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Parametric Support approach for Solving Mean-Variance Problem under General Constraints

Souhaib Boudjelda [1,*], Belkacem Brahmi [1,2]

[1] *LaMOS Research Unit, Faculty of the Exact Sciences, University of Bejaia, Algeria*
[2] *Department of Operations Research, Faculty of the Exact Sciences, University of Bejaia, Algeria*

Abstract The intuitive and natural formulation of the Mean-Variance (MV) model has attracted the attention of researchers over the years. This model is typically presented as a constrained Quadratic Problem (QP), although the practical aspects of investment often require risk tolerance to be considered. In such cases, Parametric Quadratic Programming (PQP) is employed to explore all optimal solutions on the efficient frontier. In this paper, we propose a novel approach for solving the portfolio optimization problem of the mean-variance model. This problem is considered in its parametric formulation under general linear equality constraints with bounded assets. The proposed algorithm iteratively derives the exact efficient frontier by calculating all corner portfolios as a function of the risk aversion parameter. Finally, we test the computational performance of our algorithm in comparison with two state-of-the-art approaches using a set of real benchmarks. The results demonstrate the effectiveness of our approach in solving such problems and in identifying the efficient frontier. Additionally, considering large-scale randomly generated problems with dense covariance matrices, we show that our algorithm can efficiently solve this class of problems in a reasonable computation time.

Keywords Markowitz's Mean-Variance Model, Portfolio Optimization, Efficient Frontier, Parametric Quadratic Programming, Direct Support Method, Parametric Support Method

AMS 2010 subject classifications 91G10, 65K05, 90C25, 90C90

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1. Introduction

The stock markets present excellent investment opportunities. However, it is important to note that the most profitable assets are often associated with higher risk. Therefore, an investor's main objective is to construct an optimal portfolio that maximizes its expected return while minimizing potential risk.

Markowitz's Mean-Variance (MV) model [1], introduced in 1952, is the cornerstone of modern portfolio theory. This model explores how investors can allocate their capital to achieve an optimal trade-off between risk and return, through an asset selection process that maximizes a portfolio's expected return and minimizes its risk. According to Markowitz [1, 3], the term "mean" refers to the average of the observed returns of each asset and aims to maximize it. In contrast, "variance" represents the risk, being the variance or standard deviation of the returns on these assets, and aims to be minimized. Consequently, the MV model is formulated as a bi-objective problem that aims to both maximize the investor's expected return and minimize their risk. The optimal portfolio for an investor lies on the Pareto front, also known in the MV framework as the Markowitz efficient frontier, which consists of all efficient portfolios that offer the best trade-off between return and risk. This frontier has been shown to be a continuous, piecewise hyperbolic curve [11, 9, 24]. In other words, it is characterized by a series of semi-hyperbolic segments connected at extreme points called corner portfolios.

*Correspondence to: Souhaib Boudjelda (Email: souhaib.boudjelda@univ-bejaia.dz). LaMOS Research Unit, Faculty of the Exact Sciences, University of Bejaia, Targa Ouzemmour, Bejaia, Algeria (06000).

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In order to provide investors with a comprehensive view of their asset allocation, taking into account their risk tolerance, it is crucial that the efficient frontier is fully determined. Achieving this requires the use of Parametric Quadratic Programming (PQP) algorithms [20]. These involve transforming the bi-objective problem into a parametric mono-objective problem, using the risk aversion parameter. By varying this parameter, the entire efficient frontier can be derived. In this context, several PQP algorithms have been introduced, starting with the so-called Critical Line Algorithm (CLA), described in Markowitz's seminal work [2, 3] as an extension of Wolf's simplex method [21]. This algorithm was improved in Markowitz and Todd [4] and implemented in "Visual-Basic for Applications" (VBA). Furthermore, the CLA has undergone several improvements, consolidating its position as a fundamental tool for portfolio optimization in the financial field [10, 32, 30, 31]. Best [6] developed the Parametric Active Set Method (PASM) based on the active set method [22] to solve convex PQP problems, which was applied to the portfolio selection problem in his subsequent works [7, 28]. Later, Stein et al. [8] adapted the PASM to large-scale portfolio problems with a dense covariance matrix, suggesting an efficient implementation to reduce the computation time of the efficient frontier. In addition, different algorithms were developed in the literature to derive the parametrically efficient frontier using PQP. Notable contributions include the works of Steuer et al. [23, 24, 25], which extend Merton's model [13] to analytically derive the efficient frontier for multiobjective portfolio selection, Qi [11], Pang [5] and Hirschberger et al.[9].

In this paper, we focus on the mean-variance portfolio selection model, which includes general linear equality constraints with lower and upper bounds on the assets. This model is presented as a parametric quadratic programming (PQP) problem using the risk aversion parameter. We then extend the Direct Support Method (DSM) [16, 15] to solve this problem. Our algorithm iteratively identifies all corner portfolios; starting from the maximum-return portfolio, in order to determine the entire Markowitz efficient frontier. The proposed parametric support algorithm (PSA), which lies between the active set and interior point methods [26], is consistent with the MV model structure and treats all its constraints in their original formulation, thereby avoiding pre-transformations of decision variables. In contrast to other PQP algorithms, which are frequently complex and difficult to understand, our method is straightforward to apply and easy to implement. Indeed, by redefining the support concept within the context of the parametric formulation of the MV model, the size of the systems required to be solved at each iteration is reduced. This leads to a more efficient solution process, which significantly reduces the computation time needed to determine the efficient frontier.

To evaluate and analyze the performance of our method, we conducted a comparative study with the parametric active set algorithm [7] and the Matlab package. This comparison was carried out using publicly available datasets and randomly generated problems. The results obtained demonstrate that our approach can effectively compute the efficient frontier and provide all corner portfolios in a relatively short time.

The main contribution of this work is the suggestion of an exact parametric support algorithm, both efficient and easy to implement, designed to solve the mean-variance portfolio selection problem under general constraints with bounded assets. Our definition of the support concept, which is different from the classical definition [14, 16], allows us to introduce a new iterative approach based on the principles of the simplex and support methods. In addition, extensive numerical experiments conducted on several randomly generated benchmark datasets demonstrate the significant improvement of our algorithm over comparative classical approaches in solving large-scale portfolio optimization problems.

The remainder of this paper is organized as follows: Section 2 presents the mathematical background and theoretical elements of the classical Markowitz's mean-variance portfolio selection problem. Section 3 presents and formulates the proposed parametric resolution approach, along with an application example. The results of the computational experiments and performance tests of our approach are presented in Section 4. Finally, this paper is concluded in Section 5.

2. Portfolio optimization

Portfolio optimization aims to find an optimal allocation of risky assets in order to maximize an investor's expected return while minimizing his risk. This subject was first addressed by Markowitz in his well-known Mean-Variance

(MV) model [1]. In the following, we will review the mathematical formulas and the main idea of Markowitz's theory.

2.1. Portfolio return and risk

Consider a universe of n assets, and let $x = (x_1, x_2, \cdots, x_n)'$ be an n-vector representing the investment proportions, also known as a *portfolio*, where each x_i represents the proportion of the capital to be allocated to an asset i. Throughout the paper, the symbol $()'$ is the transposition operation.
Let $R = (r_1, \cdots, r_n)$ be the vector of returns on individual assets, and r_i are random variables. Thus, the portfolio return is:

$$R_p = x_1 r_1 + x_2 r_2 + \cdots + x_n r_n = \sum_{i=1}^{n} r_i x_i,$$

which represents the percentage return that will be realized during the holding period. The mean value (expected return) of the portfolio is the weighted average of the expected returns of each asset, given by:

$$\begin{aligned} \mu_p = E(R_p) &= x_1 E(r_1) + x_2 E(r_2) + \cdots + x_n E(r_n) \\ &= \sum_{i=1}^{n} \mu_i x_i, \quad \text{with } \mu_i = E(r_i), \end{aligned}$$

where $E(.)$ is the expected value operator. Thus, the matrix notation of the expected return is:

$$\mu_p = \mu' x, \quad \text{with } \mu = (\mu_1, \mu_2, \cdots, \mu_n)',$$

where each μ_i is the expected return on asset i.

The concept of portfolio risk formulated by Markowitz [1, 3] using the variance of returns is given as follows:

$$\sigma_p^2 = Var(R_p) = \sum_{i=1}^{n} \sum_{j=1}^{n} x_i x_j \sigma_{ij}, \quad \text{where } \sigma_{ij} = E((r_i - \mu_i)(r_j - \mu_j)).$$

The matrix notation of the risk is given as:

$$\sigma_p^2 = x' \Sigma x,$$

where Σ is the $(n \times n)$ variance-covariance matrix, such that:

$$\Sigma = \begin{pmatrix} \sigma_1^2 & \sigma_{12} & \cdots & \sigma_{1n} \\ \sigma_{21} & \sigma_2^2 & \cdots & \sigma_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{n1} & \sigma_{n2} & \cdots & \sigma_n^2 \end{pmatrix}.$$

Here, each $\sigma_{ij} = \sigma_{ji}$ represents the covariance between returns of assets i and j, and $\sigma_{ii} = \sigma_i^2$ is the variance of asset i's return. Note that throughout this paper Σ is assumed to be positive semidefinite.

2.2. Mean-Variance model formulation

Markowitz's Mean-Variance model, as presented by Markowitz [1], consists in minimizing the risk of a portfolio for a given level of return, which leads to the following Quadratic Programming (QP) problem :

$$\begin{cases} \min & \frac{1}{2} x' \Sigma x \\ s.t & \mu' x = \bar{\mu} \\ & e' x = 1 \\ & x \geq 0, \end{cases} \tag{1}$$

where $\bar{\mu}$ is a level of return desired by the investor, $e = (1, 1, \cdots, 1)'$ is a unit vector. The second constraint (budget constraint) implies that all available capital is to be invested, and the last constraint (no short sales) requires non-negativity of the investments.

Here, the objective is to calculate the portfolio with the minimum risk for a fixed return, while the investor's interest is to be provided with the set of all efficient portfolios that present the best trade-off between risk and return, which is inconvenient with this formulation. For this purpose, another equivalent formulation is widely used [2, 3, 12, 6], based on the weighting method, employing Parametric Quadratic Programming (PQP) as follows:

$$\begin{cases} \min & \frac{1}{2}x'\Sigma x - \lambda\mu' x \\ s.t & e'x = 1 \\ & x \geq 0, \end{cases} \tag{2}$$

where the parameter λ, lies in the interval $[0, \infty)$, is interpreted as the investor's risk tolerance (risk aversion), reflecting their propensity to avoid risk. The higher the value of λ, the more risk-averse the investor. In contrast, a lower value of λ, indicates a lower risk tolerance. Solving Problem (2) parametrically for different values of λ, enables us to determine the set of optimal portfolios (as a function of λ), that provide the best trade-off between risk and return. These solutions form the Markowitz efficient frontier [2, 6, 9].

This paper examines the MV model as a convex PQP problem under general linear constraints with lower and upper bounds on assets, given as follows :

$$\begin{cases} \min & \frac{1}{2}x'\Sigma x - \lambda\mu' x \\ s.t & Ax = b \\ & l \leq x \leq u, \end{cases} \tag{3}$$

where A is an $(m \times n)$ equality-constraint matrix [†], b is an m right-hand-side vector, l and u are n-vectors representing the lower and upper bounds of x, respectively.

2.3. *Karush-Kuhn-Tucker optimality conditions*

To derive the optimal solutions of Problem (3) and compute its efficient frontier, we use the Karush-Kuhn-Tucker (KKT) conditions. Given that Problem (3) is convex (Σ is assumed to be a positive semi-definite matrix and the constraints are convex), therefore, the first-order KKT-conditions are both necessary and sufficient for the optimality of the point $x \in \mathbb{R}^n$. Let $L(x, y, \nu_1, \nu_2)$ be the Lagrangian function of Problem (3), such as:

$$L(x, y, \nu_1, \nu_2) = \frac{1}{2}x'\Sigma x - \lambda\mu' x + y'(Ax - b) - \nu_1'(x - l) + \nu_2'(x - u), \tag{4}$$

where $y \in \mathbb{R}^m$ is the multiplier vector associated with general equality constraints, ν_1 and ν_2 are n-vectors associated respectively with lower and upper bounds constraints. The following relations summarize the KKT optimality conditions:

$$\frac{\partial L}{\partial x} = \Sigma x - \lambda\mu + A'y - \nu_1 + \nu_2 = 0, \tag{5a}$$

$$\frac{\partial L}{\partial y} = Ax - b = 0, \quad l \leq x \leq u, \tag{5b}$$

$$\nu_1'(x - l) = 0, \quad \nu_2'(x - u) = 0, \tag{5c}$$

$$\nu_1 \geq 0, \quad \nu_2 \geq 0. \tag{5d}$$

[†] The budget constraint $e'x = 1$ is included and imposed in the general constraints.

3. Parametric support procedure for the general MV portfolio optimization problem

In this section, we present our parametric support approach to solve the MV model under general linear equality constraints with limitation on portfolios. This approach extends the DSM proposed by Gabasov at al. [16, 14] to solve Problem (3) in a PQP framework. We note that the DSM is a variant of the revised simplex method [27] for solving general convex QP problems. Before describing the steps in the solution process, we first provide some notations and define some fundamental concepts related to our method.

Consider a universe of n assets, and let $I = \{1, 2, \cdots, m\}$ be the set of constraints indices, and $J = \{1, .., n\}$ is the set of decision variables indices, with $J = J_B \cup J_N$ and $J_B \cap J_N = \emptyset$. Following the partition of the indices set J, we decompose the vectors and matrices of our problem in the following way:

$x = \begin{pmatrix} x_B \\ x_N \end{pmatrix}, \;\; x_B = x(J_B) = (x_j, j \in J_B), \;\; x_N = x(J_N) = (x_j, j \in J_N),$

$\mu = \begin{pmatrix} \mu_B \\ \mu_N \end{pmatrix}, \;\; \mu_B = \mu(J_B) = (\mu_j, j \in J_B), \;\; \mu_N = \mu(J_N) = (\mu_j, j \in J_N),$

$A = (A_B \mid A_N), \;\; A_B = A(I, J_B), \;\; A_N = A(I, J_N),$

$\Sigma = \begin{pmatrix} \sigma_B & \sigma_{BN} \\ \sigma_{NB} & \sigma_N \end{pmatrix}, \sigma_B = \Sigma(J_B, J_B), \sigma_{BN} = \Sigma(J_B, J_N), \sigma_N = \Sigma(J_N, J_N),$

given that Σ is symmetric, then $\sigma_{NB} = \sigma'_{BN}$.

Definition 1
A portfolio $x \in \mathbb{R}^n$ satisfying all the constraints of the PQP (3) is called a feasible solution.

Definition 2
A feasible solution $\bar{x}$ is said to be optimal for Problem (3), if

$$f(\bar{x}) = \frac{1}{2}(\bar{x})'\Sigma\bar{x} - \lambda\mu'\bar{x} = \min_x f(x),$$

where $\bar{x}$ is chosen within the set of feasible solutions of the problem, with respect to λ.

Following the structure of PQP (3), we redefine the *Support* concept of the problem as follows:

Definition 3
A nonempty subset of indices $J_B \subset J$, such that $\mid J_B \mid \geq m$, is called *support* of the PQP (3) if and only if the submatrix $\sigma_B = \Sigma(J_B, J_B)$ is nonsingular, and the columns of the submatrix $A_B = A(I, J_B)$ are linearly independent.

Definition 4
The pair $\{x, J_B\}$ formed by the feasible solution x and the support J_B is called a *support feasible solution* (SFS), and is said to be *non-degenerate* if:

$$l_j < x_j < u_j, \forall j \in J_B.$$

According to the KKT-optimality conditions (5), we define the reduced costs vector as follows:

$$E = \begin{pmatrix} E_B \\ E_N \end{pmatrix} = \Sigma x - \lambda\mu + A'y. \quad (6)$$

The following theorem restates the KKT-optimality conditions for the problem (3). These optimality conditions are expressed with respect to the support set J_B, and can be proved analogously to the works [14, 16, 15].

Theorem 1
Let $\{x, J_B\}$ be an SFS for the PQP (3). Then, the following relations:

$$\begin{cases} E_j \geq 0 & \text{if} \quad x_j = l_j, \\ E_j \leq 0 & \text{if} \quad x_j = u_j, \\ E_j = 0 & \text{if} \quad l_j < x_j < u_j \;\; \forall j \in J_N, \end{cases} \quad (7)$$

are sufficient for the optimality of x and are also necessary if the SFS $\{x, J_B\}$ is non-degenerate.

3.1. Initial portfolio

As with all parametric approaches to solving the MV model [6, 4, 10], our approach needs to know at least one portfolio on the efficient frontier. In general, the easiest portfolio to identify is the portfolio with the highest expected return or the maximum expected return portfolio (MRP), which leads to solving the following bounded Linear Problem (LP): ‡

$$\begin{cases} \max & \mu' x, \\ s.t & Ax = b, \\ & l \leq x \leq u. \end{cases} \tag{8}$$

Problem (8) can be solved using any Simplex-type method for linear programming [26, 19, 17, 18], which enabled us to obtain the maximum return portfolio x^0 and the support set J_B^0.

The rest of the parametric approach proposed in this work aims to compute the other efficient portfolios $\{x^k, J_B^k\},\ k = 1, 2, \cdots$, starting from the optimal SFS $\{x^0, J_B^0\}$ of LP (8), by identifying the corresponding pivot points, such as :

$$\lambda^0 > \lambda^1 > \lambda^2 > \cdots > \lambda^\rho = 0, \quad \text{with } \rho \text{ is the number of pivot points.}$$

Each two adjacent pivot points; form an interval $[\lambda^k, \lambda^{k+1}]$; and represent a distinct segment of the efficient frontier that is characterized by a unique support set J_B^k, where the optimal solution $x^k(\lambda)$ at its extremities is called a *corner portfolio*. In other words, a solution $x^k(\lambda)$ of PQP (3) is a corner portfolio if there exists in its neighborhood another portfolio corresponding to a different support set.

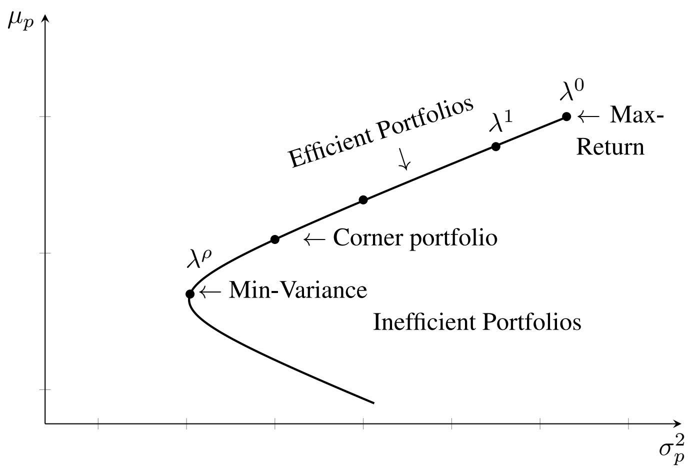

Figure 1. Parametric efficient frontier for Problem (3) and the corner portfolios that constitute its segments.

In order to construct the fully parametric efficient frontier of Problem (3), the KKT system (5) is solved for all values of λ. Thus, the explicit expressions of the optimal solution and associated multipliers obtained as linear functions of parameter λ, given in the following theorem:

Theorem 2
For all parameter λ between two adjacent pivot point, $\lambda^{k+1} \leq \lambda \leq \lambda^k; k = 0, 1, 2, \cdots$, the optimal solution of the PQP (3), its multipliers vector and its reduced costs vector are linear functions depending on the parameter λ, such that :

$$y(\lambda) = \alpha_1 + \lambda\alpha_2, \quad x_B(\lambda) = \gamma_1 + \lambda\gamma_2 \quad \text{and} \quad E_N(\lambda) = \beta_1 + \lambda\beta_2, \tag{9}$$

‡In some situations, where not all assets have upper and lower bounds, it is possible that problem (8) has no solution, so it is necessary to start the algorithm from the minimum variance portfolio as described in [28].

such that

$$\alpha_1 = (-A_B\sigma_B^{-1}\sigma_{BN}x_N + A_Nx_N - b)\delta^{-1}, \qquad \alpha_2 = (A_B\sigma_B^{-1}\mu_B)\delta^{-1}. \tag{10}$$

$$\gamma_1 = -\sigma_B^{-1}(A'_B\alpha_1^k + \sigma_{BN}x_N), \qquad \gamma_2 = \sigma_B^{-1}(\mu_B - A'_B\alpha_2^k). \tag{11}$$

$$\beta_1 = \sigma_{NB}\gamma_1^k + A'_N\alpha_1^k + \sigma_Nx_N, \qquad \beta_2 = \sigma_{NB}\gamma_2 + A'_N\alpha_2^k - \mu_N, \tag{12}$$

where $\delta = (A_B\sigma_B^{-1}A'_B)$ represent the basic matrix associate to Problem (3).

Proof
Let $\{x, J_B\}$ be an optimal SFS of Problem (8), and according to equation (6) we define the reduced cost vector $E = (E_B, E_N)'$ by:

$$\begin{cases} E_B &= \sigma_Bx_B + \sigma_{BN}x_N - \lambda\mu_B + A'_By, \\ E_N &= \sigma'_{BN}x_B + \sigma_Nx_N - \lambda\mu_N + A'_Ny, \end{cases} \tag{13}$$

with $E_B = (E_j, j \in J_B) = 0$ (by definition of an SFS), and y is the potentials vector. From the first part of system (13), we have:

$$x_B = \sigma_B^{-1}(\lambda\mu_B - \sigma_{BN}x_N - A'_By), \tag{14}$$

To obtain the expression of y as a function of λ, we use the equality constraint $Ax = b$, and the first part of system (13), which leads to solve the following system of equations:

$$\begin{pmatrix} \sigma_B & \sigma_{BN} \\ A_B & A_N \end{pmatrix}\begin{pmatrix} x_B \\ x_N \end{pmatrix} + y\begin{pmatrix} A'_B \\ 0 \end{pmatrix} = \lambda\begin{pmatrix} \mu_B \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ b \end{pmatrix},$$

upon substitution of x_B with its expression (14), and after simplification we obtain:

$$\begin{aligned} y(\lambda) &= \underbrace{(-A_B\sigma_B^{-1}\sigma_{BN}x_N + A_Nx_N - b)}_{\alpha_1}\delta^{-1} + \lambda\underbrace{(A_B\sigma_B^{-1}\mu_B)}_{\alpha_2}\delta^{-1}, \\ &= \alpha_1 + \lambda\alpha_2. \end{aligned} \tag{15}$$

with $\delta = (A_B\sigma_B^{-1}A'_B)$ is a square matrix of order n_B[§], where $n_B =| J_B |$.

Once the potentials vector $y(\lambda)$ is determined, the final expression of the optimal solution x_B as a function of λ is obtained using equation (14):

$$\begin{aligned} x_B(\lambda) &= \sigma_B^{-1}(\lambda\mu_B - \sigma_{BN}x_N - A'_B(\alpha_1 + \lambda\alpha_2)) \\ &= \underbrace{-(\sigma_B^{-1}A'_B\alpha_1^k + \sigma_B^{-1}\sigma_{BN}x_N)}_{\gamma_1} + \lambda\underbrace{(\sigma_B^{-1}(\mu_B - A'_B\alpha_2^k)}_{\gamma_2} \\ &= \gamma_1 + \lambda\gamma_2. \end{aligned} \tag{16}$$

Having known $x(\lambda)$ and $y(\lambda)$, the reduced costs vector $E_N(\lambda)$ is obtained by substitution in the second entity of system (13) which yields:

$$\begin{aligned} E_N(\lambda) &= \sigma'_{BN}(\gamma_1 + \lambda\gamma_2) + \sigma_Nx_N - \lambda\mu_N + A'_N(\alpha_1 + \lambda\alpha_2) \\ &= \underbrace{\sigma'_{BN}\gamma_1^k + A'_N\alpha_1^k + \sigma_Nx_N}_{\beta_1} + \lambda\underbrace{\sigma_{NB}\gamma_2^k + A'_N\alpha_2^k - \mu_N}_{\beta_2} \\ &= \beta_1 + \lambda\beta_2. \end{aligned} \tag{17}$$

□

[§]By the definition of the support set J_B, it is necessary to assume that problem (3) has no redundant constraints in order to ensure the invertibility of δ.

3.2. The expected return and the risk of corner portfolios

In the risk-return space, the expected return $\mu_p = \mu' x$ and the associated risk (variance) σ_p^2 for each corner portfolio can be obtained as a function of parameter λ, using the optimal solution $x(\lambda)$ of the PQP (3). The following theorem presents the explicit expressions of the expected return and the risk of any corner portfolio as a function of λ.

Theorem 3
Let $\{x(\lambda), J_B\}$ be an optimal SFS to the PQP (3), where $x(\lambda) = (x_B(\lambda),\ x_N(\lambda))'$, such that $x_B(\lambda)$ is defined by relation (16). For all λ between two adjacent pivot points, $\lambda^{k+1} \leq \lambda \leq \lambda^k; k = 0, 1, 2, \cdots$, the expected return and the associated risk of each corner portfolio are functions depending on the parameter λ, such that:

$$\mu_p(\lambda) = \omega_1 + \lambda\omega_2, \quad \text{and} \quad \sigma_p^2(\lambda) = \xi_1 + \lambda\xi_2 + \lambda^2\xi_3, \tag{18}$$

where

$$\begin{aligned} \omega_1 &= \mu_B'\gamma_1 + \mu_N' x_N, & \omega_2 &= \mu_B'\gamma_2. \\ \xi_1 &= \gamma_1'\sigma_B\gamma_1 + 2\gamma_1'\sigma_{BN}x_N + x_N'\sigma_{BN}x_N, & \xi_2 &= 2\gamma_1'\sigma_B\gamma_2 + 2\gamma_2'\sigma_{BN}x_N, \\ \text{and} \quad \xi_3 &= \gamma_2'\sigma_B\gamma_2. \end{aligned}$$

Proof
Let $\{x(\lambda), J_B\}$ be an optimal SFS of the PQP (3), where:

$$x(\lambda) = \begin{pmatrix} x_B(\lambda) \\ x_N(\lambda) \end{pmatrix}, \quad \text{with} \left\{ \begin{array}{lcl} x_B(\lambda) &=& \gamma_1 + \lambda\gamma_2, \\ x_N(\lambda) &=& x_j,\ j \in J_N. \end{array} \right. \tag{19}$$

For each corner portfolio, the expected return is given by:

$$\mu_p(\lambda) = \mu' x(\lambda),$$

substituting $x(\lambda)$ in the last formula of $\mu_p(\lambda)$ yields:

$$\begin{aligned} \mu_p(\lambda) &= (\mu_B,\ \mu_N)' \begin{pmatrix} \gamma_1 + \lambda\gamma_2 \\ x_N \end{pmatrix} \\ &= \mu_B'(\gamma_1 + \lambda\gamma_2) + \mu_N' x_N \\ &= \underbrace{\mu_B'\gamma_1 + \mu_N' x_N}_{\omega_1} + \lambda \underbrace{\mu_B'\gamma_2}_{\omega_2} \\ &= \omega_1 + \lambda\omega_2. \end{aligned} \tag{20}$$

The associated risk of the corner portfolios is calculated as follows:

$$\sigma_p^2(\lambda) = x'(\lambda)\Sigma x(\lambda),$$

substituting $x(\lambda)$ into the variance equation yields:

$$\begin{aligned} \sigma_p^2(\lambda) &= (\gamma_1 + \lambda\gamma_2,\ x_N)' \begin{pmatrix} \sigma_B & \sigma_{BN} \\ \sigma_{BN}' & \sigma_N \end{pmatrix} \begin{pmatrix} \gamma_1 + \lambda\gamma_2 \\ x_N \end{pmatrix} \\ &= ((\gamma_1 + \lambda\gamma_2)'\sigma_B + x_N'\sigma_{BN}',\ (\gamma_1 + \lambda\gamma_2)'\sigma_{BN} + x_N'\sigma_N)' \begin{pmatrix} \gamma_1 + \lambda\gamma_2 \\ x_N \end{pmatrix} \\ &= ((\gamma_1 + \lambda\gamma_2)'\sigma_B + x_N'\sigma_{BN}')(\gamma_1 + \lambda\gamma_2) + ((\gamma_1 + \lambda\gamma_2)'\sigma_{BN} + x_N'\sigma_N)x_N, \end{aligned}$$

upon simplification and using the symmetry property of Σ, the final expression of the variance becomes:

$$\begin{aligned}\sigma_p^2(\lambda) &= \underbrace{\gamma_1'\sigma_B\gamma_1 + 2\gamma_1'\sigma_{BN}x_N + x_N'\sigma_{BN}x_N}_{\xi_1} + \lambda\underbrace{2\gamma_1'\sigma_B\gamma_2 + 2\gamma_2'\sigma_{BN}x_N}_{\xi_2} + \lambda^2\underbrace{\gamma_2'\sigma_B\gamma_2}_{\xi_3} \\ &= \xi_1 + \lambda\xi_2 + \lambda^2\xi_3. \end{aligned} \tag{21}$$

□

From formula (18), we can see that the expected return is a linear function, while the risk is quadratic with respect to λ. In his analytical study of the efficient frontier and its properties, Best [28] provided the relationship between the expected return and the variance in the following expression:

$$(\sigma_p^2(\lambda) - \xi_1) = \frac{(\mu_p(\lambda) - \omega_1)^2}{\xi_3}, \tag{22}$$

this relation defines the set of points belonging to the k-th interval of the efficient frontier $[\lambda^k, \lambda^{k+1}]$, established using the following important properties [29]:

$$\omega_2 = \xi_3, \text{ and } \xi_2 = 0. \tag{23}$$

Proposition 1
For all values of λ between two adjacent pivot points, $\lambda^{k+1} \leq \lambda \leq \lambda^k; k = 0, 1, 2, \cdots$, the efficient frontier curve of PQP (3) is continuously differentiable with a positive slop given by:

$$\frac{\partial\mu_p(\lambda)/\partial\lambda}{\partial\sigma_p^2(\lambda)/\partial\lambda} = \frac{1}{2\lambda}. \tag{24}$$

Proof
This follows from substituting the properties (23) into the expressions of the expected return and the variance (18), which results in:

$$\begin{cases} \mu_p(\lambda) &= \omega_1 + \lambda\xi_3, \\ \sigma_p^2(\lambda) &= \xi_1 + \lambda^2\xi_3. \end{cases} \tag{25}$$

Differentiating the two entities with respect to λ, we obtain:

$$\begin{aligned}\frac{\partial\mu_p(\lambda)}{\partial\sigma_p^2(\lambda)} &= \frac{\partial(\omega_1 + \lambda\xi_3)}{\partial(\xi_1 + \lambda^2\xi_3)}, \\ &= \frac{\xi_3}{2\lambda\xi_3}, \quad \text{with } \lambda > 0 \text{ and } \xi_3 \neq 0, \\ &= \frac{1}{2\lambda}. \end{aligned}$$

□

3.3. *Determination of the next pivot point*

As described above, the proposed algorithm identifies all the pivot points of Problem (3), which forms the ends of the parabolic segments of the efficient frontier. During each iteration k, the objective is to move downwards from the current pivot point λ^k to the other end of the segment λ^{k+1}, ensuring that the KKT conditions remain satisfied. The algorithm stops once the minimum variance portfolio $\lambda^\rho = 0$ is reached. The new pivot point is determined as follows:

$$\lambda^{k+1} = \max\{\lambda_{j_0}^k, \lambda_{j_1}^k\}.$$

The step $\lambda_{j_0}^k$ is the smallest value of λ for which $x_B^k(\lambda)$ remains feasible, and $j_0 \in J_B$ is the index of the basic variable that wants to become nonbasic:

- **Case (a):** $x_{j_0}^k$ moves to the lower bound, i.e: $x_{j_0}^{k+1} = l_{j_0}$:

$$\gamma_{1,j}^k + \lambda_j^k \gamma_{2,j}^k = l_j \Leftrightarrow \lambda_j^k = \frac{\gamma_{1,j}^k - l_j}{\gamma_{2,j}^k}, \quad \gamma_{2,j}^k \neq 0.$$

- **Case (b):** $x_{j_0}^k$ moves to the upper bound, i.e: $x_{j_0}^{k+1} = u_{j_0}$:

$$\gamma_{1,j}^k + \lambda_j^k \gamma_{2,j}^k = l_j \Leftrightarrow \lambda_j^k = \frac{\gamma_{1,j}^k - u_j}{\gamma_{2,j}^k}, \quad \gamma_{2,j}^k \neq 0.$$

Therefore, the step $\lambda_{j_0}^k$ will be calculated by combining the two previous cases:

$$\lambda_{j_0}^k = \max_{j \in J_B^k} \{\lambda_j^k\}; \lambda_j^k = \begin{cases} \dfrac{\gamma_{1,j}^k - l_j}{\gamma_{2,j}^k}, & \text{if } \gamma_{2,j}^k < 0, \quad \text{and } \lambda_j^k < \lambda^k, \\ \dfrac{\gamma_{1,j}^k - u_j}{\gamma_{2,j}^k}, & \text{if } \gamma_{2,j}^k > 0, \quad \text{and } \lambda_j^k < \lambda^k, \\ 0, & \text{otherwise.} \end{cases} \tag{26}$$

Analogously, the step $\lambda_{j_1}^k$ represents the smallest value of λ for which the reduced costs vector $E_N^k(\lambda)$ does not change the sign, and $j_1 \in J_N$ is the index of the nonbasic variable that wants to become basic:

$$\beta_{1,j}^k + \lambda_j^k \beta_{2,j}^k = 0 \Leftrightarrow \lambda_j^k = \frac{-\beta_{1,j}^k}{\beta_{2,j}^k}, \quad \beta_{2,j}^k \neq 0.$$

Therefore, the step $\lambda_{j_1}^k$ will be calculated as follows:

$$\lambda_{j_1}^k = \max_{j \in J_N^k} \{\lambda_j^k\}; \lambda_j^k = \begin{cases} \dfrac{-\beta_{1,j}^k}{\beta_{2,j}^k}, & \text{if } \beta_{2,j}^k \neq 0, \quad \text{and } \lambda_j^k < \lambda^k, \\ 0, & \text{otherwise.} \end{cases} \tag{27}$$

3.4. Updating the support

Once the new pivot point is determined, we change the support of the problem, invoking the previous cases.

- **Case (1):** If $\lambda^{k+1} = 0$, then the minimum variance portfolio is reached, and the efficient frontier is fully obtained. Thus, we stop the algorithm.

Otherwise, the two sets J_B^k and J_N^k will be updated as follows:

- **Case (2):** If $\lambda^{k+1} = \lambda_{j_0}^k$, then:

$$J_B^{k+1} \leftarrow J_B^k \setminus j_0, \quad J_N^{k+1} \leftarrow J_N^k \cup j_0. \tag{28}$$

- **Case (3):** If $\lambda^{k+1} = \lambda_{j_1}^k$, then:

$$J_B^{k+1} \leftarrow J_B^k \cup j_1, \quad J_N^{k+1} \leftarrow J_N^k \setminus j_1. \tag{29}$$

After this step, the resolution process is iterated until all corner portfolios are completely determined.

Remark 1
As the algorithm processes iteratively, where the support set J_B^k is updated at each iteration k, which implies that there can be at most $(\rho - 1)$ distinct sets, where each one corresponds to a segment on the efficient frontier $[\lambda^k, \lambda^{k+1}]$, thus guaranteeing the termination of the solving process after ρ iterations.

Algorithm 1: ***Parametric Support Method for the general MV model***

Data: $\Sigma,\ \mu,\ A,\ b,\ l,\ u$;

Result: $\lambda^k,\ \{x^k(\lambda), J_B^k\},\ \ k=0,1,\cdots,\rho$; with ρ is the maximum numbre of corner portfolios;

- **Initialization:**
  - ▷ Set $k=0$ and $\lambda^k=\infty$;
  - ▷ Let $\{x^0, J_B^0\}$ be the optimal SFS of the MRP problem (8);
- **Step 1:**
  - ▷ Calculate the vectors $\alpha_1^k, \alpha_2^k, \gamma_1^k, \gamma_2^k, \beta_1^k$, and β_2^k according to formulas (10), (11) and (12) from Theorem 3.
  - ▷ Compute the next pivot point $\lambda^{k+1}=\max\{\lambda_{j0}^k, \lambda_{j1}^k\}$, where $\lambda_{j_0}^k$ and $\lambda_{j_1}^k$ are computed using formulas (26) and (27) respectively;
  - ▷ Calculate the vectors:

$$y^k(\lambda)=\alpha_1^k+\lambda^{k+1}\alpha_2^k;$$
$$x_B^k(\lambda)=\gamma_1^k+\lambda^{k+1}\gamma_2^k;$$
$$E_N^k(\lambda)=\beta_1^k+\lambda^{k+1}\beta_2^k;$$

- **Step 2:**
  ▷ Calculate the expected return and variance of the corner portfolio, using formulas given in (18);
- **Step 3:**
  - ▷ Update the support of the problem according to the following cases:

    if $\lambda^{k+1}=0$ **then**

    The efficient frontier is fully obtained, and stop the algorithm;

    else

    Case (a): if $\lambda^{k+1}=\lambda_{j_0}^k$ **then**

    $J_B^{k+1}\longleftarrow J_B^k\setminus j_0;\ \ J_N^{k+1}\longleftarrow J_N^k\cup j_0;$

    Case (b): if $\lambda^{k+1}=\lambda_{j_1}^k$ **then**

    $J_B^{k+1}\longleftarrow J_B^k\cup j_1;\ \ J_N^{k+1}\setminus J_N^k\cup j_0;$

    end

▷ Increment $k\longleftarrow k+1$, and go to Step 1;

3.5. *The Algorithm*

In the following, we provide our algorithm schematic with the various steps to solve the MV model under general linear constraints with bounded assets.

3.6. *Numerical example*

To validate and illustrate the effectiveness of our algorithm, we present two numerical examples. The first example provides a detailed step-by-step illustration to show the algorithm's robustness and the precise way it works. The second example uses a well-known data set to confirm the algorithm's performance.

3.6.1. Example 1. To further show the steps and operation of our algorithm, we consider a portfolio composed of 3 assets, with the following data:

$$\Sigma = \begin{pmatrix} 0.4032 & 0.2174 & 0.3308 \\ 0.2174 & 0.2262 & 0.2926 \\ 0.3308 & 0.2926 & 0.4044 \end{pmatrix}, \ \mu = \begin{pmatrix} 0.8627 \\ 0.4843 \\ 0.8449 \end{pmatrix}, \ A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \end{pmatrix}, \ b = \begin{pmatrix} 1 \\ 0.8 \end{pmatrix}, \ l = \begin{pmatrix} 0.1 \\ 0 \\ 0.1 \end{pmatrix}, \ u = \begin{pmatrix} 0.8 \\ 1 \\ 0.9 \end{pmatrix}.$$

- **Initialization:**
  - ▷ We set $k = 0$ and $\lambda^0 = \infty$;
  - ▷ We solve the MRP problem using any Simplex-type method for linear programming. The optimal SFS obtained $\{x^0, J_B^0\}$ is $x^0 = (0.6, 0.3, 0.1)'$, with $J_B^0 = \{1, 2\}$, and $J_N^0 = \{3\}$.
- **Iteration 1.**
  - **Step 1:**
    - ▷ We calculate the vectors $\alpha_1^0, \alpha_2^0, \gamma_1^0, \gamma_2^0, \beta_1^0,$ and β_2^0, such that :

$$\alpha_1^0 = \begin{pmatrix} -0.2276 \\ -0.1127 \end{pmatrix}, \ \alpha_2^0 = \begin{pmatrix} 0.4843 \\ 0.3784 \end{pmatrix}, \ \gamma_1^0 = \begin{pmatrix} 0.6000 \\ 0.3000 \end{pmatrix}, \ \gamma_2^0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \ \beta_1^0 = -0.1262, \text{ and } \beta_2^0 = 0.3962.$$

    - ▷ We compute the next pivot point $\lambda^1 = \max\{\lambda_{j0}^0, \lambda_{j1}^0\}$, where:

$$\bullet \ \lambda_{j_0}^0 = \max_{j \in J_B^0} \{\lambda_j\} = \left\{ \begin{array}{c} \dfrac{\gamma_{1,j}^0 - l_j}{\gamma_{2,j}^0}, \\ \dfrac{\gamma_{1,j}^0 - u_j}{\gamma_{2,j}^0}, \end{array} \right\}, \text{ for all } \lambda_j \in]\lambda^0, 0].$$

Thus, $\lambda_{j_0}^0 = \max\{0, 0\} \Leftrightarrow \lambda_{j_0}^0 = 0$.

$$\bullet \ \lambda_{j_1}^0 = \max_{j \in J_N^0} \lambda_j = \left\{ \dfrac{-\beta_{1,j}^0}{\beta_{2,j}^0}, \right\}, \text{ for all } \lambda_j \in]\lambda^0, 0].$$

Thus, $\lambda_{j_1}^0 = 0.3185$ and $j_1 = 3$. Therefore, the next pivot point is $\lambda^1 = \max\{0, 0.3185\} = 0.3185$.

    - ▷ We calculate the vectors $x_B^0(\lambda)$ and $E_N^0(\lambda)$, where:

$$x_B^0(\lambda) = \gamma_1^0 + \lambda^1 \gamma_2^0 = \begin{pmatrix} 0.6000 \\ 0.3000 \end{pmatrix} + 0.3185 \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0.6000 \\ 0.3000 \end{pmatrix}.$$

$$E_N^0(\lambda) = \beta_1^0 + \lambda^1 \beta_2^0 = -0.1262 + 0.3185(0.3962) = 0.$$

  - **Step 2:**
    ▷ We calculate the expected return and variance of the first corner portfolio $x^0(\lambda) = \begin{pmatrix} x_B^0(\lambda) \\ x_N^0(\lambda) \end{pmatrix}$:

$$\mu_p^0(\lambda) = \mu' x^0(\lambda) = 0.7474,$$

$$\sigma_p^2(\lambda) = (x^0(\lambda))' \Sigma x^0(\lambda) = 0.3051.$$

  - **Step 3:**
    ▷ As $\lambda^1 = \lambda_{j_0}^0$, we update the support of the problem as follows:

$$J_B^1 \leftarrow J_B^0 \cup j_1 = \{1, 2, 3\}, \quad J_N^1 \leftarrow J_N^0 \setminus j_1 = \emptyset.$$

▷ We increment $k \longleftarrow 1$, and go to Step 1.

- **Iteration 2.**
  - **Step 1:**
    ▷ We calculate the vectors $\alpha_1^1, \alpha_2^1, \gamma_1^1, \gamma_2^1$, such that:

$$\alpha_1^1 = \begin{pmatrix} -0.2442 \\ -0.0447 \end{pmatrix}, \; \alpha_2^1 = \begin{pmatrix} 0.5366 \\ 0.1652 \end{pmatrix}, \; \gamma_1^1 = \begin{pmatrix} 0.2031 \\ 0.4985 \\ 0.2985 \end{pmatrix}, \; \gamma_2^0 = \begin{pmatrix} 1.2463 \\ -0.6232 \\ -0.6232 \end{pmatrix}.$$

    ▷ We compute the next pivot point $\lambda^2 = \max\{\lambda_{j0}^1, \lambda_{j1}^1\}$, where:

$$\bullet \; \lambda_{j_0}^1 = \max_{j \in J_B^1}\{\lambda_j\} = \left\{ \begin{array}{c} \dfrac{\gamma_{1,j}^1 - l_j}{\gamma_{2,j}^1}, \\ \dfrac{\gamma_{1,j}^1 - u_j}{\gamma_{2,j}^1}, \end{array} \right\}, \text{ for all } \lambda_j \in]\lambda^1, 0].$$

    Thus, $\lambda_{j_0}^0 = \max\{0, 0, 0\} \Leftrightarrow \lambda_{j_0}^0 = 0$.
    As $J_N^1 = \emptyset$, then we set $\lambda_{j_1}^0 = 0$. Therefore, the next pivot point is $\lambda^2 = 0$.

    ▷ We calculate the vectors $x_B^1(\lambda)$ and $E_N^1(\lambda)$, where:

$$x_B^1(\lambda) = \gamma_1^1 + \lambda^2\gamma_2^1 = \begin{pmatrix} 0.2031 \\ 0.4985 \\ 0.2985 \end{pmatrix} + 0 \begin{pmatrix} 1.2463 \\ -0.6232 \\ -0.6232 \end{pmatrix} = \begin{pmatrix} 0.2031 \\ 0.4985 \\ 0.2985 \end{pmatrix}.$$

    As $J_N^1 = \emptyset$, then $E_N^1(\lambda) = \emptyset$.
  - **Step 2:**
    ▷ We calculate the expected return and variance of the second corner portfolio $x^1(\lambda) = \begin{pmatrix} x_B^1(\lambda) \\ x_N^1(\lambda) \end{pmatrix}$:

$$\mu_p^1(\lambda) = \mu' x^1(\lambda) = 0.6688,$$

$$\sigma_p^2(\lambda) = (x^0(\lambda))'\Sigma x^0(\lambda) = 0.2800.$$

  - **Step 3:**
    As $\lambda^2 = 0$, then the minimum variance portfolio is reached, and consequently the efficient frontier is fully determined.

3.6.2. Example 2. To ensure the accuracy and reliability of our algorithm, we consider the same numerical example used in [4] and [30]. This example considers a universe of 10 assets, whose expected returns and covariance matrix data are given in Table 1. Moreover, to verify that the algorithm works properly, we have modified the upper and lower bounds of the example, while preserving the budget constraint.

The first step of the algorithm is to initialize $k = 0$, and solve the maximum-return portfolio problem using the DSM. The optimal SFS $\{x^0, J_B^0\}$ obtained is $x^0 = (0.1, 0.5, 0.1, 0, 0, 0.1, 0, 0.1, 0.1, 0)'$ with $J_B^0 = \{2\}$. The rest of the execution results are summarized in Table 2, which provides a detailed view of the different iterations. For each iteration, the algorithm outputs the pivot point λ^k associated with the corner portfolio $x^k = (x_1, x_2, \cdots, x_{10})'$ and the support set J_B^k, along with the expected return and risk μ_p^k and $(\sigma_p^k)^2$ respectively.

The efficient frontier of the example, illustrated in Figure 3, represents the set of optimal portfolios that provide the best trade-off between return and risk, with the corner portfolios that constitute it.

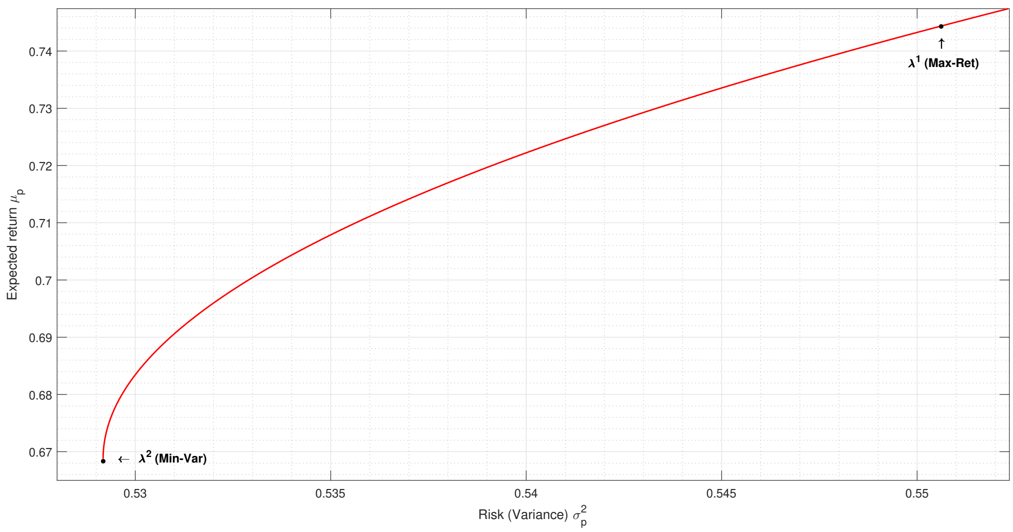

Figure 2. Parametric efficient frontier for the Example 1 with associated corner portfolios.

Table 1. Expected returns, variance-covariance matrix, and asset bounds vectors.

Stock	1	2	3	4	5	6	7	8	9	10
μ	1.175	1.19	0.396	1.12	0.346	0.679	0.089	0.73	0.481	1.08
	0.4075									
	0.0317	0.9063								
	0.0518	0.0314	0.1949							
	0.0566	0.0269	0.0441	0.1953						
Σ	0.0330	0.0192	0.0301	0.0278	0.3406					
	0.0083	0.0093	0.0132	0.0053	0.0078	0.1598				
	0.0216	0.0249	0.0352	0.0137	0.0207	0.0210	0.6806			
	0.0133	0.0076	0.0115	0.0078	0.0074	0.0052	0.0138	0.9553		
	0.0343	0.0287	0.0427	0.0291	0.0254	0.0172	0.0463	0.0106	0.3168	
	0.0225	0.0134	0.0206	0.0164	0.0128	0.0072	0.0193	0.0076	0.0185	0.1108
l_i	0.1	0.2	0.1	0	0	0.1	0	0.1	0.1	0
u_i	0.8	1	0.5	0.9	1	0.8	0.8	1	1	0.8

4. Computational results

In this section, we present the computational results of experiments performed on nine publicly available benchmark sets. Five of these datasets are as described by Chang et al. [33], and are available through Beasley's OR library [37]. The other four are provided by Cesarane et al. [34] and are available on [36]. These benchmarks include return vectors and covariance matrices for various stock indices, including the Hang Seng, DAX 100, FTSE 100, EuroStoxx 50, S&P 100, Nikkei 225, US S&P 100, US S&P 500, and Euro-American NASDAQ.

In the following, we present a comparative study of our Parametric Support Algorithm (PSA) with the Parametric Active Set Method (PASM) [6] and the Matlab portfolio optimization package. The goal of this comparison is to measure and analyze the execution time performance of these different methods. Our PSA was implemented using Matlab R2019b. For the PASM algorithm, we used Matlab code provided by the author in [7]. The tests were performed on the same machine running Windows 11, equipped with an Intel Core i5-8250U 1.8 GHz processor and 8 GB of RAM.

The algorithms are evaluated in a simplified context where only the budget constraint is considered ($e'x = 1$), with the lower bound set to zero and the upper bound set to one ($0 \leq x \leq 1$). Note that the algorithms provide the

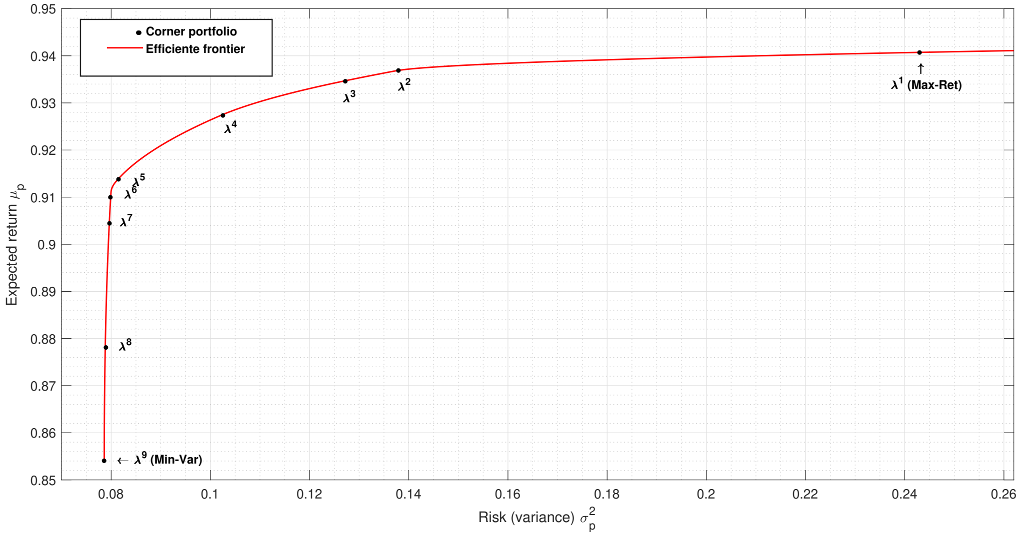

Figure 3. Parametric efficient frontier for the Example 1 with associated corner portfolios.

exact efficient frontier, as shown in Figure 4. However, unlike the Matlab package, which only computes a fixed number of corner portfolios, the PSA and PASM algorithms comprehensively provide all corner portfolios on the frontier.

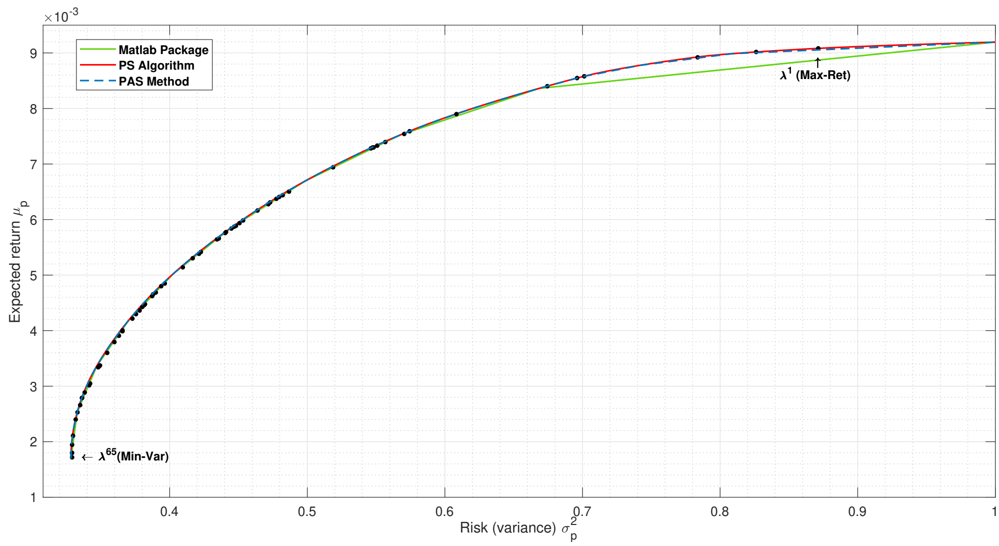

Figure 4. The efficient frontier plotted by the three algorithms for the S&P 100 stock index (98 assets), along with the corner portfolios provided by the PSA method.

Table 3 lists the results of the comparison of the execution times (in seconds) between the three algorithms; PSA, PASM and the Matlab package, on the nine benchmarks. Each dataset is characterized by the name of the

Table 2. Overview of the algorithm iterations and corner portfolios characteristics.

Ieration	Pivot points	Corner portfolios										Return	Risk	Support
k	λ^k	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	μ_p^k	$(\sigma_p^k)^2$	J_B^k
1	26.4414	0.1000	0.5000	0.1000	0.000	0.000	0.1000	0.000	0.1000	0.1000	0.0000	0.9411	0.2620	$\{2\}$
2	2.5265	0.3869	0.2131	0.1000	0.0000	0.0000	0.1000	0.0000	0.1000	0.1000	0.0000	0.9368	0.1373	$\{2, 1\}$
3	1.2820	0.3633	0.2000	0.1000	0.0367	0.0000	0.1000	0.0000	0.1000	0.1000	0.0000	0.9346	0.1266	$\{2, 1, 4\}$
4	1.0123	0.2307	0.2000	0.1000	0.1693	0.0000	0.1000	0.0000	0.1000	0.1000	0.0000	0.9273	0.1020	$\{1, 4\}$
5	0.3938	0.1000	0.2000	0.1000	0.1270	0.0000	0.1000	0.0000	0.1000	0.1000	0.1730	0.9132	0.0809	$\{1, 4, 10\}$
6	0.0307	0.1000	0.2000	0.1000	0.0739	0.0000	0.1000	0.0000	0.1000	0.1000	0.2261	0.9111	0.0800	$\{4, 10\}$
7	0.0227	0.1000	0.2000	0.1000	0.0677	0.0000	0.1148	0.0000	0.1000	0.1000	0.2175	0.9049	0.0796	$\{4, 10, 6\}$
8	0.0087	0.1000	0.2000	0.1000	0.0494	0.0244	0.1339	0.0000	0.1000	0.1000	0.1922	0.8785	0.0788	$\{4, 10, 6, 5\}$
9	0.0000	0.1000	0.2000	0.1000	0.0364	0.0384	0.1428	0.0099	0.1000	0.1000	0.1726	0.8545	0.0786	$\{4, 10, 6, 5, 7\}$

stock index and the number of assets it contains, denoted by n. The columns "CPU-Time" and "ρ" for the PSA and PASM indicate the execution time for deriving the efficient frontier for each stock index, and the number of corner portfolios ρ obtained by both approaches, respectively. In addition, the last column reports the CPU time of the Matlab package, if available ¶

Table 3. Comparison results in terms of CPU time (in seconds) for stock indices.

	PS Algorithm		PAS Method		Matlab Package
Stock indices (n)	CPU-Time	ρ	CPU-Time	ρ	CPU-Time
Hang Seng (31)	0.0328	17	0.0973	17	0.9539
DAX 100 (85)	0.0385	37	0.1735	37	1.2411
FTSE 100 (89)	0.0410	50	0.2093	50	1.4704
S&P 100 (98)	0.0436	65	0.2437	65	1.7592
Nikkei 500 (225)	0.0402	44	0.9901	44	1.9605
EuroStoxx50 (48)	0.0331	25	0.1103	25	0.9444
FTSE 100 (79)	0.0369	44	0.2043	44	1.3039
S&P 500 (476)	0.1651	99	8.5162	99	51.3856
NASDAQ (2196)	0.3498	396	-	-	-

The results presented in Table 3 demonstrate the effectiveness of our PSA algorithm in comparison with PASM and Matlab package, particularly for datasets with large numbers of assets. For example, for the S&P 500 index, the PSA achieved results in 0.1651 seconds, whereas the PASM algorithm required 8.5162 seconds and the Matlab package took over 51 seconds. Moreover, for NASDAQ index with 2195 assets, neither PASM nor Matlab package were able to obtain results in less than 10 minutes, while our algorithm required less than 0.3 seconds.

In order to confirm these results and further evaluate the effectiveness of our PSA approach to solve large-scale portfolio optimization problems, we generated random test problems following two procedures. The first procedure was based on the methodology described by Niedermayer et al. in [10]. This involved generating a series of random problems comprising 50, 100, 150, and 200 assets. Furthermore, to test the performance of our algorithm on large-scale problems with dense covariance matrices, additional datasets comprising 500, 1000, 2000, 3000, and 5000 assets were generated, following the procedure described by Hirschberger et al. in [35].

In addition to execution time, we evaluated the temporal complexity of the three methods to better understand their effectiveness in relation to the problem size. Table 4 summarizes the results of the comparison in terms of CPU time for each set of generated problems, where ”Avr-Time” represents the average of 10 executions. In addition, the last row of the table shows the estimated time complexity of the algorithms as a function of the number of assets involved in each problem.

The obtained results confirm that our PSA algorithm outperforms PASM and the Matlab package for randomly generated problems, particularly for medium-sized problems. Once the problem size exceeds 2000 assets, the PSA continues to show its superior performance, although the average time is slightly higher, which can be explained by the dense covariance matrices generated by the second procedure. In contrast, PASM and Matlab package are impractical for problems with more than 1000 assets and are unable to provide results within the 10-minute time limit. In context, the estimated temporal complexity of PSA, $O(n^{2.2})$, is significantly better than that of PASM and Matlab. Note that this complexity can be reduced to $O(n^{1.3})$ if we consider only the sets generated by Niedermayer's procedure.

In conclusion, the computational results presented confirm that the approach proposed in this study is an efficient method for deriving the Markowitz efficient frontier and providing all its corner portfolios in a reasonable time.

¶To ensure the efficiency and the accuracy of the analysis, the algorithms are set to automatically stop after 10 minutes of execution without producing any concrete results.

Table 4. Comparison results in terms of CPU time (in seconds) for randomly generated problems.

	PS Algorithm	PAS Method	Matlab Package
Problem size	Avr-Time	Avr-Time	Avr-Time
50	0.0016	0.0445	0.0473
100	0.0023	0.1031	0.1662
150	0.0040	0.2619	0.2242
200	0.0106	0.6651	0.5217
500	0.2087	12.5769	89.9587
1000	0.7196	98.1911	504.5293
2000	3.0209	-	-
3000	11.0567	-	-
5000	30.8725	-	-
Estimate	$O(n^{2.2})$	$O(n^{2.7})$	$O(n^{3.4})$

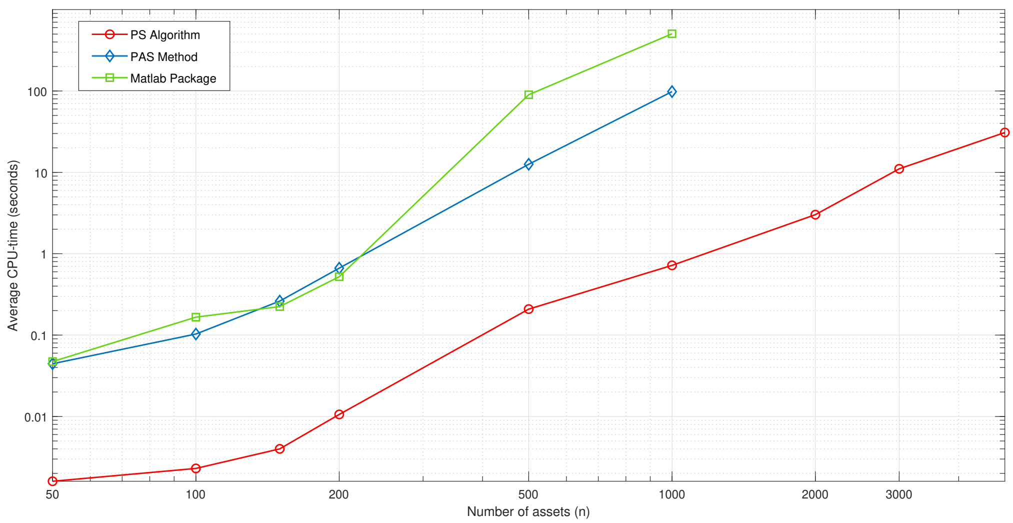

Figure 5. Evolution of the algorithms' performance as a function of problem size for different algorithms.

Furthermore, the method demonstrates its ability to solve large-scale portfolio optimization problems, including those with dense covariance matrices, while maintaining significantly superior computational performance compared to existing methods.

5. Conclusion

In this paper, we have presented a novel Parametric Support Algorithm (PSA) for solving the mean-variance portfolio optimization model with general linear constraints and portfolio limitations. This model is formulated as a convex PQP problem using the risk aversion parameter. Our approach extends the DSM algorithm [14, 16] to solve the problem in a PQP framework. The proposed algorithm enables us to trace the entire Markowitz efficient frontier (Pareto front) by iteratively identifying all corner portfolios. In addition, it determines the associated support sets, providing a comprehensive and detailed analysis of efficient investment trade-offs. The parametric solving process of our PSA algorithm is shown to be easy to understand and simple to implement, offering a practical alternative to the classical parametric approaches in portfolio optimization.

To test the efficiency of our PSA algorithm, we compared it to Matlab's portfolio optimization package and the parametric active set algorithm (PASM)[6, 7]. The results show that our algorithm outperforms both in terms of execution time. Moreover, it demonstrated exceptional computational performance for large-scale portfolio optimization problems involving over 2000 assets, including those with dense covariance matrices.

For future perspectives, we plan to extend our approach to address portfolio optimization problems by incorporating additional real-world constraints, such as transaction costs and the cardinality constraint, as well as other risk measures like semivariance and Value at Risk (VaR).

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