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A Generalized Mixture of standard Logistic and skew Logistic distributions: Properties and applications

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Abstract In this paper, the generalized mixture of standard logistic and skew logistic is introduced as a new class of distribution. Some important mathematical properties of this novel distribution are discussed along with a graphical presentation of the density function. These properties include moment generating function, m^{th} order moment, mean deviation, characteristic function, entropy, among others. Moreover, a location scale type extension of the proposed distribution is considered, and the maximum likelihood estimation method for this model is presented. To examine the performance of the estimated parameters of the proposed distribution, a simulation study is also conducted using the rejection sampling method. Furthermore, an application using two real-life data sets are also illustrated. Finally, the likelihood ratio test is performed to study the discrepancies between proposed model with their counterparts.

Keywords Logistic distribution, Mixture distribution, Skew Logistic distribution, maximum likelihood estimator, Likelihood ratio test

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1. Introduction

The normal density has been widely used in probability theory for its broad versatility. Several topics in statistics remain incomplete when relying solely on the commonly used normal distribution. Nevertheless, this renowned distribution is applicable primarily in symmetric scenarios. When an asymmetry arises in the data, then a normal distribution behaves only as an ideal one. To overcome these situations, [6] proposed the skew-normal density adding the shape parameter to regulates asymmetry. The skew-normal probability density function (pdf) is

$$f(x;\lambda) = 2\phi(x)\Phi(\lambda x); \quad x \in \mathbb{R},\, \lambda \in \mathbb{R}, \tag{1}$$

where, $\phi(\cdot)$ and $\Phi(\cdot)$ are respectively the usual pdf and cumulative distribution function (cdf) of standard normal distribution.

In real life situations, it is necessary to tackle asymmetry in data and not assume symmetry. Therefore, the skew-normal pdf is a reasonable path and, in addition, to opening up a new path towards the study of distribution theory. Subsequent research in the literature was carried out extensively into the skew normal distribution. Some notable contributions includes the works of [29], [54], [6], [11], [1], and many others. However, [7] showed the inadequacy of the normal as well as skew-normal pdf for describing situations of plurimodality. To accommodate these kind of

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situations, [31] introduced the generalized mixture of standard normal and skew-normal (GMNSND) distribution. The GMNSND pdf is given by

$$f(x;\alpha,\lambda) = \frac{2}{2+\alpha}\phi(x)\Big[1+\alpha\Phi(\lambda x)\Big], \quad x \in \mathbb{R},\, \alpha > -2,\, \lambda \in \mathbb{R}. \tag{2}$$

Posteriorly, [53] corrected $\alpha > -1$ instead of $\alpha > -2$.

On the other hand, [36] provided a comprehensive discussion of a new skew distribution namely skew logistic (SLG) one. This distribution replaces the normal pdf and normal cdf in (1) with pdf and cdf of a logistic one, respectively. The skew-logistic pdf is given by

$$f(x;\lambda) = 2\left[\frac{e^{-x}}{(1+e^{-x})^2}\right]\left[\frac{1}{1+e^{-x\lambda}}\right], \quad x \in \mathbb{R},\, \lambda \in \mathbb{R}. \tag{3}$$

[36] also discussed some properties of the skew-logistic distribution, and also stated that logistic distribution has several interesting applications in various fields such as in geology, medicine, psychology, and several ones. As in skew-normal case, many research were also conducted on skew-logistic distribution considering different situations and aspects. [8] introduced a new class of skewed version of logistic distribution using a a non-cdf skew function. [22], [45], and [35] also notably contributed with a new class of asymmetric logistic distribution. Moreover, skew-Laplace and skew-uniform distribution were other two approaches of asymmetric versions of the Laplace and uniform ones proposed by [5] and [37], respectively.

Despite the unimodality modelling of skew distributions, there are some other new skew distributions proposed to supports data with uni-bimodality. For example, the alpha skew-normal distribution [16], the alpha skew-logistic distribution [26], the alpha skew-Laplace distribution [24], the generalized alpha skew-normal distribution [51], the two parameter bimodal skew-normal distribution [19], are some of the most popular models supporting data with uni-bimodality. For modelling uni-bimodality data, some other class of probability distributions have been presented under Balakrishnan mechanism [4] including Balakrishnan alpha skew-normal distribution [27], Balakrishnan alpha skew-logistic distribution [47], Log-Balakrishnan alpha skew-normal distribution [48], Balakrishnan Alpha Skew-Generalized t distribution [39], among others. Not only uni-bimodality has been considered, some new proposals of skew distribution were also reported which allows to fit data with trimodal feature as well as uni-bimodality. Alpha beta skew-normal distribution [46], alpha beta skew-logistic distribution [19], alpha beta generalized t distribution [33], and generalized alpha beta skew-normal distribution [50] are some remarkable works towards this approach. Some new classes of distributions for fitting both bimodal as well as trimodal data were also introduced by [34]. They introduced some new class of symmetric distributions known as flexible normal distribution, trimodal normal distribution, and so on. Posteriorly, using the concepts of those distributions some new classes of asymmetric distribution were introduced which includes trimodal skew-logistic distribution [38], flexible alpha skew-normal distribution [15], among others. Furthermore, some new family of distributions were also reported in the literature to handle data with multimodality. Multimodal skew-normal distribution [10], multimodal alpha skew-normal distribution [28], multimodal Balakrishnan alpha skew-normal distribution [50], multimodal alpha skew-Laplace distribution [9], and so on, were some of the most popular research works towards this multimodal approach.

In this article, a new class of probability distribution is proposed following the idea of generalized mixture of standard normal and skew-normal distribution given by [31]. During this study, a generalized mixture of standard logistic ($LG(0,1)$) distribution and skew-logistic distribution [36] is considered. Moreover, a new continuous probability distribution namely generalised skew logistic is introduced with a discussion regarding some important mathematical properties. Furthermore, using real life data, the adaptability of the novel logistic distribution is checked with other competitors.

The following sections of the article are organized as follows: Section 2 includes the new family of skew-logistic distributions along with some pictorial visualizations of the pdfs as well as special cases. Some statistical properties of the distribution are also included in Section 2. Section 3 addressed a location scale extension and parameter estimation of the new distribution. Section 4 provides the results of simulation while applications of the proposed distribution using two real-life data sets are considered in Section 5. The summary of the hypothesis testing results are also listed in Section 5. Finally, Section 6 concludes the article.

2. The Generalized Mixture of standard logistic and skew-logistic distribution

A generalized mixture of standard logistic and skew-logistic distribution [36] is proposed in this Section, resulting a new generalized skew logistic distribution.

Definition 1
A random variable X is said to have generalized skew-logistic distribution if its pdf is given by

$$f(x;\alpha,\lambda) = \frac{2}{2+\alpha}\left[\frac{e^{-x}}{(1+e^{-x})^2}\right]\left[1+\alpha\left(\frac{1}{1+e^{-x\lambda}}\right)\right], \tag{4}$$

where $x \in \mathbb{R}$, $\lambda \in \mathbb{R}$ and $\alpha \geq -1$. This probability distribution is denoted as $GSLG(\alpha,\lambda)$.

Remark 1
The pdf of $GSLG(\alpha,\lambda)$ can be written as

$$\begin{aligned} f(x;\alpha,\lambda) &= \frac{2}{2+\alpha}\left[\frac{e^{-x}}{(1+e^{-x})^2} + \alpha\left(\frac{e^{-x}}{(1+e^{-x})^2}\frac{1}{1+e^{-x\lambda}}\right)\right] \\ &= \frac{2}{2+\alpha}\left[g(x) + \frac{\alpha}{2}g(x,\lambda)\right], \end{aligned} \tag{5}$$

where $g(x)$ and $g(x,\lambda)$ are the pdf of standard logistic distribution and skew-logistic distribution [36], respectively.

Some special properties of GSLG distribution are:

i. when $\alpha = 0$ and/or when $\lambda = 0$ then, $GSLG(\alpha,\lambda)$ reduces to standard logistic ($LG(0,1)$) distribution;
ii. when $\alpha = -1$, then $GSLG(\alpha,\lambda)$ reduces to $SLG(-x)$; and
iii. when $X \sim GSLG(\alpha,\lambda)$, then $-X \sim GSLG(\alpha,-\lambda)$.

Some visual representations of the GSLG distribution pdf are presented in Figure 1 for various parameter choices. It is clear from Figure 1 that positive skeweness exists for certain positive values of λ (see Figure 1 (a)). Similarly, the pdf of the proposed distribution exhibits negative skewed behavior if negative values of λ are assumed (see Figure 1 (b)). Furthermore, from Figure 1 (c) it can be observed that if $\alpha = \lambda = 0$, then the pdf reduces to standard logistic distribution and it exhibit negative skeweness for increasing values of α, considering λ to be fixed (see Figure 1 (c)). Additionally, as can be seen in Figure 1 (d), the suggested distribution's pdf simplifies to a standard logistic distribution for $\lambda = 0$, while it becomes a skew logistic distribution for $\alpha = -1$.

Several important issues of the new distribution are discussed in the next sections.

2.1. Cumulative distribution function

Lemma 1
The cdf of the $X \sim GSLG(\alpha,\lambda)$ random variable is given as

$$F(x) = \frac{2}{2+\alpha}\begin{cases} G(x) + \frac{\alpha}{2}\left[1 - 2\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\binom{-1}{j}\binom{-2}{k}\frac{\exp(-(C_1 x))}{C_1}\right], & x \geq 0, \\ G(x) + \alpha\left[\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\binom{-1}{j}\binom{-2}{k}\frac{\exp(-(C_1 x))}{C_1}\right], & x < 0, \end{cases} \tag{6}$$

where $G(x)$ is the cdf of standard logistic distribution and $C_1 = 1 + \lambda j + k$.

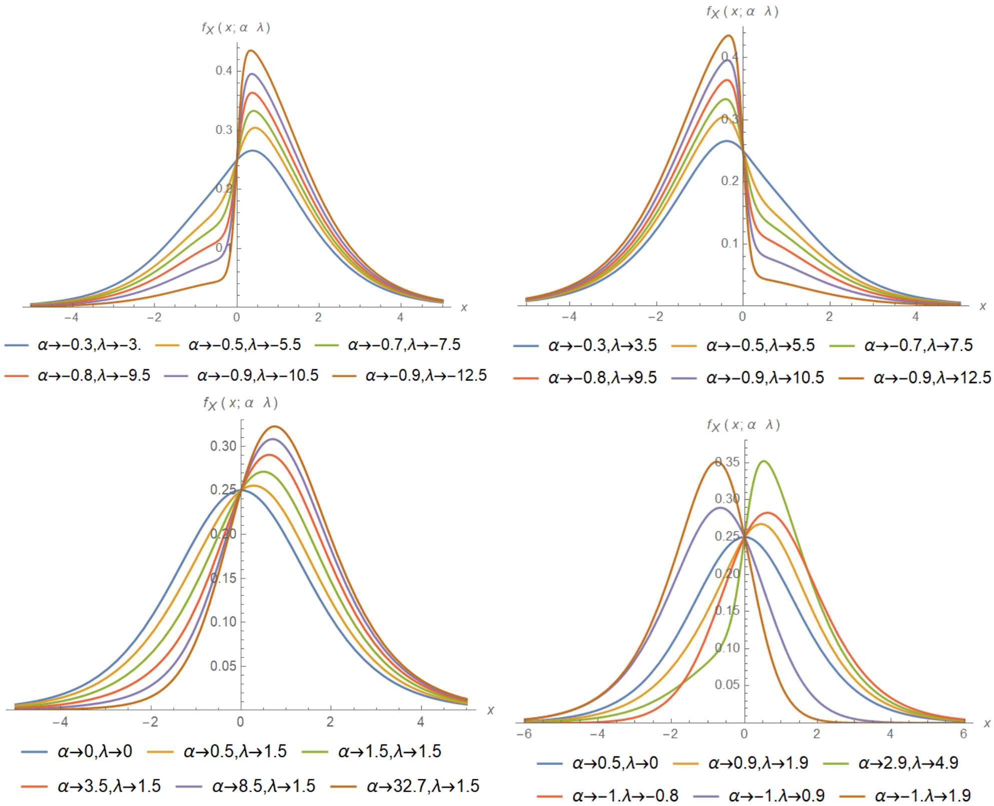

Figure 1. Pdf of $GSLG(\alpha, \lambda)$ for several α and λ parameters.

Proof. Using the expression of the density function mention in Remark 1, cdf is defined as

$$
\begin{aligned}
F(x) &= \int_{-\infty}^{x} f(x;\alpha,\lambda)dx \qquad (7)\\
&= \int_{-\infty}^{x} \frac{2}{2+\alpha}\left[g(x) + \frac{\alpha}{2}g(x,\lambda)\right]dx\\
&= \frac{2}{2+\alpha}\left[\int_{-\infty}^{x} g(x)dx + \frac{\alpha}{2}\int_{-\infty}^{x} g(x,\lambda)dx\right]\\
&= \frac{2}{2+\alpha}\left[I_1 + I_2\right]. \qquad (8)
\end{aligned}
$$

Now, I_1 is the cdf of the standard logistic distribution and I_2 can be calculated using the skew-normal pdf given by [36]. Then,

$$
F(x) = \frac{2}{2+\alpha}\begin{cases} G(x) + \dfrac{\alpha}{2}\left[1 - 2\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\begin{pmatrix} -1 \\ j \end{pmatrix}\begin{pmatrix} -2 \\ k \end{pmatrix}\dfrac{\exp(-(C_1 x))}{C_1}\right], & x \geq 0,\\[2ex] G(x) + \alpha\left[\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\begin{pmatrix} -1 \\ j \end{pmatrix}\begin{pmatrix} -2 \\ k \end{pmatrix}\dfrac{\exp(-(C_1 x))}{C_1}\right], & x < 0. \end{cases}
$$

2.2. Moment generating function

Lemma 2
The moment generating function (mgf) of a $X \sim GSLG(\alpha, \lambda)$ random variable is given as

$$M(t) = \frac{2}{2+\alpha}\Bigg[\pi t \csc \pi t + \alpha \Bigg(\sum_{j=0}^{\infty} \binom{-1}{j} \Big(1 + (t - \lambda j)\delta(1 - t + \lambda j) - (\lambda + t + \lambda j)\delta(1 + \lambda + t + \lambda j)\Big)\Bigg)\Bigg], \quad (9)$$

where $\delta(\cdot)$ is the Euler psi function given by

$$\delta(a) = \frac{1}{2}\left[\psi\left(\frac{1+a}{2}\right) - \psi\left(\frac{a}{2}\right)\right],$$

$\psi(a) = \dfrac{d \log \Gamma(a)}{da}$ is the digamma function, and $\Gamma(a) = \int_0^\infty t^{a-1} \exp(-t) dt$.

Proof. Using Remark 1, the mgf of a $X \sim GSLG(\alpha, \lambda)$ random variable is defined as

$$\begin{aligned} M(t) &= E[\exp(xt)] \\ &= \int_{-\infty}^{\infty} \exp(xt) f(x; \alpha, \lambda) dx \\ &= \frac{2}{2+\alpha}\left[\int_{-\infty}^{\infty} \exp(xt) g(x) dx + \frac{\alpha}{2}\int_{-\infty}^{\infty} \exp(xt) g(x, \lambda) dx\right] \\ &= \frac{2}{2+\alpha}\Big[I_3 + I_4\Big]. \end{aligned} \quad (10)$$

From (10), it can be seen that that I_3 is the mgf of the standard logistic distribution. On the other hand, I_4 can be calculated by using the mgf of the skew-logistic distribution given by [36]. Therefore, using the results of the last two expressions of (10), we have that

$$M(t) = \frac{2}{2+\alpha}\Bigg[\pi t \csc \pi t + \alpha \Bigg(\sum_{j=0}^{\infty} \binom{-1}{j} \Big(1 + (t - \lambda j)\delta(1 - t + \lambda j) - (\lambda + t + \lambda j)\delta(1 + \lambda + t + \lambda j)\Big)\Bigg)\Bigg].$$

Remark 2
Replacing t by (it) in (9), the characteristic function of a $X \sim GSLG(\alpha, \lambda)$ random variable is obtained as

$$\phi(t) = \frac{2}{2+\alpha}\Bigg[\pi it \csc \pi it + \alpha \Bigg(\sum_{j=0}^{\infty} \binom{-1}{j} \Big(1 + (it - \lambda j)\delta(1 - it + \lambda j) - (\lambda + it + \lambda j)\delta(1 + \lambda + it + \lambda j)\Big)\Bigg)\Bigg]. \quad (11)$$

2.3. Moments

Lemma 3
The m^{th} order moments a $X \sim GSLG(\alpha, \lambda)$ random variable are given by

$$E(X^m) = \frac{2}{2+\alpha}\left[\frac{2\alpha m!}{2}\left(\left(1-2^{1-m}\right)\zeta(m) + \frac{1}{2^m\lambda^{m+1}}\sum_{j=0}^{\infty}(-1)^j(j+1)\xi(j,m+1)\right)\right], \tag{12}$$

if m is odd, and

$$E(X^m) = 2m!\left(1-2^{1-m}\right)\zeta(m), \tag{13}$$

if m is even, where $\zeta(a) = \sum_{j=0}^{\infty}\frac{1}{j^a}$ is the Riemann's zeta function [12], $\zeta(a,q) = \sum_{j=0}^{\infty}\frac{1}{(q+j)^a}$, and $\xi(j,k) = \zeta\left(k, \frac{1+2\lambda+j}{2\lambda}\right) - \zeta\left(k, \frac{1+\lambda+j}{2\lambda}\right)$.

Proof. From Remark 1, the m^{th} order moment of a $X \sim GSLG(\alpha, \lambda)$ random variable can be defined as

$$\begin{aligned} E(X^m) &= \int_{-\infty}^{\infty} x^m f(x;\alpha,\lambda)dx \\ &= \frac{2}{2+\alpha}\left[\int_{-\infty}^{\infty} x^m g(x)dx + \frac{\alpha}{2}\int_{-\infty}^{\infty} x^m g(x,\lambda)dx\right] \\ &= \frac{2}{2+\alpha}\left[I_5 + I_6\right]. \end{aligned} \tag{14}$$

From (14), I_5 is the m^{th} order moment of a standard logistic random variable and I_6 can be obtained from the m^{th} order moment of a standard skew-logistic random variable provided by [36].

Therefore, when m is odd, then by theorem of logistic distribution $I_5 = 0$. By using again I_6 from [36], the expression for the m^{th} order moment can be calculated as

$$E(X^m) = \frac{2}{2+\alpha}\left[\frac{2\alpha m!}{2}\left(\left(1-2^{1-m}\right)\zeta(m) + \frac{1}{2^m\lambda^{m+1}}\sum_{j=0}^{\infty}(-1)^j(j+1)\xi(j,m+1)\right)\right].$$

If m is even, by the Lemma 2 of [21] and Example 23.11 of [30], an expression for the m^{th} order moment is obtained. Therefore, by putting the results in (14), the m^{th} order moment of a $X \sim GSLG(\alpha, \lambda)$ random variable can be obtained as

$$E[X^m] = 2m!\left(1-2^{1-m}\right)\zeta(m).$$

Remark 3
Using Lemma 3, the first four moments of $GSLG(\alpha, \lambda)$ distribution can be derived as

$$\begin{aligned} E[X] &= \frac{\alpha}{\lambda^2(2+\alpha)}\sum_{j=0}^{\infty}(-1)^j(j+1)\xi(j,2), \\ E[X^2] &= \frac{\pi^2}{3}, \\ E[X^3] &= \frac{\alpha}{2+\alpha}\left[-\frac{9\psi''(1)}{2} + \frac{3}{2\lambda^4}\sum_{j=0}^{\infty}(-1)^j(j+1)\xi(j,4)\right], \\ E[X^4] &= \frac{7\pi^4}{15}. \end{aligned}$$

Table 1. Mean and variance of a $X \sim GSLG(\alpha, \lambda)$ random variable for different values of parameters.

$\lambda \rightarrow$	-2		-1		1		3		4	
$\alpha \downarrow$	$E[X]$	$Var(X)$	$E[X]$	$Var(X)$	$E[X]$	$Var(X)$	$E[X]$	$Var(X)$	$E[X]$	$Var(X)$
-1	0.616	2.910	0.500	3.039	-0.500	3.039	-0.654	2.862	-0.669	2.842
1	-0.205	3.248	-0.167	3.262	0.167	3.262	0.218	3.242	0.223	3.240
2	-0.308	3.195	-0.250	3.227	0.250	3.227	0.327	3.183	0.335	3.178
3	-0.370	3.153	-0.300	3.199	0.300	3.199	0.392	3.136	0.402	3.128
4	-0.411	3.121	-0.333	3.179	0.333	3.179	0.436	3.099	0.447	3.090

Hence variance of a $GSLG(\alpha, \lambda)$ random variable can be calculated as

$$Var(X) = \frac{\pi^2}{3} - \frac{\alpha^2}{\lambda^4(2+\alpha)^2}\left[\sum_{j=0}^{\infty}(-1)^j(j+1)\xi(j,2)\right]^2. \tag{15}$$

The four moments values of the zeta function can be used from Section 8.17 of [20].

Additionally, the mean as well as variance of $GSLG(\alpha, \lambda)$ also can be evaluated numerically using the above special results for some particular values of the parameter. Hence mean and variance of the said distribution are calculated for different choices of the parameters and listed in Table 1.

2.4. Mean deviation

Lemma 4
The mean deviation of a $X \sim GSLG(\alpha, \lambda)$ random variable about the mean μ is given by

$$\mu_1(x) = \frac{2}{2+\alpha}\left[2\log 2 + \frac{\alpha}{2}\left(4\mu\sum_{j=1}^{\infty}\sum_{k=0}^{\infty}(-1)^{1+j+k}\frac{1+k}{C_1}\exp(C_1\mu)\right)\right], \quad for \;\; \mu \leq 0, \tag{16}$$

and

$$\mu_2(x) = \frac{2}{2+\alpha}\left[2\log 2 + \frac{\alpha}{2}\left(-4\log 2 + 4\sum_{j=0}^{\infty}(-1)^j\left(\mu + \frac{1}{1+j}\exp((1+j)\mu)\right)\right)\right.$$
$$\left. + 4\beta\sum_{j=1}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^{j+k}(1+k)}{C_1^2}\Big(\exp(-C_1\mu) - 2\Big)\right], \quad for \;\; \mu \geq 0. \tag{17}$$

Proof. Remark 1 is used to calculate the mean deviation about mean defined as

$$\begin{aligned}\mu(x) &= \int_{-\infty}^{\infty}|x-\mu|f(x;\alpha,\lambda)dx \\ &= \frac{\alpha}{2+\alpha}\Big[\int_{-\infty}^{\infty}|x-\mu|g(x)dx + \frac{\alpha}{2}\int_{-\infty}^{\infty}|x-\mu|g(x,\lambda)dx\Big] \\ &= \frac{\alpha}{2+\alpha}\Big[I_7 + I_8\Big].\end{aligned} \tag{18}$$

From (18), it can be observed that I_7 is the mean deviation of standard logistic distribution given by $2\log 2$. On the other hand, I_8 is the mean deviation of skew-logistic distribution given by [36]. Using these values, the integrals related to (18) can be evaluated for both the cases, i.e, when $\mu \leq 0$ and $\mu \geq 0$. Therefore, the final results of the mean deviation about mean of a $X \sim GSLG(\alpha, \lambda)$ random variable can be obtained as

$$\mu_1(x) = \frac{2}{2+\alpha}\left[2\log 2 + \frac{\alpha}{2}\left(4\mu\sum_{j=1}^{\infty}\sum_{k=0}^{\infty}(-1)^{1+j+k}\frac{1+k}{C_1}\exp(C_1\mu)\right)\right], \quad for \;\; \mu \leq 0, \tag{19}$$

and

$$\mu_2(x) = \frac{2}{2+\alpha}\Bigg[2\log 2 + \frac{\alpha}{2}\Bigg(-4\log 2 + 4\sum_{j=0}^{\infty}(-1)^j\Bigg(\mu + \frac{1}{1+j}\exp((1+j)\mu)\Bigg)\Bigg) + 4\beta\sum_{j=1}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^{j+k}(1+k)}{C_1^2}\Big(\exp(-C_1\mu)-2\Big)\Bigg], \quad for\ \ \mu \geq 0. \tag{20}$$

Remark 4
Replacing μ by M in (16) and (17), one can subsequently obtain the results of the mean deviation about median of a $X \sim GSLG(\alpha, \lambda)$ random variable.

2.5. Rényi entropy

Definition 2
For a random variable X, the Rényi entropy of order γ [42] is defined as

$$R_\gamma(X) = \frac{1}{1-\gamma}\log\int f(x)^\gamma dx, \tag{21}$$

where $\gamma > 1$ and $\gamma \neq 1$. Shannon entropy is a special case of Rényi one an is obtained in the limit $\lim_{\gamma\to 1} R_\gamma(X)$ [13].

The Rényi entropy of random variable X can also be defined as a measure of the uncertainty or randomness associated with X [1]. The following Lemma considers the results of the Rényi entropy for a GSLG random variable.

Lemma 5
The Rényi entropy of a $X \sim GSLG(\alpha, \lambda)$ random variable is

$$R_\gamma(X) = \frac{\alpha}{(2+\alpha)(1-\gamma)}\log\Bigg[\frac{2}{\lambda}{}_2F_1(\lambda, 2\lambda, 1+\lambda, -1) + \frac{\alpha}{2}\Bigg(\frac{1}{2} + 2\sum_{j=0}^{\infty}\binom{-\gamma}{j}\Bigg(\frac{{}_2F_1(2\gamma, \gamma+\lambda j, 1+\lambda+\lambda j, -1)}{\gamma+\lambda j} + \frac{{}_2F_1(2\gamma, \gamma+\lambda j+\gamma j, 1+\lambda+\lambda j, -1}{\gamma, \lambda\gamma+\lambda j}\Bigg)\Bigg], \tag{22}$$

where

$${}_2F_1(a, b; c; x) = \sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{(c)_k}\frac{x_k}{k!}$$

is the Gauss hypergeometric function [20] and $(x)_k = x(x+1)\cdots(x+k-1)$ is the ascending factorial.

Proof. Using the Remark 1, one can write

$$\begin{aligned} R_\gamma(X) &= \frac{1}{1-\gamma}\int_{-\infty}^{\infty} f(x;\alpha,\lambda)^\gamma dx \\ &= \frac{\alpha}{(2+\alpha)(1-\gamma)}\Bigg[\int_{-\infty}^{\infty} g(x)^\gamma dx + \int_{-\infty}^{\infty} g(x;\lambda)^\gamma dx\Bigg] \\ &= \frac{\alpha}{(2+\alpha)(1-\gamma)}\Big[I_9 + I_{10}\Big]. \end{aligned} \tag{23}$$

From Definition 2, we see that I_9 is the Rényi entropy of the standard logistic distribution which can be defined as

$$I_9 = \frac{\alpha}{(2+\alpha)(1-\gamma)} \log \int_{-\infty}^{\infty} \left(\frac{e^{-x}}{(1+e^{-x})^2} \right)^{\lambda} dx$$

Now, putting $y = e^{-x}$ and subsequently applying the equations (3.194.1)-(3.194.2) of [20], I_9 can be calculated as

$$I_9 = \frac{\alpha}{(2+\alpha)(1-\gamma)} \log \left[\frac{2}{\lambda} {}_2F_1(\lambda, 2\lambda, 1+\lambda, -1) \right].$$

On the other hand, I_{10} is the Rényi entropy of a skew-logistic random variable which was already calculated by [36]. Then substituting the results of I_9 and I_{10} in (23), final expression for Rényi entropy of X is obtained.

3. Maximum likelihood estimation

We proposed a location-scale-type extension for the GSLG distribution considering the location (μ) and scale (β) parameters. The transformation used to do this is $Y = \mu + \beta X$. The, the pdf of the location-scale-type extension of GSLG one is

$$f(x;\mu,\beta,\alpha,\lambda) = \frac{2}{2+\alpha} \left[\frac{\exp\left(-\frac{x-\mu}{\beta}\right)}{\left(1+\exp\left(-\frac{x-\mu}{\beta}\right)\right)^2 \beta} \right] \left[1+\alpha \left(\frac{1}{1+\exp\left(-\frac{\lambda(x-\mu)}{\beta}\right)} \right) \right], \tag{24}$$

where $x \in \mathbb{R}$, $\mu \in \mathbb{R}$, $\beta > 0$ and $\alpha \geq -1$. If X has the pdf (24), then is denoted as $X \sim EGSLG(\mu,\beta,\alpha,\lambda)$.

Estimation of parameter vector $\theta = (\mu,\beta,\alpha,\lambda)$ of $EGSLG(\mu,\beta,\alpha,\lambda)$ pdf is discussed next. Let $x_1, x_2, x_3, \ldots, x_n$ be a set of n independently and identically distributed random variables drawn from the $EGSLG(\mu,\beta,\alpha,\lambda)$ one, then the log-likelihood equations for the three parameters is

$$\begin{aligned} l(\theta) = {} & n\log(2) - n\log(2+\alpha) - n\log(\beta) + \frac{1}{\beta}\sum_{i=1}^{n}(x_i-\mu) - 2\sum_{i=1}^{n}\log\left[1+\exp\left(-\frac{x_i-\mu}{\beta}\right)\right] \\ & + \sum_{i=1}^{n}\left[1+\alpha\left(\frac{1}{1+\exp\left(-\frac{\lambda(x_i-\mu)}{\beta}\right)}\right)\right]. \end{aligned} \tag{25}$$

Now, differentiating the equation (25) with respect to the set of parameters, one can have the likelihood equations given as

$$\frac{\partial l(\theta)}{\partial \mu} = -\frac{n}{\beta} - \frac{\alpha\lambda}{\beta}\sum_{i=1}^{n}\frac{\exp(-Y(\mu,\beta,\lambda))}{\Big(1+\exp(-Y(\mu,\beta,\lambda))\Big)^2\Big(1+\frac{\alpha}{1+\exp(-Y(\mu,\beta,\lambda))}\Big)}$$
$$-\frac{2}{\beta}\sum_{i=1}^{n}\frac{\exp(-Y(\mu,\beta))}{1+\exp(-Y(\mu,\beta))}$$

$$\frac{\partial l(\theta)}{\partial \beta} = -\frac{n}{\beta} - \frac{1}{\beta^2}\sum_{i=1}^{n}(y_i-\mu) - \frac{2}{\beta^2}\sum_{i=1}^{n}\frac{(y_i-\mu)\exp\left(\frac{\mu}{\beta}\right)}{\exp\left(\frac{\mu}{\beta}\right)+\exp\left(\frac{y_i}{\beta}\right)}$$
$$+\frac{\alpha\lambda}{2\beta^2}\sum_{i=1}^{n}\frac{\mathrm{Sech}\left[\frac{\lambda(x_i-\mu)}{2\beta}\right](x_i-\mu)}{\left[-2-\alpha+\alpha\tanh\left[\frac{\lambda(x_i-\mu)}{2\beta}\right]\right]}$$

$$\frac{\partial l(\theta)}{\partial \alpha} = \frac{n}{2+\alpha} + \sum_{i=1}^{n}\frac{1}{1+\exp(-Y(\mu,\beta,\lambda))+\alpha}$$

$$\frac{\partial l(\theta)}{\partial \lambda} = -\frac{\alpha}{\beta}\sum_{i=1}^{n}\frac{\mathrm{Sech}\left[\frac{\lambda(x_i-\mu)}{2\beta}\right]^2(x_i-\mu)}{\left[4+2\alpha-2\alpha\tanh\left[\frac{\lambda(x_i-\mu)}{2\beta}\right]\right]}.$$

The estimates of the four parameters can be obtained after solving the latter equations. Explicit solutions can not directly obtained, therefore numerical procedure could be implemented using the `GenSA` package of `R` software.

4. Simulation Study

A simulation study is considered in this Section to evaluate the effectiveness of maximum likelihood estimates (MLEs) for $GSLG(\mu,\beta,\alpha,\lambda)$ model parameters. To generate the random sample from $GSLG(\mu,\beta,\alpha,\lambda)$ distribution, we employ the relationship given in Remark 1. The "*rlogis*" function is used to generate a random sample from $g(x)$ in Remark 1 and to generate a random sample from $g(x,\lambda)$, the rejection sampling method (see [44]) has been applied. The simulation process is replicated 10,000 times, incorporating three distinct sample sizes ($n = 100, 300$, and 500). Subsequently, the `GenSA` package of `R` software [40] is used to compute maximum likelihood estimates for each generated sample. Finally, the are evaluated in terms of bias and mean square error (MSE):

$$Bias(\hat{\theta}) = E(\hat{\theta})-\theta,$$
$$MSE(\hat{\theta}) = V(\hat{\theta})+Bias(\hat{\theta})^2,$$

respectively, where $\hat{\theta}=(\hat{\mu},\hat{\sigma},\hat{\lambda})$.

From Tables 2-5, it can be see that the MLEs perform well in estimated parameters of the GSLS model. In addition, the results also showed that with an increase in sample size, bias and MSE of the MLEs decrease, indicating an asymptotic consistency of the MLEs of $GSLG(\alpha,\lambda)$ random variable.

5. Applications

This section examines the applicability and adaptability of the proposed distribution to be compared with some competitor models. For this comparison, normal distribution $N(\mu,\beta)$, Laplace distribution $La(\mu,\beta)$, logistic distribution $LG(\mu,\beta)$, skew normal distribution $SN(\mu,\beta,\lambda)$ and skew-logistic distribution $SLG(\mu,\beta,\lambda)$ are considered. Akaike information Criteria (AIC) and Bayesian information criteria (BIC) are used for the comparisons. Additionally, as the newly proposed distribution looks like a heavy-tailed unimodal skew distribution, thus different real-life data sets exhibiting heavy-tailed behavior are taken into consideration for applications in real life.

Table 2. Simulation results.

			$\mu = 0,\quad \beta = 1$							
			μ		β		λ		α	
α	λ	n	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
		100	-0.0399	0.0214	-0.1096	0.0734	-0.0650	0.0498	-0.0447	0.0649
	-2	300	-0.0239	0.0137	-0.0650	0.0530	-0.0440	0.0390	0.0391	0.0239
		500	-0.0099	0.0196	0.0398	0.0369	0.0160	0.0190	0.0194	0.0260
		100	0.0498	0.0381	-0.0447	0.0331	-0.0641	0.0361	-0.0599	0.0431
	-1	300	-0.0190	0.0329	-0.0194	0.0195	0.0329	0.0332	-0.0247	0.0340
		500	0.0189	0.0110	0.0187	0.0145	0.0207	0.0232	-0.0189	0.0321
		100	0.0398	0.0338	-0.0499	0.0323	-0.0810	0.0588	0.0498	0.0506
-1	0	300	-0.0215	0.0189	-0.0267	0.0198	-0.0568	0.0340	-0.0345	0.0245
		500	0.0163	0.0120	0.0173	0.0109	-0.0210	0.0194	0.0254	0.0430
		100	0.0827	0.0387	-0.0533	0.0419	0.0470	0.0498	-0.0879	0.0321
	1	300	0.0485	0.0350	0.0274	0.0289	-0.0332	0.0286	0.0340	0.0156
		500	-0.0189	0.0162	-0.0137	0.0180	-0.0233	0.0187	0.0189	0.0199
		100	-0.0750	0.0898	0.0890	0.0651	-0.1000	0.0120	0.0992	0.0855
	2	300	0.0636	0.0550	-0.0540	0.0480	0.0755	0.0844	-0.0755	0.0650
		500	0.0404	0.0393	-0.0424	0.0390	-0.0665	0.0498	-0.0331	0.0265

Table 3. Simulation results.

			$\mu = 0,\quad \beta = 1$							
			μ		β		λ		α	
α	λ	n	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
		100	-0.0990	0.0678	0.0465	0.0631	-0.0487	0.0355	0.0396	0.0468
	-2	300	0.0565	0.0500	-0.0327	0.0344	-0.0310	0.0296	-0.0320	0.0356
		500	-0.0320	0.0238	-0.0256	0.0325	-0.0268	0.0208	0.0129	0.0271
		100	0.0509	0.0556	-0.0924	0.1056	0.0237	0.0312	0.0750	0.0986
	-1	300	-0.0464	0.0345	-0.0645	0.0480	-0.0267	0.0109	0.0489	0.0498
		500	0.0195	0.0199	-0.0222	0.0164	0.0198	0.0109	-0.0259	0.0232
		100	-0.0699	0.0840	-0.0689	0.0190	-0.0351	0.0240	0.0672	0.0380
1	0	300	-0.0361	0.0389	0.0595	0.0281	-0.0198	0.0098	-0.0587	0.0406
		500	0.0285	0.0291	-0.0297	0.0100	-0.0100	0.0100	-0.0321	0.0144
		100	-0.0859	0.0592	-0.0433	0.0459	-0.0778	0.0823	-0.1157	0.0987
	1	300	0.0627	0.0442	-0.0365	0.0235	0.0870	0.0670	-0.0636	0.0646
		500	-0.0351	0.0297	-0.0301	0.0154	0.0332	0.0160	-0.0444	0.0460
		100	0.1301	0.0940	0.1160	0.2301	0.0450	0.0459	-0.0957	0.0900
	2	300	-0.1054	0.0359	-0.0950	0.0497	-0.0410	0.0541	-0.1201	0.0784
		500	0.0452	0.0335	-0.0620	0.0166	-0.0197	0.0197	0.0618	0.0609

Table 4. Simulation results.

			$\mu = 0,\quad \beta = 1$							
			μ		β		λ		α	
α	λ	n	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
		100	-0.0520	0.0599	-0.0650	0.0551	-0.0990	0.1090	-0.1266	0.0686
	-2	300	0.0499	0.0463	-0.0450	0.0342	0.0623	0.0609	-0.0694	0.0577
		500	0.0318	0.0323	0.0349	0.0268	-0.0198	0.0450	-0.0422	0.0358
		100	0.0639	0.0461	-0.2004	0.1009	0.0478	0.0777	-0.0775	0.0766
	-1	300	0.0457	0.0381	0.1087	0.0658	-0.0407	0.0289	-0.0398	0.0524
		500	-0.0302	0.0213	0.0499	0.0421	0.0370	0.0380	-0.0520	0.0649
		100	-0.0630	0.0739	-0.0660	0.0732	0.0498	0.0740	0.0810	0.0987
-1	0	300	-0.0232	0.0655	-0.0358	0.0250	-0.0360	0.0390	0.0344	0.0458
		500	-0.0241	0.0323	-0.0250	0.0295	-0.0298	0.0378	-0.0279	0.0254
		100	-0.0920	0.1027	-0.0859	0.0659	-0.0987	0.0598	-0.0990	0.0990
	1	300	-0.0879	0.0723	0.0633	0.0237	0.0613	0.0457	-0.0546	0.0784
		500	0.0330	0.0290	-0.0219	0.0240	-0.0198	0.0201	0.0164	0.0238
		100	-0.1000	0.0823	-0.0390	0.0457	-0.0390	0.0455	-0.0681	0.0409
	2	300	-0.0650	0.0562	-0.0229	0.0288	0.0265	0.0422	0.0370	0.0481
		500	0.0601	0.0335	-0.0297	0.0212	0.0309	0.0350	-0.0353	0.0378

5.1. Distant galaxy dataset

This application considers the dataset of velocities of 82 distant galaxies, diverging from Milky Way galaxy. The data set was reported earlier by [43] and is available at `http://www.stats.bris.ac.uk/~peter/mixdata`. The results of the maximum likelihood estimators (MLEs), log-likelihood, AIC and BIC of the models are reported in the Table 6.

From Table 6, it is clear to see that $GSLG(\mu, \beta, \alpha, \lambda)$ distribution is more appropriate and better fitted among the other competitors in terms of log-likelihood, AIC and BIC. The flexibility of the new distribution can also be visualized from Figure 2.

Table 5. Simulation results.

			$\mu=0,\quad\beta=1$							
			μ		β		λ		α	
α	λ	n	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
		100	0.0989	0.0756	0.0650	0.0520	0.0786	0.0675	0.0872	0.0511
	-2	300	-0.0660	0.0698	-0.0566	0.0327	0.0491	0.0239	-0.0755	0.0497
		500	0.0331	0.0387	-0.0320	0.0168	0.0259	0.0209	0.0397	0.0458
		100	-0.0655	0.0488	0.0997	0.0974	-0.0741	0.0650	0.0776	0.0560
	-1	300	0.0270	0.0401	-0.0730	0.0569	0.0389	0.0397	0.0591	0.0489
		500	0.0236	0.0247	-0.0386	0.0382	-0.0206	0.0247	0.0304	0.0199
		100	-0.0799	0.0658	0.0655	0.0594	0.0656	0.0297	0.0689	0.0490
1	0	300	-0.0682	0.0459	0.0455	0.0394	0.0500	0.0552	-0.0438	0.0424
		500	0.0541	0.0198	-0.0194	0.0197	0.0279	0.0323	-0.0200	0.0248
		100	-0.0859	0.0686	0.0660	0.0897	-0.0850	0.0598	-0.0960	0.0982
	1	300	0.0655	0.0566	0.0434	0.0557	0.0569	0.0424	-0.0698	0.0397
		500	-0.0279	0.0354	-0.0390	0.0223	0.0293	0.0329	-0.0209	0.0361
		100	-0.2465	0.0990	-0.1298	0.0890	-0.0630	0.0864	0.1456	0.0870
	2	300	-0.0967	0.0564	-0.0950	0.0595	0.0497	0.0553	-0.0960	0.0845
		500	-0.0857	0.0510	0.0397	0.0348	-0.0247	0.0398	0.0197	0.0634

Table 6. MLEs, log-likelihood, AIC and BIC of models fitted to 82 distant galaxies.

Distributions	μ	β	α	λ	$logL$	AIC	BIC
$N(\mu,\beta)$	20.832	4.540	–	–	-240.410	484.820	489.633
$La(\mu,\beta)$	20.834	2.997	–	–	-228.830	461.660	466.470
$LG(\mu,\beta)$	21.074	2.204	–	–	-233.649	471.298	476.111
$SN(\mu,\beta,\lambda)$	24.610	5.907	–	-1.394	-239.210	484.420	491.640
$SLG(\mu,\beta,\lambda)$	21.532	2.219	–	-0.154	-233.314	472.628	479.848
$GSLG(\mu,\beta,\alpha,\lambda)$	18.596	2.789	-0.867	-17.909	-222.316	452.632	462.258

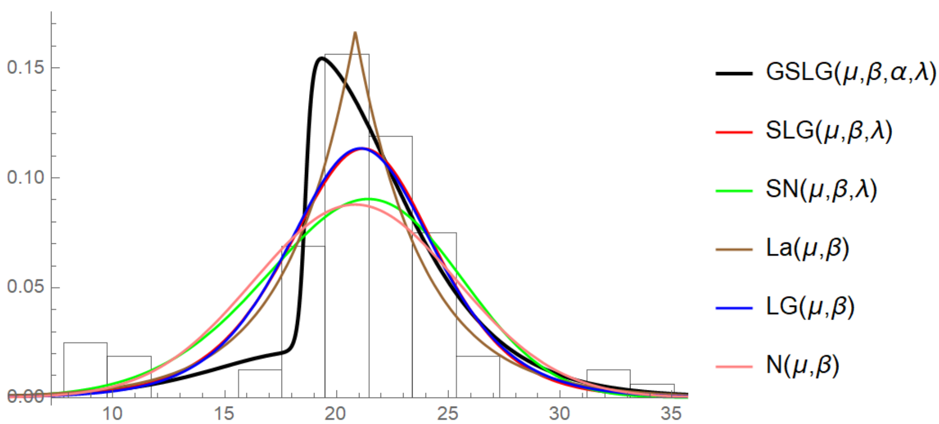

Figure 2. Plots of observed and expected densities of 82 distant galaxies.

5.2. Acidity Index of Lakes dataset

For this application, the dataset consisting in an acidity index measured in a sample of 155 lakes from Northeastern US is considered, which was previously considered for a mixture of normal pdfs on the log-scale by [14]. The results of the maximum likelihood estimators (MLEs), log-likelihood, AIC and BIC of the models are presented in the Table 7.

From Table 7 it is clear to observe that $GSLG(\mu,\beta,\alpha,\lambda)$ distribution is more appropriate and better fitted among the other competitors in terms of log-likelihood, AIC and BIC. The flexibility of the new distribution can also be visualized from Figure 3.

5.3. Hypothesis Testing

This section performs the likelihood ratio test (LRT) to discriminate between $GSLG(\mu,\beta,\alpha,\lambda)$ and some other nested models. The statistic considered for this test is

Table 7. MLEs, log-likelihood, AIC and BIC of models fitted to acidity index measured in a sample of 155 lakes in the Northeastern United States.

Distributions	μ	β	α	λ	$logL$	AIC	BIC
$N(\mu,\beta)$	5.105	1.038	–	–	-225.785	455.570	461.656
$La(\mu,\beta)$	4.727	0.892	–	–	-244.649	493.298	499.384
$LG(\mu,\beta)$	5.023	0.631	–	–	-232.790	469.580	475.486
$SN(\mu,\beta,\lambda)$	4.061	1.473	–	1.937	-220.618	447.236	456.366
$SLG(\mu,\beta,\lambda)$	3.845	0.941	–	7.681	-210.306	426.612	435.742
$GSLG(\mu,\beta,\alpha,\lambda)$	3.849	0.938	-0.992	-13.594	-206.589	421.178	433.350

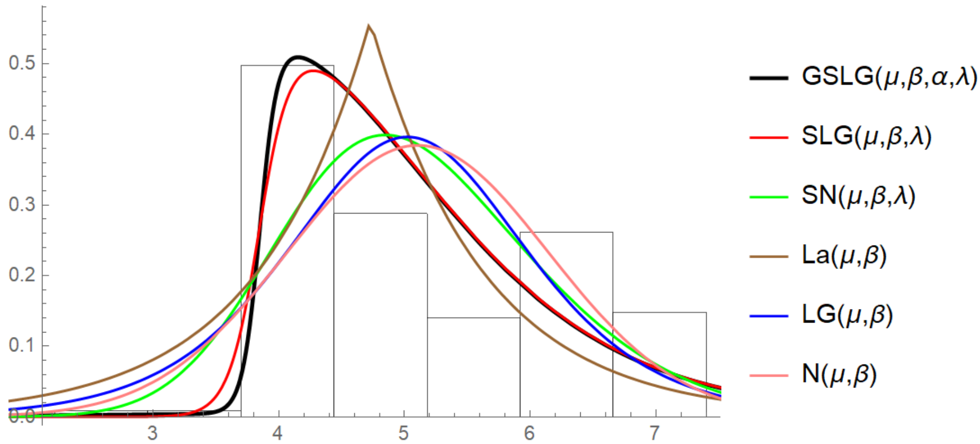

Figure 3. Plots of observed and expected densities of 155 acidity index of lakes

(i) To discriminate $LG(\mu,\beta)$ from $GSLG(\mu,\beta,\alpha,\lambda)$ distribution, the null hypothesis $H_0: \alpha=0,\lambda=0$ is considered against $H_1: \alpha\neq 0,\lambda\neq 0$. Then the test statistic is

$$-2\log(LRT) = -2[\log\ L(\hat{\mu}_1,\hat{\beta}_1,\alpha=0,\lambda=0|x) - \log\ L(\hat{\mu}_2,\hat{\beta}_2,\hat{\alpha}_2,\hat{\lambda}_2)] \sim \chi^2_2,$$

where $(\hat{\mu}_1,\hat{\beta}_1)$ and $(\hat{\mu}_2,\hat{\beta}_2,\hat{\lambda}_2,\hat{\alpha}_2)$ are the MLEs of $LG(\mu,\beta)$ and $GSLG(\mu,\beta,\lambda,\alpha)$ distributions, respectively; and $r=2$ is the difference between the number of parameters.

(ii) To discriminate $SLG(\mu,\beta,\lambda)$ from $GSLG(\mu,\beta,\alpha,\lambda)$ distribution, the null hypothesis $H_0: \alpha=0$ is considered against $H_1: \alpha\neq 0$. Then, the test statistic is

$$-2\log(LRT) = -2[\log L(\hat{\mu}_1,\hat{\beta}_1,\hat{\lambda}_1,\alpha=0|x) - \log L(\hat{\mu}_2,\hat{\beta}_2,\hat{\alpha}_2,\hat{\lambda}_2)] \sim \chi^2_1,$$

where $(\hat{\mu}_1,\hat{\beta}_1,\hat{\lambda}_1)$ and $(\hat{\mu}_2,\hat{\beta}_2,\hat{\lambda}_2,\hat{\alpha}_2)$ are the MLEs of $SLG(\mu,\beta,\lambda)$ and $GSLG(\mu,\beta,\lambda,\alpha)$ distributions, respectively; and $r=1$ is the difference between the number of parameters.

The results of the LRT are listed in Table 8 for both Aircraft Windshield dataset and Acidity Index of Lakes dataset. From Table 8, it can be seen that for both null hypothesis, LRT statistic is higher than tabulated critical value at the 5% level of significance. Therefore, it can be concluded that null hypothesis are rejected indicating that data comes from the novel family of distribution rather than simpler distributions.

Table 8. The value of LR test statistic for different hypotheses for the data set I and data set II.

Hypothesis	LRT statistic		d.f.	Critical Values at 5 %
	Dataset I	Dataset II		
$H_0 : \alpha = 0, \lambda = 0$ Vs $H_1 : \alpha \neq 0, \lambda \neq 0$	22.666	52.420	2	5.990
$H_0 : \alpha = 0$ Vs $H_1 : \alpha \neq 0$	21.996	7.434	1	3.841

6. Conclusion

In this paper, a new family of continuous probability distributions were proposed by considering a generalized mixture of standard logistic and skew-logistic distribution [36]. The graphical visualization of the new model was checked for different parameters. Some important mathematical properties of proposed probability distribution were also discussed. A simulation study was also conducted using rejection sampling method where was observed that the estimated parameter were asymptotically consistent with the increasing number of sample size. Furthermore, application of proposed distribution to real-life datasets reveals that proposed model was much flexible and useful compared to some competitors. Executing likelihood ratio test, differences between the new distributions were checked with their counterparts showing a better goodness of fit.

Further work can be developed, by comparing the proposed model with existing ones [32, 41, 25, 52]; and under risk analysis, VaR modelling and insurance data approaches [23, 2, 3, 18].

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Appendix

`R` codes used in this study are available at `github` repository: `https://github.com/jondeep98/FASLa.Code`

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