STATISTICS, OPTIMIZATION AND INFORMATION COMPUTING
Stat., Optim. Inf. Comput., Vol. **13**, February 2025, pp 716–740.
Published online in International Academic Press (www.IAPress.org)

A New Power Xgamma Distribution: Statistical Properties, Estimation Methods and Application

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Abstract This paper introduces a novel probability distribution called the power xgamma distribution. We investigate several statistical properties essential for its characterization, including moments, moment generating function, quantile function, and order statistics. Estimation methods are explored to determine the parameters and characteristic functions of this distribution through a comprehensive simulation study. To illustrate its practical applicability, a real-world data example is provided, which demonstrates the effectiveness and relevance of the proposed model in empirical contexts.

Keywords New power xgamma distribution, order statistics, Renyi entropy function, inverse moment.

AMS 2010 subject classifications: 60E05; 62G30; 62F10

DOI: 10.19139/soic-2310-5070-2116

1. Introduction

In many disciplines, including statistics, mathematics, science, engineering, economics, and more, probability distributions are essential, the following are some key justifications for the importance of probability distributions:

- With probability distributions, uncertainty may be modeled and quantified. Results are frequently probabilistic rather than deterministic in real-world scenarios. Predicting stock prices, for instance, in finance entails evaluating the probability distribution of potential price changes.
- Probability distributions serve as the foundation for inference and hypothesis testing in statistics. Through an understanding of a dataset's distribution, statisticians are able to infer relevant details about the population that the data originates from.
- Predictive modeling in data science and machine learning uses probability distributions. They support the estimation of various outcomes' likelihood, which is crucial for tasks like regression and classification.
- Probability distributions play a vital role in risk assessment and uncertain decision-making. Making well-informed decisions is aided by their assistance in assessing possible hazards and their likelihoods.
- Probability distributions are used to mimic random events and optimize processes in operations research and simulation studies. This is especially crucial for industries like logistics, where efficiency can be increased by comprehending the stochastic nature of supply and demand.
- Probability distributions are used in manufacturing and quality control to keep an eye on procedures and make sure final goods fulfill requirements. They support the establishment of suitable quality levels and the comprehension of variation.
- In economics and finance, probability distributions are essential for simulating asset values, interest rates, and other economic variables. They offer perceptions of the return and risk characteristics of investments.

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ISSN 2310-5070 (online) ISSN 2311-004X (print)

Given the importance of probability distribution in lifetime data, many researchers have been intrested in studing and appling it, like the xgamma distribution that is a mixture of exponantial and gamma distribution, [1] introduced this distribution and give it's different properties, also, he demonstrate the flexibility of this new distribution. Many authors have been interested in new distributions based on xgamma ditribution, [2] studied the inference of a new distribution called the log-xgamma distribution. [3] considered the weighted xgamma distribution that is an other vesrion of xgamma distribution. A generalisation and application in bladder cancer data were proposed by [4] named quasi xgamma distributin. [5] presented the different mathematical and statistical properties of the new inverse xgamma distribution. The bivariate xgamma distribution is defined by [6] and the truncated version of this distribution is introduced by [7], Another variant, known as the xgamma Exponential distribution, is validated by [8].
In another way, several works have dealt with the estimation of the xgamma distribution by Bayesian method, [9] calculate the Bayesian estimators of xgamma distribution under type II hybrid censored data, Also, under type I hybrid censored data and assymetric loss function [10] found the Bayesian estimators of the parameters and reliability function of the xgamma distribution.
we have decided to examine the power xgamma distribution due to its significant properties, considering that:

- The flexibility of its PDF allows it to assume a diverse array of forms (negative skewed, symmetrical skewed, positively skewed, J-shaped) depending on the choice of parameters. This flexibility enables the distribution to model a range of variation of data sets.
- This novel distribution provides several advantages, including a range of parameters (2) that can be utilized for modeling in survival analysis and others fields.
- • This distribution has a closed CDF form and this facilitates calculations and analysis making simulation more efficient.
- • This distribution can accommodate both monotonic and non-monotonic hazard function, making it applicable to a wide range of real world data.
- • This distribution provides better adjustment than some well-known distribution as: xgamma, gamma, Weibull, gamma-Lindley, Lomax and Lindley distributions.

We consider Y a random variable follows the one-parameter xgamma distribution, whose the cumulative distribution function is given by the following formula

$$G(y,\theta) = 1 - \frac{1+\theta+\theta y+\frac{\theta^2}{2}y^2}{1+\theta}e^{-\theta y}, \quad y>0,\, \theta>0.$$

Let the following transformation $X = \left(\frac{1}{Y}\right)^{\beta}$, therefore, the variable X follows the new power xgamma distribution with two parameters θ and β, and its cumulative distribution function (CDF) is

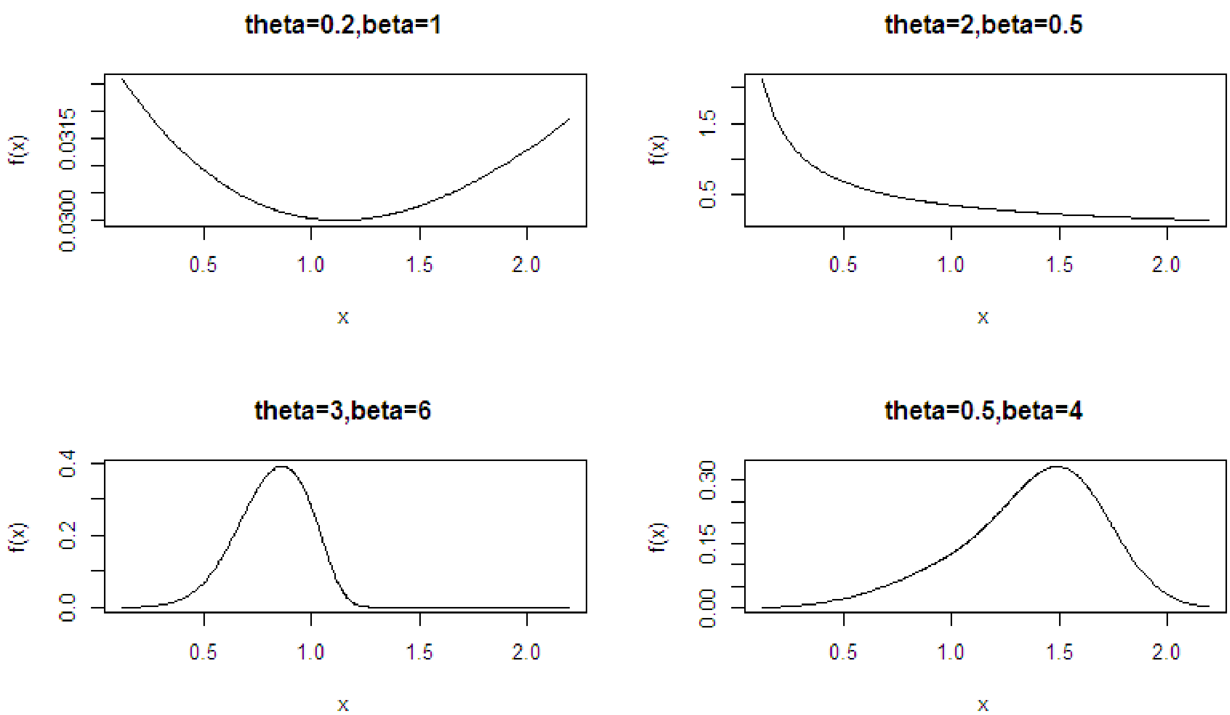
$$F(x;\theta,\beta) = 1 - \frac{1+\theta+\theta x^{\beta}+\frac{\theta^2}{2}x^{2\beta}}{1+\theta}e^{-\theta x^{\beta}}, \;\; x>0, \theta, \beta>0. \tag{1}$$

For more details (see [20]), by deriving $F(x;\theta,\beta)$ with respect to x, we obtain the formula for the probability density function (PDF) given by

$$f(x;\theta,\beta) = \frac{\theta^2\beta}{1+\theta}x^{\beta-1}\left(\frac{\theta^2}{2}x^{2\beta}+1\right)e^{-\theta x^{\beta}}, \;\; x>0, \theta, \beta>0. \tag{2}$$

The following graphs present the probability density function (PDF) of the power xgamma distribution with different values of θ and β.

Figure 1. The PDF of power xgamma for different values of θ and β



The rest of this paper is organized as follows: The survival carachteristics functions are calculated in section 2. Differents moments are presented in section 3. Order statistics and quantile function are considered in sections 4 and 5 respectively. Lorenz and Bonferroni curves are discussed in section 6. The Renyi entropy function is obtained in section 7. Section 8 addressed the differents estimation methods of the parameters, reliability and failure rate functions. A simulation study is applied in section 10 to illustrate the results obtained in section 9. We ended by an exemple to demonstrate the application of the new power xgamma distribution with real data.

2. Survival analysis

The reliability function $R(x;\theta,\beta)$ of the new power-xgamma distribution is given by

$$R(x;\theta,\beta) = 1 - F(x;\theta,\beta)$$

Replacing F with its expression given in (1), we obtain

$$R(x;\theta,\beta) = \frac{1+\theta+\theta x^{\beta}+\frac{\theta^2}{2}x^{2\beta}}{1+\theta}e^{-\theta x^{\beta}}, \quad x>0,\ \theta,\beta>0. \tag{3}$$

The failure rate function $h(x;\theta,\beta)$ of the new power xgamma distribution calculate using the following formula:

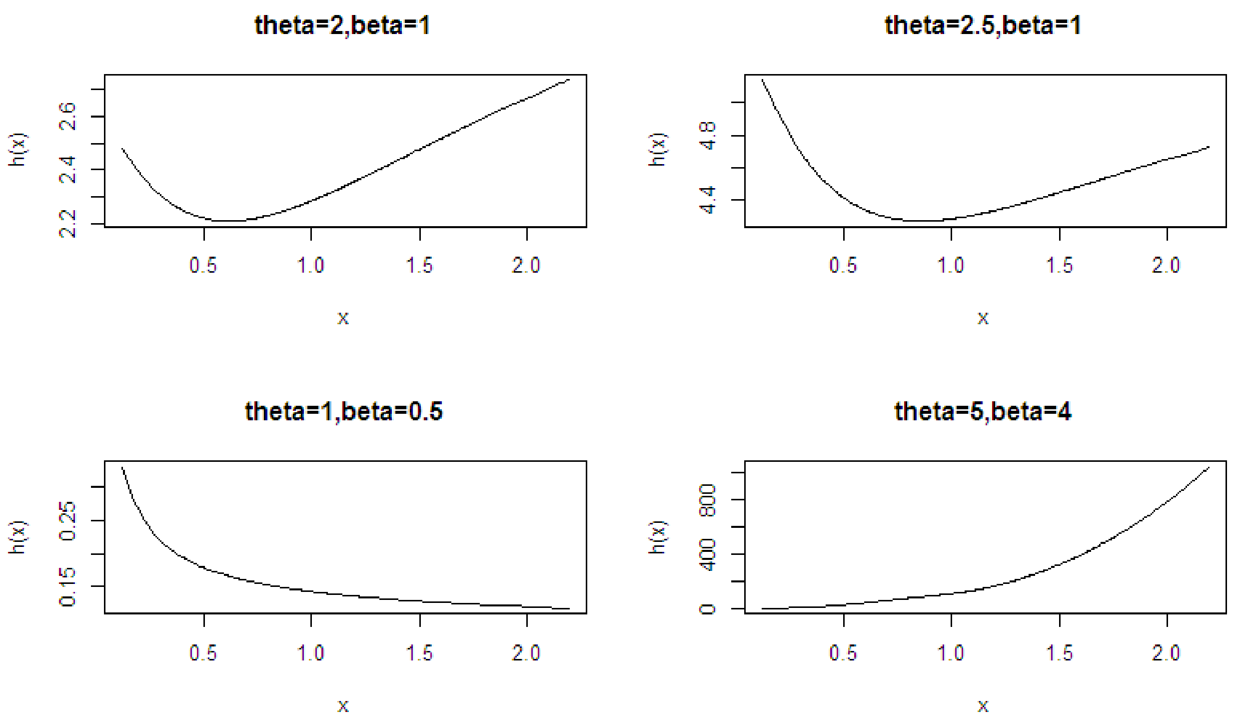
$$h(x;\theta,\beta) = \frac{f(x;\theta,\beta)}{1-F(x;\theta,\beta)} = \frac{f(x;\theta,\beta)}{R(x;\theta,_b eta)}$$

Replacing f and R with the expressions (2) and (3) respectively, we obtain

$$h(x;\theta,\beta)=\frac{\theta^2\beta x^{\beta-1}\left(\frac{\theta^2}{2}x^{2\beta}+1\right)}{1+\theta+\theta x^{\beta}+\frac{\theta^2}{2}x^{2\beta}},\ \ x>0,\ \ \theta,\beta>0. \tag{4}$$

The following graphs presents the reliability function of the new power xgamma distribution for differents values θ and β.

Figure 2. The reliability function of power xgamma for different values of θ and β



3. Moments and other measures

In this section, we present the moments and some other statistical properties as inverse moments, moment generating function,... of the new power xgamma distribution.

3.1. The k^{th} moments

Proposition 1
The k^{th} moment of the new power xgamma distribution is given by

$$m_k=\frac{\theta^{1-\frac{k}{\beta}}}{(1+\theta)}\Gamma(\frac{k}{\beta}+1)\left[\frac{1}{2}\left(\frac{k}{\beta}+2\right)\left(\frac{k}{\beta}+1\right)+1\right],\quad k=1,2,3,...$$

Proof
Using the definition of the k^{th} moment we have:

$$
\begin{aligned}
m_k &= \int_0^{+\infty} x^k f(x;\theta,\beta)dx \\
&= \frac{\theta^2\beta}{1+\theta}\int_0^{+\infty} x^{k+\beta-1}\left(\frac{\theta^2}{2}x^{2\beta}+1\right)e^{-\theta x^{\beta}}dx \\
&= \frac{\theta^2\beta}{1+\theta}\left[\frac{\theta^2}{2}I_1+I_2\right],
\end{aligned}
$$

by the following change of variables $Y = x^{\beta}$, we obtain

$$I_1 = \frac{1}{\beta}\int_0^{+\infty} y^{\frac{k}{\beta}+2}e^{-\theta y}dy = \frac{1}{\beta}\frac{\Gamma\left(\frac{k}{\beta}+3\right)}{\theta^{\frac{k}{\beta}+3}}.$$

and

$$I_2 = \frac{1}{\beta}\int_0^{+\infty} y^{\frac{k}{\beta}}e^{-\theta y}dy = \frac{1}{\beta}\frac{\Gamma\left(\frac{k}{\beta}+1\right)}{\theta^{\frac{k}{\beta}+1}}.$$

So

$$
\begin{aligned}
m_k &= \frac{\theta^2\beta}{1+\theta}\left[\frac{\theta^2}{2}\frac{1}{\beta}\frac{\Gamma\left(\frac{k}{\beta}+3\right)}{\theta^{\frac{k}{\beta}+3}}+\frac{1}{\beta}\frac{\Gamma\left(\frac{k}{\beta}+1\right)}{\theta^{\frac{k}{\beta}+1}}\right] \\
&= \frac{\theta^{1-\frac{k}{\beta}}}{(1+\theta)}\Gamma(\frac{k}{\beta}+1)\left[\frac{1}{2}\left(\frac{k}{\beta}+2\right)\left(\frac{k}{\beta}+1\right)+1\right]
\end{aligned}
$$

□

Thus, the first four moments of the power xgamma distribution are obtained when we replace k with 1, 2, 3, 4 respectively, so

$$
\begin{aligned}
k=1,\ \ E(x) &= \frac{\theta^{1-\frac{1}{\beta}}}{(1+\theta)}\Gamma(\frac{1}{\beta}+1)\left(\frac{1}{2}\left(\frac{1}{\beta}+2\right)\left(\frac{1}{\beta}+1\right)+1\right) \\
k=2,\ \ E(x^2) &= \frac{\theta^{1-\frac{2}{\beta}}}{(1+\theta)}\Gamma(\frac{2}{\beta}+1)\left(\frac{1}{2}\left(\frac{2}{\beta}+2\right)\left(\frac{2}{\beta}+1\right)+1\right) \\
k=3,\ \ E(x^3) &= \frac{\theta^{1-\frac{3}{\beta}}}{(1+\theta)}\Gamma(\frac{3}{\beta}+1)\left(\frac{1}{2}\left(\frac{3}{\beta}+2\right)\left(\frac{3}{\beta}+1\right)+1\right) \\
k=4,\ \ E(x^4) &= \frac{1-\theta^{\frac{4}{\beta}}}{(1+\theta)}\Gamma(\frac{4}{\beta}+1)\left(\frac{1}{2}\left(\frac{4}{\beta}+2\right)\left(\frac{4}{\beta}+1\right)+1\right)
\end{aligned}
$$

The variance ($Var(x)$), the skewness ($\sqrt{\delta_1}$), the kurtosis (δ_2) and the coefficient of variation ($c.v$) of the new power xgamma distribution are:

$$Var(x) = \frac{\theta^{1-\frac{2}{\beta}}}{(1+\theta)}\left[\Gamma(\frac{2}{\beta}+1)\left(\frac{1}{2}\left(\frac{2}{\beta}+2\right)\left(\frac{2}{\beta}+1\right)+1\right)-\frac{1}{1+\theta}\Gamma(\frac{2}{\beta}+1)^2\psi(\theta,\beta)\right]$$

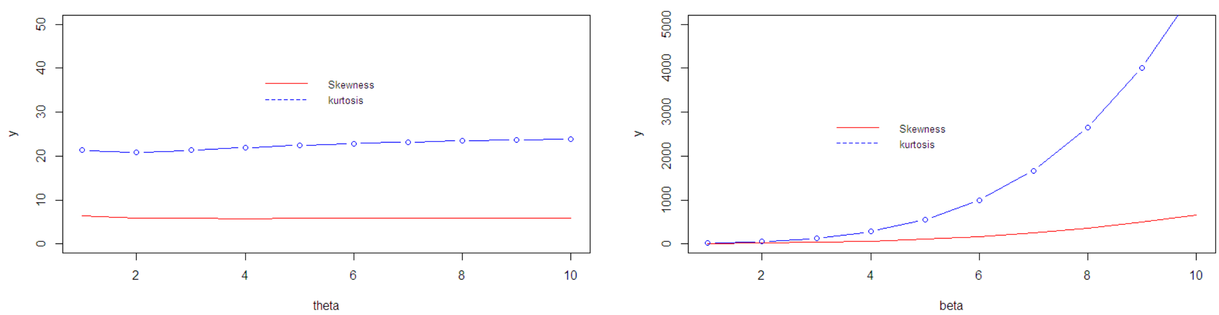
$$\sqrt{\delta_1} = \frac{E(x^3)}{var(x)^{\frac{3}{2}}} = \frac{(1+\theta)^{\frac{1}{2}}\Gamma(\frac{3}{\beta}+1)\left(\frac{1}{2}\left(\frac{3}{\beta}+2\right)\left(\frac{3}{\beta}+1\right)+1\right)}{\left[\Gamma(\frac{2}{\beta}+1)\left(\frac{1}{2}\left(\frac{2}{\beta}+2\right)\left(\frac{2}{\beta}+1\right)+1\right)-\frac{1}{1+\theta}\Gamma(\frac{2}{\beta}+1)^2\psi(\theta,\beta)\right]^{\frac{3}{2}}}$$

$$\delta_2 = \frac{E(x^4)}{var(x)^2} = \frac{(1+\theta)\Gamma(\frac{4}{\beta}+1)\left(\frac{1}{2}\left(\frac{4}{\beta}+2\right)\left(\frac{3}{\beta}+1\right)+1\right)}{\left[\Gamma(\frac{4}{\beta}+1)\left(\frac{1}{2}\left(\frac{2}{\beta}+2\right)\left(\frac{2}{\beta}+1\right)+1\right)-\frac{1}{1+\theta}\Gamma(\frac{2}{\beta}+1)^2\psi(\theta,\beta)\right]^2}$$

$$c.v = \frac{\sqrt{var(x)}}{E(x)} = \frac{\sqrt{(1+\theta)(\Gamma(\frac{2}{\beta}+1)\left(\frac{1}{2}\left(\frac{2}{\beta}+2\right)\left(\frac{2}{\beta}+1\right)+1\right)-\frac{1}{1+\theta}\Gamma(\frac{2}{\beta}+1)^2\psi(\theta,\beta))}}{\Gamma(\frac{1}{\beta}+1)\left(\frac{1}{2}\left(\frac{1}{\beta}+2\right)\left(\frac{1}{\beta}+1\right)+1\right)}$$

Where $\psi(\theta,\beta) = \left(\frac{1}{4}\left(\frac{2}{\beta}+2\right)^2\left(\frac{2}{\beta}+1\right)^2+1+\left(\frac{2}{\beta}+2\right)\left(\frac{2}{\beta}+1\right)\right)$.
The following graphs shows some values of skewness and kurtosis for different values of θ and β.

Figure 3. The skewness and the kurtosis of the new power xgamma distribution for different values of θ and β



3.2. The inverse moment

Proposition 2
Let X be a real random variable following the new power xgamma distribution. The inverse moment is given by

$$\mu_k = \frac{\theta^{\frac{k}{\beta}+1}}{1+\theta}\Gamma\left(1-\frac{k}{\beta}\right)\left[\frac{1}{2}\left(2-\frac{2}{k}\right)\left(1-\frac{k}{\beta}\right)+1\right],\quad k=1,2,3,...$$

Proof
Using the definition of the inverse moment, we have

$$\mu_k = E(x^{-k}) = \int_0^{+\infty} x^{-k} f(x;\theta,\beta)dx$$

replacing f by the expression (2), we obtain

$$\begin{aligned}\mu_k &= \frac{\theta^2\beta}{1+\theta}\left[\frac{\theta^2}{2}\int_0^{+\infty} x^{3\beta-k+1}e^{-\theta x^\beta}dx + \int_0^{+\infty} x^{\beta-k-1}e^{-\theta x^\beta}dx\right] \\ &= \frac{\theta^2}{1+\theta}\left[\frac{\theta^2}{2}\int_0^{+\infty} y^{2-\frac{k}{\beta}}e^{-\theta y}dy + \int_0^{+\infty} y^{-\frac{k}{\beta}}e^{-\theta y}dy\right], \;\; y = x^\beta \\ &= \frac{\theta^2}{1+\theta}\left[\frac{\theta^2}{2}\frac{\Gamma(3-\frac{k}{\beta})}{\theta^{3-\frac{k}{\beta}}} + \frac{\Gamma(1-\frac{k}{\beta})}{\theta^{1-\frac{k}{\beta}}}\right] \\ &= \frac{\theta^{\frac{k}{\beta}+1}}{1+\theta}\Gamma\left(1-\frac{k}{\beta}\right)\left[\frac{1}{2}\left(2-\frac{2}{k}\right)\left(1-\frac{k}{\beta}\right)+1\right], \;\; k = 1,2,3,...\end{aligned}$$

□

3.3. Moment generating function

Proposition 3
The moment generating function , of the new power xgamma distribution is

$$\phi(x,s) = \frac{1}{1+\theta}\sum_{k=0}^{\infty}\frac{s^k}{k!}\theta^{1-\frac{k}{\beta}}\Gamma(\frac{k}{\beta}+1)\left(\frac{1}{2}\left(\frac{k}{\beta}+2\right)\left(\frac{k}{\beta}+1\right)+1\right), \quad k = 1,2,3...$$

Proof
We have

$$\phi(x,s) = \int_0^{+\infty} e^{sx}f(x;\theta,\beta)dx,$$

we know that $e^{sx} = \sum_{k=0}^{+\infty}\frac{(sx)^k}{k!}$, so, we can write

$$\begin{aligned}\phi(x,s) &= \sum_{k=0}^{\infty}\frac{s^k}{k!}\int_0^{+\infty} x^k f(x;\theta,\beta)dx \\ &= \frac{1}{1+\theta}\sum_{k=0}^{\infty}\frac{s^k}{k!}\theta^{1-\frac{k}{\beta}}\Gamma(\frac{k}{\beta}+1)\left(\frac{1}{2}\left(\frac{k}{\beta}+2\right)\left(\frac{k}{\beta}+1\right)+1\right).\end{aligned}$$

□

3.4. Mean and median deviation

The dispersion or spread of data points around a central value, such as the mean or median, is quantified using the mean and median deviation.
A central value, typically the mean x of the data collection, is used to calculate the average absolute distance of data points from the mean deviation, also called average absolute deviation. The mean of the new power xgamma

distribution provided by:

$$\begin{aligned}\eta_m &= \int_0^{+\infty} |x-m| f(x;\theta,\beta)dx \\ &= \int_0^{m} (m-x) f(x;\theta,\beta)dx + \int_x^{+\infty} (x-m) f(x;\theta,\beta)dx \\ &= 2mF(m) - 2\int_0^{x} x f(x;\theta,\beta)dx \\ &= \frac{2}{1+\theta}\left[\left(2\,(1+\theta) - \left(1+\theta+\theta m^{\beta} + \frac{\theta^2}{2} m^{2\beta}\right) e^{-\theta m^{\beta}}\right)\right. \\ &\left. - \left(\frac{1}{\theta}\right)^{\frac{1}{\beta}+1} \left(\frac{1}{\theta}\gamma\left(\frac{1}{\beta}+3, \theta m^{\beta}\right) + \gamma\left(\frac{1}{\beta}+1, \theta m^{\beta})\right)\right)\right]\end{aligned}$$

The median absolute difference between each data point and the median of the data set is measured by the median deviation. The median deviation of new power xgamma distribution is:

$$\begin{aligned}\varsigma_m &= m - 2\int_0^{m} x f(x;\theta,\beta)dx \\ &= \frac{1}{1+\theta}\left[(m+m\theta) - 2\left(\frac{1}{\theta}\right)^{\frac{1}{\beta}+1}\left(\frac{1}{2\theta}\gamma(\frac{1}{\beta}+3, \theta m^{\beta}) + \gamma(\frac{1}{\theta}+1, \theta m^{\beta})\right)\right]\end{aligned}$$

4. Order statistics

The sample $X = (x_1, x_2, ..., x_n)$ is assumed to be independent and identically distributed according to the new power xgamma distribution of probability density function $f(x;\theta,\beta)$ and cumulative distribution function $F(x;\theta,\beta)$, then the density of the density of j^{th} order statistics is

$$f_{X_{(j)}}(x) = \frac{n!}{(j-1)!(n-j)!} F(x;\theta,\beta)^{j-1} R(x;\theta,\beta)^{n-j} f(x;\theta,\beta),$$

by replacing f, F and R with its formulas, we obtain

$$\begin{aligned}f_{X_{(j)}}(x) &= \frac{n!}{(j-1)!(n-j)!}\left(1 - \frac{1+\theta+\theta x^{\beta} + \frac{\theta^2}{2}x^{2\beta}}{1+\theta} e^{-\theta x^{\beta}}\right)^{(j-1)} \left(\frac{1+\theta+\theta x^{\beta} + \frac{\theta^2}{2}x^{2\beta}}{1+\theta} e^{-\theta x^{\beta}}\right)^{(n-j)} \\ &\frac{\theta^2\beta}{1+\theta} x^{\beta-1}\left(\frac{\theta^2}{2}x^{2\beta}+1\right)e^{-\theta x^{\beta}}\end{aligned}$$

Using the binomial theorem, we have

$$\left(1 - \frac{1+\theta+\theta x^{\beta} + \frac{\theta^2}{2}x^{2\beta}}{1+\theta} e^{-\theta x^{\beta}}\right)^{(j-1)} = \sum_{i=1}^{j-1}\binom{j-1}{i}\left(-\frac{1+\theta+\theta x^{\beta} + \frac{\theta^2}{2}x^{2\beta}}{1+\theta} e^{-\theta x^{\beta}}\right)^{j-1-i}$$

so

$$\begin{aligned}f_{X_{(j)}}(x) &= \frac{n!(1+\theta+\theta x^{\beta} + \frac{\theta^2}{2}x^{2\beta})^{n-j}\theta^2\beta x^{\beta-1}(\frac{\theta}{2}x^{2\beta}+1)}{(j-1)!(n-j)!(1+\theta)^{n-j+1}} \\ &\sum_{i=1}^{j-1}\binom{j-1}{i}\left(-\frac{1+\theta+\theta x^{\beta} + \frac{\theta^2}{2}x^{2\beta}}{1+\theta}\right)^{j-1-i} e^{-\theta(j-1-i)x}.\end{aligned}$$

The density of the first and last order statistics of the new power xgamma distribution are respectively given by

$$f_{X_{(1)}}(x) = \frac{n!(1+\theta+\theta x^{\beta}+\frac{\theta^2}{2}x^{2\beta})^{n-1}\theta^2\beta x^{\beta-1}(\frac{\theta}{2}x^{2\beta}+1)}{(1+\theta)^n}$$

and

$$f_{X_{(n)}}(x) = \frac{n\theta^2\beta x^{\beta-1}(\frac{\theta}{2}x^{2\beta}+1)}{(1+\theta)}\sum_{i=1}^{j-1}\binom{j-1}{i}\left(-\frac{1+\theta+\theta x^{\beta}+\frac{\theta^2}{2}x^{2\beta}}{1+\theta}\right)^{j-1-i} e^{-\theta(j-1-i)x}.$$

5. Quantile function

Let X be a random variable following the new power xgamma distribution, the quantile function $G(p)$, $0 < p < 1$ is the solution of the nonlinear equation $F(G(p)) = p$ i.e

$$log\left(1+\theta+\theta G(p)^p+\frac{\theta^2}{2}G(p)^{2\beta}\right) - \theta G(p)^{\beta} - log(1-p) - log(1+\theta) = 0$$

The next table give some values of the quantile function for different values of θ and β and p.
We notice that when the values of p increases, the values of the quantile function also increases.

Table 1. Values of the quantile function

p	$(\theta,\beta)=(1,2)$	$(\theta,\beta)=(2,3)$	$(\theta,\beta)=(3,1)$	$(\theta,\beta)=(0.5,0.8)$	$(\theta,\beta)=(2,0.5)$
0.1	1.425089	0.988775	0.627785	6.099662	0.934587
0.2	1.482162	1.016072	0.6823365	6.70388	1.100476
0.3	1.543113	1.044821	0.7431316	7.388059	1.301038
0.4	1.609393	1.075625	0.8120622	8.177857	1.548626
0.5	1.682972	1.10927	0.8920671	9.112839	1.862998
0.6	1.767178	1.147149	0.9882328	10.25961	2.27865
0.7	1.868082	1.191671	1.109698	11.74379	2.863885
0.8	1.998698	1.248082	1.277218	13.84912	3.779497
0.9	2.19868	1.331929	1.556013	17.49044	5.583355
0.95	2.37734	1.404591	1.827815	21.18894	1.678944
0.99	2.737779	1.54557	2.441297	29.99929	13.6303

6. Lorenz and Bonferroni curves

In statistics, two distinct graphical tools are used: Lorenz and Bonferroni curves. In conclusion, the Bonferroni curve helps to control the total type I error rate in multiple hypothesis testing, while the Lorenz curve aids in visualizing income or wealth inequality. The Lorenz curve of the new power xgamma distribution is calculated as:

$$
\begin{aligned}
L(p) &= \frac{1}{m}\int_0^k xf(x;\theta,\beta)dx \\
&= \frac{1}{m}\int_0^k \frac{\theta^2\beta}{1+\theta}x^{\beta}\left(\frac{\theta^2}{2}x^{2\beta}+1\right)exp(-\theta x^{\beta})dx \\
&= \frac{\theta^2\beta}{m(1+\theta)}\left[\frac{\theta^2}{2}\int_0^k x^{3\beta}exp(-\theta x^{\beta})dx + \int_0^k x^{\beta}exp(-\theta x^{\beta})dx\right] \\
&= \frac{\theta^2\beta}{m(1+\theta)}\left[\frac{\theta^2}{2}I_1 + I_2\right]
\end{aligned}
$$

where

$$
\begin{aligned}
I_1 &= \int_0^k x^{3\beta}exp(-\theta x^{\beta})dx = \frac{1}{\beta\theta^{\frac{1}{\beta}+3}}\int_0^{\theta k^{\beta}} u^{\frac{1}{\beta}+2}exp(-u)du \\
&= \frac{1}{\beta\theta^{\frac{1}{\beta}+3}}\gamma(\theta k^{\beta},\frac{1}{\beta}+3), \quad u=\theta x^{\beta} \\
I_2 &= \frac{1}{\beta\theta^{\frac{1}{\beta}+1}}\gamma(\theta k^{\beta},\frac{1}{\beta}+1)
\end{aligned}
$$

Consequently, the Lorenz curve of the new power xgamma is

$$
L(p) = \frac{\theta^{-\frac{1}{\beta}-1}}{m(1+\theta)}\left[\gamma\left(\theta k^{\beta},\frac{1}{\beta}+3\right)+\theta^2\gamma\left(\theta k^{\beta},\frac{1}{\beta}+1\right)\right]
$$

In the same way, we calculate the Bonferroni curve, we obtain

$$
\begin{aligned}
B(p) &= \frac{1}{pm}\int_0^k xf(x;\theta,\beta)dx \\
&= \frac{\theta^{-\frac{1}{\beta}-1}}{pm(1+\theta)}\left[\gamma\left(\theta k^{\beta},\frac{1}{\beta}+3\right)+\theta^2\gamma\left(\theta k^{\beta},\frac{1}{\beta}+1\right)\right]
\end{aligned}
$$

7. Renyi entropy

Renyi entropy is a quantity that is highly significant, particularly in the ecology and statistics domains. [17] introduced it, and the expression its formula is

$$
R_{\lambda} = \frac{1}{1-\lambda}log(f(x;\theta,\beta))^{\lambda}dx
$$

The Renyi entropy of the new power xgamma distribution is

$$
\begin{aligned}
R_{1-\lambda} &= \frac{1}{1-\lambda}log\left[\int_0^{+\infty}\frac{\theta^{2\lambda}\beta^{\lambda}}{(1+\theta)^{\lambda}}\left(x^{\beta-1}\left(\frac{\theta^2}{2}x^{2\beta}+1\right)\right)^{\lambda}exp(-\lambda\theta x^{\beta})dx\right] \\
&= \frac{1}{1-\lambda}log\left[\frac{\theta^{2\lambda}\beta^{\lambda}}{(1+\theta)^{\lambda}}\left[\int_0^{+\infty}\left(\frac{\theta^2}{2}x^{3\beta-1}+x^{\beta-1}\right)^{\lambda}exp(-\lambda\theta x^{\theta})dx\right]\right]
\end{aligned}
$$

Using the binomial theorem

$$\left(\frac{\theta^2}{2}x^{3\beta-1}+x^{\beta-1}\right)^{\lambda}=\sum_{j=0}^{\lambda}\binom{\lambda}{j}\left(\frac{\theta^2}{2}x^{3\beta-1}\right)^{\lambda-j}(x^{\beta-1})^{j}exp(-\lambda\theta x^{\beta})$$

$$\begin{aligned}R_{\lambda}&=\frac{1}{1-\lambda}log\left[\frac{\theta^{2\lambda}\beta^{\lambda}}{(1+\theta)^{\lambda}}\sum_{j=0}^{\lambda}\binom{\lambda}{j}\frac{\theta^{2(\lambda-j)}}{2^{\lambda-j}}\int_0^{+\infty}x^{3\beta\lambda-2\beta j-\lambda}exp(-\lambda\theta x^{\theta})dx\right]\\&=\frac{1}{1-\lambda}log\left[\frac{\theta^{2\lambda}\beta^{\lambda}}{(1+\theta)^{\lambda}}\sum_{j=0}^{\lambda}\binom{\lambda}{j}\frac{\theta^{2(\lambda-j)}}{2^{\lambda-j}}\int_0^{+\infty}y^{3\lambda-2j-\frac{\lambda-1}{\beta}-1}exp(-\lambda\theta y)dy\right],\ y=x^{\beta}\\&=\frac{1}{1-\lambda}log\left[\frac{\beta^{\lambda-1}}{(1+\theta)^{\lambda}}\sum_{j=0}^{\lambda}\binom{\lambda}{j}\theta^{\lambda+\frac{\lambda-1}{\beta}}2^{j-\lambda}\lambda^{2j-3\lambda+\frac{\lambda-1}{\beta}}\Gamma\left(3\lambda-2j-\frac{\lambda-1}{\beta}\right)\right]\end{aligned}$$

8. Estimation methods

In this section, we consider two estimation methods: the maximum likelihood method and the Bayesian method with two loss functions (squared loss function and Linex loss funtion), to estimate the parameters (θ,β), the raliability function and the failure rate function of the new power xgamma distribution.

8.1. Maximum likelihood estimation method

Let $\underline{x}=(x_1,x_2,...,x_n)$ be an n-sample of the new power xgamma distribution with two parameters θ and β. The likelihood function is :

$$\begin{aligned}L(\underline{x};\theta,\beta)&=\prod_{i=1}^{n}f(x_i;\theta,\beta)\\&=\frac{\theta^{2n}\beta^{n}}{(1+\theta)^{n}}\prod_{i=1}^{n}x_i^{\beta-1}\prod_{i=1}^{n}\left(\frac{\theta^2}{2}x_i^{2\beta}+1\right)e^{-\theta\sum_{i=1}^{n}x_i^{\beta}}\end{aligned}$$

The log likelihood function is

$$l(\underline{x};\theta,\beta)=2nlog(\theta)+nlog(\beta)-nlog(1+\theta)+(\beta-1)\sum_{i=1}^{n}log(x_i)+\sum_{i=1}^{n}log\left(\frac{\theta^2}{2}x_i^{2\beta}+1\right)-\theta\sum_{i=1}^{n}x_i^{\beta}.$$

$\hat{\theta}_{mle}$ and $\hat{\beta}_{mle}$ the maximum likelihood of the parameters θ and β are the solutions of the system of equations

$$\begin{cases}\frac{2n}{\theta}+\frac{n}{1+\theta}+\sum_{i=1}^{n}\left(\frac{\theta x_i^{2\beta}}{\frac{\theta^2}{2}x_i^{2\beta}+1}\right)-\sum_{i=1}^{n}x_i^{\beta}=0\\ \frac{n}{\beta}+\sum_{i=1}^{n}log(x_i)+\sum_{i=1}^{n}\left(\frac{\theta^2 log(x_i)x_i^{2\beta}}{\frac{\theta^2}{2}x_i^{2\beta}+1}\right)-\theta\sum_{i=1}^{n}log(x_i)x_i^{\beta}=0\end{cases}$$

We apply numerical methods like EM (Expectation-Maximisation) algorithm (for more details see [14]), to obtain the numerical values of the estimators for the parameters α and β

The maximum likelihood estimators of the reliability and failure rate functions are obtain when we replace θ and β by $\hat{\theta}_{mle}$ and $\hat{\beta}_{mle}$ in the expressions (3) and (4) respectively, so

$$\hat{R}_{mle}(t)=\frac{1+\hat{\theta}_{mle}+\hat{\theta}_{mle}t^{\hat{\beta}_{mle}}+\frac{\hat{\theta}_{mle}^2}{2}t^{2\hat{\beta}_{mle}}}{1+\hat{\theta}_{mle}}e^{-\hat{\theta}_{mle}x^{\hat{\beta}_{mle}}}$$

$$\hat{h}_{mle}(t) = \frac{\hat{\theta}^2_{mle}\hat{\beta}_{mle}t^{\hat{\beta}_{mle}-1}\left(\frac{\hat{\theta}^2_{mle}}{2}t^{2\hat{\beta}_{mle}}+1\right)}{1+\hat{\theta}_{mle}+\hat{\theta}_{mle}t^{\hat{\beta}_{mle}}+\frac{\hat{\theta}^2_{mle}}{2}t^{2\hat{\beta}_{mle}}}.$$

8.2. Bayesian estimation

In this subsection, we consider the Bayesian estimation of the parameters θ and β, the reliability funtcion $R(x;\theta,\beta)$ and the failure rate function $h(x;\theta,\beta)$, using a symmetrical loss function (squared loss function) then asymmetric loss function (Linex loss function).
Let $X = (X_1, X_2, ..., X_n)$ an n-sample from new power xgamma distribution. The posterior density is calculated as follows:

$$\begin{aligned}\pi(\theta,\beta|x) &= \frac{L(x;\theta,\beta)\pi(\theta,\beta)}{\int\int L(x;\theta,\beta)\pi(\theta,\beta)d\theta d\beta}\\ &= K^{-1}L(x;\theta,\beta)\pi(\theta,\beta)\end{aligned}$$

Where $K = \int\int L(x;\theta,\beta)\pi(\theta,\beta)d\theta d\beta$ is the normalisation constant.
We assume that the parameters θ and β are independent, and we take a natural conjugate prior distribution those of gamma

$$\theta \sim Gamma(a_1,b_1) \Rightarrow \pi(\theta) = \frac{b_1^{a_1}}{\Gamma(a_1)}\theta^{a_1-1}exp(-b_1\theta)$$

$$\beta \sim Gamma(a_2,b_2) \Rightarrow \pi(\beta) = \frac{b_2^{a_2}}{\Gamma(a_2)}\beta^{a_2-1}exp(-b_2\beta)$$

The prior distribution of (θ,β) is

$$\pi(\theta,\beta) = \pi(\theta)\pi(\beta) = \frac{b_1^{a_1}b_2^{a_2}}{\Gamma(a_1)\Gamma(a_2)}\theta^{a_1-1}\beta^{a_2-1}exp(-a_1\theta-b_2\beta)$$

So, the posterior density is

$$\pi(\theta,\beta|x) = K^{-1}\frac{\theta^{2n+a_1-1}\beta^{n+a_2-1}}{(1+\theta)^n}\prod_{i=1}^{n}x_i^{\beta-1}\prod_{i=1}^{n}\left(\frac{\theta^2}{2}x_i^{2\beta}+1\right)exp\left(-\theta\left(\sum_{i=1}^{n}x_i^{\beta}+b_1\right)-b_2\beta\right)$$

Bayesian estimation under squared loss function
This loss function is defined by $L_s(t,\delta) = (t-\delta)^2$, under this loss function, the Bayesian estimator is the posterior mean:

$$\hat{\delta}_B = E_\pi(t|x).$$

The Bayesian estimators of the parameters, reliability and failure rate functions of the new power xgamma distribution undthe er squared loss function are

$$\begin{aligned}\hat{\theta}_{BS} &= E_\pi(\theta|x)\\ &= \int_0^{+\infty}\int_0^{+\infty}\theta\pi(\theta,\beta|x)d\theta d\beta\\ &= K^{-1}\int\int\frac{\theta^{2n+a_1}\beta^{n+a_2-1}}{(1+\theta)^n}\prod_{i=1}^{n}x_i^{\beta-1}\prod_{i=1}^{n}\left(\frac{\theta^2}{2}x_i^{2\beta}+1\right)exp\left(-\theta\left(\sum_{i=1}^{n}x_i^{\beta}+b_1\right)-b_2\beta\right)d\theta d\beta\end{aligned}$$

$$\begin{aligned}\hat{\beta}_{BS} &= E_\pi(\beta|x)\\ &= \int_0^{+\infty}\int_0^{+\infty}\beta\pi(\theta,\beta|x)d\theta d\beta\\ &= K^{-1}\int\int\frac{\theta^{2n+a_1-1}\beta^{n+a_2}}{(1+\theta)^n}\prod_{i=1}^{n}x_i^{\beta-1}\prod_{i=1}^{n}\left(\frac{\theta^2}{2}x_i^{2\beta}+1\right)exp\left(-\theta\left(\sum_{i=1}^{n}x_i^{\beta}+b_1\right)-b_2\beta\right)d\theta d\beta\end{aligned}$$

$$\begin{aligned}\hat{R}_{BS}(t) &= E_\pi(R(t)|x)\\ &= \int_0^{+\infty}\int_0^{+\infty} R(t)\pi(\theta,\beta|x)d\theta d\beta\\ &= K^{-1}\int\int \frac{\theta^{2n+a_1-1}\beta^{n+a_2-1}}{(1+\theta)^{n+1}}\prod_{i=1}^{n}x_i^{\beta-1}\prod_{i=1}^{n}\left(\frac{\theta^2}{2}x_i^{2\beta}+1\right)\left(1+\theta+\theta t^\beta+\frac{\theta^2}{2}t^{2\beta}\right)\\ &* exp\left(-\theta\left(\sum_{i=1}^{n}x_i^\beta+b_1+t^\beta\right)-b_2\beta\right)d\theta d\beta\end{aligned}$$

$$\begin{aligned}\hat{h}_{BS}(t) &= E_\pi(h(t)|x)\\ &= \int_0^{+\infty}\int_0^{+\infty} h(t)\pi(\theta,\beta|x)d\theta d\beta\\ &= K^{-1}\int\int \frac{\theta^{2n+a_1+1}\beta^{n+a_2}}{(1+\theta)^{n}}\prod_{i=1}^{n}x_i^{\beta-1}\prod_{i=1}^{n}\left(\frac{\theta^2}{2}x_i^{2\beta}+1\right)\frac{t^{\beta-1}\left(\frac{\theta^2}{2}t^{2\beta}+1\right)}{1+\theta+\theta t^\beta+\frac{\theta^2}{2}t^{2\beta}}\\ &* exp\left(-\theta\left(\sum_{i=1}^{n}x_i^\beta+b_1\right)-b_2\beta\right)d\theta d\beta\end{aligned}$$

Bayesian estimation under Linex loss function

Introduced by [16], this loss function almost exponentially on one side of zero, given by:

$$L_l(t,\delta) \propto exp(w(t-\delta)) - w(t-\delta) + 1, \quad w \neq 0.$$

The Bayesian estimator under this loss function is

$$\hat{\delta} = \frac{-1}{w}log(E_\pi(exp(-w\theta)|x))$$

The Bayesian estimators of the parameters, reliability and failure rate functions pf the new power xgamma distribution under Linex loss function are given by:

$$\begin{aligned}\hat{\theta}_{BL} &= \frac{-1}{w}log(E_\pi(exp(-w\theta)|x))\\ &= \frac{-1}{w}log\left(\int_0^{+\infty}\int_0^{+\infty} exp(-w\theta)\pi(\theta,\beta|x)d\theta d\beta\right)\\ &= \frac{-1}{w}log\left(K^{-1}\int_0^{+\infty}\int_0^{+\infty}\frac{\theta^{2n+a_1-1}\beta^{n+a_2-1}}{(1+\theta)^n}\prod_{i=1}^{n}x_i^{\beta-1}\prod_{i=1}^{n}\left(\frac{\theta^2}{2}x_i^{2\beta}+1\right)\right.\\ &\left. exp\left(-\theta\left(\sum_{i=1}^{n}x_i^\beta+b_1+w\right)-b_2\beta\right)d\theta d\beta\right)\end{aligned}$$

$$\begin{aligned}\hat{\beta}_{BL} &= \frac{-1}{w}log(E_\pi(exp(-w\beta)|x))\\ &= \frac{-1}{w}log\left(\int_0^{+\infty}\int_0^{+\infty} exp(-w\beta)\pi(\theta,\beta|x)d\theta d\beta\right)\\ &= \frac{-1}{w}log\left(K^{-1}\int_0^{+\infty}\int_0^{+\infty}\frac{\theta^{2n+a_1-1}\beta^{n+a_2-1}}{(1+\theta)^n}\prod_{i=1}^{n}x_i^{\beta-1}\prod_{i=1}^{n}\left(\frac{\theta^2}{2}x_i^{2\beta}+1\right)\right.\\ &\left. exp\left(-\theta\left(\sum_{i=1}^{n}x_i^\beta+b_1\right)-(b_2+w)\beta\right)d\theta d\beta\right)\end{aligned}$$

$$\begin{aligned}\hat{R}_{BL}(t) &= \frac{-1}{w} log(E_\pi(exp(-wR(t))|x)) \\ &= \frac{-1}{w} log\left(\int_0^{+\infty}\int_0^{+\infty} exp(-wR(t))\pi(\theta,\beta|x)d\theta d\beta\right) \\ &= \frac{-1}{w} log\left(K^{-1}\int_0^{+\infty}\int_0^{+\infty} \frac{\theta^{2n+a_1-1}\beta^{n+a_2-1}}{(1+\theta)^n}\prod_{i=1}^n x_i^{\beta-1}\prod_{i=1}^n\left(\frac{\theta^2}{2}x_i^{2\beta}+1\right)\right. \\ &\left. exp\left(-\theta\left(\sum_{i=1}^n x_i^\beta+b_1\right)-b_2\beta\right) exp\left(-w\frac{1+\theta+\theta t^\beta+\frac{\theta^2}{2}t^{2\beta}}{1+\theta}exp(-\theta t^\beta)\right)d\theta d\beta\right)\end{aligned}$$

$$\begin{aligned}\hat{h}_{BL}(t) &= \frac{-1}{w} log(E_\pi(exp(-wh(t))|x)) \\ &= \frac{-1}{w} log\left(\int_0^{+\infty}\int_0^{+\infty} exp(-wh(t))\pi(\theta,\beta|x)d\theta d\beta\right) \\ &= \frac{-1}{w} log\left(K^{-1}\int_0^{+\infty}\int_0^{+\infty} \frac{\theta^{2n+a_1-1}\beta^{n+a_2-1}}{(1+\theta)^n}\prod_{i=1}^n x_i^{\beta-1}\prod_{i=1}^n\left(\frac{\theta^2}{2}x_i^{2\beta}+1\right)\right. \\ &\left. exp\left(-\theta\left(\sum_{i=1}^n x_i^\beta+b_1\right)-b_2\beta\right) exp\left(-w\frac{\theta^2\beta t^{\beta-1}\left(\frac{\theta^2}{2}t^{2\beta}+1\right)}{1+\theta+\theta t^\beta+\frac{\theta^2}{2}t^{2\beta}}\right)d\theta d\beta\right)\end{aligned}$$

9. MCMC method

The MCMC method is employed in this section to derive the Bayesian estimators, the Metropolis-Hastings algorithm is taken into consideration in order to produce samples from the conditional posterior distributions and subsequenty obtained the Bayesian estimates. Samples are generated from an arbitrary proposal distibution using the Metropolis-Hastings algorithm (for more details see [15]). The following is the intire contitional posterior PDF of θ and β

$$\pi_{(*)}(\theta|\beta) = \frac{\theta^{2n+a_1-1}}{(1+\theta)^n}\prod_{i=1}^n\left(\frac{\theta^2}{2}x_i^{2\beta}+1\right)e^{-\theta\sum_{i=1}^n x_i^\beta-b_1\theta} \tag{5}$$

and

$$\pi_{(*)}(\beta|\theta) = \beta^{b+a_1-1}\prod_{i=1}^n x_i^{\beta-1}\prod_{i=1}^n\left(\frac{\theta^2}{2}x_i^{2\beta}+1\right)e^{-\theta\sum_{i=1}^n x_i^\beta-b_2\beta} \tag{6}$$

Since the full conditional posterior PDFs of θ and β cannot be reduced to a known distribution, as can be seen from equation (5) and (6), we adopt a normal distribution as the proposal distribution.

Algorithm

1. Begin with the initial values of θ and β (we pose: $\theta_{(0)} = \hat{\theta}_{mle}$ and $\beta_{(0)} = \hat{\beta}_{mle}$
2. $k = 1$
3. We generate $\theta_{(k)}$ from (5) and $\theta_{(*)}$ follows a normal distribution (proposal distribution).
4. The acceptance probability is calculated as the following

$$s(\theta_{(k-1)},\theta_{(*)}) = min\left[1, \frac{\pi_{(*)}(\theta_{(*)}|\beta_{(k-1)})}{\pi_{(*)}(\theta_{(k-1)}|\beta_{(k-1)})}\right]$$

5. Let $u \sim \mathbf{U}_{[0,1]}$

6.
$$\begin{cases} if \;\; u < s(\theta_{(k-1)}, \theta_{(*)}) : & \theta_{(*)} = \theta_{(k)} \\ else & \theta_{(k-1)} = \theta_{(k)} \end{cases}$$

7. generate $\beta_{(k)}$ from (6) and $\beta_{(*)}$ follows a normal distribution.
8. Repeat steps 4 to 6 for β
9. $R(t)$ and $h(t)$ are calculated as

$$R_{(k)}(t) = \frac{1 + \theta_{(k)} + \theta_{(k)} x^{\beta_{(k)}} + \frac{\theta_{(k)}^2}{2} x^{2\beta_{(k)}}}{1 + \theta_{(k)}} e^{-\theta_{(k)} x^{\beta_{(k)}}}$$

and

$$h_{(k)}(t) = \frac{\theta_{(k)}^2 \beta_{(k)} x^{\beta_{(k)}-1} \left(\frac{\theta_{(k)}^2}{2} x^{2\beta_{(k)}} + 1 \right)}{1 + \theta_{(k)} + \theta_{(k)} x^{\beta_{(k)}} + \frac{\theta_{(k)}^2}{2} x^{2\beta_{(k)}}}$$

10. Repeat the steps from 3 to 10 N times.
11. The Bayesian estimators of θ, β, $R(t)$ and $h(t)$ under squared loss function are respectively given by

$$\hat{\theta}_{BS} = \frac{1}{N-K} \sum_{k=K+1}^{n} \theta_{(k)}$$

$$\hat{\beta}_{BS} = \frac{1}{N-K} \sum_{k=K+1}^{n} \beta_{(k)}$$

$$\hat{R}_{BS}() = \frac{1}{N-K} \sum_{k=K+1}^{n} R_{(k)}(t)$$

and

$$\hat{h}_{BS} = \frac{1}{N-K} \sum_{k=K+1}^{n} h_{(k)}(t)$$

12. The Bayesian estimators of θ, β, $R(t)$ and $h(t)$ under Linex loss functio are given by

$$\hat{\theta}_{BL} = \frac{-1}{w} log \left[\frac{1}{N-K} \sum_{k=K+1}^{N} e^{-w\theta_{(k)}} \right], \;\; w \neq 0.$$

$$\hat{\beta}_{BL} = \frac{-1}{w} log \left[\frac{1}{N-K} \sum_{k=K+1}^{N} e^{-w\beta_{(k)}} \right], \;\; w \neq 0.$$

$$\hat{R}_{BL}(t) = \frac{-1}{w} log \left[\frac{1}{N-K} \sum_{k=K+1}^{N} e^{-wR_{(k)}(t)} \right], \;\; w \neq 0.$$

and

$$\hat{h}_{BL}(t) = \frac{-1}{w} log \left[\frac{1}{N-K} \sum_{k=K+1}^{N} e^{-wh_{(k)}(t)} \right], \;\; w \neq 0.$$

N stand for the total number of draws and K is a burn-in phase that is optional.

10. Simulation study

In this section, we examine how well the estimation procedures employed in this paper perform. To achieve this goal, we generate $N = 1000$ samples of differents size $(n = 10, 30, 50, 100, 250, 500)$ of the new power xgamma distribution with the parameters $(\theta, \beta) = (0.8, 0.5)$ then, $(\theta, \beta) = (3, 0.8)$. For each estimator, we calculate
The average biais

$$AV(\hat{\theta}) = \frac{1}{N}\sum_{i=1}^{N}(\hat{\theta}_i - \theta), \quad AV(\hat{\beta}) = \frac{1}{N}\sum_{i=1}^{N}(\hat{\beta}_i - \beta)$$

$$AV(\hat{R}(t)) = \frac{1}{N}\sum_{i=1}^{N}(\hat{R}_i(t) - R(t)), \quad AV(\hat{h}(t)) = \frac{1}{N}\sum_{i=1}^{N}(\hat{h}_i(t) - h(t))$$

and the mean squared error

$$MSE(\hat{\theta}) = \frac{1}{N}\sum_{i=1}^{n}(\hat{\theta}_i - \theta)^2, \quad MSE(\hat{\beta}) = \frac{1}{N}\sum_{i=1}^{n}(\hat{\beta}_i - \beta)^2$$

$$MSE(\hat{R}(t)) = \frac{1}{N}\sum_{i=1}^{n}(\hat{R}_i(t) - R(t))^2, \quad MSE(\hat{h}(t)) = \frac{1}{N}\sum_{i=1}^{n}(\hat{h}_i(t) - h(t))^2$$

The results obtained from the maximum likelihood estimators, Bayesian estimators under squared and Linex loss functions are summarized in the following tables

Table 2. Maximum likelihood estimation of θ, β, R and h when $(\theta, \beta) = (0.8, 0.5)$

n	Criteria	$\hat{\theta}_{mle}$	$\hat{\beta}_{mle}$	$\hat{R}_{mle}(t)$	$\hat{h}_{mle}(t)$
n=10	Estimator	1.0201	0.4085	0.7012	0.2815
	AV	-0.2201	0.0915	0.1781	0.1197
	MSE	0.0212	0.0075	0.0143	0.0261
n=30	Estimator	0.9941	0.4114	0.7186	0.3041
	AV	-0.1941	0.0887	0.1607	0.0971
	MSE	0.0203	0.0072	0.0101	0.0198
n= 50	Estimator	0.9405	0.4331	0.7305	0.3200
	AV	-0.1405	0.0669	0.1488	0.0812
	MSE	0.0118	0.0065	0.0090	0.0105
n=100	Estimator	0.9018	0.4501	0.7771	0.3754
	AV	-0.1018	0.0499	0.1022	0.0258
	MSE	0.0091	0.0057	0.0081	0.0091
n=250	Estimator	0.8844	0.4809	0.8001	0.4043
	Av	-0.0844	0.0191	0.0729	-0.0031
	MSE	0.0074	0.0032	0.0067	0.0078
n=500	Estimator	0.8284	0.5081	0.8118	0.4241
	AV	-0.0284	-0.0081	0.0675	0.0229
	MSE	0.058	0.0021	0.0047	0.0051

Table 3. Bayesian estimation of θ, β, R and h under squared loss function when $(\theta, \beta) = (0.8, 0.5)$

n	Criteria	$\hat{\theta}_{BS}$	$\hat{\beta}_{BS}$	$\hat{R}_{BS}(t)$	$\hat{h}_{BS}(t)$
n=10	Estimator	0.9789	0.4181	0.7911	0.3028
	AV	-0.1789	0.0819	0.0882	0.0984
	MSE	0.0184	0.0071	0.0108	0.0186
n=30	Estimator	0.9501	0.4309	0.7984	0.3119
	AV	-0.1501	0.0691	0.0809	0.0893
	MSE	0.0172	0.0067	0.0096	0.0145
n= 50	Estimator	0.9108	0.4513	0.8042	0.3418
	AV	-0.1108	0.0487	0.0373	0.0594
	MSE	0.0128	0.0061	0.0095	0.0093
n=100	Estimator	0.8814	0.4824	0.8488	0.3801
	AV	-0.0814	0.0176	0.0305	0.0211
	MSE	0.0086	0.0054	0.0083	0.0088
n=250	Estimator	0.8209	0.5016	0.8801	0.4012
	AV	-0.0209	-0.0016	-0.0008	-0.0100
	MSE	0.0058	0.0026	0.0060	0.0065
n=500	Estimator	0.7845	0.5310	0.9009	0.4187
	AV	0.0155	-0.0310	-0.0054	-0.0175
	MSE	0.0040	0.0019	0.0035	0.0032

Table 4. Bayesian estimation of θ, β, R and h under Linex loss function when $(\theta, \beta) = (0.8, 0.5)$

n	Criteria	$\hat{\theta}_{BL}$	$\hat{\beta}_{BL}$	$\hat{R}_{BL}(t)$	$\hat{h}_{BL}(t)$
n=10	Estimator	1.1877	0.3214	0.6810	0.2595
	AV	-0.3877	0.1786	0.1989	0.1417
	MSE	0.0338	0.0107	0.0209	0.0288
n=30	Estimator	1.1058	0.3542	0.7001	0.2708
	AV	-0.3058	0.1458	0.1792	0.1304
	MSE	0.0319	0.0097	0.0191	0.0242
n=50	Estimator	1.0187	0.3941	0.7454	0.3785
	AV	-0.2187	0.1059	0.1336	0.0827
	MSE	0.0203	0.0080	0.0095	0.0132
n=100	Estimator	0.9728	0.4185	0.7901	0.3488
	AV	-0.1728	0.0814	0.0892	0.0524
	MSE	0.0203	0.0080	0.0095	0.0132
n=250	Estimator	0.9104	0.4312	0.8122	0.3881
	AV	-0.1104	0.0688	0.0671	0.0130
	MSE	0.0138	0.0068	0.0074	0.0118
n=500	Estimator	0.8704	0.4627	0.8289	0.4028
	AV	-0.0704	0.0373	0.0504	-0.0016
	MSE	0.0107	0.0061	0.0060	0.0099

Table 5. MLE estimation of θ, β, R and h when $(\theta, \beta) = (3, 0.8)$

n	Creteria	$\hat{\theta}_{mle}$	$\hat{\beta}_{mle}$	$\hat{R}_{mle}$	$\hat{h}_{mle}$
n=10	Estimator	2.6482	0.9948	0.5124	3.1427
	AV	0.3518	-0.1948	-0.1908	-0.2915
	MSE	0.0192	0.0181	0.0218	0.0394
n=30	Estimator	2.6904	0.9788	0.4874	3.0908
	AV	0.3096	-0.1788	-0.1658	-0.2399
	MSE	0.0178	0.0161	0.0186	0.0307
n=50	Estimator	2.8008	0.8844	0.4616	2.9112
	AV	0.1992	-0.0844	-0.1400	-0.0603
	MSE	0.0141	0.0130	0.0144	0.0274
n=100	Estimator	2.8955	0.8202	0.4100	2.8922
	AV	0.1045	-0.0202	-0.0884	-0.0413
	MSE	0.0102	0.0100	0.0108	0.0221
n=250	Estimator	2.9344	0.8198	0.3755	2.8748
	AV	0.0656	-0.0198	-0.0539	-0.0239
	MSE	0.0088	0.0081	0.0094	0.0193
n=500	Estimator	3.0031	0.8001	0.3234	2.8033
	AV	0.0031	-0.0001	-0.0021	0.0476
	MSE	0.0042	0.0073	0.0070	0.0124

Table 6. Bayesian estimation of θ, β, R and h when $\theta = 3$, $\beta = 0.8$ under squared loss function when $(\theta, \beta) = (3, 0.8)$

n	creteria	$\hat{\theta}_{BS}$	$\hat{\beta}_{BS}$	$\hat{R}_{BS}$	$\hat{h}_{BS}$
n=10	Estimator	2.7248	0.9879	0.4813	3.0202
	AV	0.2752	-0.1879	-0.1597	-0.1693
	MSE	0.0134	0.0128	0.0184	0.0212
n=30	Estimator	2.7828	0.9650	0.4646	2.9870
	AV	0.2172	-0.1650	-0.1430	-0.1361
	MSE	0.0120	0.0112	0.0134	0.0195
n=50	Estimator	2.8992	0.9001	0.4309	2.9013
	AV	0.1008	-0.1001	-0.1093	-0.0504
	MSE	0.0095	0.0107	0.0105	0.0148
n=100	Estimator	3.0032	0.8605	0.4008	2.8791
	AV	0.0032	-0.0605	-0.0792	-0.0282
	MSE	0.0041	0.0084	0.0091	0.0123
n=250	Estimator	3.1080	0.8010	0.3641	2.8516
	AV	-0.1080	-0.0010	-0.0425	-0.0007
	MSE	0.0022	0.0070	0.0072	0.0104
n=500	Estimator	3.2237	0.7907	0.3392	2.8304
	AV	-0.2237	0.0093	-0.0176	0.0205
	MSE	0.0018	0.0061	0.0067	0.0098

Table 7. Bayesian estimation of θ, β, R and h when $\theta = 3$, $\beta = 0.8$ under Linex loss function when $(\theta, \beta) = (3, 0.8)$

n	creteria	$\hat{\theta}_{BL}$	$\hat{\beta}_{BL}$	$\hat{R}_{BL}$	$\hat{h}_{BL}$
n=10	Estimator	2.5288	1.0020	0.5322	3.2809
	AV	0.4712	-0.2020	-0.2106	-0.4300
	MSE	0.0204	0.0198	0.0304	0.0408
n=30	Estimator	2.6010	0.9702	0.5048	3.1207
	AV	0.3990	-0.1702	-0.1832	-0.2698
	MSE	0.0198	0.0173	0.0207	0.0388
n=50	Estimator	2.7404	0.9001	0.4841	2.9705
	AV	0.2596	-0.1001	-0.1625	-0.1196
	MSE	0.0143	0.0125	0.0188	0.0346
n=100	Estimator	2.8424	0.8802	0.4595	2.9108
	AV	0.1776	-0.0802	-0.1379	-0.0599
	MSE	0.0094	0.0098	0.0145	0.0298
n=250	Estimator	3.0076	0.8101	0.4287	2.8890
	AV	-0.0076	-0.0101	-0.1071	-0.0381
	MSE	0.0054	0.0082	0.0125	0.0243
n=500	Estimator	3.0012	0.7905	0.4001	2.8402
	AV	0.0012	0.0095	-0.0785	0;0107
	MSE	0.0039	0.0055	0.0010	0.0204

Results The previous tables present the estimators of the parameters θ, β, the reliability function $R(t)$, and the failure rate function using the maximum likelihood method, as well as the Bayesian method under both the sqaured and Linex loss functions. They also include the values of corresponding Av and MSE.

- for the tables $2, 3$ and 4 we took the parameter values $(\theta, \beta) = (0.8, 0.5)$, we note that for large values of n , the value of MSE tends towards 0 for all the estimators, and this indicates that the estimators obtained are consistent.
- for tables $5, 6$ and 7, we took the values of parameters $(\theta, \beta) = (3, 0.8)$. From these tables we can notice that the estimators are very close to the true values of the parameters, and also that the values of AV and MSE decrease when the size of sample increases, therefore, also in this case the estimators are consistent.
- According to the results appearing in the previous tables, we can notice that the smallest values of AV and MSE are obtained when we apply the Bayesian method with the squared loss function, therefore this method is the most powerful method among the methods used.

11. Application with real data

The analysis of three real data sets is conducted to demonstrate the applicability of the proposed distribution in modeling various phenomena and to assess the performance of the methods used for estimating the unknown parameters.
The suggested distribution is compatible with xgamma, gamma, Weibull, gamma-Lindley, Lomax, and Lindley. The R software is used to compute the analytical metrics for identifying the best-fitting model, which depends on: Akkaike information criterion:

$$AIC = -2l + 2p$$

Corrected akaike information criterion

$$CAIC = -2l + \frac{2p}{n-p-1}$$

Bayesian information criterion:

$$BIC = -2l + plog(n)$$

Hannan-Quin information criterion:

$$HQIC = -2l + 2plog(log(n))$$

Whith l is the log likelihood function and p is the number of parameter.
then, the Anderson-Darling (AD) and the Cramer-Von-Mises (CVM) statistics are used to compare the fitted distributions. Also, for each distribution, we calculate the Kolmogorov statistics (K.S) , the maximum likelihood estimators of the parameters with their associated squared errors.

11.1. First data set: repair times for an airborne communication transceiver

In this subsection, we consider the real data present by [11], The 40 observations are given by:
0.50, 0.60, 0.60, 0.70, 0.70, 0.70, 0.80, 0.80,1.00, 1.00, 1.00, 1.00, 1.10, 1.30, 1.50, 1.50, 1.50, 1.50, 2.00,2.00, 2.20, 2.50, 2.70, 3.00, 3.00, 3.30, 4.00, 4.00, 4.50, 4.70,5.00, 5.40, 5.40, 7.00, 7.50, 8.80, 9.00, 10.20, 22.00, 24.50.

Table 8. The statistics AIC, CAIC, BIC, HQIC, AD and CVM for the fisrt data set

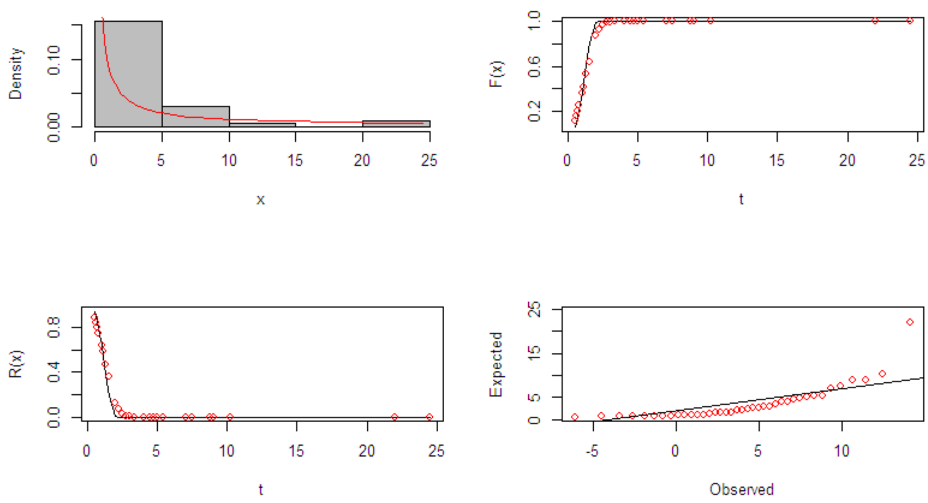
Distribution	AIC	CAIC	BIC	HQIQ	AD	CVM
Power xgamma	249.2266	245.334	252.6043	250.4479	0.105	0.815
xgamma	454.2786	450.3867	457.6564	455.4999	0.145	0.890
gamma	324.556	320.447	328.678	236.4423	0.210	0.995
Weibull	317	320.8919	313.6222	315.7787	0.190	0.940
Gamma-Lindley	276.3694	272.4775	279.7471	277.5906	0.270	1.155
Lomax	391.335	393.4056	400.0087	398.3467	0.235	1.120
Lindley	274.3694	272.422	276.0582	274.98	0.255	1.135

Table 9. Mle of the parameters for the proposed distribution based on first data set

Distribution	K.S	p-value	$\hat{\theta}_{mle}$	SE($\hat{\theta}_{mle}$)	$\hat{\beta}_{mle}$	SE($\hat{\beta}_{mle}$)
Power xgamma	0.225	0.570	1.2319	0.001	0.4240	0.141
xgamma	0.275	0.515	1.5452	0.4287		
gamma	0.375	0.490	1.8909	0.6008	1.4217	0.4048
Weibull	0.355	0.510	2.9639	3.1113	2.0024	1.4457
Gamma-Lindley	0.525	0.310	1.2927	0.0085	1.1170	0.1783
Lomax	0.455	0.470	0.5689	0.3699	0.9675	0.4674
Lindley	0.475	0.390	0.4242	0.6018		

The following graph give the histogramm with the probability density function, the cumulative distribution function, reliabilty function and the p-p plots of the first real data set.

Figure 4. The histogramm with pdf, cdf, reliability function and p-p plots for the first real data set



11.2. secod data set: Times of failure of fatigue fracture of Kevlar 373/epoxy

In this subsection, we present the data set used by [18], the observations are: 1.2985, 1.3211, 1.3503, 1.3551, 1.4595, 1.4880, 1.5728, 1.5733, 1.7083, 1.7263, 1.7460, 1.7630, 1.7746, 1.8275, 0.0251, 0.0886, 0.0891, 0.2501, 0.3113, 0.3451, 0.4763, 0.5650, 0.5671, 0.6566, 0.6748, 0.6751, 0.6753, 0.7696, 0.8375, 0.8391, 0.8425, 0.8645, 0.8851, 0.9113, 0.9120, 0.9836, 1.0483, 1.0596, 1.0773, 1.1733, 1.2570, 2.2100, 3.7455, 3.9143, 4.8073, 5.4005, 5.4435, 5.5295, 6.5541, 9.0960, 2.2460, 2.2878, 2.3203, 2.3470, 2.3513, 2.4951, 2.5260, 2.9911, 3.0256, 3.2678, 3.4045, 3.4846, 3.7433, 1.2766, 1.8375, 1.8503, 1.8808, 1.8878, 1.8881, 1.9316, 1.9558, 2.0048, 2.0408, 2.0903, 2.1093, 2.1330. The different mle of the parameters for the proposed distribution are in the folowwing table

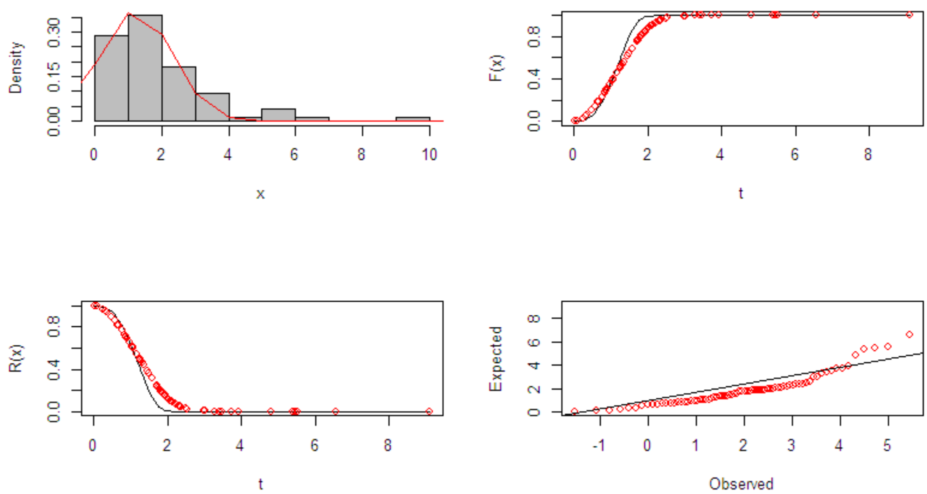
Table 10. The statistics AIC, CAIC, BIC, HQIC, AD and CVM for the second data set

Distribution	AIC	CAIC	BIC	HQIQ	AD	CVM
Power xgamma	276.1246	266.1794	274.8761	271.9876	0.3241	0.7121
xgamma	343.8371	339.892	348.4986	345.7001	0.3708	0.7708
gamma	381.2435	377.2014	386.9087	383.0867	0.4118	0.8102
Weibull	427.217	431.1622	422.5556	425.3541	0.3945	0.7924
Gamma-Lindley	407.8128	403.8689	412.4762	409.6777	0.5314	0.9132
Lomax	504.7896	501.8456	509.5632	506.7332	0.5524	0.9314
Lindley	624.0857	622.1127	626.4164	625.0171	0.5208	0.8912

Table 11. Mle of the parameters for the proposed distribution based on second data set

Distribution	K.S	p-value	$\hat{\theta}_{mle}$	SE($\hat{\theta}_{mle}$)	$\hat{\beta}_{mle}$	SE($\hat{\beta}_{mle}$)
Power xgamma	0.02631	0.8944	1.2310	0.0009	0.7138	0.0070
xgamma	0.03289	0.8322	1.0331	0.0270		
gamma	0.04502	0.7841	0.7880	0.0514	0.5213	0.0180
Weibull	0.04342	0.8017	2.3099	1.2318	2.0901	1.6643
Gamma-Lindley	0.05921	0.6523	1.3358	0.0184	1.0204	0.2204
Lomax	0.06902	0.6192	1.1589	0.0897	1.44003	0.5687
Lindley	0.04605	0.7081	0.7947	0.1642		

Figure 5. The histogramm with pdf, cdf, reliability function and p-p plots of the second real data set



11.3. Third data set: The lengths of remission times

In this subsection, we used the data set present by [19], the observations are: 0.08, 4.87, 6.94, 8.66, 2.09, 3.48, 13.11, 23.63, 0.20, 13.80, 25.74, 0.50, 2.23, 3.52, 4.98, 6.97, 9.02, 3.88, 5.32, 7.39, 10.34, 13.29, 0.40, 2.26, 9.22, 2.46, 3.64, 5.09, 7.26, 0.51, 2.54, 14.76, 26.31, 0.81, 3.70, 5.17, 7.28, 9.47, 14.24, 25.82, 9.74, 2.62, 3.82, 5.32, 2.69, 4.23, 5.41, 7.62, 7.32, 10.06, 14.77, 32.15, 2.64, 14.83, 34.26, 4.33, 5.49, 7.66, 0.90, 5.34, 7.59, 10.66, 15.96, 2.69, 4.18, 36.66, 12.05, 10.75, 16.62, 43.01, 5.41, 7.63, 11.25, 17.14, 79.05, 17.12, 46.12, 4.40, 5.85, 8.26, 1.26, 2.83, 1.35, 2.87, 5.62, 1.19, 2.75, 4.26, 7.87, 11.64, 4.34, 5.71, 7.93, 11.79, 17.36, 1.40, 3.02, 18.10, 1.46, 11.98, 19.13, 12.02, 2.02, 3.31, 1.76, 3.25, 12.03, 20.28, 2.02, 4.50, 6.25, 8.37, 4.51, 6.54, 8.53, 3.36, 6.76, 8.65, 12.63, 22.69, 12.07, 21.73, 3.57, 5.06, 7.09, 2.07, 3.36, 6.93

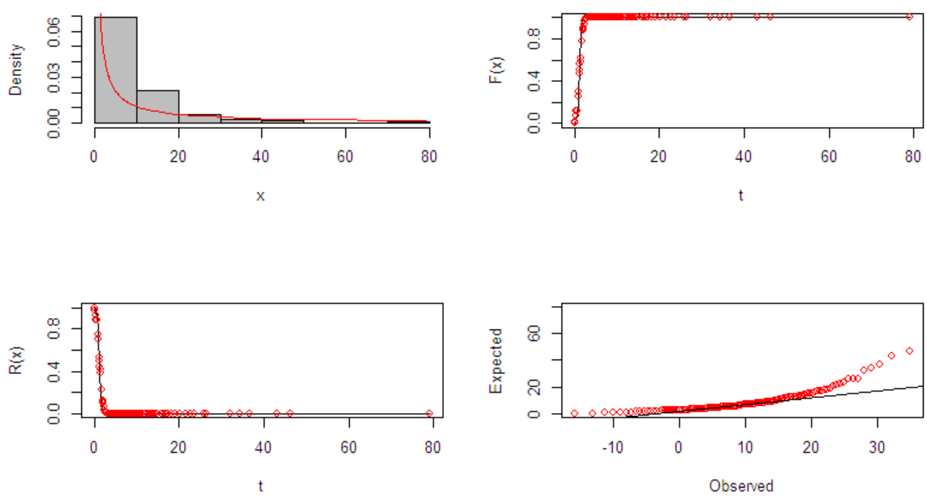
Table 12. The statistics AIC, CAIC, BIC, HQIC, AD and CVM for the third real data set

Distribution	AIC	CAIC	BIC	HQIQ	AD	CVM
power xgamma	1078.027	1074.059	1083.344	1080.344	0.5004	0.0614
xgamma	3816.778	3812.81	3822.482	3820.096	0.6187	0.0887
gamma	4231.763	4225.809	4234.897	4232.543	0.5128	0.0722
Weibull	14935.95	14939.91	14930.24	14933.63	0.5209	0.0836
Gamma-Lindley	4254.786	4250.818	4260.49	4257.104	0.6769	0.1278
Lomax	9876.2234	9871.315	9882.543	1989.6345	0.6304	0.1088
Lindley	6538.435	6536.451	6541.287	6539.593	0.6547	0.1220

Table 13. Mle of the parameters for the proposed distribution based on third data set

Distribution	K.S	p-value	$\hat{\theta}_{mle}$	SE($\hat{\theta}_{mle}$)	$\hat{\beta}_{mle}$	SE($\hat{\beta}_{mle}$)
Power xgamma	0.67188	0.1203	1.2378	0.0163	0.6401	0.0255
xgamma	0.82811	0.3405	0.8571	0.1108		
gamma	0.7203	0.2388	0.6578	0.0897	0.8801	0.2089
Weibull	0.73438	0.2707	1.6015	0.1612	1.9194	1.2530
Gamma-Lindley	0.89844	0.3813	1.4846	0.0809	1.0494	0.0622
Lomax	0.83451	0.3486	1.6734	0.2322	1.7098	0.3002
Lindley	0.83594	0.3622	0.9943	0.0423		

Figure 6. The histogramm with pdf, cdf, reliability function and p-p plots of the real data set



The tables 8, 10 and 12 present the AIC, CAIC, BIC, HQIC, AD and CVM statistics for the different distributions based on first, second and third real data set respectively. The numerical values of this statistics shows that the power xgamma model provides the smallest values of AIC, CAIC, BIC, HQIC, AD and CVM, so, the power xgamma

model can be selected as the preferred model for the tree real data sets.
The tables 9, 11 and 13 present the K.S statistics (p-value) thus the the mle of the parameters with their squared errors. We note that power xgamma model has the lowest values of K.s, $SE(\hat{\theta}_{mle})$, $SE(\hat{\beta}_{mle})$ and the biggest p-value compared to the other models, which confirms that the power xgamma model fits this tree data sets better that all other models.

12. Conclusion

In this study, we explored the new power xgamma distribution, an extension of the xgamma distribution, focusing on its survival characteristics and statistical properties. We presented and demonstrated various aspects of this distribution to understand its applicability in statistical modeling.

To estimate the parameters of the power xgamma distribution, we employed two estimation methods: maximum likelihood estimation (MLE) and Bayesian methods utilizing squared loss and linex loss functions. Our comparative analysis revealed that the Bayesian method with the linex loss function yielded the smallest errors among the methods considered. This result highlights the effectiveness of incorporating the linex loss function in Bayesian estimation for optimizing parameter estimation accuracy.

Furthermore, we applied the power xgamma distribution to a real dataset, illustrating its practical utility in modeling real-world data scenarios. This empirical validation underscores the robustness and relevance of the new distribution in statistical applications.

In conclusion, the power xgamma distribution emerges as a promising extension of the xgamma distribution, offering enhanced flexibility and statistical properties. The preference for the Bayesian method with the linex loss function underscores the importance of choosing appropriate estimation techniques tailored to the characteristics of the data and the distribution under study. Future research can further explore advanced Bayesian methodologies and expand the application domains of the power xgamma distribution in diverse fields of study.

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