3. MLE of $R = P(X > t)$.

Suppose that $X \sim BP(\alpha, \beta, \theta, k)$. That it can be shown that,

$$R = P(X > t) = F_B(\beta, \alpha; W).$$

Where $F_B(\alpha, \beta; W) = I_W(\alpha, \beta)$ is the distribution function of the beta distribution with $W = (\frac{t}{\theta})^{-k}$ and in other words $I_W(\alpha, \beta)$ is the incomplete beta function in point $W = (\frac{t}{\theta})^{-k}$. By setting $z = (\frac{x}{\theta})^{-k}$, $x \sim PD$ white parameters (θ, k) and $W = (\frac{t}{\theta})^{-k}$, it can be easily seen from (2)

$$R = P(X > t) = \int_t^{\infty} f_X(x)dx = \int_t^{\infty} \frac{k}{\theta B(\alpha, \beta)} \left[1 - \left(\frac{x}{\theta}\right)^{-k}\right]^{\alpha-1} \left(\frac{x}{\theta}\right)^{-k\beta-1} dx$$

$$= \int_0^W \frac{1}{B(\alpha, \beta)} [1 - z]^{\alpha-1} (z)^{\beta-1} dz = F_B(\beta, \alpha; W). \tag{23}$$

Now to compute the MLE of $R(t)$, we use the estimates of parameters that were calculated in the previous section. Therefore, according to the reliability property of MLE, we can write,

$$\hat{R(t)} = R(\hat{\alpha}, \hat{\beta}, \hat{\theta}, \hat{k}). \tag{24}$$

2.4. MLE of $P = P(Y < X)$.

The MLE approach is a statistical method used to estimate the parameters of a model. In the context of stress-strength reliability, MLE can be applied to estimate the parameters of the distributions for stress and strength. In summary, MLE is a powerful technique for estimating the parameters of the stress and strength distributions in stress-strength reliability (SSR) analysis. By maximizing the likelihood function based on collected data, you can derive reliable estimates of both stress-strength performance and overall reliability. This is crucial for engineering applications where safety and performance are dependent on the relationship between stress and strength. MLE provides estimates that have desirable properties when the sample size is large. Specifically, MLE estimates are asymptotically unbiased and efficient, achieving the lowest possible variance among unbiased estimators (Cramér-Rao lower bound). Using MLE for estimating Stress-Strength Reliability allows analysts to leverage a powerful

statistical tool that provides robust, flexible, and efficient estimates of reliability under realistic conditions. This makes MLE a preferred choice for practitioners facing the challenges of characterizing the interaction between stress and strength in engineering and reliability contexts. The motivation for using maximum likelihood estimation (MLE) for stress-strength reliability (SSR) estimation is multifaceted and rooted in various statistical advantages. A few of them were mentioned above.
Suppose that $X \sim BP(\alpha, \beta, \theta, k)$ and $Y \sim BP(c, d, \sigma, l)$ are independent. That it can be shown that,

$$P = P(Y < X) = E[P(Y < X|X)] = E[\int_{\sigma}^{X} f_Y(y)dy].$$

Let $P^* = 1 - P$ then P^* for the BPD with density function given in Equation (2) is,

$$\begin{aligned} P^* &= 1 - E[\int_{\sigma}^{X} f_Y(y)dy] = E[1 - \int_{\sigma}^{X} f_Y(y)dy] \\ &= E[\int_{X}^{\infty} f_Y(y)dy] = E[\int_{X}^{\infty} \frac{l}{\sigma B(c,d)}[1-(\frac{y}{\sigma})]^{c-1}(\frac{y}{\sigma})^{-ld-1}dy] \end{aligned}$$

By setting $z = (\frac{y}{\sigma})^{-l}$ where $W = (\frac{X}{\sigma})^{-l}$ and $X \sim BPD$ white parameters $(\alpha, \beta, \theta, k)$, therfor the above integration becomes,

$$\begin{aligned} P^* &= E[\int_{0}^{W} \frac{1}{B(c,d)}[1-z]^{c-1}(z)^{d-1}dz] \\ &= E\big[F_B(d,c;z)\big|_0^W\big] = E[F_B(d,c;W) - F_B(d,c;0)] = E[F_B(d,c;W)]. \end{aligned}$$

Where $F_B(d, c; W) = I_W(d, c)$ is distribution function of the beta distribution with $W = (\frac{X}{\sigma})^{-l}$ and $X \sim BPD$ white parameters $(\alpha, \beta, \theta, k)$.

$$P^* = \sum_{m}^{\infty} \frac{\binom{c-1}{m}(-1)^m}{B(c,d)(d+m)} E(W^{d+m}). \tag{25}$$

To calculate $E(W^{d+m})$ we act as follows:

$$\begin{aligned} E(W^{d+m}) &= \frac{k}{\theta B(\alpha,\beta)} \int_{\theta}^{\infty} W^{d+m} \left[1 - \left(\frac{x}{\theta}\right)^{-k}\right]^{\alpha-1} \left(\frac{x}{\theta}\right)^{-k\beta-1} \\ &= \frac{k}{\theta B(\alpha,\beta)} \int_{\theta}^{\infty} (\frac{x}{\sigma})^{-l(d+m)} \left[1 - \left(\frac{x}{\theta}\right)^{-k}\right]^{\alpha-1} \left(\frac{x}{\theta}\right)^{-k\beta-1} \end{aligned} \tag{26}$$

As for the moments of the beta-Pareto distribution white parameters $(\alpha, \beta, \theta, k)$, we simply have:

$$E(\frac{X}{\theta})^r = \theta^r \frac{B(\alpha, \beta - \frac{r}{k})}{B(\alpha,\beta)}$$

Using the above relation and (26) and performing mathematical operations, the following result is easily obtained,

$$P(\alpha, \beta, \theta, k, c, d, \sigma, l) = 1 - \frac{(\frac{\sigma}{\theta})^{l(d+m)}}{B(c,d)B(\alpha,\beta)} \sum_{m}^{\infty} \frac{\binom{c-1}{m}(-1)^m B(\alpha, \beta + \frac{l(d+m)}{k})}{(d+m)}. \tag{27}$$

If c is a positive integer number, then the upper limit of summation stops at $c - 1$. Now to compute the MLE of P, we use the estimates of parameters that were calculated in the previous section. Therefore, according to the reliability property of MLE, we can write,

$$\hat{P} = P(\hat{\alpha}, \hat{\beta}, \hat{\theta}, \hat{k}, \hat{c}, \hat{d}, \hat{\sigma}, \hat{l}). \tag{28}$$

It should be noted that apart from the limitation of the parameters in the relevant distributions for calculating P, there is no limitation in the selection of parameters.

3. Numerical Experiments and Discussions

A simulation study has been performed to compare the performance of $\hat{R(t)}$ for different sample sizes in this section. we generate 10,000 samples each of size n from the BP distribution and repeat this procedure for several values of $R(t)$. Then, for equal parameters and different n values, we have calculated the index Mean Squared Error (MSE) and presented it in the graph. Mean Squared Error (MSE) is a common measure used to assess the accuracy of an estimator or a predictive model. It quantifies the average squared difference between the observed values and the values predicted by the model. MSE is defined as:

$$MSE = \frac{1}{n}\sum_{i:1}^{n}(y_i - \hat{y}_i)^2.$$

In figure (1) the value of $R(t) = P(X < t)$ where $X \sim BP(5, 8, 0.8, 3)$ with $t = 6$ and in figure (2) the value of $R(t) = P(X < t)$ where $X \sim BP(5, 8, 0.8, 3)$ with $t = 8$ has been investigated. From these figures we note that the MSE of the MLE of $R(t)$ is always greater than when n is less than 20, however for large sample sizes (more than 20) this estimator of $R(t)$ is better and almost efficient. As the sample size increases, MSE becomes a more reliable estimate of a model's predictive performance due to the Law of Large Numbers. The influence of outliers diminishes, and the average converges to the expected value. Also, with the increase in the t, the value $R(t)$ increased and has an upward trend.

In Table (1) the estimation of $P = P(Y < X)$, when $X \sim BP(1, 1, 2, 2)$ and $Y \sim BP(1, 1, 2, 3)$ are independent random variables from BP distribution using RSS and SRS has been compared. We can see from Table(1) that the RSS produces smaller absolute biases and MSEs compared to SRS for sample size 50 or less. For large sample size (say, 100), their performance is not significantly different. In large samples, many statistical properties hold (e.g., normality due to the Central Limit Theorem), making MSE a robust metric for comparing model performance. Beyond a certain sample size, improvements in MSE may become marginal. Larger datasets can help refine the model, but the marginal gain might not justify the added complexity or cost of data collection.

The plots in Figure (3) show the process of change $Bias$ and MSE in P when $X \sim BP(1, 1, 2, 2)$ and $Y \sim BP(1, 1, 2, 3)$ are independent random variables from BP distribution. According to the graphs, it can be seen that in n less than 50, the performance of the RSS estimator is better than the SRS estimator. With increasing n, there is no significant difference between the performance of estimators in these two methods.

Table 1. Estimation of $P = P(Y < X)$

$SamplingMethod$	$Results$	$n = 9$	$n = 12$	$n = 15$	$n = 20$	$n = 50$	$n = 100$
$MLE(SRS)$	P	0.5808	0.5808	0.5808	0.5808	0.5808	0.5808
	$\hat{P}$	0.4235	0.4596	0.4790	0.5027	0.5439	0.5542
	$Bias(P)$	0.1572	0.1211	0.1018	0.0780	0.0368	0.0266
	$MSE(P)$	0.0420	0.0286	0.0249	0.0156	0.0069	0.0057
$MLE(RSS)$	P	0.5808	0.5808	0.5808	0.5808	0.5808	0.5808
	$\hat{P}$	0.4409	0.4697	0.4853	0.5115	0.5431	0.5481
	$Bias(P)$	0.1398	0.1110	0.0955	0.0692	0.0377	0.0327
	$MSE(P)$	0.0338	0.0244	0.0195	0.0122	0.0061	0.0075

In Table(2) the estimation of $P = P(Y < X)$, when $X \sim BP(3, 2.5, 2, 4)$ and $Y \sim BP(1, 5, 2, 1.5)$ are independent random variables from BP distribution using RSS and SRS has been compared. We can see from Table(2) that the RSS produces smaller biases and MSEs compared to SRS for sample size 20 or less. For large sample size (say, 50 or 100), their performance is not significantly different.

The plots in Figure (4) show the process of change $Bias$ and MSE in P when $X \sim BP(3, 2.5, 2, 4)$ and $Y \sim BP(1, 5, 2, 1.5)$ are independent random variables from BP distribution. According to the graphs, it can be

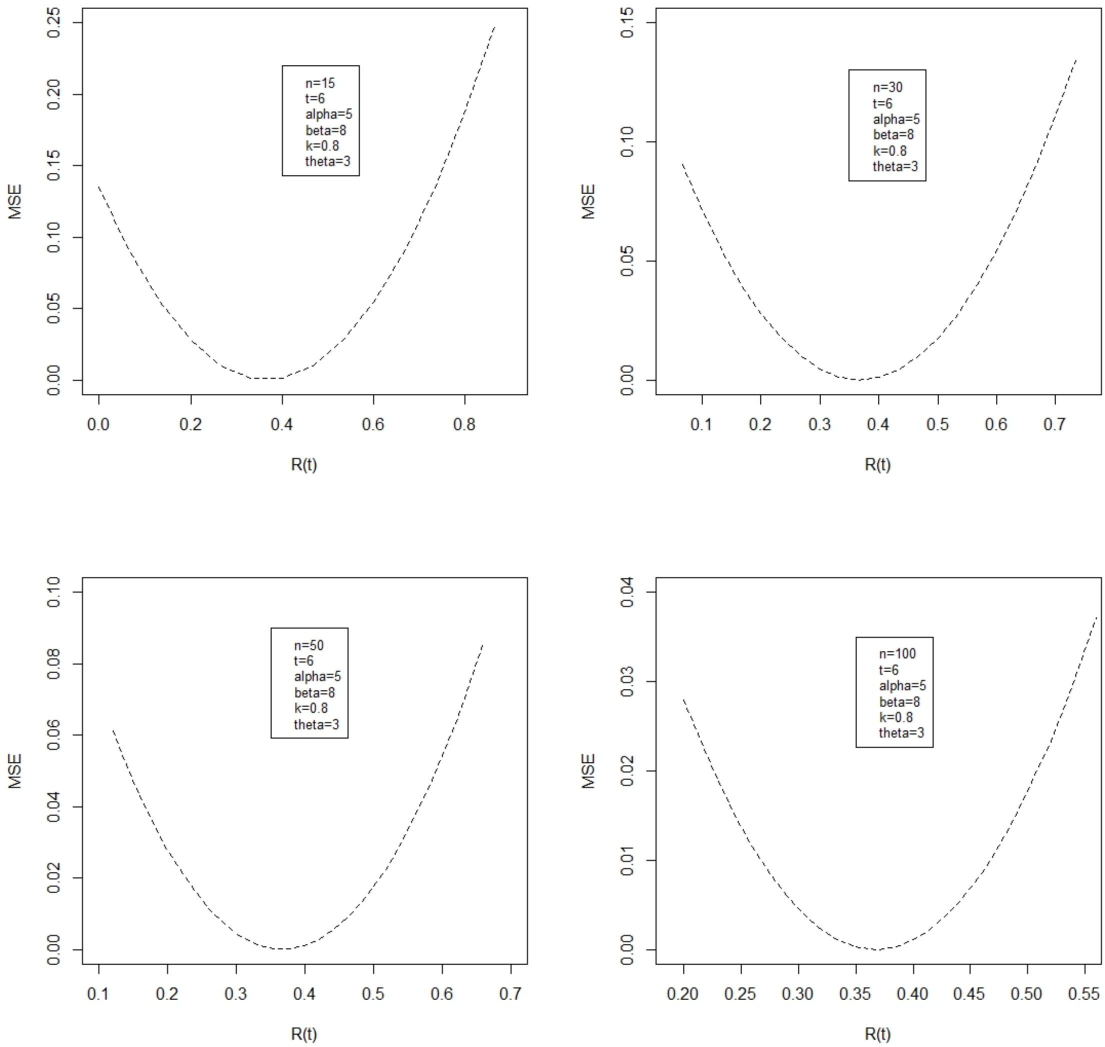

Figure 1. The performance of MSE of $R(t)$ by the MLE for different sample sizes n.

seen that in n less than 40, the performance of the RSS estimator is better than the SRS estimator. With increasing n, there is no significant difference between the performance of estimators in these two methods.

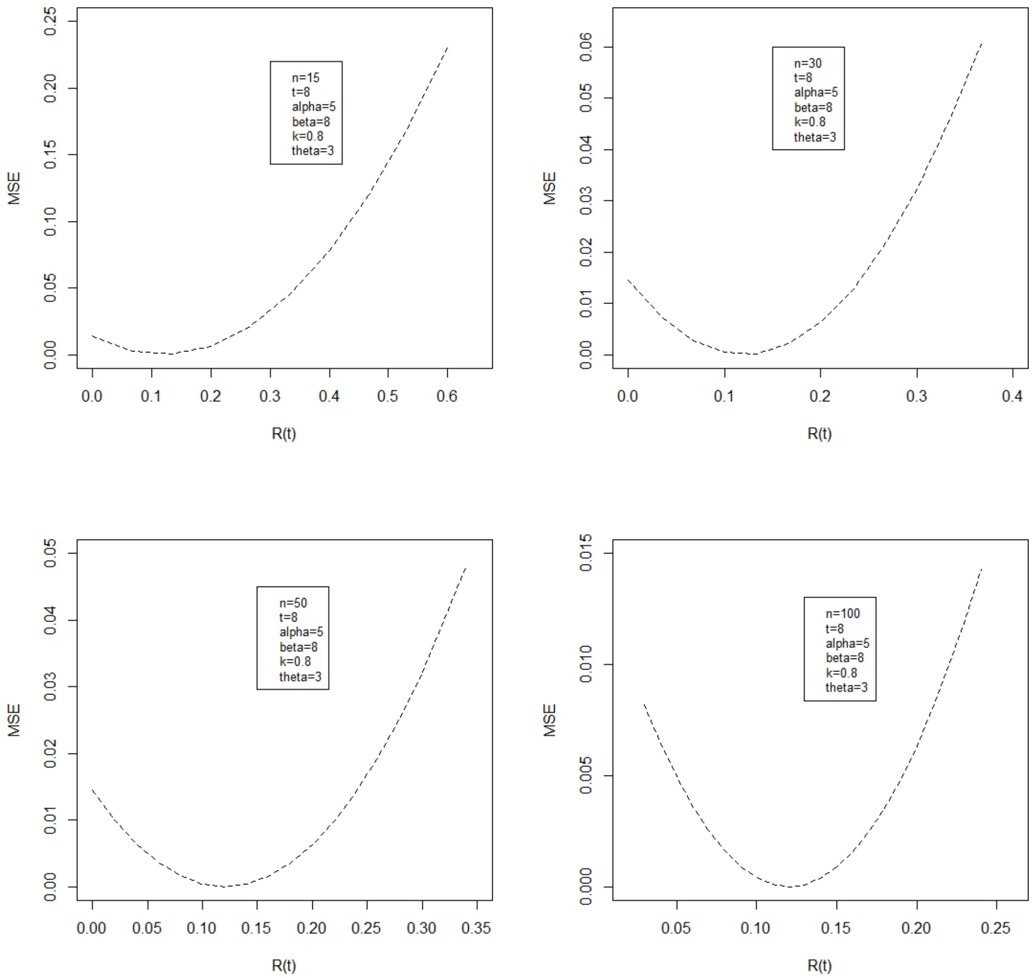

Figure 2. The performance of MSE of $R(t)$ by the MLE for different sample sizes n.

Table 2. Estimation of $P = P(Y < X)$

$SamplingMethod$	$Results$	$n = 9$	$n = 12$	$n = 15$	$n = 20$	$n = 50$	$n = 100$
$MLE(SRS)$	P	0.7006	0.7006	0.7006	0.7006	0.7006	0.7006
	$\hat{P}$	0.2923	0.3174	0.5182	0.3718	0.4703	0.5304
	$Bias(P)$	0.4082	0.3831	0.3542	0.3287	0.2223	0.1702
	$MSE(P)$	0.1822	0.1645	0.1451	0.1267	0.0695	0.0471
$MLE(RSS)$	P	0.7006	0.7006	0.7006	0.7006	0.7006	0.7006
	$\hat{P}$	0.2955	0.3252	0.3374	0.3811	0.4703	0.5304
	$Bias(P)$	0.4050	0.3753	0.3632	0.3195	0.2303	0.1701
	$MSE(P)$	0.1803	0.1586	0.1449	0.1205	0.0715	0.0477

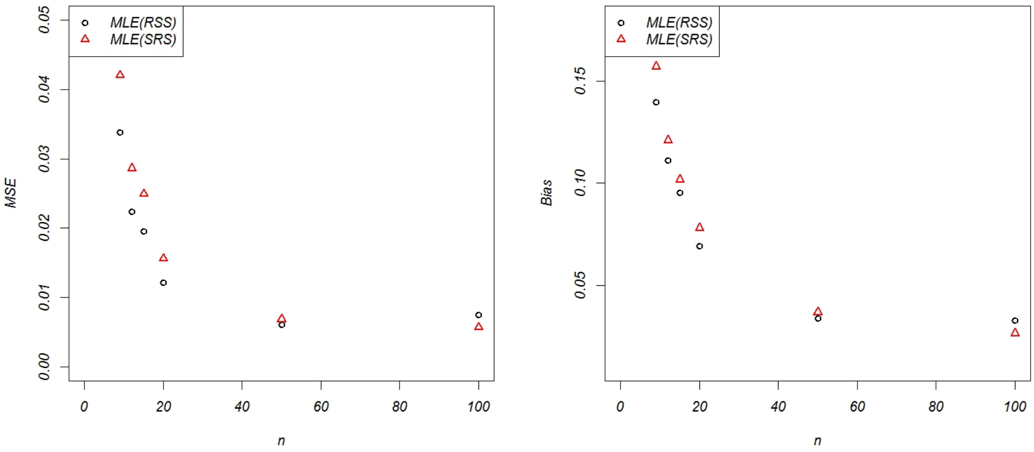

Figure 3. performance of $Bias$ and MSE in P

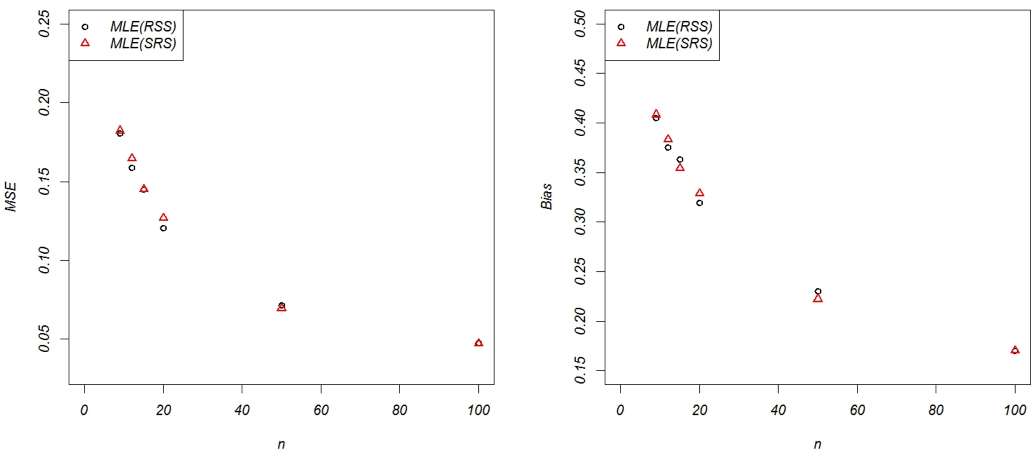

Figure 4. performance of $Bias$ and MSE in P

In Table (3) the estimation of $P = P(Y < X)$, when $X \sim BP(2, 4, 3, 2)$ and $Y \sim BP(1, 2.5, 3, 4)$ are independent random variables from BP distribution using RSS and SRS has been compared. We observed from Table (3) that the RSS produces smaller compared to SRS for sample size 50 for all sample sizes.

Figure (5) show the process of change $Bias$ and MSE in P when $X \sim BP(2, 4, 3, 2)$ and $Y \sim BP(1, 2.5, 3, 4)$ are independent random variables from BP distribution. From these figures we note that the MSE of P of the sampling RSS is always greater that of the sampling SRS. However, for large sample sizes these estimators of are better and almost equally efficient. Considering that in this research, the comparison between RSS and SRS estimators was done considering different parameters and no restrictions were applied in the selection of parameters, it can be concluded that in general, for n less than 50, the error of the RSS method is equal to . It is less and therefore more suitable. Given that large sample sizes can impose large costs on researchers, it is worthwhile to use estimators that provide better results in small sample sizes.

Table 3. Estimation of $P = P(Y < X)$

$SamplingMethod$	$Results$	$n = 9$	$n = 12$	$n = 15$	$n = 20$	$n = 50$	$n = 100$
$MLE(SRS)$	P	0.7450	0.7450	0.7450	0.7450	0.7450	0.7450
	$\hat{P}$	0.3518	0.3794	0.4037	0.4396	0.5342	0.5836
	$Bias(P)$	0.3931	0.3656	0.3412	0.3053	0.2108	0.1613
	$MSE(P)$	0.1742	0.1526	0.1368	0.1110	0.0605	0.0428
$MLE(RSS)$	P	0.7450	0.7450	0.7450	0.7450	0.7450	0.7450
	$\hat{R}$	0.3520	0.3832	0.4004	0.4500	0.5433	0.5797
	$Bias(P)$	0.3930	0.3618	0.3446	0.2915	0.2017	0.1653
	$MSE(P)$	0.1713	0.1496	0.1355	0.1053	0.0580	0.0441

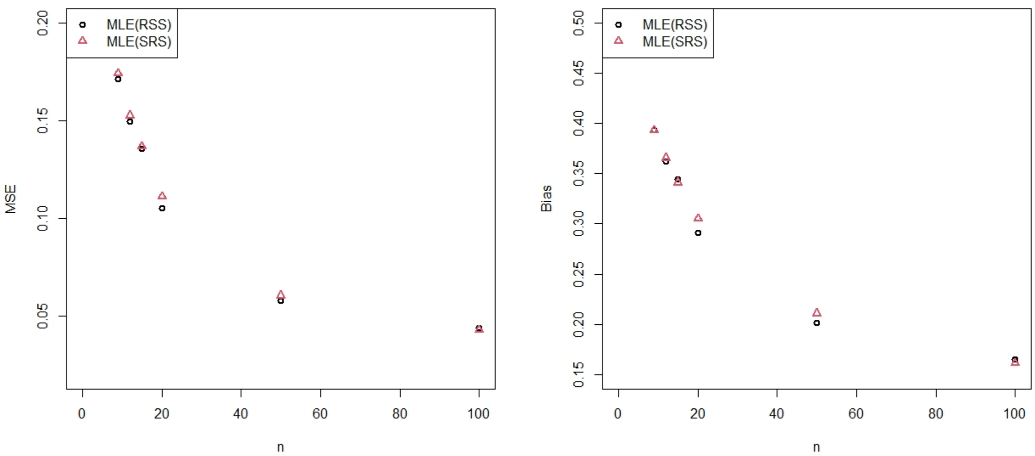

Figure 5. performance of $Bias$ and MSE in P

4. Real data analysis

In this section, the BP distribution is fitted to two data sets from the Wheaton River. These are the exceedances of flood peaks, discussed in [15]. These sets of data are fitted by using the Pareto distribution and the BP distribution. The data are the exceedances of flood peaks (in m^3/s) of the Wheaton River near Carcross in Yukon Territory, Canada. This data set represents 72 excrescences of flood peaks for the years 1958–1984 (rounded to one decimal place) of flood peaks (in m3 per s) of the Wheaton River near Carcross in Yukon Territory, Canada. While the exact method is not specified, it likely involved river gauging stations to measure water flow rates during flood events. The collection of such data typically involves continuous monitoring of river levels and flow rates, with particular attention paid to periods of high water flow. The focus on exceedances suggests that a baseline threshold was established, and only flood events exceeding this threshold were included in the dataset. These are exceedances, meaning they represent flood peaks that exceeded a certain threshold. This approach is common in extreme value analysis for hydrological events. This dataset has been used in multiple studies to test and compare different statistical distributions for modeling flood data. The distribution is highly skewed to the right. The data is suitable for various statistical analyses and modeling of extreme hydrological events. It represents extreme events (flood peaks), which is useful for studying the river's flood behavior. Recently, it is used by Mohamed [29] and fitted a Marshall-Olkin extended Gompertz Makeham model. The data was also used to compare the Composite Fréchet

Generalized Modified Weibull Exponential Modified (CFGMWEM) distribution with other common hydrological statistical distributions [30]. This dataset is particularly useful for studying extreme flood events and testing the fit of various probability distributions to hydrological data. It represents real-world observations of flood peaks, which are crucial for flood risk assessment and water resource management. This dataset is valuable for researchers and practitioners in hydrology, environmental science, and statistical modeling, particularly those focusing on extreme value analysis and flood frequency estimation. This type of data is crucial for flood risk assessment, water resource management, and understanding long-term hydrological patterns in the region. These data were analyzed in [7] and are given Tables (4) and (6). The data are fitted by using the Pareto, and the BP distributions. The Kolmogorov – Smirnov $(K-S)$ goodness - of - fit statistic is used for the comparison of the fits. The parameters are estimated by the maximum likelihood technique. The MLEs and the $p-values$ based on the $(K-S)$ goodness - of - fit statistics are given and presented in Table (5). According to the figures (6) and (7), and Tables (5) and (7), it is clear that our distributions have a good fit on these data sets.

Table 4. The flood levels data set (I)

1.7	2.2	14.4	1.1	0.4	20.6	5.3	0.7	1.9	13.0
12.0	9.3	1.4	18.7	8.5	25.5	11.6	14.1	22.1	1.1
2.5	14.4	1.7	37.6	0.6	2.2	39.0	0.3	15.0	27.0
11.0	7.3	22.9	1.7	0.1	1.1				

Table 5. The parameters estimates and goodness of fit criteria for data set (I).

Distribution	MLE(SRS)	$(K-S)$ statistics	p-value
Pareto	$\hat{k}=0.2438$	2.7029	0.000
	$\hat{\theta}=0.1$		
BP	$\hat{\alpha}=6.695$	1.2534	0.0864
	$\hat{\beta}=74.751$		
	$\hat{k}=0.021$		
	$\hat{\theta}=0.1$		

Table 6. The flood levels data set (II).

20.1	0.4	2.8	14.1	9.9	10.4	10.7	30.0	3.6	5.6
30.8	13.3	4.2	25.5	3.4	11.9	21.5	27.6	36.4	2.7
1.5	2.5	27.4	1.0	27.1	20.2	16.8	5.3	9.7	27.5
64.0	2.5	0.6	1.7	7.0	0.9				

The MLEs and the $p-values$ based on the $(K-S)$ goodness - of - fit statistics are given and presented in Table (7). In Tables (8) and, (9) the observed $R(t)$ values for data set (I) and data set (II), and their predicted values are calculated based on the parameters estimated in Tables (5) and (7) for different t. Also, the values of bias and MSE have been calculated and included.

Figures (8) and (9) show the MSE and $Bias$ of Predicted $R(t)$ for data set (I) and data set (II).

Figures (10) shows the MSE and $Bias$ of Predicted $R(t)$ for data set (I) and data set (II).

Now, for the above two data sets, we obtain estimators of $P=P(Y<X)$ for BP distribution, and the results are presented in Table (10). Figure (11) shows the CDF for fitted distribution function of the BP model according to data set (I) and data set (II).

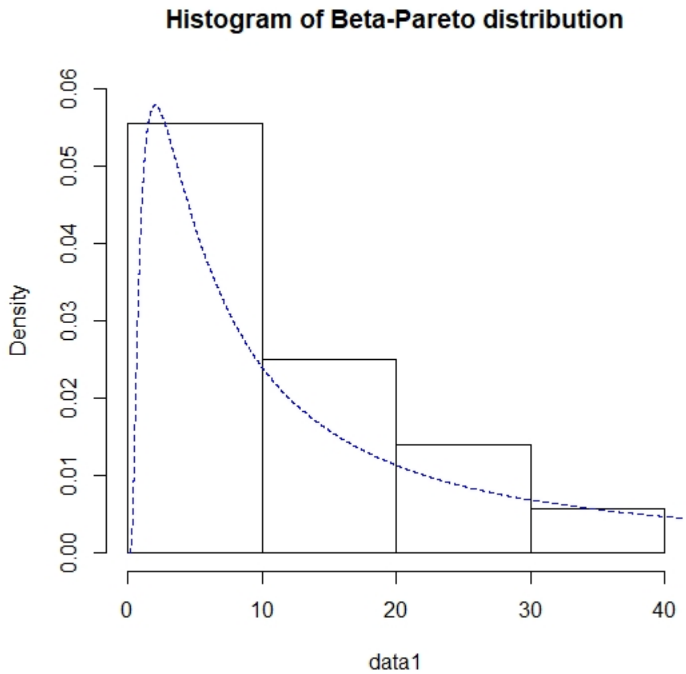

Figure 6. Plot of the PDF for BP based on data set (I).

Table 7. The parameters estimates and goodness of fit criteria for data set (II).

Distribution	MLE(SRS)	$(K-S)$ statistics	p-value
Pareto	$\hat{k}=1.367$	1.9505	0.000
	$\hat{\theta}=0.4$		
BP	$\hat{\alpha}=7.018$	0.9838	0.8849
	$\hat{\beta}=72.261$		
	$\hat{k}=0.029$		
	$\hat{\theta}=0.4$		

Table 8. Observed $R(t)$ and their predicted values for data set (I).

$Results$	$t=0.2$	$t=1$	$t=5$	$t=10$	$t=30$	$t=50$
$ObservedR(t)$	0.972	0.861	0.555	0.472	0.055	0.000
$PredictedR(t)$	0.994	0.891	0.498	0.416	0.053	0.003
$Bias$	0.022	0.030	0.057	0.014	0.002	0.003
MSE	0.0004	0.0009	0.0032	0.0001	0.0000	0.0000

5. Some Concluding Remarks

This paper discusses the estimation of the stress-strength reliability when X is distributed as the BP distribution. We obtained the MLEs of the parameters $R(t)$ and P under both SRS and RSS. A simulation study has been

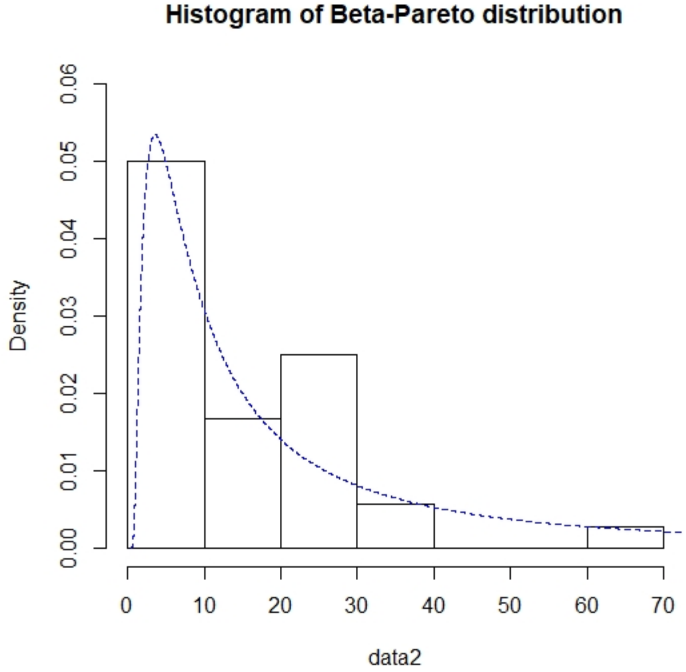

Figure 7. Plot of the PDF for BP based on data set (II).

Table 9. Observed $R(t)$ and their predicted values for data set (II).

$Results$	$t = 0.8$	$t = 2$	$t = 5$	$t = 10$	$t = 30$	$t = 50$
$Observed R(t)$	0.921	0.833	0.611	0.500	0.111	0.027
$Predicted R(t)$	0.993	0.862	0.687	0.531	0.124	0.013
$Bias$	0.072	0.029	0.076	0.031	0.013	0.014
MSE	0.0051	0.0008	0.0057	0.0009	0.0001	0.0001

Table 10. The MLE and of $P = P(Y < X)$.

Distribution	$\hat{P}$
BP	0.781

conducted to compare the performance of the estimators. From simulation study it is evident that the proposed estimators under the RSS performed better than SRS in most of the cases. Two real life data are analyzed to support the simulation results.
Based on the analysis of the provided data sets (data set (I) and data set (II)) using the BP distribution, the following conclusions can be drawn:

- Goodness - of - fit: The BP distribution provides a good fit for both data sets, as indicated by the $K - S$ goodness - of - fit statistics and $p - values$ presented in Tables (5) and (7).
- Parameter estimates: The MLE technique was used to estimate the parameters of the BP distribution for each data set. The estimated parameter values are presented in Tables (5) and (7).

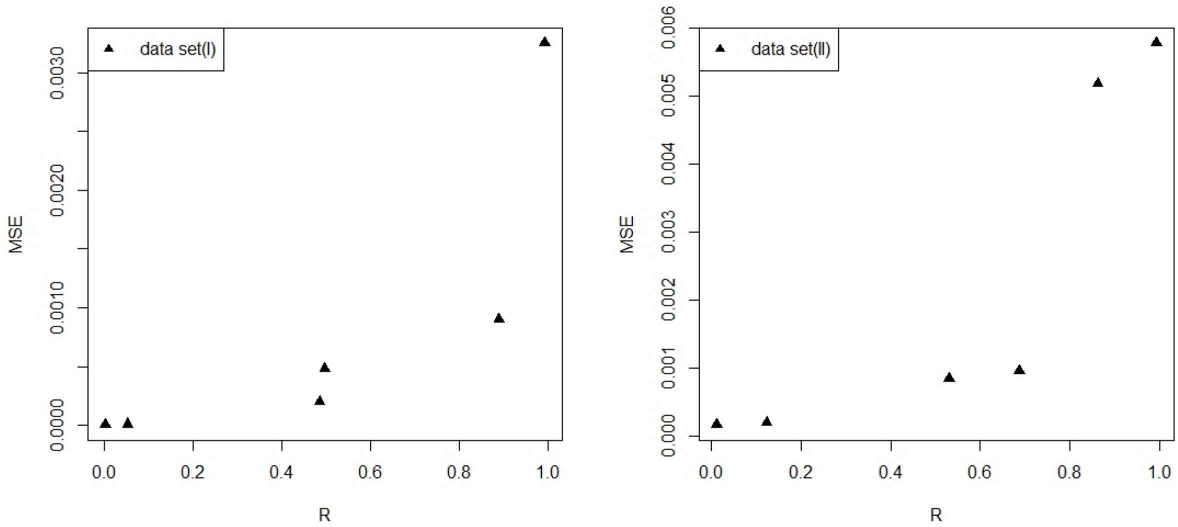

Figure 8. The performance of MSE of $R(t)$ for data set (I) and data set (II).

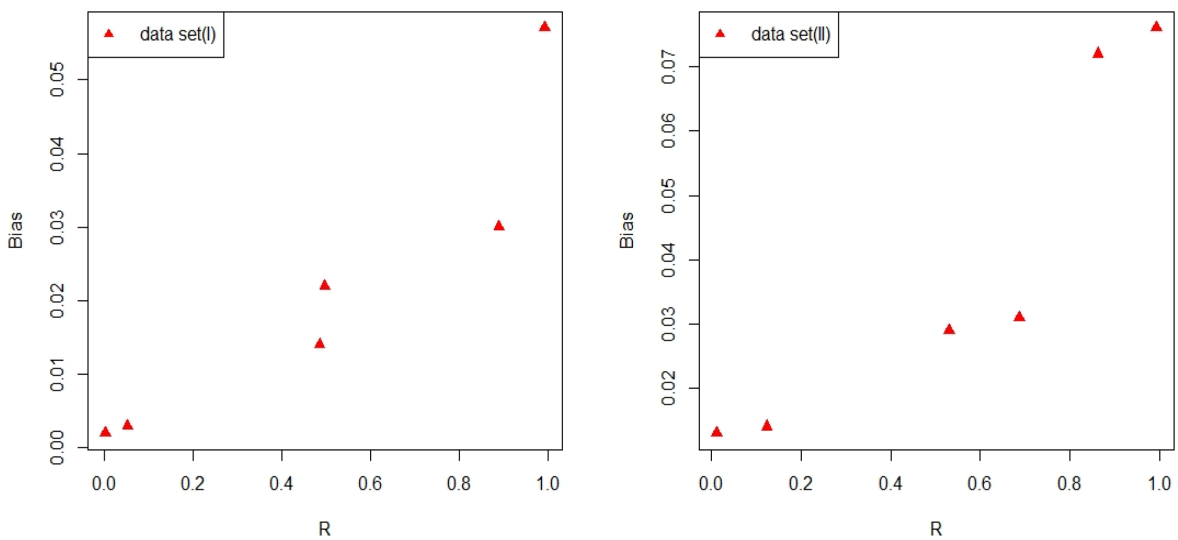

Figure 9. The performance of $Bias$ of $R(t)$ for data set (I) and data set (II).

- Predicted $R(t)$: The observed and predicted values of the reliability function $R(t)$ for different time points (t) were calculated based on the estimated parameters. Tables (8) and (9) show the observed and predicted $R(t)$ values for data set (I) and data set (II), respectively. The bias and MSE of the predictions were also calculated.
- Performance evaluation: Figures (6) - (10) provide visual representations of the performance of the predicted $R(t)$ values in terms MSE and bias for both data sets.

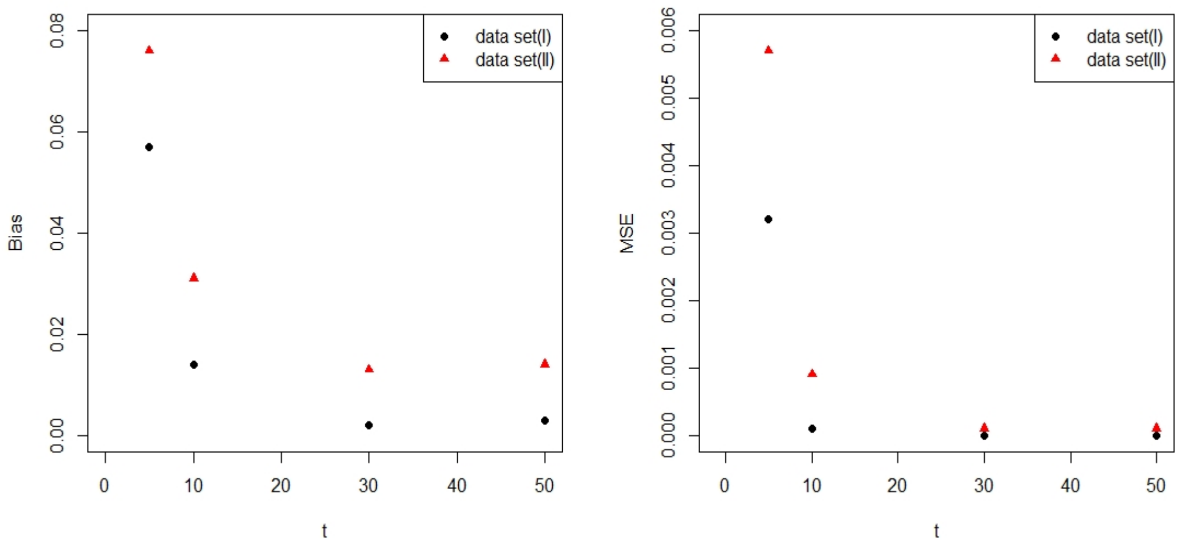

Figure 10. The performance of MSE and $Bias$ of $R(t)$ for different value t for data set (I) and data set (II).

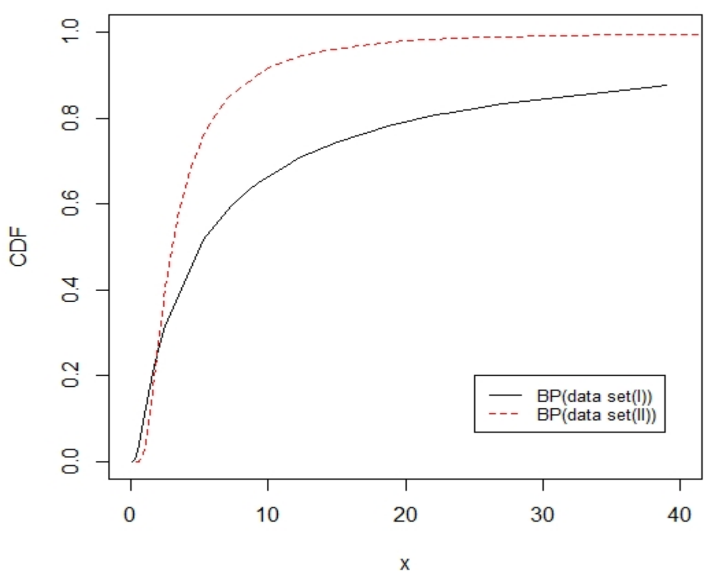

Figure 11. The empirical CDF of $BP(\hat{\alpha}, \hat{\beta}, \hat{k}, \hat{\theta})$ for data set (I) and data set (II).

Overall, the results suggest that BP distribution provide a good fit to the exceedances of flood peaks data for the Wheaton River, and the estimated parameters and reliability function can be used for further analysis and modeling.

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