STATISTICS, OPTIMIZATION AND INFORMATION COMPUTING
Stat., Optim. Inf. Comput., Vol. **13**, June 2025, pp 2406–2411.
Published online in International Academic Press (www.IAPress.org)

On Derivability Criteria of $h-$Convex Functions

Mousaab Bouafia[1,2,*], Adnan Yassine[3], Thabet Abdeljawad[4,5,6]

[1] *University of 8 May 1945 Guelma. BP 401, 24000 Guelma, Algerie*
[2] *LMAH UR 3821, 76600 Le Havre, France*
[3] *Université Le Havre Normandie, Normandie Univ., LMAH UR 3821, 76600 Le Havre, FRANCE*
[4] *Department of Mathematics and Sciences, Prince Sultan University, Riyadh, 11586, Saudi Arabia*
[5] *Department of Mathematics and Applield Mathematics, Sefako Makgatho, Health Sciences University, South Africa*
[6] *Center for Applied Mathematics and Bioinformatics (CAMB), Gulf University for Science and Technology, Hawally, 32093, Kuwait*

Abstract This study pursues two main objectives. First, we aim to generalize the Criterion of Derivability for convex functions, which posits that for a specific type of mathematical function defined on an interval, the function is convex if and only if its rate of change (first derivative) is monotonically increasing across that interval. We aim to expand this concept to encompass the realm of 'h-convexity' which generalizes convexity for nonnegative functions by allowing a function h to act on the right hand side of the convexity inequality. Additionally, we delve into the second criterion of convexity, which asserts that for a similar type of function on an interval, the function is convex if and only if its second derivative remains non-negative across the entire interval, adhering to the conventional definition of convexity. Our goal is to reinterpret this criterion within the framework of 'h-convexity'. Furthermore, we prove that if a certain non-zero function defined on the interval $[0, 1]$ is non-negative, concave, and bounded above by the identity function, then this function is fixing the end point of the interval if and only if it is the identity function. Finally, we also provide a negative response to the conjecture given by Mohammad W. Alomari (See [4]) by providing two counterexamples.

Keywords Convex function, $h-$Convex function, First criterion of $h-$convexity, Second criterion of convexity, Second criterion of $h-$convexity

AMS 2010 subject classifications 46N10, 26B25, 32F10

DOI:2096-SOIC 10.19139/soic-2310-5070-2096

1. Introduction

The study of convexity and generalized convexity stands as a cornerstone in the realm of mathematical programming. These subjects represent an ongoing trend in addressing a wide range of optimization problems, expanding the scope from classical convex programs to more diverse optimization challenges. In this paper, we delve into properties and relationships related to the derivability of h-Convex functions. We aim to broaden the scope of applicability and explore new avenues for optimization in scientific and engineering domains. For a comprehensive understanding of the basic elements of convex analysis and its applications in specific scientific fields, the derivability of convex functions is crucial in optimization and mathematical analysis. It allows for establishing optimality conditions, characterizing global minima, enabling efficient optimization algorithms, and advancing the field of convex analysis. We recommend referring to the works of Mordukhovich and Nam [5], Niculescu and Persson [6], Roberts and Varberg [7], Sezer et *al*. [9], and other relevant references provided therein. We consider the definition of h-convexity proposed by Sanja Varosanec (See [11]), which encompasses

*Correspondence to: Mousaab Bouafia (Email: bouafia.mousaab@univ-guelma.dz). University of 8 May 1945 Guelma. BP 401, 24000 Guelma, Algerie. LMAH UR 3821, 76600 Le Havre, France.

ISSN 2310-5070 (online) ISSN 2311-004X (print)

h-convex functions, generalizations of convexity (See [1, 11, 10]), s-convexity, s-Godunova-Levin functions, and P-functions.

Definition 1.1 (h-Convexity (See [11]))
Let $C \subseteq \mathbb{R}^n$ be a convex set. A non-negative function $f : C \to \mathbb{R}$ is called h-convex in C if it satisfies the inequality:

$$f(\alpha x + (1-\alpha)y) \leq h(\alpha)f(x) + h(1-\alpha)f(y), \ \ \forall x, y \in C, \tag{1.1}$$

for all $x, y \in C$ and all $\alpha \in [0, 1]$, where $h : [0, 1] \to \mathbb{R}$ is a non-negative function with $h \neq 0$.

If the function f is not necessarily non-negative and $h(t) = t$, then we say that f is convex [8, 7].
Special cases of the function h yield various types of convexity:

- When $h(t) = t$, function f becomes a non-negative convex function.
- For $h(t) = t^s$, where $s \in (0, 1)$, f becomes a non-negative s-convex function [9].
- When $h(t) = \frac{1}{t^s}$, where $s \in (0, 1)$, f takes the form of a non-negative s-Godunova-Levin function [10].
- With $h(t) = 1$, f becomes a non-negative P-convex function [3].
- For $h(t) = t^p(1-t)^q$, where $p, q > -1$, f embodies a non-negative β-convex function [10].

Convexity is a fundamental concept in mathematical analysis and optimization. In this work, we aim to generalize the derivability convexity theorem using the framework of so-called h-convexity (See [11]). This extension opens up new avenues for understanding and addressing optimization problems across various domains.

Below are the preliminary results from literature that will be essential to present our main results.

Lemma 1.2 (Necessary condition of convex function [2])
Let $I \subset \mathbb{R}$ be an interval in $\mathbb{R}$ and let the function $f : I \to \mathbb{R}$ be convex. Then for any $a, b \in I$ such that $a < b$, and any $x \in \mathbb{R}$ such that $a < x < b$ we have the following inequality:

$$\frac{f(x) - f(a)}{x - a} \leq \frac{f(b) - f(a)}{b - a} \leq \frac{f(b) - f(x)}{b - x} \tag{1.2}$$

Theorem 1.3 (First criterion of convexity [8])
Let I be an open interval and let $f : I \to \mathbb{R}$ be a derivable function. Then $f(t)$ is convex if and only if the first derivative $f^{(1)}(t)$ is increasing on I.

Theorem 1.4 (Second criterion of convexity [8])
Let I be an open interval and let $f : I \to \mathbb{R}$ be a function. Suppose $f(t)$ is twice derivable on I. Then $f(t)$ is convex if and only if the second derivative $f^{(2)}(t) \geq 0$ for all $t \in I$.

2. Main Results

This section presents the first and second derivability criteria in the scope of h-convexity.

Lemma 2.1 (Necessary Condition of h-Convex Function)
Let $I \subset \mathbb{R}$ be an interval in $\mathbb{R}$, and let the function $f : I \to \mathbb{R}$ be h-convex with $h(t) \leq t$. Then, for any $a, b \in I$ with $a < b$ and any $x \in \mathbb{R}$ such that $a < x < b$, the following inequality holds:

$$\frac{f(x) - f(a)}{x - a} \leq \frac{f(b) - f(a)}{b - a} \leq \frac{f(b) - f(x)}{b - x} \tag{2.1}$$

Proof: Let $I \subset \mathbb{R}$ be an interval in $\mathbb{R}$, and let the function $f : I \to \mathbb{R}$ be h-convex. This implies that f is a **positive** function on I, and we have $h(t) \leq t$. Consider a and b in I such that $a < b$, and let $x \in \mathbb{R}$ be such that $a < x < b$.

This implies that $0 < x - a < b - a$. If we define $t = \frac{x-a}{b-a}$, then $t \in (0, 1)$. By the convexity of f, we have:

$$\begin{aligned} f(x) &= f(a + (x - a)) \\ &= f(a + \frac{x-a}{b-a}(b - a)) \\ &= f(a + t(b - a)) \\ &= f(tb + (1 - t)a) \\ &\leq h(t)f(b) + h(1 - t)f(a) \\ &\leq tf(b) + (1 - t)f(a) \end{aligned}$$

This leads to $f(x) \leq tf(b) + (1 - t)f(a)$. By subtracting $f(a)$ from both sides, we obtain:

$$f(x) - f(a) \leq t(f(b) - f(a)) = \frac{x-a}{b-a}(f(b) - f(a)).$$

Dividing both sides by $x - a$, we arrive at the first inequality:

$$\frac{f(x) - f(a)}{x - a} \leq \frac{f(b) - f(a)}{b - a}.$$

Now, let's prove the second inequality. From the previous derivation, we have $f(x) \leq tf(b) + (1 - t)f(a)$. Subtracting $f(b)$ from both sides yields:

$$\begin{aligned} f(x) - f(b) &\leq (t - 1)(f(b) - f(a)) \\ &= \left(\tfrac{x-a}{b-a} - 1\right)(f(b) - f(a)) \\ &= \left(\tfrac{x-b}{b-a}\right)(f(b) - f(a)). \end{aligned}$$

Since $x - b < 0$, dividing both sides by $x - b$ gives us:

$$\frac{f(b) - f(a)}{b - a} \leq \frac{f(x) - f(b)}{x - b},$$

which completes the proof. □

Theorem 2.2 (First Criterion of h-Convexity)
Let I be an open interval, and let $f : I \to \mathbb{R}$ be a differentiable function.

1. If $h : [0, 1] \to \mathbb{R}$ is a non-negative function with $h \neq 0$ and $h(t) \leq t$, then: If $f(t)$ is h-convex, then the first derivative $f^{(1)}(t)$ is increasing on I.
2. If $h : [0, 1] \to \mathbb{R}$ is a non-negative, differentiable, concave function with $h \neq 0$ and $h(0) = 0, h(1) = 1$, then: If the first derivative $f^{(1)}(t)$ is increasing on I, then $f(t)$ is h-convex.

Proof:

1. Let $I \subset \mathbb{R}$ be an interval in $\mathbb{R}$. Suppose the function $f : I \to \mathbb{R}$ is h-convex on I with $h(t) \leq t$. We aim to prove that $f^{(1)}(t)$ is increasing on I. Take $a, b \in I$ such that $a < b$. By Lemma 2.1, we have:

$$\frac{f(x) - f(a)}{x - a} \leq \frac{f(b) - f(a)}{b - a}$$

for all $a < x < b$. Then, as f is differentiable, we can take the limit as x approaches a^+:

$$\lim_{x \to a^+} \frac{f(x) - f(a)}{x - a} \leq \frac{f(b) - f(a)}{b - a}.$$

By the differentiability of f, this is equivalent to:

$$f^{(1)}(a) \leq \frac{f(b)-f(a)}{b-a}.$$

Similarly, we have:

$$\frac{f(b)-f(a)}{b-a} \leq \frac{f(b)-f(x)}{b-x} = \frac{f(x)-f(b)}{x-b}$$

for all $a < x < b$. Taking the limit as x approaches b^-:

$$\frac{f(b)-f(a)}{b-a} \leq \lim_{x\to b^-} \frac{f(b)-f(x)}{b-x}.$$

And again, by the differentiability of f, we get:

$$\frac{f(b)-f(a)}{b-a} \leq f^{(1)}(b).$$

Thus, $f^{(1)}(a) \leq f^{(1)}(b)$, indicating that $f^{(1)}(t)$ is non-decreasing.

2. To prove the converse, assume that $f^{(1)}(t)$ is increasing on I, and $h : [0,1] \to \mathbb{R}$ is a non-negative, differentiable, concave function with $h \neq 0$ and $h(0) = 0, h(1) = 1$. We need to show that $f(t)$ is h-convex on I. Fix $x, y \in I$ and $t \in [0,1]$. Define:

$$\begin{aligned} g(t) &= h(t)f(y) + h(1-t)f(x) - f(ty + (1-t)x) \\ &= h(t)f(y) + h(1-t)f(x) - f(x + t(y-x)). \end{aligned}$$

Since $h(0) = 0$ and $h(1) = 1$, we have $g(0) = g(1) = 0$. We can calculate the derivative of $g(t)$:

$$g^{(1)}(t) = h^{(1)}(t)f(y) - h^{(1)}(1-t)f(x) - (y-x)f^{(1)}(x + t(y-x)).$$

As $h^{(1)}(t)$ is decreasing on $[0,1]$, for any $t_1, t_2 \in [0,1]$ with $t_1 < t_2$, and $x \neq y$, we have:

$$\begin{gathered} g^{(1)}(t_1) - g^{(1)}(t_2) \\ = \\ (h^{(1)}(t_1) - h^{(1)}(t_2))f(y) - (h^{(1)}(1-t_1) - h^{(1)}(1-t_2))f(x) \\ +(y-x)^2(t_2-t_1)\left(\frac{f^{(1)}(x+t_2(y-x)) - f^{(1)}(x+t_1(y-x))}{(t_2-t_1)(y-x)}\right) \\ \leq \\ 0 \end{gathered}$$

Thus, $g^{(1)}(t)$ is decreasing on $[0,1]$. By Rolle's theorem, as $g(t)$ is continuous on a proper closed interval $[0,1]$, differentiable on the open interval $(0,1)$, and $g(0) = g(1)$, there exists at least one c in the open interval $(0,1)$ such that $g^{(1)}(c) = 0$. This implies $g(t) \geq 0$ on $[0,1]$, which is equivalent to $f(t)$ being h-convex. This completes the proof.

□

Example 2.3
We provide two examples below to support the theoretical results:

1. For all $t \in [0,1]$, let $h(t) = t^\beta, \beta \geq 1$. We see that $h(t)$ is a non-negative function with $h \neq 0$ and $h(t) \leq t$, for all $x \in]0,+\infty[$. The function $f(x) = -x^\alpha, 0 < \alpha < 1$ is $h-$convex (see example 7 in [11]), and the first derivative $f^{(1)}(x) = -\alpha x^{\alpha-1}, 0 < \alpha < 1$ is increasing on $]0,+\infty[$.
2. For all $x \in]0,1[$, $f(x) = x^\alpha, \alpha > 1$ is $h-$convex with $h(t) = \sqrt{t}$ is a non-negative, differentiable, concave function, $h \neq 0$ and $h(0) = 0, h(1) = 1$. Further, we have $f^{(1)}(x) = \alpha x^{\alpha-1}$ is increasing on $]0,1[$.

Theorem 2.4 (Second Criterion of h-Convexity)
Let I be an open interval, and let $f : I \to \mathbb{R}$ be a function that is twice differentiable on I.

1. If $h : [0, 1] \to \mathbb{R}$ is a non-negative function with $h \neq 0$ and $h(t) \leq t$, then: If $f(t)$ is h-convex, then the second derivative $f^{(2)}(t) \geq 0$ for all $t \in I$.
2. If $h : [0, 1] \to \mathbb{R}$ is a non-negative, differentiable, concave function with $h \neq 0$ and $h(0) = 0, h(1) = 1$, then: If the second derivative $f^{(2)}(t) \geq 0$ for all $t \in I$, then $f(t)$ is h-convex.

Proof: Apply Theorem 2.2 with $f(t)$ being twice differentiable on I. □

Corollary 2.5
Let $h : [0, 1] \to \mathbb{R}$ be a non-negative concave function with $h \neq 0$ and $h(t) \leq t$ for all $t \in [0, 1]$. $h(1) = 1$ if and only if it is a $h(t) = t$ for all $t \in [0, 1]$.

Proof
Suppose that $h(t)$ is a non-negative concave function with $h \neq 0$ and $h(t) \leq t$ for all $t \in [0, 1]$. Then $h(0) = 0$, and

$$th(1) + (1 - t)h(0) \leq h(t(1) + (1 - t)(0)) \leq t \text{ for all } t \in [0, 1]$$

According to $h(1) = 1$, we have

$$h(t) = t \text{ for all } t \in [0, 1]$$

If $h(t) = t$ for all $t \in [0, 1]$, then $h(t)$ is a non-negative concave function with $h \neq 0$ and $h(t) \leq t$ for all $t \in [0, 1]$ and $h(1) = 1$. □

We will respond in the following to the conjecture given by Mohammad W. Alomari (See [4]) that it is incorrect with two counterexamples

Conjecture 2.6
(two counter examples disproving the two implications) The conjecture states that: For $h : [0, 1] \to \mathbb{R}$ a non-negative function with $h \neq 0$ and $h(t) \geq t$ for all $t \in]0, 1[$, a function $f : I \to \mathbb{R}$, which is twice differentiable, is $h-$convex if and only if $f''(x) \geq 1 - 2h(\frac{1}{2})$.

Proof
($\Rightarrow$)
For all $x \in [0, +\infty[$, the function $f(x) = \sqrt{x}$ is $h-$convex with $h(t) = \sqrt{1 + t} \geq t$. Further, we have $f''(x) = -\frac{1}{4x\sqrt{x}}$, $1 - 2h(\frac{1}{2}) = 1 - 2\sqrt{\frac{3}{2}}$, and

$$\lim_{x \to 0^+} f''(x) = -\infty < 1 - 2h(\frac{1}{2}) = 1 - 2\sqrt{\frac{3}{2}}$$

($\Leftarrow$)
For all $x \in] - 1, +1[$, let $f(x) = \frac{1-2h(\frac{1}{2})}{4}x^2 - \frac{1-2h(\frac{1}{2})}{4} > 0$, and

$$h(t) = \begin{cases} t & if \quad 0 \leq t < \frac{1}{4} \\ 2t & if \quad \frac{1}{4} \leq t \leq 1 \end{cases}$$

$h(t) \geq t$ for all $t \in]0, 1[$, we have

$$f''(x) = \frac{1 - 2h(\frac{1}{2})}{2} \geq 1 - 2h(\frac{1}{2}).$$

On the other hand, for all $x, y \in] - 1, +1[$ and $\alpha \in]0, \frac{1}{4}[$, we have

$$f(\alpha x + (1 - \alpha)y) > \alpha f(x) + (1 - \alpha)f(y) = h(\alpha)f(x) + h(1 - \alpha)f(y),$$

and hence $f(x)$ is not h-convex. □

3. Conclusion

In conclusion, our work has uncovered valuable insights in the realm of h-convexity. We have established:

- A necessary condition for h-convex functions, shedding light on their properties.
- Two criteria for h-convexity, showing the relationship between function behavior and h-convexity.
- The criterion for $h(t) = t$ as a special case of h-convexity.
- We have disproved the conjecture given by Mohammad W. Alomari (See [4]) by giving two counterexamples.

These findings expanded our understanding of optimization problems within the framework of h-convexity, demonstrated its versatility and relevance in various mathematical contexts.

Acknowledgement

The authors are very grateful and would like to thank the Editor-in-Chief and the anonymous referees for their suggestions and helpful comments which significantly improved the the presentation of this paper. The author T. Abdeljawad would like to thank Prince sultan University for the support through TAS research lab.

REFERENCES

1. W.W. Breckner,, *Stetigkeitsaussagenf ureine Klass ever all gemeinerter konvexer funktionen in topologisc henlinearen Raumen*, Pupl. Inst. Math., vol. 23, pp. 13–20, 1978.
2. Boris S. Mordukhovich, and Nguyen Mau Nam, *An Easy Path to Convex Analysis and Applications, Part of: Synthesis Lectures on Mathematics and Statistics*, Publisher: Morgan and Claypool Publishers; 1st edition, (48 books), December 1, 2013.
3. S. Dragomir, J. Pecaric, and L. Persson, *Some inequalities of Hadamard type*, Soochow J.Math., vol. 21, pp. 335–341, 1995.
4. Mohammad W. Alomari, *A note on $h-$convex functions*, e-Journal of Analysis and Applied Mathematics, vol. 23, pp. 55–67, 2019.
5. B.S. Mordukhovich, and N.M. Nam, *An Easy Path to Convex Analysis and Applications (Synthesis Lectures on Mathematics and Statistics)*, Springer Nature 2013.
6. C.P. Niculescu, and L.E. Persson, *Convex Functions and Their Applications: A Contemporary Approach*, Springer Science & Business Media, New York 2006.
7. A.W. Roberts, and D.E. Varberg, *Convex Functions*, Academic Press, New York, 1973.
8. R. T. Rockafellar, *Convex Analysis*, Princeton University Press, 1970.
9. S. Sezer, Z. Eken, G. Tinaztepe, and G. Adilov, *$p-$Convex Functions and Some of Their Properties*, Numerical Functional Analysis and Optimization, pp. 443–459, 2021.
10. M. Tunc, U. Sanal, and E. Gov, *Some Hermite-Hadamard inequalities for $beta-$convex and its fractional applications*, New Trends in Mathematical Sciences, vol. 4, pp. 18–33, 2015.
11. S. Varosanec, *On $h-$convexity*, J. Math. Anal. Appl., vol. 326, pp. 303–311, 2007.