Addendum to "Log-ergodicity: A New Concept for Modeling Financial Markets"

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Introduction

This addendum serves to address certain errors identified in the proofs presented in our original publication. We sincerely thank the reviewers for their constructive feedback, which has allowed us to refine and improve the mathematical rigor of our work. Specifically, we revisit the derivations related to the wide-sense stationarity (WSS) property of Z_δ, the covariance structure, and the ergodicity argument. Additionally, we clarify the implications of our assumptions and provide alternative explanations where necessary.

δ As a Portion of The Time Interval $[0, T]$

In this document, we define the time interval length δ as a portion of the interval $[0, T]$, proportional to the inhibition degree β, with the assumption:

$$\delta = k \cdot \beta, \tag{1}$$

where:

- δ: The time interval length under consideration,
- β: The inhibition degree,
- k: A proportionality constant, reflecting the relative contribution of the inhibition degree to the interval length.

This assumption links δ to β, ensuring that the time intervals are dynamically aligned with the system's responsiveness or resistance.

Definition of the Inhibition Degree β

The inhibition degree β is defined as:

$$\beta := \begin{cases} \alpha, & \text{if } \alpha > \frac{3}{2}, \\ \frac{3}{2} + |\alpha|, & \text{if } \alpha < \frac{3}{2}, \end{cases}$$

where:

- α: A parameter capturing dynamic properties of the system, such as responsiveness or resistance to external changes.

The inhibition degree β reflects the system's ability to absorb shocks or deviations, with larger values of β corresponding to higher resistance or slower adjustments.

Economic Reasoning for $\delta = k \cdot \beta$

The relationship $\delta = k \cdot \beta$ is motivated by economic principles and observations, as outlined below:

Proportional Contribution of δ

In financial models, the time interval δ represents the horizon over which observations are made or changes are measured. Scaling δ proportionally to β ensures:

- **Shorter Intervals for Faster Reactions:** When the system exhibits high responsiveness (low β), δ becomes smaller, capturing short-term fluctuations in the process.
- **Longer Intervals for Higher Resistance:** When the system exhibits high resistance (large β), δ becomes longer, reflecting more aggregated behavior over time.

Role of the Inhibition Degree β

The inhibition degree β encapsulates the system's ability to resist external changes, with implications for δ:

- **Economic Shocks:** A smaller β indicates a highly reactive system, resulting in shorter intervals δ to capture rapid adjustments.
- **Stability and Aggregation:** A larger β indicates a more stable system, allowing longer intervals δ to capture aggregated effects over time.

Time-Sensitivity in Financial Markets

Financial markets exhibit varying time sensitivities based on prevailing conditions:

- During periods of heightened activity (e.g., after major news releases), δ decreases to capture rapid reactions.
- During periods of stability (e.g., low market volatility), δ increases to reflect longer-term trends.

By linking δ to β, the model dynamically adapts to these conditions, ensuring realistic representation of market behavior.

Practical Implications of the Assumption

The assumption $\delta = k \cdot \beta$ has the following implications:

- **Dynamic Interval Lengths:** The relationship between δ and β allows the model to dynamically adjust time intervals based on the system's responsiveness or resistance.
- **Alignment with Economic Reality:** Scaling δ with β reflects empirical observations in financial markets, where time horizons are influenced by market activity and system stability.
- **Model Flexibility:** The proportionality constant k provides flexibility to calibrate δ based on specific economic scenarios or modeling requirements.

The assumption $\delta = k \cdot \beta$ establishes a meaningful relationship between the time interval length δ and the inhibition degree β, reflecting the system's responsiveness or resistance. This assumption provides a dynamic and realistic foundation for modeling financial processes, ensuring that time intervals are aligned with economic behavior.

Time-Lag-Dependent Assumptions on Drift and Volatility

First, we outline the economic reasoning and justification behind the assumptions:

$$\mu_t = \mu_0 + g(\tau), \quad \sigma_t = \sigma_0 + h(\tau),$$

where:

- μ_t: The drift of the process at time t,
- σ_t: The volatility of the process at time t,

- μ_0, σ_0: The baseline (initial) drift and volatility of the process,
- $g(\tau), h(\tau)$: Time-lag-dependent bounded adjustment functions, where τ is the time lag between t and $t + \tau$.

These assumptions are designed to reflect the evolving behavior of financial processes, capturing market dynamics such as changing expectations, responses to external shocks, and clustering of volatility.

Definition of the Assumptions

We define the drift and volatility as follows:

1. Drift:

$$\mu_t = \mu_0 + g(\tau), \tag{2}$$

   where:

   - μ_0: Represents the baseline drift, which can be interpreted as the long-term average growth rate of the process,
   - $g(\tau)$: Models the adjustments in drift over the time lag τ, driven by economic factors such as changes in market expectations or external shocks.

2. Volatility:

$$\sigma_t = \sigma_0 + h(\tau), \tag{3}$$

   where:

   - σ_0: Represents the baseline volatility, which reflects the inherent uncertainty or risk level of the process,
   - $h(\tau)$: Models the adjustments in volatility over the time lag τ, reflecting phenomena like volatility clustering or market turbulence.

Economic Reasoning

Drift Assumption: $\mu_t = \mu_0 + g(\tau)$

The drift μ_t represents the deterministic trend or growth rate of the process, which is influenced by several economic factors:

- **Baseline Drift (μ_0):** The baseline drift μ_0 captures the long-term average return or growth rate of the process, which reflects stable economic conditions or intrinsic properties of the financial asset (e.g., risk-free rate or average market return).
- **Adjustment Function ($g(\tau)$):** The adjustment $g(\tau)$ reflects deviations from the baseline drift over the time lag τ, due to:
  - *Changing Market Expectations:* Over time, market participants update their expectations based on new information (e.g., earnings reports, policy changes, or macroeconomic indicators), causing temporary deviations in drift.
  - *External Shocks:* Events like geopolitical crises, regulatory changes, or technological disruptions introduce changes in the growth trajectory over specific time intervals.
  - *Time-Lagged Adjustments:* Financial markets often exhibit delayed reactions to news, resulting in time-lagged effects on drift.

Volatility Assumption: $\sigma_t = \sigma_0 + h(\tau)$

The volatility σ_t represents the uncertainty or variability of the process, which evolves due to various market dynamics:

- **Baseline Volatility (σ_0):** The baseline volatility σ_0 represents the inherent risk level of the asset or process under normal market conditions (e.g., day-to-day price fluctuations).
- **Adjustment Function ($h(\tau)$):** The adjustment $h(\tau)$ models changes in volatility over the time lag τ, caused by:
  - *Volatility Clustering:* Financial markets often exhibit periods of high or low volatility clustered together, driven by herding behavior, market sentiment, or structural changes.
  - *Market Turbulence:* Events like financial crises or speculative bubbles can cause short-term spikes in volatility, which decay over time.
  - *Time-Lagged Volatility Adjustments:* The volatility of an asset may adjust gradually over time as market participants reassess risk following major events.

Implications of the Assumptions

The assumptions $\mu_t = \mu_0 + g(\tau)$ and $\sigma_t = \sigma_0 + h(\tau)$ have important implications for modeling financial processes:

- They enable the process to capture time-lag-dependent effects, making it more realistic and suitable for analyzing financial markets where drift and volatility evolve over time.
- The adjustment functions $g(\tau)$ and $h(\tau)$ provide flexibility to model various economic scenarios, such as mean reversion, structural breaks, or cyclical behavior.
- They ensure that the process reflects empirical observations, such as volatility clustering and delayed market reactions.

By defining the drift and volatility as $\mu_t = \mu_0 + g(\tau)$ and $\sigma_t = \sigma_0 + h(\tau)$, we incorporate economic reasoning into the modeling framework, making the process more realistic and aligned with financial market behavior. These assumptions allow us to capture time-lag effects and ensure the model's applicability in various economic contexts.

Periodicity and Stationarity

Periodicity in μ_s and σ_s ensures that the integrals $\int_0^\delta \sigma_s^2 ds$ and $\int_0^\delta \mu_s ds$ exhibit consistent behavior over repeated intervals. This regularity guarantees that the covariance remains a function of τ alone, preserving WSS. Therefore, if σ_s and μ_s are time-independent or periodic, the process Z_δ satisfies both conditions for WSS.

Wide-Sense Stationarity (WSS) of Z_δ

In the original manuscript, we claimed that the process Z_δ satisfies WSS. However, it is now evident that this claim is not valid for processes starting from $Z_0 = 0$. The critical issue lies in the implication:

$$\mathrm{Cov}(Z_\delta, Z_0) = 0 \implies \mathrm{Var}(Z_\delta) = 0,$$

which results in $Z_\delta = 0$ almost surely—a contradiction with the intended dynamics of Z_δ.

Proof Adjustment and Assumption of $Z_0 \neq 0$: In Lemma 3.1, to prove that the process Z_δ constructed using the ergodic maker operator (EMO) satisfies wide-sense stationarity under the new assumption $Z_0 \neq 0$:

Non-Zero Initial Condition: Assuming $Z_0 \neq 0$ ensures that the process begins with a meaningful state, avoiding trivial or degenerate behavior. This assumption aligns with financial applications where initial conditions matter.

Asymptotic Stationarity: By following the derivation for Z_δ, it can be shown that both the mean and autocovariance of Z_δ depend only on the time difference δ, not on the absolute time t or s, ensuring wide-sense stationarity. The added non-zero initial condition solidifies the argument and ties to the definition of asymptotic WSS processes in ergodic theory.

Revised Definition

To address this, we propose a redefinition of Z_δ. Instead of starting with a constant initial condition ($Z_0 = 0$), we suggest a stochastic initialization such as $Z_0 \sim \mathcal{N}(\mu_0, \sigma_0^2)$, where $\mu_0 \neq 0$. This ensures a non-degenerate covariance structure and consistency with stationarity.

Statement of the Lemma 3.1. Under New Assumptions

If σ_s and μ_s are time-independent (or periodic), then under the assumptions 2 and 3 $Z_\delta = \xi^{\beta}_{\delta, W_\delta}[Y'_t]$ is a wide-sense stationary stochastic process.

Corrected Proof

Under this revised definition, Z_δ satisfies the following WSS conditions:

$$E[Z_\delta] = \text{constant for all } \delta, \quad \mathrm{Cov}(Z_\delta, Z_{\delta+\tau}) = f(\tau),$$

where $f(\tau)$ is a function of the time lag τ only.

Wide-Sense Stationarity (WSS) of Z_δ

We aim to prove that the process Z_δ satisfies wide-sense stationarity (WSS) under the assumption $Z_0 \sim N(\mu_0, \sigma_0^2)$, where $\mu_0 \neq 0$.

Definition of Z_δ

The process Z_δ is defined as:

$$Z_\delta = Z_0 + \frac{1}{T^\beta} \int_0^\delta \sigma_s dW_s + \frac{W_T}{T^\beta} \int_0^\delta \mu_s ds,$$

where:

- $Z_0 \sim N(\mu_0, \sigma_0^2)$, with $\mu_0 \neq 0$,
- W_t is a standard Wiener process,
- σ_s and μ_s are stochastic volatility and drift terms, respectively.

Mean of Z_δ

The mean is given by:

$$\mathbb{E}[Z_\delta] = \mathbb{E}[Z_0] + \frac{1}{T^\beta}\mathbb{E}\left[\int_0^\delta \sigma_s dW_s\right] + \frac{1}{T^\beta}\mathbb{E}\left[W_T \int_0^\delta \mu_s ds\right].$$

1. $\mathbb{E}[Z_0] = \mu_0$, as $Z_0 \sim N(\mu_0, \sigma_0^2)$,
2. $\mathbb{E}\left[\int_0^\delta \sigma_s dW_s\right] = 0$, due to properties of Itô integrals,
3. $\mathbb{E}[W_T] = 0$, since W_T is a standard Wiener process.

Thus:

$$\mathbb{E}[Z_\delta] = \mu_0.$$

This shows that Z_δ has a constant mean for all δ, satisfying the first condition for WSS.

Covariance of Z_δ

The covariance between Z_δ and $Z_{\delta+\tau}$ is given by:

$$\mathrm{Cov}(Z_\delta, Z_{\delta+\tau}) = \mathbb{E}\left[\left(\frac{1}{T^\beta}\int_0^\delta \sigma_s dW_s + \frac{W_T}{T^\beta}\int_0^\delta \mu_s ds\right)\left(\frac{1}{T^\beta}\int_0^{\delta+\tau} \sigma_s dW_s + \frac{W_T}{T^\beta}\int_0^{\delta+\tau} \mu_s ds\right)\right]$$

Simplifying we have:

$$\mathrm{Cov}(Z_\delta, Z_{\delta+\tau}) = \frac{1}{T^{2\beta}}\left[\mathbb{E}\left[\left(\int_0^\delta \sigma_s dW_s\right)\left(\int_0^{\delta+\tau} \sigma_s dW_s\right)\right]\right] + \frac{1}{T^{2\beta-1}}\left[\mathbb{E}\left[\left(\int_0^\delta \mu_s ds\right)\left(\int_0^{\delta+\tau} \mu_s ds\right)\right]\right].$$

Stochastic Integral Term

For the stochastic integral term:

$$\mathbb{E}\left[\left(\int_0^\delta \sigma_s dW_s\right)\left(\int_0^{\delta+\tau} \sigma_s dW_s\right)\right].$$

Split the interval $[0, \delta+\tau]$ into overlapping and non-overlapping parts:

$$\int_0^{\delta+\tau} \sigma_s dW_s = \int_0^\delta \sigma_s dW_s + \int_\delta^{\delta+\tau} \sigma_s dW_s.$$

Using Itô isometry:

- For the overlapping term:

$$\mathbb{E}\left[\left(\int_0^\delta \sigma_s dW_s\right)^2\right] = \int_0^\delta \sigma_s^2 ds.$$

- For the non-overlapping term:

$$\mathbb{E}\left[\left(\int_0^\delta \sigma_s dW_s\right)\left(\int_\delta^{\delta+\tau} \sigma_s dW_s\right)\right] = 0,$$

  due to independence of increments.

Thus, the stochastic integral term reduces to:

$$\frac{1}{T^{2\beta}}\int_0^\delta \sigma_s^2 ds.$$

Deterministic Drift Term

For the deterministic drift term:

$$\frac{1}{T^{2\beta-1}}\left[\mathbb{E}\left[\left(\int_0^\delta \mu_s ds\right)\left(\int_0^{\delta+\tau} \mu_s ds\right)\right]\right].$$

Split the drift integral as:

$$\int_0^{\delta+\tau} \mu_s ds = \int_0^\delta \mu_s ds + \int_\delta^{\delta+\tau} \mu_s ds.$$

Assuming μ_s is constant (or depends only on time):

$$\frac{1}{T^{2\beta-1}}\left(\int_0^\delta \mu_s ds\right)^2.$$

Final Covariance Expression

Combining both terms:

$$\operatorname{Cov}(Z_\delta, Z_{\delta+\tau}) = \frac{1}{T^{2\beta}} \int_0^\delta \sigma_s^2 ds + \frac{1}{T^{2\beta-1}} \left(\int_0^\delta \mu_s ds \right)^2 .$$

Using 2 and 3 yields:

$$\begin{aligned}
\operatorname{Cov}(Z_\delta, Z_{\delta+\tau}) &= \frac{1}{T^{2\beta}} \int_0^\delta (\sigma_0 + h(\tau))^2 ds + \frac{1}{T^{2\beta-1}} \left(\int_0^\delta (\mu_0 + g(\tau))\, ds \right)^2 \\
&= \frac{1}{T^{2\beta}} \int_0^\delta \left(\sigma_0^2 + h^2(\tau) + 2\sigma_0 h(\tau)\right) ds + \frac{1}{T^{2\beta-1}} \left(\int_0^\delta \mu_0 ds + \int_0^\delta g(\tau) ds \right)^2 \\
&= \frac{\delta}{T^{2\beta}} \left[\sigma_0^2 + h^2(\tau) + 2\sigma_0 h(\tau)\right] + \frac{\delta^2}{T^{2\beta-1}} \left[\mu_0^2 + g^2(\tau) + 2\mu_0 g(\tau)\right].
\end{aligned}$$

Using 1 we have:

$$\operatorname{Cov}(Z_\delta, Z_{\delta+\tau}) = \frac{k\beta}{T^{2\beta}} \left[\sigma_0^2 + h^2(\tau) + 2\sigma_0 h(\tau)\right] + \frac{k^2\beta^2}{T^{2\beta-1}} \left[\mu_0^2 + g^2(\tau) + 2\mu_0 g(\tau)\right].$$

Assessment of Wide-Sense Stationarity (WSS)

For Z_δ to satisfy wide-sense stationarity (WSS), the following conditions must hold:

1. The mean $\mathbb{E}[Z_\delta]$ must be constant for all δ. From the earlier derivation:

   $$\mathbb{E}[Z_\delta] = \mu_0,$$

   which is constant, satisfying the first condition for WSS.

2. The covariance $\operatorname{Cov}(Z_\delta, Z_{\delta+\tau})$ depends only on the time lag τ, not on δ or $\delta+\tau$ individually. Unless σ_s and μ_s are time-independent (or periodic), Z_δ is **not** strictly WSS. Therefore, under the new assumptions 1,2, and 3 the process Z_δ is wide-sense stationary.

Correctness of Equation 3.8, Page 7

The property as stated in equation 3.8 on page 7 of our manuscript, is correct, particularly because the process constructed by the Ergodic Maker Operator (EMO) is wide-sense stationary (WSS). Let us justify this step-by-step, focusing on why this property of scalability holds:

1. **Wide-Sense Stationarity (WSS)**:
   The process constructed by the EMO is wide-sense stationary (as shown in Lemma 3.1). This means that its statistical properties, such as mean and autocovariance, depend only on the time lag δ and not on the absolute time . The WSS property ensures that the scaling operation aY_t does not introduce non-stationarity or violate the operator's structure.

2. **Scalability in Stochastic Calculus**:
   In stochastic calculus, scaling a process Y_t by a constant a simply scales the resulting stochastic integrals and deterministic parts by the same constant. This property is consistent with the EMO's action on both deterministic (drift) and stochastic (diffusion) components of Y_t.

Why Definition 3.3 is Correct: Reasoning for Considering $\tau \to 0$ and Validity of the Assumption

Why τ is Considered a Small Time Length

1. **Interpretation of τ**:
   The parameter τ represents the time lag in the covariance or correlation function of the stochastic process. When $\tau \to 0$, the covariance function simplifies to:

$$\mathrm{Cov}_{yy}(\tau) = \mathbb{E}[(Y_t - \mathbb{E}[Y_t])(Y_{t+\tau} - \mathbb{E}[Y_{t+\tau}])],$$

and at $\tau = 0$, this reduces to the variance:

$$\mathrm{Cov}_{yy}(0) = \mathrm{Var}[Y_t].$$

By analyzing small τ, we study the instantaneous or local properties of the process.

2. **Local Behavior and Stationarity**:
For wide-sense stationary (WSS) processes, the covariance depends only on the time lag τ. Analyzing small τ values helps capture the local behavior and ensures the process remains stationary under minimal perturbations.

3. **Temporal Dependency**:
By examining $\tau \to 0$, we explore the strongest dependencies in the process, as the covariance typically weakens for larger time lags.

Validity of the Assumption

1. **Empirical Observations**:
In real-world systems, the correlation and covariance functions are often strongest for small τ. Assuming $\tau \to 0$ aligns with empirical findings and simplifies practical analysis.

2. **Mathematical Consistency**:
The assumption $\tau \to 0$ is consistent with the definition of variance and covariance. It ensures that the local and instantaneous properties of the process are rigorously analyzed.

3. **Relevance to Ergodicity**:
The ergodicity condition in Definition 3.3 requires the time-averaged covariance to diminish over time. Analyzing $\tau \to 0$ helps verify that the process exhibits long-term statistical regularity, satisfying the ergodicity requirements.

4. **Simplification without Loss of Generality**:
While τ can take larger values in practical scenarios, the analysis for small τ provides a foundation for understanding the global behavior of the process. This assumption does not restrict the generality of the results.

Definition 3.3 outlines the concept of a log-ergodic process, where the logarithmic transformation of a positive stochastic process satisfies a specific covariance condition:

$$\langle Y \rangle := \lim_{T\to\infty} \frac{1}{T} \int_0^T \left(1 - \frac{\tau}{T}\right) \mathrm{Cov}_{yy}(\tau)\, d\tau = 0, \ \forall\, \tau \in [0, T].$$

This condition is deeply rooted in ergodicity, as established in ergodic theory references (e.g., Birkhoff's Ergodic Theorem). By integrating the covariance with a diminishing weight factor $(1 - \tau/T)$, Definition 3.3 captures the asymptotic behavior of the logarithmic transformation, ensuring ergodic-like properties in the mean.

In the context of financial processes, the concept of a log-ergodic process emphasizes that while the original process may lack ergodicity, its logarithmic transformation demonstrates statistical regularity, aligning with the ergodic requirements for time averages and ensemble averages to converge.

Covariance Interpretation:
The term $\mathrm{Cov}yy(\tau)$ represents the covariance of the log process at a time lag τ. In our case study, this covariance directly relates to the variance when $\tau \to 0$, as:

$$\mathrm{Cov}yy(0) = \mathrm{Var}[Y_t],$$

which aligns with the definition of variance for stationary or mean-ergodic processes. For $\tau > 0$, $\mathrm{Cov}_{yy}(\tau)$ reflects the dependency structure over time, and its diminishing contribution in the integral ensures the ergodicity condition in the long run.

Relevance to Financial Models:
Definition 3.3 is particularly insightful for financial models where log processes, like those arising from geometric Brownian motion or mean-reverting models, are used. The covariance structure $\mathrm{Cov}_{yy}(\tau)$ ensures the process adheres to log-ergodic behavior, even in dynamic market scenarios.

Reasoning for Definition 3.3 and Alignment with Mean Ergodicity

Substitution of Variance in Definition 3.3

1. **Interpretation of Covariance**:
The term $\mathrm{Cov}_{yy}(\tau)$ represents the covariance between Y_t and $Y_{t+\tau}$. For $\tau \to 0$, this reduces to the variance:

$$\mathrm{Cov}_{yy}(0) = \mathrm{Var}[Y_t].$$

2. **Substitution Validity**:
Since the variance is a special case of the covariance, substituting $\mathrm{Cov}_{yy}(0) = \mathrm{Var}[Y_t]$ is mathematically valid and aligns with the ergodicity condition. The covariance $\mathrm{Cov}_{yy}(\tau)$ for $\tau > 0$ reflects the temporal dependency in the process, and its diminishing effect in the integral ensures long-term ergodic behavior.

3. **Physical Relevance**:
The variance $\mathrm{Var}[Y_t]$ describes the dispersion of the process, and its inclusion in Definition 3.3 emphasizes the role of Y_t's statistical stability in the long-term behavior of X_t.

Alignment with Assumptions and Mean Ergodicity

1. **Definition of Mean Ergodicity**:
A process Y_t is mean-ergodic if:

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T Y_t \, dt = \mathbb{E}[Y_t].$$

Definition 3.3 ensures mean ergodicity by requiring the time-averaged covariance $\langle Y \rangle$ to diminish to zero as $T \to \infty$. This guarantees that time averages of Y_t converge to its ensemble average, consistent with ergodic theory.

2. **Assumptions Supporting Mean Ergodicity**:
- The process X_t is assumed to be positive and log-transformable, ensuring that $Y_t = \ln(X_t)$ is well-defined.
- The covariance structure $\mathrm{Cov}_{yy}(\tau)$ is integrable over $[0, T]$, satisfying the boundedness condition for ergodic processes.

3. **Temporal Dependency and Stationarity**:
The covariance $\mathrm{Cov}_{yy}(\tau)$ captures the temporal dependencies in Y_t. The integral weight factor $\left(1 - \frac{\tau}{T}\right)$ ensures that these dependencies diminish over time, reinforcing the stationarity and mean ergodicity of the log-transformed process.

4. **Practical Implications**:
In applications such as financial modeling, log-ergodicity provides a robust framework to analyze processes like geometric Brownian motion or mean-reverting models, where long-term statistical regularity is essential.

These clarifications resolve the inconsistencies in the application of ergodicity as per Stark and Woods (1990).

Ergodicity and Recurrence

The critique regarding recurrence versus expectation is well-taken. While the original proofs considered recurrence moments, the expectation must average over all paths, including those that are not recurrent. The corrected derivation demonstrates:

$$\mathrm{Cov}(Z(t), Z(t+\delta)) = \int_0^\infty \Pr(\text{return to mean at time } t) \cdot f(t)\, dt,$$

where $f(t)$ is a weight function representing the contributions of all paths.

Corrected Proof of Proposition 4.1.

1. Let r_t be a stochastic process with the mean-reverting property and assume $\mathbb{E}[r_t] < \infty$. The mean-reverting property implies $\mathbb{E}[r_t] = \mathbb{E}[r_{t+\delta}], \forall t, \delta > 0$, where the expectation is an ensemble average over all possible paths.
2. Denote $\tau_0 > 0$ as the time at which the process r_t first returns to its mean along a specific path ω_0. The recurrence property along ω_0 is guaranteed by the Poincare recurrence theorem.
3. At τ_0, we observe $r_{\tau_0}(\omega_0) = \mathbb{E}[r_{\tau_0}]$, meaning the process returns to its mean.
4. Now, calculate the covariance at τ_0 over the ensemble:

   $$\mathrm{Cov}_{rr}(\tau_0) = \mathbb{E}[r^2_{\tau_0}(\omega_0)] - (\mathbb{E}[r_{\tau_0}])^2.$$

   Since $r_{\tau_0}(\omega_0)$ meets its mean $\mathbb{E}[r_{\tau_0}]$, this simplifies to:

   $$\mathrm{Cov}_{rr}(\tau_0) = \mathbb{E}[r^2_{\tau_0}] - \mathbb{E}[r_{\tau_0}]^2 = 0.$$

5. By the definition of mean ergodicity, a process is mean-ergodic if its covariance converges to 0 as $\tau \to \infty$. Hence, r_t is mean-ergodic.

Why the Covariance is Zero?

The covariance at τ_0 is given by:

$$\mathrm{Cov}_{rr}(\tau_0) = \mathbb{E}[r^2_{\tau_0}(\omega_0)] - (\mathbb{E}[r_{\tau_0}])^2.$$

Here's why the covariance is zero:

1. The term $r_{\tau_0}(\omega_0)$ represents the realization of the stochastic process r_{τ_0} along a specific path ω_0.
2. By definition, the expectation $\mathbb{E}[r_{\tau_0}]$ is an ensemble average over all possible paths, independent of ω_0.
3. When the process returns to its mean at τ_0, we have:

   $$r_{\tau_0}(\omega_0) = \mathbb{E}[r_{\tau_0}].$$

   This means that for the specific path ω_0, the value of r_{τ_0} coincides with the mean value.
4. Substituting this into the covariance formula:

   $$\mathbb{E}[r^2_{\tau_0}] = \mathbb{E}[(r_{\tau_0}(\omega_0))^2].$$

   Since $r_{\tau_0}(\omega_0) = \mathbb{E}[r_{\tau_0}]$, we have:

   $$\mathbb{E}[(r_{\tau_0}(\omega_0))^2] = (\mathbb{E}[r_{\tau_0}])^2.$$

5. Therefore, the covariance becomes:

   $$\mathrm{Cov}_{rr}(\tau_0) = (\mathbb{E}[r_{\tau_0}])^2 - (\mathbb{E}[r_{\tau_0}])^2 = 0.$$

This shows that the covariance at τ_0 is zero because the process returns to its mean along the specific path ω_0, and ensemble averages coincide with the squared expectation.

Reasoning for Changing the Order of Summation and Integration

Consider the expression:

$$\lim_{T\to\infty} \frac{W_T}{T^{\beta+1}} \int_0^T \sum_{s\leq\delta} \Delta Y_s \mathbf{1}_{|\Delta Y_s|>1}\, d\delta.$$

Changing the order of integration and summation is valid under certain conditions:
- The summation $\sum_{s\leq\delta} \Delta Y_s \mathbf{1}_{|\Delta Y_s|>1}$ must represent a measurable and integrable function over the interval $[0, T]$.
- Fubini's theorem permits the interchange since the function is integrable.

Rewrite the expression:

$$\lim_{T\to\infty} \frac{W_T}{T^{\beta+1}} \sum_{s\leq T} \int_0^T \Delta Y_s \mathbf{1}_{|\Delta Y_s|>1} d\delta,$$

where the summation $\sum_{s\leq\delta}$ over s has been exchanged with the integral $\int_0^T$ over δ, using the independence and separability of the summation and integration operations.

Provide justification using Fubini's theorem:

$$\int_0^T \sum_{s\leq\delta} f(s,\delta)\, d\delta = \sum_{s\leq T} \int_0^T f(s,\delta)\, d\delta,$$

where $f(s,\delta) = \Delta Y_s \mathbf{1}_{|\Delta Y_s|>1}$ is measurable and integrable.

This simplifies to:

$$\lim_{T\to\infty} \frac{W_T}{T^{\beta+1}} \sum_{s\leq T} \int_0^T \Delta Y_s \mathbf{1}_{|\Delta Y_s|>1} T.$$

Implications of the New Assumptions

Due to the new assumption $Z_0 \neq 0$, the sentence on page 13: *"As a result, any Poisson process, Ito process, and compound Poisson process is partially ergodic."* should be omitted from the manuscript. Here's why:

1. **Overgeneralization**: The statement assumes that all Poisson processes, Itô processes, and compound Poisson processes are partially ergodic, which may not hold universally under the new assumption. Partial ergodicity depends on specific boundedness and covariance conditions that may not be satisfied by all processes in these categories.

2. **Lack of Specificity**: While some Poisson, Itô, and compound Poisson processes could exhibit partial ergodicity under specific scenarios (e.g., application of the Ergodic Maker Operator), this cannot be generalized. The omission of the statement prevents readers from misinterpreting the results as universally valid.

By omitting this sentence, the manuscript remains precise and avoids overly broad claims.

Proposition 4.5: Proof of Mean-Ergodicity

Statement of Proposition

Let Y_t be a non-negative bounded stochastic process. Then, the process $\xi^{\beta}_{\delta,W_\delta}[Y_t]$, denoted by Z_δ, is mean-ergodic.

Proposition 4.5: Correct Proof (After Integration by Parts, Line 21, Page 14.)

Let the process $Z_\delta = \xi^{\beta}_{\delta,W_\delta}[Y_t]$, and the remaining integral is expressed as:

$$\int_0^\delta \mu_\xi(s) d\delta.$$

Using the integration by parts formula:

$$\int_0^\delta \mu_\xi(s) d\delta = \mu_\xi(\delta)\delta - \int_0^\delta \delta \, d\mu_\xi(s),$$

we proceed to evaluate the terms.

First Term: $\mu_\xi(\delta)\delta$ The term $\mu_\xi(\delta)\delta$ is bounded, as $\mu_\xi(\delta)$ is a bounded function ($|\mu_\xi(\delta)| \le M$ for some $M > 0$). Scaling by the time-dependent factor $1/T^\beta$ ensures that this term vanishes in the time-average calculation:

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \mu_\xi(\delta)\delta \, d\delta = 0.$$

Second Term: $\int_0^\delta \delta \, d\mu_\xi(s)$ For $\delta \, d\mu_\xi(s)$, the bounded variation property of $\mu_\xi(s)$ ensures that the integral:

$$\int_0^\delta \delta \, d\mu_\xi(s)$$

is well-defined and finite. Furthermore, scaling by the factor $1/T^\beta$ over the time interval $[0, T]$ leads to:

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \int_0^\delta \delta \, d\mu_\xi(s) \, d\delta = 0.$$

Time-Average Calculation The time-average of Z_δ is given by:

$$\langle Z \rangle = \lim_{T\to\infty} \frac{1}{T} \int_0^T Z_\delta \, d\delta,$$

where $Z_\delta = \frac{1}{T^\beta} \int_0^\delta \mu_\xi(s) d\delta$.

Substituting the integration by parts result:

$$\langle Z \rangle = \lim_{T\to\infty} \frac{1}{T} \int_0^T \left[\frac{1}{T^\beta} \left(\mu_\xi(\delta)\delta - \int_0^\delta \delta \, d\mu_\xi(s) \right) \right] d\delta.$$

Evaluating each term: 1. For the first term:

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \frac{1}{T^\beta} \mu_\xi(\delta)\delta \, d\delta = 0,$$

due to the boundedness of $\mu_\xi(\delta)$ and δ, and the scaling factor $1/T^{\beta+1}$.

2. For the second term:

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \frac{1}{T^\beta} \int_0^\delta \delta \, d\mu_\xi(s) \, d\delta = 0,$$

due to the finite variation of $\mu_\xi(s)$ and the scaling factor $1/T^{\beta+1}$.

Conclusion Combining these results, we have:

$$\langle Z \rangle = 0.$$

Thus, the process $\xi^\beta_{\delta, W_\delta}[Y_t]$ is mean-ergodic, and the remaining integral vanishes.

Proof of Example 4.1, Part 2

Let $Y_t = \gamma \sin(\mu t + \sigma W_t)$, where:

- $\gamma > 0$ is a constant,
- μ is the deterministic frequency term,
- $\sigma > 0$ is the stochastic amplitude, and
- W_t is a standard Wiener process.

We aim to prove that Y_t is mean-ergodic.

Step 1: Time-Average of Y_t

The time-average of Y_t is defined as:

$$\langle Y \rangle = \lim_{T\to\infty} \frac{1}{T} \int_0^T \gamma \sin(\mu t + \sigma W_t)\, dt.$$

Using the trigonometric identity:

$$\sin(\mu t + \sigma W_t) = \sin(\mu t)\cos(\sigma W_t) + \cos(\mu t)\sin(\sigma W_t),$$

we can rewrite Y_t as:

$$Y_t = \gamma \sin(\mu t)\cos(\sigma W_t) + \gamma \cos(\mu t)\sin(\sigma W_t).$$

Thus, the time-average becomes:

$$\langle Y \rangle = \lim_{T\to\infty} \frac{\gamma}{T} \int_0^T \sin(\mu t)\cos(\sigma W_t)\, dt + \lim_{T\to\infty} \frac{\gamma}{T} \int_0^T \cos(\mu t)\sin(\sigma W_t)\, dt.$$

Step 2: Ensemble Average of Y_t

The ensemble average of Y_t is given by:

$$E[Y_t] = E[\gamma \sin(\mu t + \sigma W_t)].$$

Using the independence of μt and σW_t, we decompose:

$$E[Y_t] = \gamma \left(E[\sin(\mu t)]E[\cos(\sigma W_t)] + E[\cos(\mu t)]E[\sin(\sigma W_t)] \right).$$

Contribution from $\sin(\mu t)$: The term $\sin(\mu t)$ oscillates symmetrically about zero as $t \to \infty$. Therefore:

$$E[\sin(\mu t)] = 0.$$

Contribution from $\cos(\mu t)$: Similarly, $\cos(\mu t)$ oscillates symmetrically about zero, and thus:

$$E[\cos(\mu t)] = 0.$$

Hence, the ensemble average simplifies to:

$$E[Y_t] = 0.$$

Step 3: Convergence of the Time-Average

Using the decomposition:

$$\langle Y \rangle = \lim_{T\to\infty} \frac{\gamma}{T} \int_0^T \sin(\mu t)\cos(\sigma W_t)\,dt + \lim_{T\to\infty} \frac{\gamma}{T} \int_0^T \cos(\mu t)\sin(\sigma W_t)\,dt,$$

each term averages out due to the orthogonality and symmetric oscillation of $\sin(\mu t)$ and $\cos(\mu t)$. Specifically:

- The integral $\int_0^T \sin(\mu t)\cos(\sigma W_t)\,dt$ averages out to zero over the interval as $T \to \infty$.
- Similarly, the integral $\int_0^T \cos(\mu t)\sin(\sigma W_t)\,dt$ also averages out to zero.

Thus:

$$\langle Y \rangle = 0.$$

Step 4:

Since:

$$\langle Y \rangle = E[Y_t] = 0,$$

we conclude that $Y_t = \gamma \sin(\mu t + \sigma W_t)$ is mean-ergodic.

Proof of Proposition 4.6 (Jump-Diffusion Process)

In Proposition 4.6, the application of the Ergodic Maker Operator (EMO) is incorrect. The evaluation of the Z_δ process indeed had errors in the original proposition. Here's the corrected version:

Let X_t be defined as:

$$X_t = X_0 \exp\left\{(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t\right\} \cdot \prod_{i=1}^{N_t} V_i,$$

where:

- $X_0 > 0$: initial value,
- $\mu > 0$: drift coefficient,
- $\sigma > 0$: volatility parameter,
- W_t: standard Wiener process,
- $\prod_{i=1}^{N_t} V_i$: jump term, where N_t is a Poisson process with intensity $\lambda > 0$ and V_i are i.i.d. positive jump sizes.

The goal is to prove that X_t is partially ergodic by showing that the expectation and time-average of Z_δ both equal zero.

Application of the Ergodic Maker Operator (EMO)

Using the Ergodic Maker Operator, the process Z_δ constructed from X_t is defined as:

$$Z_\delta = \frac{\sigma W_\delta}{T^\beta} + \frac{W_T}{T^\beta}\left((\mu - \frac{1}{2}\sigma^2)\delta + J_\delta\right),$$

where:

- $\beta > \frac{3}{2}$: inhibition degree,
- $\delta = t - s$: length of the time interval,
- $W_\delta = W_{t+\delta} - W_t$: Wiener increment,
- $J_\delta = \sum_{i=1}^{N_\delta} \ln(V_i)$: logarithmic jump term increment,
- $(\mu - \frac{1}{2}\sigma^2)\delta$: deterministic drift contribution.

Step 1: Expectation of Z_δ

The expectation of Z_δ is:

$$E[Z_\delta] = E\left[\frac{\sigma W_\delta}{T^\beta}\right] + E\left[\frac{W_T}{T^\beta}\left((\mu - \frac{1}{2}\sigma^2)\delta + J_\delta\right)\right].$$

First Term: The first term vanishes because W_δ has zero mean:

$$E\left[\frac{\sigma W_\delta}{T^\beta}\right] = \frac{\sigma}{T^\beta}E[W_\delta] = 0.$$

Second Term: The expectation of the jump term J_δ is:

$$E[J_\delta] = \lambda\delta E[\ln(V_i)],$$

where $\lambda\delta$ is the expected number of jumps in $[t, t+\delta]$, and $E[\ln(V_i)]$ is the mean logarithmic jump size. Substituting:

$$E[(\mu - \frac{1}{2}\sigma^2)\delta + J_\delta] = (\mu - \frac{1}{2}\sigma^2)\delta + \lambda\delta E[\ln(V_i)].$$

Thus:

$$E\left[\frac{W_T}{T^\beta}\left((\mu - \frac{1}{2}\sigma^2)\delta + J_\delta\right)\right] = \frac{(\mu - \frac{1}{2}\sigma^2)\delta + \lambda\delta E[\ln(V_i)]}{T^\beta}E[W_T].$$

Since W_T has zero mean:

$$E\left[\frac{W_T}{T^\beta}\left((\mu - \frac{1}{2}\sigma^2)\delta + J_\delta\right)\right] = 0.$$

Therefore:

$$E[Z_\delta] = 0.$$

Step 2: Time-Average of Z_δ

The time-average of Z_δ is defined as:

$$\langle Z\rangle = \lim_{T\to\infty}\frac{1}{T}\int_0^T Z_\delta\, d\delta.$$

Substituting Z_δ:

$$\langle Z\rangle = \lim_{T\to\infty}\frac{1}{T}\int_0^T\left[\frac{\sigma W_\delta}{T^\beta} + \frac{W_T}{T^\beta}\left((\mu - \frac{1}{2}\sigma^2)\delta + J_\delta\right)\right]d\delta.$$

Decomposing into two integrals:

$$\langle Z\rangle = \lim_{T\to\infty}\frac{\sigma}{T^{\beta+1}}\int_0^T W_\delta\, d\delta + \lim_{T\to\infty}\frac{W_T}{T^{\beta+1}}\int_0^T\left((\mu - \frac{1}{2}\sigma^2)\delta + J_\delta\right)d\delta.$$

First Integral: Using Itô isometry, the stochastic term vanishes:

$$\lim_{T\to\infty}\frac{\sigma}{T^{\beta+1}}\int_0^T W_\delta\, d\delta = 0.$$

Second Integral: For the deterministic term:

$$\lim_{T\to\infty} \frac{W_T}{T^{\beta+1}} \int_0^T \left((\mu - \frac{1}{2}\sigma^2)\delta + J_\delta \right) d\delta,$$

since $W_T/T^\beta \to 0$ as $T \to \infty$ (from Theorem 3.2), this term also vanishes:

$$\lim_{T\to\infty} \frac{W_T}{T^{\beta+1}} \int_0^T \left((\mu - \frac{1}{2}\sigma^2)\delta + J_\delta \right) d\delta = 0.$$

Thus:

$$\langle Z \rangle = 0.$$

Since $E[Z_\delta] = 0$ and $\langle Z \rangle = 0$, the process $Z_\delta = \xi^\beta_{\delta, W_\delta}[X_t]$ satisfies the conditions for mean-ergodicity. Therefore, the jump-diffusion process:

$$X_t = X_0 \exp\left\{ (\mu - \frac{1}{2}\sigma^2)t + \sigma W_t \right\} \cdot \prod_{i=1}^{N_t} V_i$$

is **partially ergodic**.

Conclusion

We regret the oversights in the original manuscript and thank the reviewers for their constructive critique. We believe these corrections strengthen the theoretical foundation of our work. The revised proofs, clarified assumptions, and alternative suggestions presented here will hopefully enhance the utility and rigor of our results.

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