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Estimating the Parameters of the Odd Lomax Exponential Distribution

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Abstract In this study, we introduce two methods for estimation of the unknown parameters of the Odd Lomax Exponential(OLE) distribution are Least Squares Estimation (LSE) and Maximum Likelihood Estimation (MLE). Some statistical functions and mathematical properties were derived for which investigated from distribution's flexibility . Through Monte Carlo simulations we investigated the performance of the estimate for these parameters, and us comparison these estimations in terms of bias and mean squared error (MSE) for various sample sizes and four different scenarios of initial parameter values for two methods. Our analysis revealed that least square estimation consistently outperformed on MLE, yielding lower MSE values. Additionally, both two methods demonstrated decreasing in criteria values with increasing sample size, indicating improved accuracy for larger datasets. To evaluate the applicability of the OLE distribution, we applied it to two types of dataset in the reliability engineering field. All computational and graphics in this work were performed in a Matlab,23b code.

Keywords odd Lomax distribution, Exponential distribution, Survival function, Hazard function, maximum likelihood method; least squares method

AMS 2010 subject classifications 62F10,60E05,97K50

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1. Introduction

Distributions provide us with high flexibility in building structural statistical models for real phenomena, which is often difficult. Additionally, these mixed distributions are easy and flexible to deals with new applications, and their purpose is to develop statistical properties such as the probability density function, cumulative distribution function, survival function, hazard rate function, and moments, to be compatible with other fields of knowledge. Statistical distributions have played an important role in theoretical statistics and its applications. Our objective in this paper was to introduce a new probability distribution family called the Odd Lomax Exponential (OLE) distribution, which is a mixture of the Lomax and Odd Exponential distributions. This distribution is also very useful in describing real data in many fields. There are many researchers who have developed various distributions in this filed, and they working in this area. The article [16] introduced the Generalized Odd Lomax-G ($GOL-G$) family, a flexible family of probability distributions that can be used to model various types of data. It is formed by adding three additional parameters to any continuous baseline distribution. [26] proposed four different methods for estimating the parameters of the Exponentialed Lomax (EL) distribution. He conducted a numerical study to compare these methods and identify the best one for estimating the distribution's parameters. [31] introduced a new probability distribution called the Weibull-Lomax distribution. This distribution is a combination of the Weibull and Lomax distributions, resulting in a flexible model with various shapes for the hazard rate function. [24] introduces an estimate of the parameters of the Lomax distribution when the available

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observations are described by fuzzy information. Sometimes, information about an underlying system might be imprecise, which can be represented using fuzzy sets. [7] introduced a new class of model called the Exponentialed Weibull Lomax, they suggested three procedures for estimating the parameters of this new model,and in an applied, they compared it with several other new models. [3] proposed a new family of continuous distributions called the odd Lomax-G class, with four special models. They used the maximum likelihood and least squares methods to estimate the model parameters. [9] presented a new modification of the Lomax distribution consisting of two parameters called Lomax exponential Distribution(LE) and he was introduced the statistical properties for the LE distribution are obtained including moments, Furthermore, asymptotic confidence intervals of the parameters, based on maximum likelihood estimation method. [17] introduced a new four-parameter distribution, called the truncated Weibull power Lomax distribution($TWPL$). [18] introduced a compound continuous distribution with four parameters for modeling life data, titled the odd generalized exponential inverse Lomax distribution,in this paper, they presented the distribution in detail, Simulation was used to verify the behavior of the proposed distribution and to use the possibility estimation method for verifying correct parameter estimation. [8] studied a new version of the Poisson Lomax distribution. The probability density function of this new distribution expresses a mixture of Lomax densities. They demonstrated that the failure rate function of the new model is steadily increasing, The new model provided a good and appropriate combination. [2] introduced new probability distribution that is mixed between Lomax distribution and Rayleigh distribution. [14] proposed the Exponential Flexible Lomax Distribution and studied some of its basic statistical properties to demonstrate its flexibility. [36] proposed a new Odd Lomax Exponential distribution, the structural properties of this model were derived and studied, and three applications were presented to illustrate the model. [28] suggested a univariate continuous probability distribution having four parameters named odd Lomax generalized exponential distribution. [22] proposed of a new extension to a new generalized family of continuous distributions from type Odd Lomax-G(OLG) derived based on the $T-X$ family and the Lomax distribution. [32] studied a new exponentialized Generalized Odd Lomax Exponential distribution, using another formula, the Odd Lomax Generator of the exponential distribution. [27] they studies The Lomax distribution, and by properly redefining its scale parameter, it was applied to wireless fading modeling, Relevant performance indicators were derived, including the amount of fading, channel capacity, outage probability, and error rate. [29] introduces a new Beta Exponentialed Lomax –Expoential distribution. [33] introduced four parameter continuous distribution using Odd Lomax Generator and exponential distribution. [15] Odd Lomax-Dagum distribution has been studied for Some characteristics of the distribution such as the quantile and hazard functions. [11] proposed the application of the Odd Lomax Log-Logistic distribution, adding flexibility and allowing it to capture all characteristics by using the $T-X$ class of distribution. The aim of this paper is to explain the method of deriving the odd exponential Lomax distribution in detail, as well as its structural properties and two methods for estimating its parameters, and we considered this through the following: In the section 1, we review the relevant literature on the research topic. In the second 2, we explained the method of deriving the probability density function and the cumulative distribution function and verifying the validity property as well as some special model additional, the survival function and the failure rate function. The section 3, introduced some mathematical properties, which include useful expansion, moments, and some statistical measurements. in section 4, consider two methods for estimating distribution parameters. the section 5, we presented a study of simulation experiments for different samples. In the section 6, we studied two sets of real data related to reliability engineering, in addition, conclusions reached by the research.

2. Odd Lomax Exponential Distribution

In this section, we introduce some characteristic properties of Odd Lomax Exponential (OLE) distribution, In addition, we derived The cumulative density function, probability density function, Survival function and Hazard function and plots the shapes of the distribution. The formula for mixing two distributions by the inserting the CDF of baseline distribution within the limits of integration of the other distribution and it

called Lomax-X family.
Let X has an exponential distribution with parameter $1/b$, And we interested in the *cdf* and *pdf* of this distribution as follows[20].

$$G(x) = 1 - exp(\frac{-x}{b}). \tag{1}$$

$$And \quad g(x) = (\frac{1}{b})exp(\frac{-x}{b}). \tag{2}$$

where $(x > 0)$, $(b > 0)$is the scale parameter,[20]. Also, a random variable X is said to have Lomax distribution if it has *cdf* and *pdf* are given by [24]

$$F(x) = 1 - (1 + \lambda x)^{-\alpha} \tag{3}$$

$$And \quad f(x) = \alpha\lambda(1 + \lambda x)^{-(\alpha+1)} \tag{4}$$

where $(x > 0)$ and $(\alpha, \lambda > 0)$ are the shape and scale parameters, respectively,[2].
Now, The *cdf* of OLE distribution can be obtained by using the following, [3], [25] and [36] :

$$F(x; \alpha, \lambda, \varphi) = \int_0^{W(G(x,\varphi))} f(x)dx \tag{5}$$

$$\begin{aligned} F(x; \alpha, \lambda, \varphi) &= \int_0^{W(G(x,\varphi))} \alpha\lambda(1 + \lambda x)^{-(\alpha+1)}\, dx \\ &= -\left.(1 + \lambda x)^{-\alpha}\right|_0^{W(G(x,\varphi))} \\ &= -\big([1 + \lambda W(G(x, \varphi))]^{-\alpha} - 1\big) \end{aligned}$$

$$F(x; \alpha, \lambda, \varphi) = 1 - (1 + \lambda W(G(x, \varphi)))^{(-\alpha)} \tag{6}$$

where $W\,(G(x), \varphi) = \frac{G(x,\varphi)}{1-G(x,\varphi)})$ is the Odd function [11],[19]. we obtain,

$$F(x; \alpha, \lambda, \varphi) = 1 - \left(1 + \lambda\frac{G(x, \varphi)}{1 - G(x, \varphi)}\right)^{-\alpha} \tag{7}$$

We know $\big(since f(x) = \frac{d}{dx}F(x)\big)$, We can obtained the probability density function of the OLE distribution as following :

$$f(x; \alpha, \lambda, \varphi) = \alpha\left(1 + \lambda\frac{G(x, \varphi)}{1 - G(x, \varphi)}\right)^{-(\alpha+1)} \frac{d}{dx}\left(1 + \lambda\frac{G(x, \varphi)}{1 - G(x, \varphi)}\right),$$

$$f(x; \alpha, \lambda, \varphi) = \alpha\left(1 + \lambda\frac{G(x, \varphi)}{1 - G(x, \varphi)}\right)^{-(\alpha+1)} \left(0 + \lambda\frac{g(x, \varphi)}{(1 - G(x, \varphi))^2}\right),$$

$$f(x; \alpha, \lambda, \varphi) = \frac{\alpha\lambda g(x, \varphi)}{(1 - G(x, \varphi))^2}\left(1 + \lambda\frac{G(x, \varphi)}{1 - G(x, \varphi)}\right)^{-(\alpha+1)}. \tag{8}$$

when $G(x, \varphi)$ and $g(x, \varphi)$ are represented *cdf* and *pdf* of the baseline(exponential)distribution in equations (1) , (2) respectively,[23],[29];[36] We obtained following:

$$F(x; \alpha, \lambda, \varphi) = 1 - \left(1 + \lambda\frac{1 - \exp(\frac{-x}{\varphi})}{1 - (1 - \exp(\frac{-x}{\varphi}))}\right)^{-\alpha}$$

$$F(x;\alpha,\lambda,\varphi) = 1 - \left(1 + \lambda(exp(\frac{x}{\varphi}) - 1)\right)^{-\alpha} \tag{9}$$

and, we can find the *pdf* from *cdf* $(since f(x;\alpha,\lambda,\varphi) =) = \frac{d}{dx}F(x;\alpha,\lambda,\varphi)$ as follows:

$$f(x;\alpha,\lambda,b) = \frac{\alpha\lambda\frac{1}{\varphi}\exp\left(-\frac{x}{\varphi}\right)}{\left(1-\left(1-\exp\left(-\frac{x}{\varphi}\right)\right)\right)^2}\left(1+\lambda\frac{1-\exp\left(-\frac{x}{\varphi}\right)}{1-\left(1-\exp\left(-\frac{x}{\varphi}\right)\right)}\right)^{-\alpha}$$

then,

$$f(x;\alpha,\lambda,\varphi) = \frac{\alpha\lambda}{\varphi}exp(\frac{x}{\varphi})\left(1 + \lambda(exp(\frac{x}{\varphi}) - 1)\right)^{-(\alpha+1)} \tag{10}$$

where $(x > 0)$ and $\lambda, \varphi > 0$ are two scale parameters, And $\alpha > 0$ is the shape parameter.

2.1. Validity of the Model

In this subsection, a random variable which it has the probability distribution is achieve the integral in equation (11) as following[11] :

$$\int_{-\infty}^{\infty} f(x)\,dx = 1 \tag{11}$$

Proof. By substituting equation (11),

$$\int_0^{\infty} f(x;\alpha,\lambda,b)\,dx = \int_0^{\infty}\frac{\alpha\lambda}{b}\exp\left(\frac{x}{b}\right)\left(1+\lambda\left(\exp\left(\frac{x}{b}\right)-1\right)\right)^{-(\alpha+1)}\,dx. \tag{12}$$

Now

$$Let : y = \exp(\frac{x}{b}) - 1 \quad \Rightarrow \qquad x = b\ln(y+1) \quad \Rightarrow \quad dx = \frac{b}{y+1}dy$$

$$\int_0^{\infty}\frac{\alpha\lambda}{b}(y+1)\,(1+\lambda y)^{-(\alpha+1)}\,\frac{b}{y+1}\,dy = 1 \tag{13}$$

$$\left.\frac{-1}{(1+\lambda y)^{\alpha}}\right|_0^{\infty} = 1 \quad \Rightarrow \quad -[0-1] = 1.$$

the Figures (1) and (2) represents *pdf* and *cdf* of the *OLE* distribution, respectively. .

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2.2. Some Special Model

Some of the sub-model of the *OLE* distribution are given in the section2 (10) :
1.When $\alpha = 1$, we obtain the *pdf* as following :

$$f(x;\lambda,b) = \frac{\lambda}{b}\exp(\frac{x}{b})\big(1+\lambda(\exp(\frac{x}{b})-1)\big)^{-(2)} \tag{14}$$

If $\lambda = 1$, then

$$f(x;b) = \frac{1}{b}\exp(\frac{-x}{b}) \tag{15}$$

The *OLE* distribution will be reduce to Standard exponential distribution, [20]with the parameter (b).
2.When $b = 1$, We obtain the *pdf* as following :

$$f(x;\alpha,\lambda) = \alpha\lambda\exp(x)\big(1+\lambda(\exp(x)-1)\big)^{-(\alpha+1)} \tag{16}$$

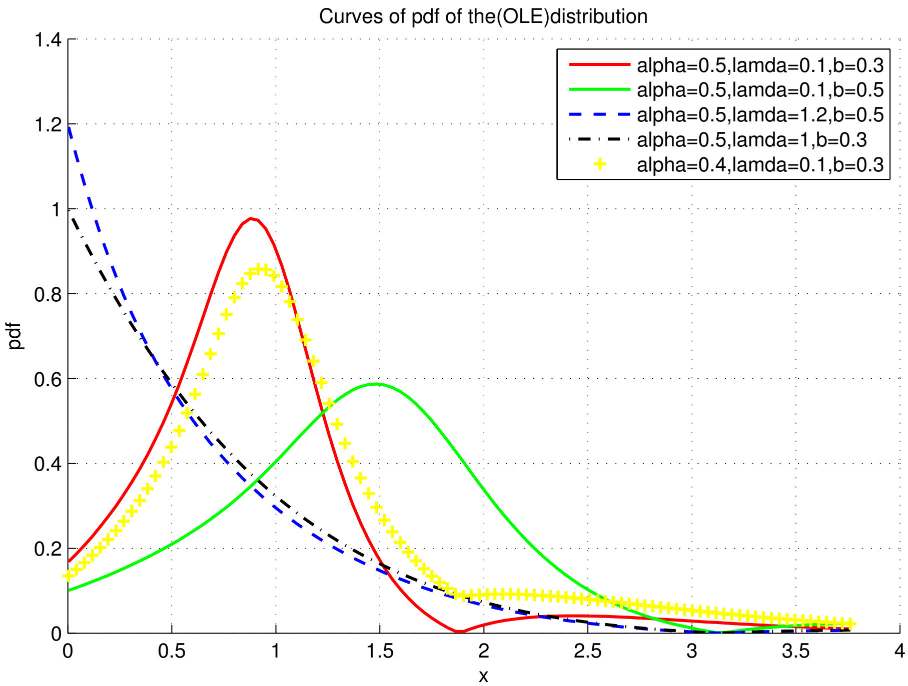

Figure 1. The Probability Density Function of the OLE Distribution

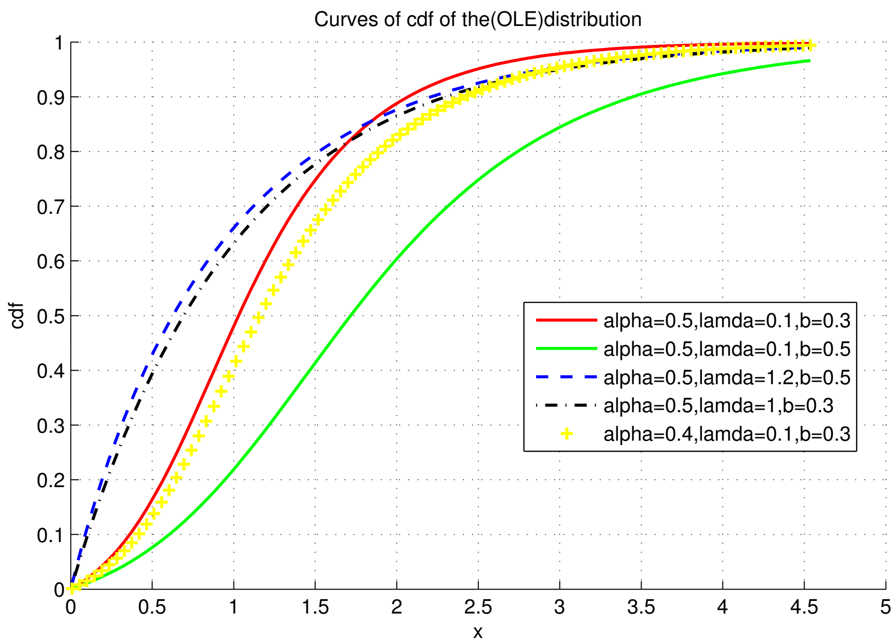

Figure 2. The Cumulative Distribution Function of the OLE Distribution

If $\lambda = 1$, then

$$f(x;\alpha) = \alpha \exp(x) \left(1 + \exp(x) - 1\right)^{-(\alpha+1)}$$

$$f(x;\alpha) = \alpha \exp(-\alpha x) \qquad \text{for } \alpha > 0 \tag{17}$$

The OLE distribution will be reduce to Negative exponential distribution $(SeeDeffinition(12), pp.112,$ [20]), with parameter $(\frac{1}{\alpha})$.
3.When $\alpha = b = 1$, we obtain,

$$f(x;\lambda) = \lambda \exp(x)\big(1 + \lambda(\exp(x) - 1)\big)^{-2}. \tag{18}$$

.

2.3. *Survival function*

In this subsection, we have derived the survival function $S(x)$ of the OLE distribution by using *cdf*, it can be written as :

$$S(x;\alpha,\lambda,b) = 1 - F_X(x;\alpha,\lambda,\varphi) \tag{19}$$

by substituting equation (9) in to (19). hence, [4].

$$S(x;\alpha,\lambda,b) = 1 - \left[1 - \left(1 + \lambda \frac{1 - \exp\left(-\frac{x}{b}\right)}{1 - \left(1 - \exp\left(-\frac{x}{b}\right)\right)}\right)^{-\alpha}\right],$$
$$= \left(1 + \lambda \frac{1 - \exp\left(-\frac{x}{b}\right)}{1 - \left(1 - \exp\left(-\frac{x}{b}\right)\right)}\right)^{-\alpha}.$$

We obtain,

$$S(x;\alpha,\lambda,b) = \left(1 + \lambda\left(\exp\left(\frac{x}{b}\right) - 1\right)\right)^{-\alpha}. \tag{20}$$

.

2.4. *Hazard function*

The hazard function of the OLE distribution is derived from this definition:

$$h(x;\alpha,\lambda,b) = \frac{f_X(x;\alpha,\lambda,\varphi)}{1 - F_X(x;\alpha,\lambda,\varphi)} = \frac{f_X(x;\alpha,\lambda,\varphi)}{S(x;\alpha,\lambda,b)} \tag{21}$$

by substituting ((9)),((10)), into the ((21)). [12],[4].hence,

$$h(x;\alpha,\lambda,b) = \frac{1}{\left(1 + \lambda(\exp(\frac{x}{b}) - 1)\right)^{-\alpha}} \frac{\alpha\lambda}{b} \exp(\frac{x}{b})\left(1 + \lambda(\exp(\frac{x}{b}) - 1)\right)^{-(\alpha+1)}$$

We obtain,

$$h(x;\alpha,\lambda,b) = \frac{\alpha\lambda}{b} \exp(\frac{x}{b})\left(1 + \lambda(\exp(\frac{x}{b}) - 1)\right)^{-1} \tag{22}$$

where : $F_X(x)$ and $f_X(x)$ are the *cdf* and *pdf* of OLE distribution in ((9)),((10)),respectively. .

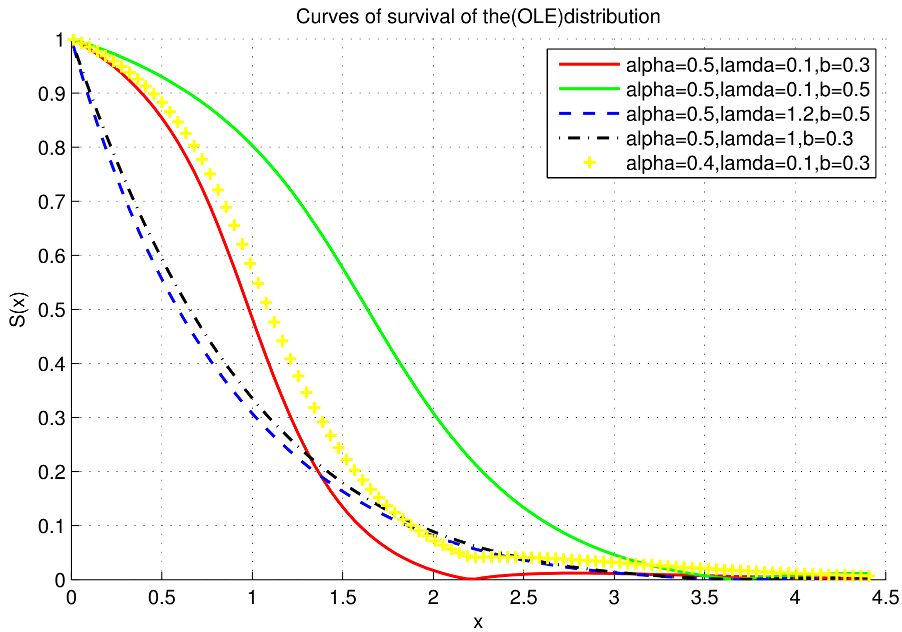

Figure 3. The Survival Function

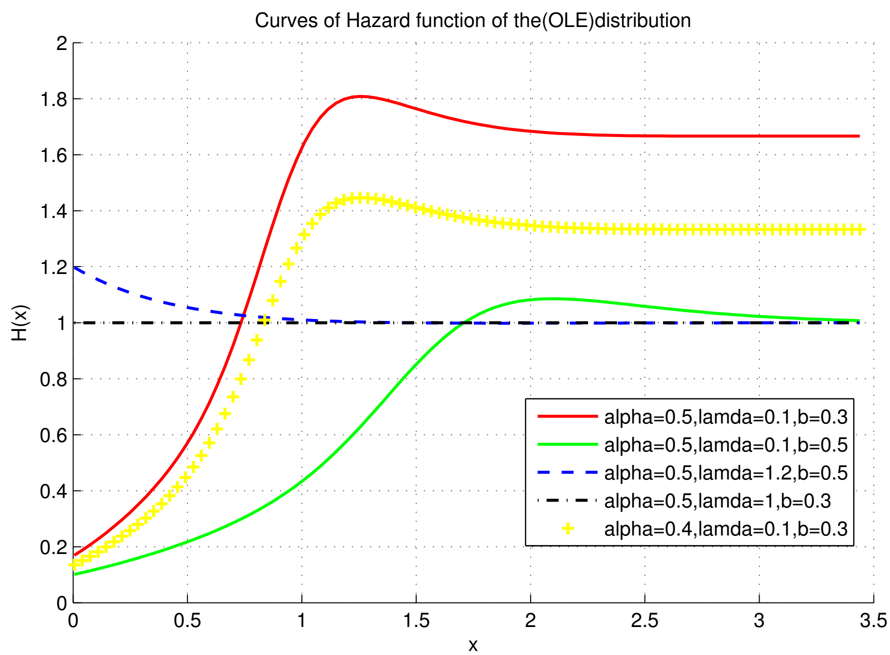

Figure 4. The Hazard Rate Function

3. Some Mathematical Properties

In this section, we will present five subsections to obtain simplest useful expansion of *pdf* , r^{th} moments, skewness and kurtosis, the mode; the median and quantile:

3.1. Useful Expansion

The expansion for *pdf* of the OLE distribution are derived, the *pdf* in equation (10), [34] which is : The expansion for *pdf* of the OLE distribution are derived, the *pdf* in equation (10) which is :

$$f(x;\alpha,\lambda,b)=\frac{\alpha\lambda\frac{1}{b}\exp\left(-\frac{x}{b}\right)}{\left(1-\left(1-\exp\left(-\frac{x}{b}\right)\right)\right)^{2}}\left(1+\lambda\frac{1-\exp\left(-\frac{x}{b}\right)}{1-\left(1-\exp\left(-\frac{x}{b}\right)\right)}\right)^{-(\alpha+1)} \tag{23}$$

In other words,

$$f(x;\alpha,\lambda,b)=\alpha\lambda\frac{1}{b}\exp\left(\frac{x}{b}\right)\left(1-\lambda\left(1-\exp\left(\frac{x}{b}\right)\right)\right)^{-(\alpha+1)}.$$

If we putting,

$$z=1-\exp\left(\frac{x}{b}\right)\quad\Rightarrow\quad 1-z=\exp\left(\frac{x}{b}\right),\ \text{then}$$

$$f(z;\alpha,\lambda,b)=\alpha\lambda\frac{1}{b}\left(1-z\right)\left(1-\lambda z\right)^{-(\alpha+1)}, \tag{24}$$

Consider the power series as following [8],[34]:

$$\left(1-\lambda z\right)^{-(\alpha+1)}=\sum_{i=0}^{\infty}\frac{\Gamma(\alpha+1+i)}{i!\Gamma(\alpha+1)}\left(\lambda z\right)^{i}\quad\text{for }\alpha>0\text{ is real non-integer and }|\lambda z|<1$$

Then, by applying the power series for the exponential function in equation (24) ,We obtain

$$f(z;\alpha,\lambda,b)=\alpha\lambda\frac{1}{b}\left(1-z\right)\sum_{i=0}^{\infty}\binom{\alpha+i}{i}\left(\lambda z\right)^{i} \tag{25}$$

We rewritten the *pdf* by $z=1-\exp(\frac{x}{b})$ yield,

$$f(x;\alpha,\lambda,b)=\alpha\lambda\frac{1}{b}\exp(\frac{x}{b})\sum_{i=0}^{\infty}\lambda^{i}\binom{\alpha+i}{i}\left(1-\exp(\frac{x}{b})\right)^{i}$$

Also, It is know that series expansion, in [1],[7] which is:

$$\left(1-\exp(\frac{x}{b})\right)^{i}=\sum_{j=0}^{\infty}\frac{-1^{j}}{j!}\frac{\Gamma(1+i)}{\Gamma(i+1-j)}\left(\exp(\frac{x}{b})\right)^{j}\quad\text{for }i>0$$

By applying the last term of the expansion series to the equation (25) yield,

$$f(x;\alpha,\lambda,b)=\alpha\lambda\frac{1}{b}\exp(\frac{x}{b})\sum_{i=0}^{\infty}\lambda^{i}\binom{\alpha+i}{i}\sum_{j=0}^{\infty}\frac{-1^{j}}{j!}\frac{\Gamma(i+1)}{\Gamma(i+1-j)}\left(\exp(\frac{x}{b})\right)^{j}, \tag{26}$$

Now we obtain,

$$f(x;\alpha,\lambda,b)=\alpha\lambda\frac{1}{b}\exp(\frac{x}{b})\sum_{i,j=0}^{\infty}\lambda^{i}(-1)^{j}\binom{\alpha+i}{i}\binom{i}{j}\left(\exp(\frac{x}{b})\right)^{j}. \tag{27}$$

3.2. The Moments

The r^{th} moment of the OLE distribution [4],[13]and [26] as following:

$$E\left(x^r\right)=\int_0^\infty x^r dx, \tag{28}$$

$$E\left(x^r\right)=\int_0^\infty \alpha\lambda\frac{1}{b}x^r\exp(\frac{x}{b})\left(\sum_{i,j=0}^{\infty}\lambda^i(-1)^j\binom{\alpha+i}{i}\binom{i}{j}\left(\exp(\frac{x}{b})\right)^j\right)dx,$$

$$E\left(x^r\right)=\alpha\lambda\frac{1}{b}\sum_{i,j=0}^{\infty}\binom{\alpha+i}{i}\binom{i}{j}(-1)^j\lambda^i\left(\int_0^\infty x^r\exp(\frac{x}{b})\left(\exp(\frac{x}{b})\right)^j dx\right), \tag{29}$$

$$Where\int_0^\infty x^r\exp\left(\frac{x}{b}\right)\left(\exp\left(\frac{x}{b}\right)\right)^j dx=\frac{1}{\left(\frac{j+1}{b}\right)^{r+1}}\Gamma(r+1).$$

then,

$$E\left(x^r\right)=\alpha\lambda\frac{1}{b}\sum_{i,j=0}^{\infty}\binom{\alpha+i}{i}\binom{i}{j}(-1)^j\lambda^i\frac{b^{r+1}}{(j+1)^{r+1}}\Gamma(r+1),\qquad\forall r=1,2,... \tag{30}$$

$$E\left(x^r\right)=\alpha\lambda\sum_{i,j=0}^{\infty}\binom{i}{j}\lambda^i\frac{\binom{\alpha+i}{i}(-1)^j b^r}{(j+1)^{r+1}}\Gamma(r+1).\quad where\quad V_{i,j}=\frac{\binom{\alpha+i}{i}(-1)^j b^r}{(j+1)^{r+1}}$$

$$E\left(x^r\right)=\alpha\lambda\sum_{i,j=0}^{\infty}\binom{i}{j}\lambda^i\quad V_{i,j}\quad\Gamma(r+1) \tag{31}$$

If $r=1$ then,

$$\mu_1{}'=E\left(x^1\right)=\alpha\lambda\sum_{i,j=0}^{\infty}\binom{i}{j}\lambda^i\frac{\binom{\alpha+i}{i}(-1)^j b}{(j+1)^2}\Gamma(2). \tag{32}$$

And,If $r=2$ then,

$$\mu_2{}'=E\left(x^2\right)=\alpha\lambda\sum_{i,j=0}^{\infty}\binom{i}{j}\lambda^i\frac{\binom{\alpha+i}{i}(-1)^j b^2}{(j+1)^3}\Gamma(3). \tag{33}$$

and from equations (32)and (33) we obtain,

$$Var(x)=\mu_2{}'-\left(\mu_1{}'\right)^2 \tag{34}$$

Additionally, the moment generating function of the OLE distribution [36] can be written by ;

$$M_r^{(t)}=\sum_{r=1}^{\infty}\frac{t^r}{r!}E\left(x^r\right)=\alpha\lambda\sum_{i,j=0}^{\infty}\sum_{r=1}^{\infty}\frac{t^r}{r!}\binom{i}{j}\lambda^i V_{i,j}\Gamma(r+1),$$

$$M_r^{(t)}=\alpha\lambda\sum_{i,j=0}^{\infty}\sum_{r=1}^{\infty}t^r\binom{i}{j}\lambda^i V_{i,j}. \tag{35}$$

(**To find the order statistics, for more details See [23]**)

3.3. Skewness and Kurtosis

The Bowley's quanlite skewness [18] as following,

$$S_k = \frac{q_{(0.75)} - 2q_{(0.5)} + q_{(0.25)}}{q_{(0.75)} - q_{(0.25)}}. \tag{36}$$

And the Octiles based kurtosis [33] defined as following:

$$K_u = \frac{q_{(0.875)} - q_{(0.625)} + q_{(0.375)} - q_{(0.125)}}{q_{(0.75)} - q_{(0.25)}}. \tag{37}$$

3.4. The Mode

The mode function can be found by using survival and hazard functions, we know that the mode is $X - value$ which corresponding first derivative to equal zero ($f'(x) = 0$):

$$\frac{d}{dx}\left(\frac{\alpha\lambda}{b}exp(\frac{x}{b})\left(1+\lambda(exp(\frac{x}{b})-1)\right)^{-(\alpha+1)}\right) = 0, \tag{38}$$

$$\frac{d}{dx}\left(\frac{\alpha\lambda}{b}\exp\left(\frac{x}{b}\right)\left(1+\lambda\left(\exp\left(\frac{x}{b}\right)-1\right)\right)^{-1}\cdot\left(1+\lambda\left(\exp\left(\frac{x}{b}\right)-1\right)\right)^{-\alpha}\right) = 0, \tag{39}$$

$$where, \quad h(x;\alpha,\lambda,b) = \frac{\alpha\lambda}{b}\exp(\frac{x}{b})\left(1+\lambda(\exp(\frac{x}{b})-1)\right)^{-1} \quad and \quad S(x;\alpha,\lambda,b) = \left(1+\lambda\left(\exp\left(\frac{x}{b}\right)-1\right)\right)^{-\alpha}$$

Since,

$$\frac{d}{dx}\left(h(x;\alpha,\lambda,b) * S(x;\alpha,\lambda,b)\right) = 0,$$

$$h'(x;\alpha,\lambda,b) * S(x;\alpha,\lambda,b) + \left(h(x;\alpha,\lambda,b)\right)^2 S(x;\alpha,\lambda,b) = 0,$$

then,

$$\left(h'(x;\alpha,\lambda,b) + h^2(x;\alpha,\lambda,b)\right)\cdot S(x;\alpha,\lambda,b) = 0. \tag{40}$$

from (39) and (40) we obtain,

$$\frac{[\frac{\lambda}{b}\left(1-exp(\frac{x}{b})+\lambda\left(exp(\frac{x}{b})-1\right)\right)+\frac{\alpha\lambda}{b}exp(\frac{x}{b})]}{\left(1+\lambda\left(exp(\frac{x}{b})-1\right)\right)} \quad \frac{\alpha\lambda}{b}exp(\frac{x}{b})\left(1+\lambda(exp(\frac{x}{b})-1)\right)^{-(\alpha+1)} = 0 \tag{41}$$

3.5. Quantile and Median

The quantile and median of the OLE distribution can be to derived by using the explicit formula[21] following:

$$F(x_q) = q \qquad 0 < q < 1 \tag{42}$$

and the *cdf* of the OLE distribution is given by equation (9), We can be that find the quantile [4] as follows,

$$1-\left(1+\lambda\left(\exp(\frac{x}{b})-1\right)\right)^{-\alpha} = q$$

$$\left(1+\lambda\left(\exp(\frac{x}{b})-1\right)\right)^{-\alpha} = 1-q$$

$$1+\lambda\left(\exp(\frac{x}{b})-1\right) = (1-q)^{\frac{-1}{\alpha}}$$

$$\lambda\left(\exp(\frac{x}{b})-1\right) = (1-q)^{\frac{-1}{\alpha}} - 1$$

$$\exp(\frac{x}{b}) = \frac{(1-q)^{\frac{-1}{\alpha}}-1}{\lambda} + 1$$

Therefore,

$$x_q = b \quad * \ln\left(\frac{(1-q)^{-(\frac{1}{\alpha})} - 1}{\lambda} + 1\right). \tag{43}$$

by setting $q = \frac{1}{2}$,

$$x_q = b \quad * \ln\left(\frac{\left(1-\frac{1}{2}\right)^{-(\frac{1}{\alpha})} - 1}{\lambda} + 1\right), \tag{44}$$

The median of the OLE distribution can be obtain by setting $q = \frac{1}{2}$ in equation (44),

$$Med = x_q = b \quad * \ln\left(\frac{\left(\frac{1}{2}\right)^{-(\frac{1}{\alpha})} - 1}{\lambda} + 1\right). \tag{45}$$

4. Estimation Methods

In this section, we introduce two estimate methods of parameters of the OLE distribution,to compare the best method for finding the best fit to the distribution curve in our experimental simulation.

4.1. Maximum Likelihood Estimation (MLE)

A random sample X is taken from the OLE distribution with probability density function (pdf) [4], [13] given as :

$$f(x;\alpha,\lambda,b) = \frac{\alpha\lambda}{b} exp(\frac{x}{b}) \left(1 + \lambda\left(\exp(\frac{x}{b}) - 1\right)\right)^{-(\alpha+1)}. \tag{46}$$

By definition, the likelihood function of OLE is given by [13]:

$$f(x;\alpha,\lambda,b) = \left(\frac{\alpha\lambda}{b}\right)^n \exp\left(\frac{\sum_{i=1}^n x_i}{b}\right) \prod_{i=1}^n \left[1 + \lambda\left(\exp\left(\frac{x_i}{b}\right) - 1\right)\right]^{-(\alpha+1)} \tag{47}$$

and the log-likelihood function [35] is given by:

$$\frac{\partial L}{\partial b} = \frac{-n}{b} - \frac{\sum_{i=1}^n x_i}{b^2} + \frac{\lambda(\alpha+1)}{b^2}\left(\sum_{i=1}^n \frac{x_i * \exp(\frac{x_i}{b})}{1 + \lambda\left(\exp(\frac{x_i}{b}) - 1\right)}\right) = 0. \tag{48}$$

$$\frac{\partial L}{\partial \alpha} = \frac{n}{\alpha} - \sum_{i=1}^n \ln\left(1 + \lambda\left(\exp(\frac{x_i}{b}) - 1\right)\right) = 0. \tag{49}$$

$$\frac{\partial L}{\partial \lambda} = \frac{n}{\lambda} - (\alpha+1)\left(\sum_{i=1}^n \frac{\exp(\frac{x_i}{b}) - 1}{1 + \lambda\left(\exp(\frac{x_i}{b}) - 1\right)}\right) = 0. \tag{50}$$

By solving equations (48),(49);(50). We can obtain the MLE for OLE distribution, The parameters $\alpha, \lambda \quad and \quad b$, represented as $\alpha_{mle}, \lambda_{mle} and \quad b_{mle}$, respectively. The Newton-Raphson algorithm has use an iterative procedure to solve numerically.

4.2. Least Square Estimation (LSE)

Initially, the least square estimation were introduced to estimate the parameters of OLE distribution,,[4][10]; [30] This technique has been used to estimate unknown parameters of the OLE distribution by minimizing the concerning parameters $\alpha, \lambda \quad and \quad b$, represented, which is :

$$U(x;k,\lambda,b) = \sum_{i=1}^{n}\left(F(x_i) - \frac{i}{n+1}\right)^2 = \sum_{i=1}^{n}\left(\left[1-\left(1+\lambda(\exp(\frac{x_i}{b})-1)\right)^{-\alpha}\right] - \frac{i}{n+1}\right)^2. \quad (51)$$

The parameter's values are obtained from the least square method by partial differentiation in (51) concerning corresponding parameters, [5].

$$Let: \xi_i = \exp(\frac{x_i}{b}), \quad \nu_i = \exp(\frac{x_i}{b}) - 1; \quad \tau_i = 1 + \lambda\left(\exp(\frac{x_i}{b}) - 1\right)$$

$$\frac{\partial U}{\partial \alpha} = 2\sum_{i=1}^{n}\tau_i^{-\alpha}\ln 1 + \lambda\nu_i\left(1 - \tau_i^{-\alpha} - \frac{i}{n+1}\right), \quad (52)$$

$$\frac{\partial U}{\partial \lambda} = 2\alpha\sum_{i=1}^{n}\nu_i\tau_i^{-(\alpha+1)}\left(1 - \tau_i^{-\alpha} - \frac{i}{n+1}\right), \quad (53)$$

$$\frac{\partial U}{\partial b} = \frac{-2\alpha\lambda}{b^2}(\sum_{i=1}^{n}x_i\xi_i\tau_i^{-(\alpha+1)}\left(1 + \tau_i^{-\alpha} - \frac{i}{n+1}\right). \quad (54)$$

We solve non-linear equations $\frac{\partial U}{\partial \alpha} = 0$, $\frac{\partial U}{\partial \lambda} = 0$; $\frac{\partial U}{\partial b} = 0$. to estimate the unknown parameters of the proposed distribution by minimizing the function concerning parameters $\alpha, \lambda \quad and \quad b$.

5. Simulation Experiments

The Monte Carlo simulations were conducted on various datasets, and we estimated the parameters of the Odd-Lomax-Exponential (OLE) distribution using maximum likelihood estimation and least squares estimation methods. The bias ($Bias$) and mean square error (MSE) were calculated for different random sample sizes ($n = 20, 50, 80, 120, 160, 200, 300, 400, 500$), which were generated with a replication number of L $= 1000$, and for different parameters $\left(\hat{\theta} = [\hat{\alpha}, \hat{\lambda}; \hat{b}]\right)$ Too. Four different scenarios were performed with various correct initial values for cases ($\alpha = [2.0, 1.5, 1.5, 1.6], \lambda = [0.5, 1.0, 0.8, 1.8]; b = [0.5, 0.5, 1.0, 0.2]$). These simulation experiments aimed to discover the behavior of the densities function and compare the two estimation methods (MLE) and (LSE) as following.

Tables [1, 2, 3; 4] represent four scenarios of the results of the simulation study to calculate estimates for parameters, bias ($Bias$), and the mean square errors ($MSEs$) criterion for each experiment with different initial correct values of the parameters . Table.[1] The Least Squares Estimator (LSE) method For the (α) shape parameter and the (λ) scale parameter consistently outperforms the Maximum Likelihood Estimator (MLE) method in terms of bias and the $MSEs$ estimates for all sample sizes, while The (MLE) method For the (b) scale parameter provides superior estimates of bias and $MSEs$ for all sample sizes, Furthermore For both methods and all estimators, the $MSEs$ estimates decrease as the sample size increases. The results in Table.[2] are similar to those in table[1] . However, the estimates converge to zero more slowly, and the differences in $MSEs$ estimates between sample sizes (n=20) and (n=500) are noticeable. Also in Table. [3], The (LSE) method consistently outperforms the (MLE) method in terms of bias and $MSEs$ estimates for all sample sizes. These estimates decrease with increasing sample size, and there are clear differences in $MSEs$ values between the two methods. As previously stated, The estimates in Table.[4] follow a pattern similar to those in tables[1] and [2]. But, the $MSEs$ estimates decrease more rapidly with increasing sample size.

Table 1. Estimation, Bias and Mean Square Error for the parameters by (MLE) and (LSE) methods of the ($\alpha = 2.0, \lambda = 0.5, b = 0.5$) for different samples size.

Sample Size	*parameters*($\hat{\theta}$)	*Estimation*(MLE)	*Bias*($\hat{\theta}$)	$MSE(\hat{\theta})$	*Estimation*(LSE)	*Bias*($\hat{\theta}$)	$MSE(\hat{\theta})$
n=20	$\hat{\alpha}$	0.0315	1.9685	3.8749	0.2322	1.7678	3.1250
	$\hat{\lambda}$	0.0014	0.4986	0.2486	0.7238	0.2238	0.0501
	$\hat{b}$	0.1395	0.3605	0.1299	1.0664	0.5664	0.3208
n=50	$\hat{\alpha}$	0.0006	1.9994	3.9976	0.2381	1.7619	3.1042
	$\hat{\lambda}$	0.0004	0.4996	0.2496	0.7855	0.2855	0.0815
	$\hat{b}$	0.0584	0.4416	0.1950	1.0734	0.5734	0.3288
n=80	$\hat{\alpha}$	0.0002	1.9998	3.9991	0.2623	1.7377	3.0195
	$\hat{\lambda}$	0.0029	0.4971	0.2471	0.8589	0.3589	0.1288
	$\hat{b}$	0.0340	0.4660	0.2172	1.1883	0.6883	0.4738
n=120	$\hat{\alpha}$	0.0011	1.9989	3.9955	0.2520	1.7480	3.0555
	$\hat{\lambda}$	0.0063	0.4937	0.2438	0.8302	0.3302	0.1091
	$\hat{b}$	0.0185	0.4815	0.2318	1.1380	0.6380	0.4070
n=160	$\hat{\alpha}$	0.0001	2.0000	3.9999	0.2443	1.7557	3.0824
	$\hat{\lambda}$	0.0062	0.4938	0.2438	0.8074	0.3074	0.0945
	$\hat{b}$	0.0124	0.4876	0.2377	1.1018	0.6018	0.3622
n=200	$\hat{\alpha}$	0.0004	1.9996	3.9984	0.2396	1.7604	3.0988
	$\hat{\lambda}$	0.0050	0.4950	0.2450	0.7950	0.2950	0.0870
	$\hat{b}$	0.0100	0.4900	0.2401	1.0786	0.5786	0.3348
n=300	$\hat{\alpha}$	0.0002	1.9998	3.9993	0.2447	1.7553	3.0812
	$\hat{\lambda}$	0.0033	0.4967	0.2467	0.8068	0.3068	0.0941
	$\hat{b}$	0.0067	0.4933	0.2434	1.1041	0.6041	0.3649
n=400	$\hat{\alpha}$	0.0001	1.9999	3.9996	0.2484	1.7516	3.0681
	$\hat{\lambda}$	0.0025	0.4975	0.2475	0.8200	0.3200	0.1024
	$\hat{b}$	0.0050	0.4950	0.2450	1.1204	0.6204	0.3849
n=500	$\hat{\alpha}$	0.0001	2.0000	4.0000	0.2424	1.7576	3.0891
	$\hat{\lambda}$	0.0015	0.4985	0.2485	0.8013	0.3013	0.0908
	$\hat{b}$	0.0045	0.4955	0.2456	1.0922	0.5922	0.3507

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Table 2. Estimation, Bias and Mean Square Error for the parameters by (MLE) and (LSE) methods of the $(\alpha = 1.5, \lambda = 1.0, b = 0.5)$ for different samples size.

Sample Size	*parameters*$(\hat{\theta})$	*Estimation*(MLE)	*Bias*$(\hat{\theta})$	$MSE(\hat{\theta})$	*Estimation*(LSE)	*Bias*$(\hat{\theta})$	$MSE(\hat{\theta})$
n=20	$\hat{\alpha}$	0.0138	1.4862	2.2089	0.3096	1.1904	1.4170
	$\hat{\lambda}$	0.0037	0.9963	0.9926	0.3371	0.6629	0.4395
	$\hat{b}$	0.2082	0.2918	0.0852	0.9289	0.4289	0.1840
n=50	$\hat{\alpha}$	0.0032	1.4968	2.2404	0.3175	1.1825	1.3983
	$\hat{\lambda}$	0.0068	0.9932	0.9864	0.3711	0.6289	0.3955
	$\hat{b}$	0.1048	0.3952	0.1562	0.9525	0.4525	0.2048
n=80	$\hat{\alpha}$	0.0021	1.4979	2.2436	0.3498	1.1502	1.3230
	$\hat{\lambda}$	0.0037	0.9963	0.9926	0.4046	0.5954	0.3545
	$\hat{b}$	0.0693	0.4307	0.1855	1.0493	0.5493	0.3017
n=120	$\hat{\alpha}$	0.0032	1.4968	2.2405	0.3360	1.1640	1.3549
	$\hat{\lambda}$	0.0002	0.9998	0.9997	0.3919	0.6081	0.3698
	$\hat{b}$	0.0458	0.4542	0.2063	1.0080	0.5080	0.2580
n=160	$\hat{\alpha}$	0.0004	1.4996	2.2489	0.3258	1.1742	1.3788
	$\hat{\lambda}$	0.0001	0.9999	0.9998	0.3815	0.6185	0.3826
	$\hat{b}$	0.0350	0.4650	0.2162	0.9773	0.4773	0.2278
n=200	$\hat{\alpha}$	0.0012	1.4988	2.2463	0.3195	1.1805	1.3935
	$\hat{\lambda}$	0.0006	0.9994	0.9987	0.3760	0.6240	0.3893
	$\hat{b}$	0.0290	0.4710	0.2218	0.9586	0.4586	0.2103
n=300	$\hat{\alpha}$	0.0006	1.4994	2.2482	0.3262	1.1738	1.3778
	$\hat{\lambda}$	0.0001	1.0000	1.0000	0.3810	0.6190	0.3832
	$\hat{b}$	0.0193	0.4807	0.2311	0.9787	0.4787	0.2291
n=400	$\hat{\alpha}$	0.0001	1.5000	2.2499	0.3312	1.1688	1.3661
	$\hat{\lambda}$	0.0001	1.0000	1.0000	0.3873	0.6127	0.3754
	$\hat{b}$	0.0146	0.4854	0.2356	0.9936	0.4936	0.2436
n=500	$\hat{\alpha}$	0.0001	1.5000	2.2500	0.3232	1.1768	1.3848
	$\hat{\lambda}$	0.0001	1.0000	1.0000	0.3788	0.6212	0.3859
	$\hat{b}$	0.0117	0.4883	0.2384	0.9697	0.4697	0.2206

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Table 3. Estimation, Bias and Mean Square Error for the parameters by (MLE) and (LSE) methods of the $(\alpha = 1.5, \lambda = 0.8, b = 1.0)$ for different samples size.

Sample Size	*parameters*$(\hat{\theta})$	*Estimation*(MLE)	*Bias*$(\hat{\theta})$	$MSE(\hat{\theta})$	*Estimation*(LSE)	*Bias*$(\hat{\theta})$	$MSE(\hat{\theta})$
n=20	$\hat{\alpha}$	0.0068	1.4932	2.2297	0.3096	1.1904	1.4170
	$\hat{\lambda}$	0.0051	0.7949	0.6319	0.4213	0.3787	0.1434
	$\hat{b}$	0.1505	0.8495	0.7216	0.4864	0.5136	0.2637
n=50	$\hat{\alpha}$	0.0001	1.4999	2.2498	0.3175	1.1825	1.3983
	$\hat{\lambda}$	0.0064	0.7936	0.6298	0.4639	0.3361	0.1130
	$\hat{b}$	0.0589	0.9411	0.8856	0.4958	0.5042	0.2542
n=80	$\hat{\alpha}$	0.0005	1.4995	2.2485	0.3498	1.1502	1.3230
	$\hat{\lambda}$	0.0117	0.7883	0.6213	0.5057	0.2943	0.0866
	$\hat{b}$	0.0298	0.9702	0.9413	0.5472	0.4528	0.2050
n=120	$\hat{\alpha}$	0.0121	1.4879	2.2137	0.3360	1.1640	1.3549
	$\hat{\lambda}$	0.0080	0.7920	0.6273	0.4898	0.3102	0.0962
	$\hat{b}$	0.0198	0.9802	0.9608	0.5251	0.4749	0.2256
n=160	$\hat{\alpha}$	0.0112	1.4888	2.2164	0.3258	1.1742	1.3788
	$\hat{\lambda}$	0.0048	0.7952	0.6324	0.4768	0.3232	0.1044
	$\hat{b}$	0.0161	0.9839	0.9680	0.5089	0.4911	0.2412
n=200	$\hat{\alpha}$	0.0090	1.4910	2.2231	0.3195	1.1805	1.3935
	$\hat{\lambda}$	0.0001	0.8000	0.6400	0.4700	0.3300	0.1089
	$\hat{b}$	0.0167	0.9833	0.9669	0.4988	0.5012	0.2512
n=300	$\hat{\alpha}$	0.0060	1.4940	2.2321	0.3262	1.1738	1.3778
	$\hat{\lambda}$	0.0019	0.7981	0.6370	0.4762	0.3238	0.1048
	$\hat{b}$	0.0093	0.9907	0.9816	0.5097	0.4903	0.2404
n=400	$\hat{\alpha}$	0.0045	1.4955	2.2366	0.3312	1.1688	1.3661
	$\hat{\lambda}$	0.0001	0.8000	0.6400	0.4841	0.3159	0.0998
	$\hat{b}$	0.0084	0.9916	0.9833	0.5173	0.4827	0.2330
n=500	$\hat{\alpha}$	0.0036	1.4964	2.2392	0.3232	1.1768	1.3848
	$\hat{\lambda}$	0.0006	0.7994	0.6390	0.4734	0.3266	0.1066
	$\hat{b}$	0.0061	0.9939	0.9879	0.5047	0.4953	0.2453

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Table 4. Estimation, Bias and Mean Square Error for the parameters by (MLE) and (LSE) methods of the ($\alpha = 1.6, \lambda = 1.8, b = 0.2$) for different samples size.

Sample Size	*parameters*($\hat{\theta}$)	*Estimation*(MLE)	*Bias*($\hat{\theta}$)	$MSE(\hat{\theta})$	*Estimation*(LSE)	*Bias*($\hat{\theta}$)	$MSE(\hat{\theta})$
n=20	$\hat{\alpha}$	0.0205	1.5795	2.4947	0.2903	1.3097	1.7153
	$\hat{\lambda}$	0.0103	1.7897	3.2030	0.1905	1.6095	2.5904
	$\hat{b}$	0.3175	1.6825	2.8307	2.1008	1.9008	3.6130
n=50	$\hat{\alpha}$	0.0032	1.5968	2.5498	0.2977	1.3023	1.6961
	$\hat{\lambda}$	0.0041	1.7959	3.2253	0.2091	1.5909	2.5311
	$\hat{b}$	0.1481	1.8519	3.4297	2.1905	1.9905	3.9620
n=80	$\hat{\alpha}$	0.0014	1.5986	2.5554	0.3279	1.2721	1.6182
	$\hat{\lambda}$	0.0018	1.7982	3.2335	0.2281	1.5719	2.4709
	$\hat{b}$	0.0965	1.9035	3.6231	2.4039	2.2039	4.8572
n=120	$\hat{\alpha}$	0.0006	1.5994	2.5579	0.3150	1.2850	1.6513
	$\hat{\lambda}$	0.0001	1.8000	3.2400	0.2208	1.5792	2.4939
	$\hat{b}$	0.0652	1.9348	3.7434	2.3151	2.1151	4.4737
n=160	$\hat{\alpha}$	0.0001	1.6000	2.5599	0.3054	1.2946	1.6760
	$\hat{\lambda}$	0.0004	1.7996	3.2384	0.2149	1.5851	2.5126
	$\hat{b}$	0.0501	1.9499	3.8022	2.2474	2.0474	4.1919
n=200	$\hat{\alpha}$	0.0115	1.5885	2.5233	0.2996	1.3004	1.6911
	$\hat{\lambda}$	0.0130	1.7870	3.1935	0.2118	1.5882	2.5225
	$\hat{b}$	0.0281	1.9719	3.8885	2.2077	2.0077	4.0307
n=300	$\hat{\alpha}$	0.0076	1.5924	2.5357	0.3058	1.2942	1.6749
	$\hat{\lambda}$	0.0001	1.7999	3.2395	0.2146	1.5854	2.5134
	$\hat{b}$	0.0272	1.9728	3.8919	2.2491	2.0491	4.1986
n=400	$\hat{\alpha}$	0.0058	1.5942	2.5416	0.3105	1.2895	1.6628
	$\hat{\lambda}$	0.0064	1.7936	3.2170	0.2182	1.5818	2.5021
	$\hat{b}$	0.0143	1.9857	3.9431	2.2841	2.0841	4.3435
n=500	$\hat{\alpha}$	0.0046	1.5954	2.5453	0.3030	1.2970	1.6821
	$\hat{\lambda}$	0.0052	1.7948	3.2214	0.2133	1.5867	2.5175
	$\hat{b}$	0.0114	1.9886	3.9545	2.2313	2.0313	4.1260

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6. Application

In this section, we considered two groups of datasets which are presented as follows:
Example (1): The first dataset comprises $n = 84$ observations, acquired from [31] and [17], representing the failure times of aircraft windshield points data. A summary of the descriptive statistics for the first group's data is presented in Table 5. Example (2): The second dataset consists of $n = 100$ observations [13] representing the fatigue life of 6061-T6 aluminum coupon points data. A summary of the descriptive statistics for the first group's data is presented in Table 6.

We estimated the unknown parameters of model using two methods, Maximum Likelihood Estimation (MLE) and Least Squares Estimation (LSE), For two groups of datasets in the reliability engineering field are the failure time of aircraft windshield and the fatigue life of $6061 - T6$ aluminum coupon. In Table [5], we focused on descriptive statistics to provide a general understanding of sample size, data concentration, dispersion, and normality. We tested normality using $Q - Q$ plots as showed in Figure(5), And Provided us estimates of unknown parameters are in Table [6]. Mean square error (MSE) is included in parentheses for the parameters, along with goodness-of-fit statistics and quality criteria for the model as Akaike Information Criterion (AIC) and Bayesian Information Criterion (BIC). Histograms were created for the first and second dataset after parameters estimation to visualize their distributions. Figure(6), representing the failure time of aircraft windshield points data , closely resembles one of the distribution forms in figure 1.and representing the fatigue life of $6061 - T6$ aluminum coupon points data, closely resembles one of the distribution forms in figure 1.

Table 5. Descriptive Statistics of each groups for real-datasets

DataSet	Sample Size	Mean	Std. Deviation	Median	Mode	skewness	kurtosis	Minimum	Maximum
Set1	84	2.352	1.359	2.385	1.281	0.087	2.365	0.040	4.663
Set2	100	133.780	22.613	132.500	142.000	0.372	70.000	0.04	212.000

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7. Discussion

In this paper, we investigated the performance of the odd-lomax-exponential distribution, with another formulas for each from exponential and lomax distribution to yield a new distribution . This distribution has three different parameters,two scale parameter and shape parameter, It has been represented probability density function as in Figure(1), cumulative distribution function as in Figure(2),survival function as in Figure(3)and hazard rate function as in Figure(4). Also, we derivative more structural properties of the model. We introduced two estimation method for parameters of the OLE distribution, and then comparison

Table 6. Estimated parameters, square error parentheses and goodness-of-fit Via MLE and LSE methods for each groups for real-datasets

dataset	**Estimation method**	$\hat{\alpha}$	$\hat{\lambda}$	$\hat{b}$	**AIC**	**BIC**
dataset1	MLE	183.8849 (2.560)	140.7923 (3.2387)	535.8905 (0.040)	3293.918	3301.246
	LSE	0.2795 (1.744)	0.1931 (2.5819)	2.0622 (3.4677)	3293.920	3301.250
dataset2	MLE	72.319 (2.560)	68.761 (3.240)	595.703 (0.041)	7589.171	7596.986
	LSE	0.0062 (2.541)	0.008 (3.237)	0.045 (0.238)	1918.159	1925.975

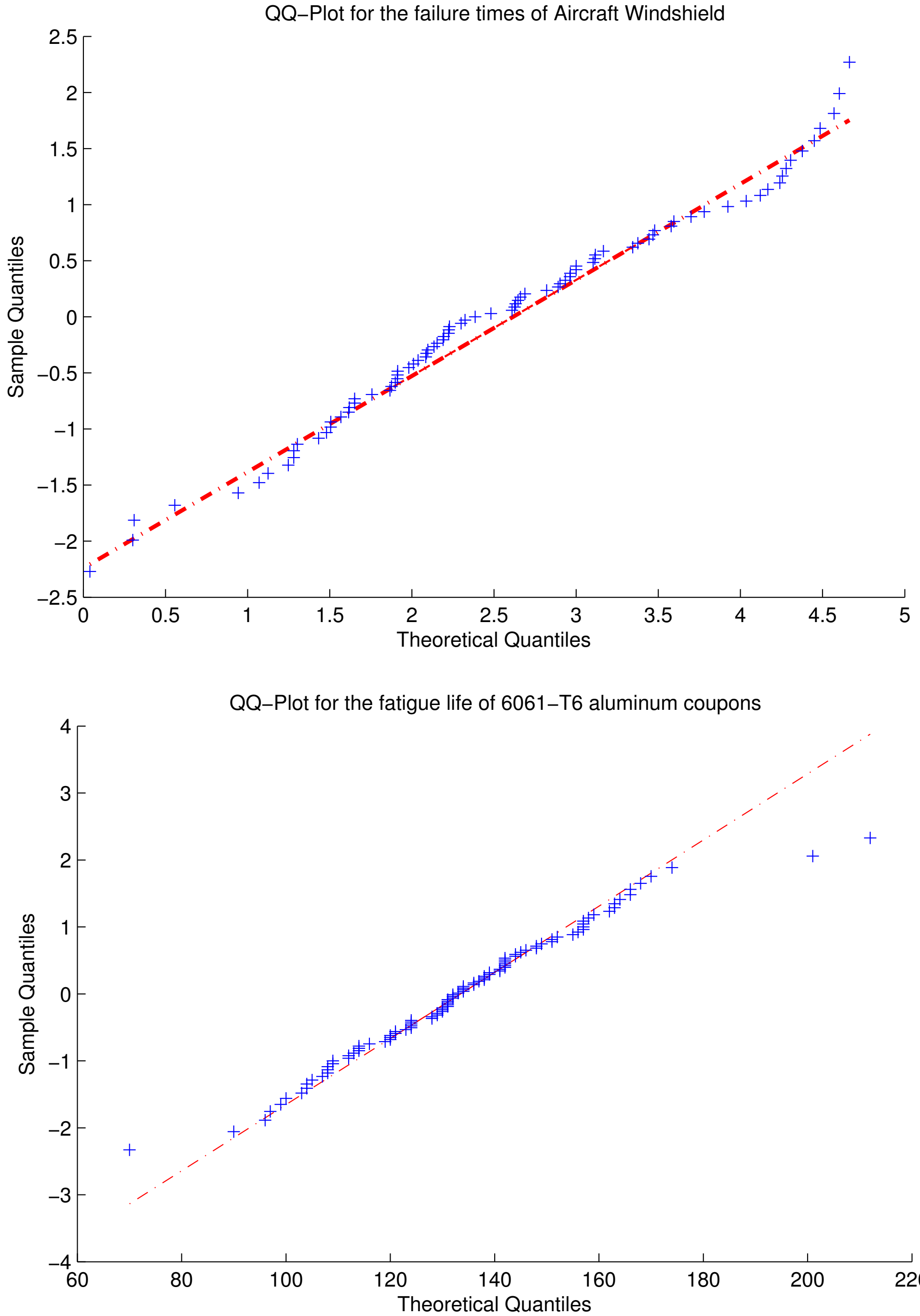

Figure 5. Represented the Expected Normal Values versus Observed Values (Normal Q-Q plot).

performance between them by Monte-Carlo simulation for four different scenarios in terms of correct initial values for the parameters is presented. And, we found through the analysis that the least squares method gives the (MSE) values less than from the maximum likelihood method for compared with a regular increase in sample sizes. We applied the model on two type datasets in the reliability engineering field which are the

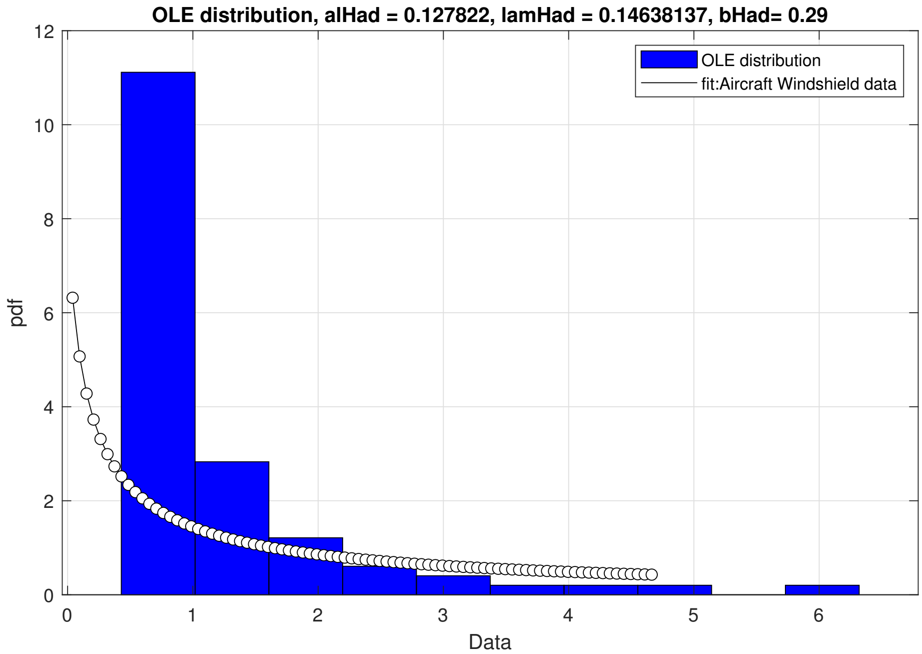

Figure 6. Plots of the histogram and fit pdfs of the failure times of aircraft windshield points data.
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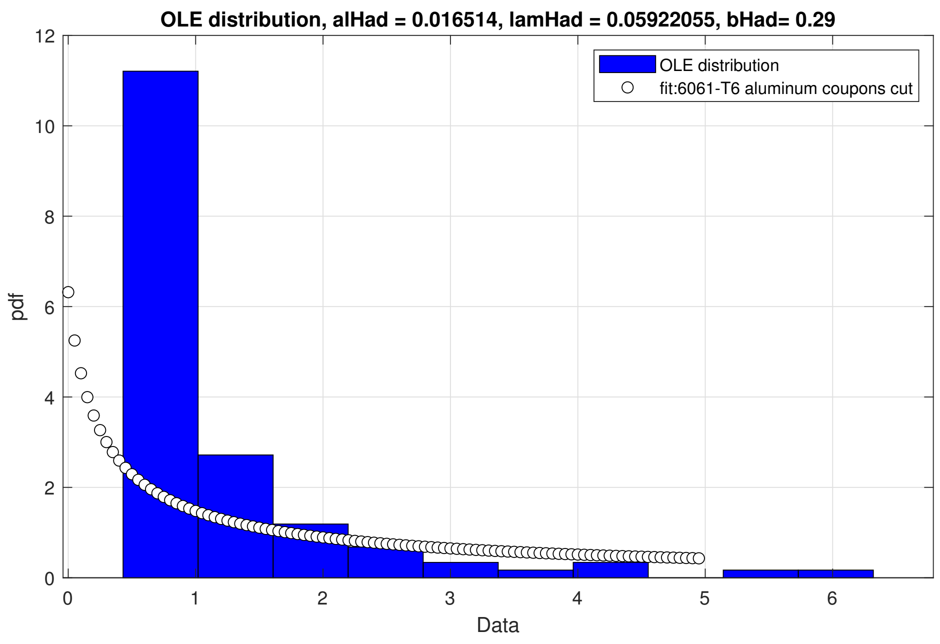

Figure 7. Plots of the histogram and fit pdfs the fatigue life of 6061-T6 aluminum coupon points data.
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failure time of aircraft windshield and the fatigue life of $6061 - T6$ aluminum coupon to evaluation the distribution.We created a Matlab,23 code to analyze the data and display the graphs.

8. Conclusion

We conclude through the study that this distribution is a flexible distribution and is well-suited for analyzing reliability engineering data. What distinguishes the distribution is that it contains three parameters to control its mathematical structure. In simulation experiments and at different initial values of the parameters, the estimates decrease with increasing sample size, and there are clear differences in $MSEs$ values between the two methods, in the applied side. The results also showed the advantages of the LSE method showed a relative advantages over using in estimation with different sample sizes.

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