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Some Properties of Dominant Local Metric Dimension

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Abstract Let G be a connected graph with vertex set V. Let W_l be an ordered subset defined by $W_l = \{w_1, w_2, \ldots, w_n\} \subseteq V(G)$. Then W_l is said to be a dominant local resolving set of G if W_l is a local resolving set as well as a dominating set of G. A dominant local resolving set of G with minimum cardinality is called the dominant local basis of G. The cardinality of the dominant local basis of G is called the dominant local metric dimension of G and is denoted by $Ddim_l(G)$. We characterize the dominant local metric dimension for any graph G and for some commonly known graphs in terms of their domination number to get some properties of dominant local metric dimension.

Keywords dominating set, local resolving set, local metric dimension, dominant local resolving set, properties.

AMS 2010 subject classifications 05C12

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1. Introduction

The dominating set and dominating number of graphs were first studied in the 1950s. These notions examined the existence of a vertex set of a graph that causes every other vertex on the graph to be neighbors with at least one element of the vertex set [1]. Generally, subsets of vertices with this property are not unique and are called dominating sets. The dominating number of a graph is the cardinality of the smallest possible dominating sets. Another notion is the metric dimension of a graph that was first introduced by Harary and Melter and mentioned by [2]. Harary and Melter defined a resolving set of a graph as a set of vertices that makes each vertex on the graph have a different representation with respect to the vertex set. The representation of a vertex for the resolving set is presented as a k-ordered pair whose elements are the distance between the vertex and the vertices on the resolving set. The resolving set that has a minimum cardinality is called a basis. If the concept of metric dimension is obtained by looking at each vertex that has a different representation of the resolving set, then the concept of the local metric dimension views that every two neighboring vertices have different representations of the local resolving set [3].

From the definition of metric dimension, several concepts have emerged such as the local metric dimension ([3],[4],[5],[6]), dominant local metric dimension ([10], [17]), multiset dimension of graphs ([18],[19]), local multiset dimension [14], and on the central-local metric dimension [16], to mention a few. For example, research conducted successfully showed similarities between the metric dimension and the local metric dimension of graph products in [6]. In addition, the commutative characterization of comb and corona product graphs based on their

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metric dimensions was established by Susilowati et al. [7]. By combining the notions of metric dimension and dominating set, a new term called the resolving dominating set was introduced by Brigham et al. [8]. For an ordered set $W = \{w_1, w_2, \cdots, w_k\}$ of vertices and a vertex v in a connected graph G, the (metric) representation of v with respect to W is the k-vector $r(v|W) = (d(v, w_1), d(v, w_2), \cdots, d(v, w_k))$, where $d(x, y)$ represents the distance between the vertices x and y. The set W is a resolving set for G if distinct vertices of G have distinct representations with respect to W. A resolving set of minimum cardinality is called a minimum resolving set or a basis and the cardinality of a basis for G is its dimension $\dim G$. A set S of vertices in G is a *dominating set* for G if every vertex of G that is not in S is adjacent to some vertex of S. The minimum cardinality of a dominating set is the *domination number* $\gamma(G)$. A set of vertices of a graph G that is both resolving and dominating is a resolving dominating set. The minimum cardinality of a resolving dominating set is called the *resolving domination number* $\gamma_r(G)$ [8].

The concept of determining the resolving dominating number has been studied in the literature and was first introduced by Slater first in 1975 [20]. Furthermore, Behrooz, et al [9] determine the relations between the metric dimension and dominating number of graphs while Henning and Oellermann combined the concept of locating dominating set and metric dimensions in metric locating dominating set [10]. However, it is important to note that the concepts introduced by Brigham et al. and Henning and Oellermann are equivalent. New research on this topic was introduced by Susilowati, et al. which determined the dominant metric dimension of some well-known graphs [11]. By referring to several concepts that have been developed regarding the concept of the metric dimension, local metric dimension, dominating set, and dominant metric dimension, it is interesting to combine the concept of the local metric dimension with the concept of dominating set which we call *dominant local metric dimension*. A dominant local metric dimension is a vertex set that is both a minimum local resolving set and a dominating set. Since the existence of this set is not unique, it is interesting to study the minimum cardinality of the dominant local resolving set, called the *dominant local metric dimension*. Thus in this paper, besides formulating the new definition, we characterize the dominant local metric dimension of graph G and also determine the dominant local metric dimension of some classes of graphs.

The following theorems on special graphs such as path, cycle, star, complete, and complete bipartite graphs; will be used in the proof of our main results for characterizing the dominant local metric dimensions of graphs. The first theorem shows the dominating number $\gamma(G)$ of these graphs while Theorem 2 describes a local metric dimension for the complete and bipartite graphs introduced by [13]. We will denote the local metric dimension of graph G by $dim_l(G)$. All graphs considered in this study are simple and connected.

Theorem 1. Let G be a connected graph. The domination number of some of the graphs is given below[12]:
a. If $G = P_m$ or $G = C_n$ with $m \geq 2$ and $n \geq 3$, then $\gamma(G) = \left\lceil \frac{|V(G)|}{3} \right\rceil$.
b. If $G = K_m$ or $G = K_{1,n-1}$ with $m \geq 1$ and $n \geq 2$, then $\gamma(G) = 1$.
c. If $G = K_{m,n}$ with $m, n \geq 3$, then $\gamma(G) = 2$.

Theorem 2. Let G be a nontrivial connected graph of order n. Then $dim_l(G) = n - 1$ if and only if $G = K_n$ and $dim_l(G) = 1$ if and only if G is bipartite.[12]

2. Results and Discussion

We start this section by presenting the definition and some of the characteristics of the dominant local metric dimension for some graphs. The results of the dominant local metric dimension of some graphs, such as path, cycle, complete, star, and complete bipartite are also presented in this section.

Definition 3. Given a connected graph G. An ordered set $W_l = \{w_1, w_2, \ldots, w_n\} \subseteq V(G)$ is called a dominant local resolving set if W_l is a local resolving set and a dominating set of G. The dominant local resolving set with minimum cardinality is called a dominant local basis. The number of vertices in a dominant local basis of G is called the dominant local metric dimension and is denoted by $Ddim_l(G)$.

The examples in Figure 1 illustrate useful observations of the dominant local resolving set of graphs. For the graph in Figure 1(a.), $\{v_1, v_3\}$ is considered to be a dominant local resolving set since it satisfies the definition of the dominant local metric dimension, where the dominant local basis equals two. However, for Figure 1(b.), $\{u_1, u_5\}$ is not a dominant local resolving set since there is a vertex that can not be dominated by either u_1 or u_5. Finally, for Figure 1(c.), $\{t_1\}$ is not a dominant local resolving set since its two adjacent vertices have the same representation.

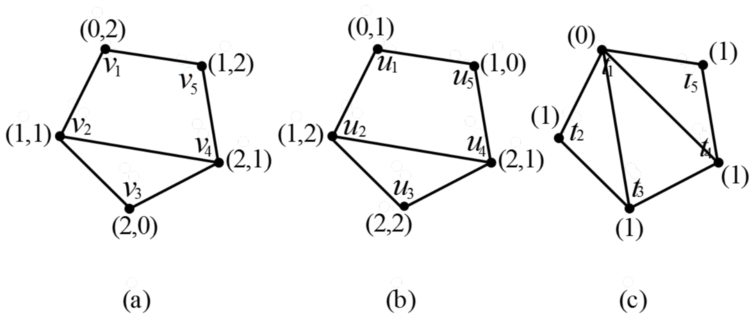

Figure 1. (a.) The dominant local resolving set of a graph (b.) The local resolving set of a graph (c.) The dominating set of a graph

The following lemma provides the properties of the local resolving set of graph G.

Lemma 4. Let G be a connected graph and $S \subseteq V(G)$ be an ordered set. Every set S containing a local resolving set is a local resolving set

Proof. Let G be a connected graph and $W_l = \{v_i | i = 1, 2, \ldots, k\} \subseteq V(G)$ be a local resolving set of G, such that $W_l \subseteq S \subseteq V(G)$. Then for every two adjacent vertices $u, v \in V(G)$, $r(u|W_l) \neq r(v|W_l)$. Since $W_l \subseteq S$, consequently $r(u|S) \neq r(v|S)$. Hence S is also a local resolving set of G.

The existence of element 0 in the representation of a vertex with respect to a local resolving set is described in Lemma 5.

Lemma 5. Let G be a connected graph and $W_l \subseteq V(G)$ be an ordered set. For every $v_i, v_j \in W_l$, $r(v_i|W_l) \neq r(v_j|W_l)$, for $i \neq j$.

Proof. Let $W_l = \{v_i | i = 1, 2, \ldots, k\} \subseteq V(G)$ be an ordered set. Since for every $v_i, v_j \in W_l$ where $i \neq j$, $d(v_i, v_i) = 0$ and $d(v_i, v_j) \neq 0$. Hence there exist 0 on i^{th} element in $r(v_i|W_l)$ for every $v_i \in W_l$. As a result, $r(v_i|W_l) \neq r(v_j|W_l)$ for $i \neq j$.

Based on Lemma 4 and Lemma 5, we can determine the lower and upper bound of a dominant local metric dimension of graph G as shown in Lemma 6.

Lemma 6. For every connected graph G of order n,

$$\max\{\gamma(G), \dim_l(G)\} \leq D\dim_l(G) \leq \min\{\gamma(G) + \dim_l(G), n-1\}.$$

Proof. Let G be a connected graph of order n. By Definition 3, $Ddim_l(G) \geq \gamma(G)$ and $Ddim_l(G) \geq dim_l(G)$, implying that $Ddim_l(G) \geq max\{\gamma(G), dim_l(G)\}$. Since the local resolving set and dominating set of a graph may have not intersected, and a set which consists $n-1$ vertices in G is always the local resolving set and dominating set of G, then $Ddim_l(G) \leq min\{\gamma(G) + dim_l(G), n-1\}$.

Graph H in Figure 2 is an example of a graph that satisfies the lower bound of the local dominant metric dimension as presented in Lemma 5.

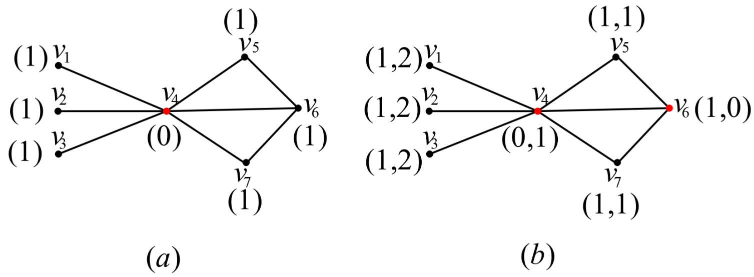

Figure 2. Graph H.

The vertex $v_4 \in V(H)$ in Figure 2(a) forms a dominating set, thus $\gamma(H) = 1$. Note also that $v_4, v_6 \in V(H)$ in Figure 2(b) form a local basis of graph H, therefore $dim_l(H) = 2$. Since a local basis of H is also a local dominant basis, then $Ddim_l(H) = 2$. Hence, H in Figure 2 satisfies the lower bound of the dominant local metric dimension.

Next, we discuss the dominant local metric dimension of the path (P_n). Note that the local resolving set of P_n presented in Lemma 7 does not necessarily imply the smallest cardinality of the local resolving set. The lemma simply shows part of the elements of the local resolving set. The complete proof for the dominant local metric dimension for a path will be presented in Theorem 8.

Lemma 7. Let P_n be a path of order $n \geq 5$ with the vertex set $V(P_n) = \{v_i | i = 1, 2, 3, \ldots, n\}$ and edge set $E(P_n) = \{v_i v_{i+1} | i = 1, 2, 3, \ldots, n-1\}$. Then $W_l = \{v_2, v_5\}$ is a local resolving set of P_n.

Proof. Choose $W_l = \{v_2, v_5\}$, then for every $v_i \in V(P_n)$ with $i = 1, 2, 3, \ldots, n$, $d(v_i, v_j) = |i - j|$, for $1 \leq i, j \leq n$. As a consequence,

$$d(v_i, v_2) = |i - 2|; 1 \leq i \leq n.$$
$$d(v_i, v_5) = |i - 5|; 1 \leq i \leq n.$$

Hence for every two adjacent vertices $v_i v_{i+1} \in E(P_n)$, $d(v_i, v_2) \neq d(v_{i+1}, v_2)$ and $d(v_i, v_5) \neq d(v_{i+1}, v_5)$ for $i = 1, 2, \ldots, n-1$. Therefore, $r(v_i | W_l) \neq r(v_{i+1} | W_l)$. Thus, it can be concluded that W_l is a local resolving set of P_n.

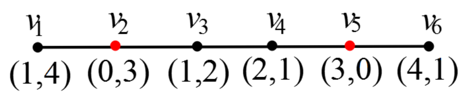

Figure 3. The local resolving set of P_6

Figure 3 shows that $\{v_2, v_5\}$ is a local resolving set of path P_6. The dominant local metric dimension of the path is presented below.

Theorem 8. Let P_n be a path of order $n \geq 2$. The dominant local metric dimension of P_n is $Ddim_l(P_n) = \gamma(P_n)$.

Proof. Let $V(P_n) = \{v_i | i = 1, 2, 3, \ldots, n\}$ and $E(P_n) = \{v_i v_{i+1} | i = 1, 2, 3, \ldots, n-1\}$. We consider two cases based on n.

a. Case 1: For $n = 2, 3, 4$
   1. If $n = 2$ or 3, choose $W_l = \{v_2\}$. It is easy to see that W_l is a dominant local resolving set of P_n. The cardinality of W_l is $|W_l| = 1 = \lceil \frac{2}{3} \rceil = \lceil \frac{n}{3} \rceil$ for $n = 2$, and $|W_l| = 1 = \lceil \frac{3}{3} \rceil = \lceil \frac{n}{3} \rceil$ for $n = 3$.
   2. If $n = 4$, choose $W_l = \{v_2, v_4\}$. Taking any two adjacent vertices in P_4, it is easy to see that one of them is an element of W_l. So, $W_l = \{v_2, v_4\}$ is a local resolving set of P_4. It is also obvious that W_l is also a dominating set. Hence, $|W_l| = 2 = \lceil \frac{4}{3} \rceil = \lceil \frac{n}{3} \rceil$.

By Theorem 1, we know that $\gamma(P_n) = \lceil \frac{n}{3} \rceil$, so W_l is a dominant local resolving set with minimum cardinality and $Ddim_l(P_n) = \lceil \frac{n}{3} \rceil = \gamma(P_n)$, for $n = 2, 3, 4$.

b. Case 2: For $n \geq 5$
Choose $W_l = \{v_2, v_5, v_8, v_{11}, \ldots, v_{3i-1}\}$ for $n \equiv 0(mod\,3)$ and $W_l = \{v_2, v_5, v_8, v_{11}, \ldots, v_{3i-1}, v_n\}$ for $n \equiv 0(mod\,3)$, then $|W_l| = \lceil \frac{n}{3} \rceil$. Since $\{v_2, v_5\} \subseteq W_l$, then by on Lemma 7 and Lemma 4, we get that W_l is a local resolving set of P_n. Next, since $E(P_n) = \{v_i v_{i+1} | i = 1, 2, 3, \ldots, n-1\}$, v_{3i-1} is adjacent to v_{3i-2} and v_{3i} is adjacent to $v_{3(i+1)-2}$. Hence, W_l is a dominating set of P_n. Based on Theorem 1, we know that $\gamma(P_n) = \lceil \frac{n}{3} \rceil$, so W_l is a minimum dominant local resolving set of P_n and $Ddim_l(P_n) = \lceil \frac{n}{3} \rceil = \gamma(P_n)$, for $n \geq 5$.

From the two cases above, it can be concluded that $Ddim_l(P_n) = \gamma(P_n)$.

The next result is the dominant local metric dimension of a cycle. By similar procedures with the dominant local metric dimension of the path, we start the proof by finding the local resolving set of a cycle as presented in Lemma 9 which does not imply a proof of the smallest cardinality of the local resolving set. The lemma shows part of the elements of the local resolving set that will be used in the proof of Theorem 10.

Lemma 9. Let C_n be a cycle with order $n \geq 4$, $E(C_n) = \{v_i v_{i+1} | i = 1, 2, 3, \ldots, n-1\} \bigcup \{v_n v_1\}$. Then $W_l = \{v_1, v_4\}$ is a local resolving set of C_n.

Proof. Let C_n be a cycle with order $n \geq 4$, $V(C_n) = \{v_i | i = 1, 2, 3, \ldots, n\}$ and edge set $E(C_n) = \{v_i v_{i+1} | i = 1, 2, 3, \ldots, n-1\} \bigcup \{v_n v_1\}$, and $W_l = \{v_1, v_4\}$.

i. if n is odd, then for every $v_i \in V(C_n)$

$$r(v_i|v_1) = \begin{cases} i-1, i = 1, 2, 3, \ldots, \lceil \frac{n}{2} \rceil \\ \lfloor \frac{n}{2} \rfloor, i = \lceil \frac{n}{2} \rceil + 1 \\ n+1-i, i = \lceil \frac{n}{2} \rceil + 2, \lceil \frac{n}{2} \rceil + 3, \lceil \frac{n}{2} \rceil + 4, \ldots, n \end{cases}$$

$$r(v_i|v_4) = \begin{cases} |i-4|, i = 1, 2, 3, \ldots, \lceil \frac{n}{2} \rceil + 3 \\ \lfloor \frac{n}{2} \rfloor, i = \lceil \frac{n}{2} \rceil + 4 \\ n+4-i, i = \lceil \frac{n}{2} \rceil + 5, \lceil \frac{n}{2} \rceil + 6, \lceil \frac{n}{2} \rceil + 7, \ldots, n \end{cases}$$

ii. if n is even, then for every $v_i \in V(C_n)$

$$r(v_i|v_1) = \begin{cases} i-1, i = 1, 2, 3, \ldots, \frac{n}{2} + 1 \\ n+1-i, i = \lceil \frac{n}{2} \rceil + 2, \lceil \frac{n}{2} \rceil + 3, \lceil \frac{n}{2} \rceil + 4, \ldots, n \end{cases}$$

$$r(v_i|v_4) = \begin{cases} |i-4|, i = 1, 2, 3, \ldots, \frac{n}{2} + 4 \\ n+4-i, i = \lceil \frac{n}{2} \rceil + 5, \lceil \frac{n}{2} \rceil + 6, \lceil \frac{n}{2} \rceil + 7, \ldots, n \end{cases}$$

Based on the description above, for every $v_i v_j \in E(C_n)$ where $i \neq j$, $r(v_i|W_l) \neq r(v_j|W_l)$. Hence, W_l is a local resolving set of C_n for $n \geq 4$.

Theorema 10. Let C_n be a cycle of order $n \geq 4$. The dominant local metric dimension of C_n is $Ddim_l(C_n) = \gamma(C_n)$.

Proof. Let $V(C_n) = \{v_i | i = 1, 2, 3, \ldots, n-1\}$ with $n \geq 4$ and $E(C_n) = \{v_i v_{i+1} | i = 1, 2, 3, \ldots, n-1\} \bigcup \{v_n v_1\}$. Choose $W_l = \{v_1, v_4, v_7, v_{10}, \ldots, v_{3i-2}\}$, for $n \equiv 0(mod\,3)$ and $W_l = \{v_1, v_4, v_7, v_10, \ldots, v_{3i-2}, v_n\}$ for $n \equiv 0(mod\,3)$. Then $|W_l| = \lceil \frac{n}{3} \rceil$. Since $\{v_1, v_4\} \subseteq W_l$, then by Lemma 9 and Lemma 4, W_l is a local resolving set of C_n. Next, since $E(C_n) = \{v_i v_{i+1} | i = 1, 2, 3, \ldots, n-1\} \bigcup \{v_n v_1\}$, we can see that v_{3i-1} is adjacent to v_{3i-2} and v_{3i} is adjacent to $v_{3(i+1)-2}$, for $i = 1, 2, 3, \ldots, n-1$. Thus W_l is a dominating set of C_n. By Theorem 1, we know that $\gamma(C_n) = \lceil \frac{n}{3} \rceil = |W_l|$, then W_l is a minimum dominant local resolving set of C_n . Therefore, it can be concluded that $Ddim_l(C_n) = \gamma(C_n)$, for $n \geq 4$.

In what follows, Theorems 11 and 12 show the characterization of a graph of order n that has the dominant local metric dimension equal to 1 or $n-1$.

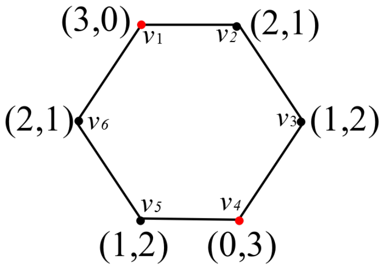

Figure 4. C_6 graph with the dominant local basis are V_1 and V_4

Theorem 11. Let G be a connected graph of order $n \geq 1$. Then $Ddim_l(G) = 1$ if and only if $G \equiv S_n$.

Proof. Let G be a connected graph of order $n \geq 1$.

1. If $Ddim_l(G) = 1$, then $G = S_n$.
   Let G be a connected graph of order $n \geq 1$ and $Ddim_l(G) = 1$, then there exists $W_l \subseteq V(G)$ as a dominant local basis of G with $|W_l| = 1$. We consider three cases below based on the value of n:
   Case 1. For $n = 1$ and 2, $G \equiv K_n$. Since $K_n \equiv S_n$, for $n = 1, 2$, $G \equiv S_n$ by Theorem 1.b.
   Case 2. For $n = 3$, suppose G is not a star graph, then G is a cycle graph C_3. Every singleton is not a local resolving set of C_3 which is contrary to $Ddim_l(G) = 1$, and so $G \equiv S_n$ by Theorem 1.b.
   Case 3. For $n > 3$, suppose G is not a star graph, then there are four possibilities for G as below:
   a. G is a cycle graph. Every singleton is not a dominating set of G. This is contrary to $Ddim_l(G) = 1$.
   b. G is a path graph, based on Theorem 8, we know that $Ddim_l(P_n) = \gamma(P_n) = \lceil \frac{n}{3} \rceil > 1$ which is contrary to $Ddim_l(G) = 1$.
   c. There exist two vertices $u, v \in V(G)$ such that $uv \in E(G)$ with $deg(u), deg(v) \geq 2$.
      – There is a vertex x such that $x \in N(u)$ and $x \in N(v)$. Since $n > 3$, it is easy to see that G is not a bipartite graph. By Theorem 1 and Lemma 6, $Ddim_l(G) > 1$. It is contrary to $Ddim_l(G) = 1$. Hence, there is no vertex x such that $x \in N(u)$ and $x \in N(v)$, and so every singleton can't be the dominating set of G. Thus, it is untrue that $Ddim_l(G) = 1$.
   d. There exist two vertices $u, v \in V(G)$ such that $uv \notin E(G)$ with $deg(u), deg(v) \geq 2$. . It is easy to see that any singleton can't be the dominating set of G when there are two vertices u, v with $d(u, v) \geq 2$. Thus, $Ddim_l(G) \neq 1$.

   All of the cases above show that if $Ddim_l(G) = 1$, $G = S_n$.
2. If $G = S_n$, then $Ddim_l(G) = 1$.
   Let $V(S_n) = \{u, v_1, v_2, v_3, \ldots, v_{n-1}\}$ and $E(S_n) = \{uv_i | i = 1, 2, 3, \ldots, n-1\}$. Choose $W_l = \{u\}$, for any two adjacent vertices $u, v_i \in E(S_n)$, with $uv_i \in E(S_n)$, $i = 1, 2, 3, \ldots, n-1$, it can be seen that $r(u|W_l) \neq r(v_i|W_l)$ and v_i adjacent to u, for every $i = 1, 2, 3, \ldots, n-1$. Hence, $Ddim_l(G) = 1$.

Based on the proof in points highlighted in (1.) and (2.) above, we have that $Ddim_l(G) = 1$ if and only if $G \equiv S_n$.

Theorem 12. Let G be a connected graph of order $n \geq 2$. Then $Ddim_l(G) = n - 1$ if and only if $G \equiv K_n$.

Proof.

a. If G be a connected graph of order $n \geq 2$ and $Ddim_l(G) = n - 1$, then $G = K_n$.
   Let $Ddim_l(G) = n - 1$. Then there exists $W_l \subseteq V(G)$ as a dominant local basis of G with $|W_l| = n - 1$. We divide the order of G in three cases below.
   Case 1. If $n = 2$, then $G \equiv K_n$ by Theorem 1.b.
   Case 2. For $n = 3$, suppose that G is not a complete graph. Then G is a path (P_3), or otherwise a disconnected graph. From Theorem 8 we know that $Ddim_l(P_n) = \gamma(P_n) = \lceil \frac{n}{3} \rceil, n > 1$. Then $Ddim_l(P_3) = 1$ which is

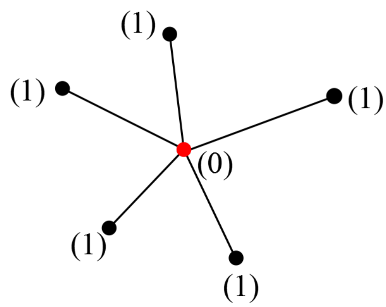

Figure 5. $Ddim_l(S_6) = 1$

contrary to the fact that $Ddim_l(G) = n - 1 = 2$. Hence, $G \equiv K_n$.
Case 3. For $n > 3$, suppose that G is not a complete graph. Then there exist two vertices $v_i, v_j \in V(G)$ such that $v_i v_j \notin E(G)$. Without loss of generality, let $E(G) = E(K_n)\{v_1 v_2\}$. Choose $S = \{v_2, v_3, v_4, \ldots, v_{n-1}\}$, so that $|S| = n - 2$. There are three possibilities for any $uv \in G$.

- By Lemma 5, $r(u|S) \neq r(v|S)$ for every $u, v \in S$.
- For $v \in V(G)\backslash S$, we can see that $r(u|S) = (2, 1, 1, 1, \ldots, 1)$ and $r(v|S) = (1, 1, 1, 1, \ldots, 1)$. Thus, $r(u|S) \neq r(v|S)$.
- For $u \in V(G)\backslash S$ with any $v \in S$: Since $v \in S$, there exist zero elements in $r(v|S)$, with $d(u, v) \leq 2$ and $u \neq S$. Then there are no zero elements in $r(u|S)$. Thus, $r(u|S) \neq r(v|S)$.

Therefore, S is a local resolving set of G. On the other hand, $v_{n-1} \in S$ is adjacent to every vertex of $V(G)$. Thus, S is also a dominating set of G. Therefore S is the dominant local resolving set. This is contrary to the fact that $Ddiml(G) = n - 1$. Therefore, $G \equiv K_n$. Based on all conditions above, we conclude that if $Ddim_l(G) = n - 1$, then $G = K_n$.

b. If $G = K_n$, then $Ddim_l(G) = n - 1$.
By using Theorem 1, we know that $dim_l(K_n) = n - 1$. Suppose $W_l = \{v_1, v_2, v_3, \ldots, v_{n-1}\}$ is a local resolving set of K_n and v_n is adjacent to all vertices of W_l. Then, W_l is also a dominating set of K_n. Next, $\gamma(K_n) = 1$ by Theorem 1.b, and Theorem 1 implies $dim_l(K_n) = n - 1$. Therefore, by the lower and upper bound of the dominant local metric dimension in Lemma 6, we can conclude that $Ddim_l(K_n) = n - 1$ for $n \geq 2$.

Therefore, based on the proof described with points (a.) and (b.), we conclude that $Ddim_l(G) = n - 1$ if and only if $G \equiv K_n$, $n \geq 2$.

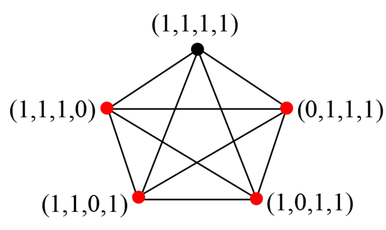

Figure 6. $Ddim_l(K_5) = 4$

The last theorem in this paper shows the dominant local metric dimension of the complete bipartite graph as follows.

Theorem 13. Let $K_{m,n}$ be a complete bipartite graph of order $m, n \geq 2$. The dominant local metric dimension of $K_{m,n}$ is $Ddim_l(K_{m,n}) = \gamma(K_{m,n})$.

Proof. Let $V(K_{m,n}) = \{a_i | i = 1, 2, 3, \ldots, m\} \bigcup \{b_j | j = 1, 2, 3, \ldots, n\}$ and $E(K_{m,n}) = \{a_i b_j | i = 1, 2, 3, \ldots, m; j = 1, 2, 3, \ldots, n\}$. Choose $W_l = \{a_1, b_1\}$ such that $|W_l| = 2$. Observe that $r(a_1|W_l) \neq r(b_1|W_l)$, and every two adjacent vertices have different representations to W_l, since $r(a_i|W_l) = (2, 1)$ and $r(b_j|W_l) = (1, 2)$ for every $V(K_{m,n})_l$. Therefore, W_l is a local resolving set of $K_{m,n}$. On the other side, since a_1 is adjacent to b_j for $j = 1, 2, \ldots, n$ and b_1 is adjacent to a_i for $i = 1, 2, \ldots, m$, we have that $W_l = \{a_1, b_1\}$ is dominating set of $K_{m,n}$. Thus, W_l is the dominant local resolving set of $K_{m,n}$. This condition satisfies Lemma 6 about the lower bound of the dominant local metric dimension with minimum cardinality since by Theorem 1, $\gamma(K_{m,n}) = 2$ and Theorem 1 shows that $dim_l(K_{m,n}) = 1$. Thus, $Ddim_l(K_{m,n}) = 2 = \gamma(K_{m,n})$ for $m, n \geq 2$. The illustration of the dominant local resolving set of a complete bipartite graph is given in Figure 7 with $m = 4$ and $n = 3$, $\{a_1, b_1\}$ form the dominant local basis of $K_{4,3}$.

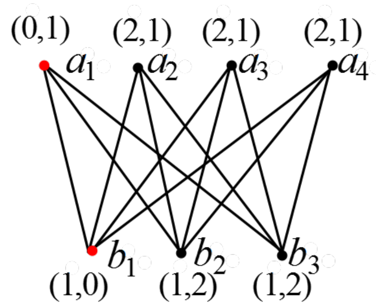

Figure 7. $Ddim_l(K_{4,3}) = 2$

3. Conclusion

We conclude this paper with some open problems as below:
Open Problem 1. Determine the dominant local metric dimension of some particular classes of graphs, such as a tree, generalized petersen graphs, and uncyclic graphs.
Open Problem 2. Generate the computer algorithm to determine the dominant local metric dimension for any graphs.
Open Problem 3. Explore some potential applications of the dominant local metric dimension in other fields, such as network analysis and data mining

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REFERENCES

1. J.Yue,et al, (2018) On the double Roman domination of graphs. Applied Mathematics and Computation 338, 669-675.
2. F. A. Pratama, L. Susilowati, M. I. Utoyo (2019) Dimensi Metrik Ketetanggaan Lokal Graf Hasil Operasi k-Comb. Contemp. Math. Appl 1,1-18.
3. F. Okamoto, L. Crosse, B. Phinezy, Z. Ping, Kalamazoo (2014) The Local Metric Dimension of Cyclic Split Graph 8, 201–205.
4. J. A. Rodríguez-Velázquez, C. García Gómez, G. A. Barragán-Ramírez (2015) Computing the local metric dimension of a graph from the local metric dimension of primary subgraphs. Int. J. Comput. Math. 92, 686–693.
5. J. A. Rodríguez-Velázquez, G. A. Barragán-Ramírez, C. García Gómez (2016) On the Local Metric Dimension of Corona Product Graphs. Bull. Malaysian Math. Sci. Soc 39, pp. 157–173.
6. L. Susilowati, Slamin, M. I. Utoyo, N. Estuningsih (2015) The similarity of metric dimension and local metric dimension of rooted product graph. Far East J. Math. Sci 97, 841–856

7. L. Susilowati, M. I. Utoyo, Slamin (2017) On commutative characterization of graph operation with respect to metric dimension. J. Math. Fundam. Sci 49, 156–170.
8. R. Brigham, G Chartrand, R.D. Dutton, P. Zhang(2019) Resolving domination in graphs. Matematica Bohemica 128, 25-36.
9. Behrooz. B. Gh., M. Jannesari, B. Omoomi (2011) Relations between Metric Dimension and Domination Number of Graphs. arXiv, Desember 2011.
10. R. Umilasari, L. Susilowati, and Slamin (2021) Dominant Local Metric Dimension of Wheel Related Graphs. IOP Conf. Ser. Mater. Sci. Eng 1115, 012029.
11. L. Susilowati, I. Sa'adah, R. Z. Fauziyyah, A. Erfanian, Slamin (2020) The dominant metric dimension of graphs. Heliyon 6, 1-6.
12. G. Chartrand, L. Eroh, M. A. Johnson, O. R. Oellermann (2000) Resolvability in graphs and the metric dimension of a graph. Discret. Appl. Math 105, 99–113.
13. F. Okamoto, L. Crosse, B. Phinezy, P. Zhang, Kalamazo (2010), The Local Metric Dimension of a Graphs.. Mathematica Bohemica 135, 239 – 255.
14. R. Alfarisi, L. Susilowati, J. Dafik, F. J. Osaye, On The Local Multiset Dimension of some Families of Graphs, WSEAS TRANSACTIONS on MATHEMATICS 22, 64 – 69.
15. F. Harary, and R. A. Melter,(1976) On the Metric Dimension of a Graph. Ars Combinatoria, 2, 191-195.
16. Y. Listiana, L. Susilowati, S. Slamin, F. J. Osaye, A central local metric dimension on acyclic and grid graph. AIMS Mathematics, 8(9): 21298–21311.
17. A. Lal, V.K. Bhat, On the dominant local metric dimension of certain polyphenyl chain graphs. Theor Chem Acc 142, 56 (2023).
18. R. Simanjuntak, T. Vetrík, P. Bintang Mulia, The multiset dimension of graphs, arXiv:1711.00225 [math.CO] (2017).
19. S. Klavžar, D. Kuziak, I.G. Yero, Further Contributions on the Outer Multiset Dimension of Graphs. Results Math 78, 50 (2023).
20. M. S. Sardar, et al, Computing Metric Dimension of Two Types of Claw-free Cubic Graphs with Applications. 119 (2024), 163–174.