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Exploring the shift in symmetry phenomenon in exponentially weighted moving average quality charts for statistics derived from beta distribution

Mohammad M. Hamasha [1,*], Ghada Shawaheen [1], Ahmad Mayyas [2]

[1] *Department of Industrial Engineering, Faculty of Engineering, The Hashemite University, Zarqa, 13133, Jordan*
[2] *Department of Management Science and Engineering, Khalifa University, Abu Dhabi, UAE*

Abstract The Exponential Weighted Moving Average (EWMA) is a statistical method used to create moving averages that assign greater weight to more recent data, frequently used in quality control. This approach, which mitigates asymmetry via the central limit theorem, encounters skewness issues majorly influenced by the lambda parameter (λ). This research investigates the impact of various EWMA smoothing factors on skewness reduction, utilizing the beta distribution, which can replicate diverse real-world distributions from heavily skewed to nearly symmetric, for data generation. With Matlab, random beta-distributed data was analyzed with the EWMA to observe changes in skewness and kurtosis. This study aids in comprehending data distributions in new or significantly modified processes, assisting in the adjustment of control chart parameters. It identifies a gap in existing literature regarding indeterminate distributions and underscores the need for further investigation in this field.

Keywords Exponential Weighted Moving Average (EWMA), Statistical Process Control (SPC), Beta Distribution, Quality Control Charts, Skewness Reduction

AMS 2010 subject classifications 93C83

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1. Introduction

Quality engineers play an important role in ensuring that industrial and service operations proceed smoothly without interruptions by monitoring the required process measures and preventing them from diverging. Statistical process control charts are commonly used to regulate the process, but errors in selecting the appropriate measures or control systems can significantly affect the quality of the final product and the efficiency of the process. Thus, statistical process control is essential to identify and manage process deviations that can compromise product quality.

Statistical Process Control (SPC) uses control charts to maintain and enhance product quality. Control charts have been successfully implemented in numerous areas such as healthcare and public health surveillance, in addition to monitoring manufacturing processes [1]. For an analytical laboratory, providing sufficient data comparability is crucial [1], as is the case in a nuclear power plant control room [2] or with coal quality [3].

Additionally, control charts are used in business quality management and control, and they can also be applied to the teaching process. They can be used to track and identify any issues that may have arisen throughout the course of the lesson with the aim of resolving or enhancing them [4].

For Statistical Process Control, the three most commonly used types of control charts are the conventional Shewhart charts, exponentially weighted moving average (EWMA), and cumulative sum (CUSUM) charts. While

*Correspondence to: Mohammad M. Hamasha (Email: mhamasha@hu.edu.jo). Department of Industrial Engineering, Faculty of Engineering, The Hashemite University, Zarqa, 13133, Jordan

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the Shewhart control charts are effective in identifying significant process shifts, they are not effective in detecting minor process changes [5]. Although the performance of EWMA and CUSUM charts is nearly identical, most researchers prefer the EWMA chart due to specific case circumstances. Nevertheless, the simplicity, ease of use, and clarity of both EWMA and CUSUM make them popular choices among quality control professionals [6, 7].

EWMA is a popular statistical method for quality control and process monitoring because it is sensitive to modest process mean changes [8]. EWMA emphasizes recent observations by exponentially reducing weights for older data, unlike Shewhart control charts, which focus on significant shifts. Manufacturing operations, public health surveillance, and financial monitoring require early identification of slight adjustments, making it useful. EWMA smooths data over time, reduces noise, and highlights major trends that may suggest process abnormalities, making it relevant to our investigation. The control chart may measure process performance even when the data does not follow a normal distribution, as in real-world applications, by using the EWMA function. The smoothing constant (λ) determines the sensitivity of the EWMA chart. A smaller λ number makes the chart more sensitive to slight changes, while a larger λ value offers a more gradual reaction. EWMA's versatility makes it essential in many industries, ensuring that processes stay within control and exceptions are quickly detected and remedied [9].

Researchers such as [10, 11, 12, 13] have developed various methods to improve the EWMA chart's ability to detect small movements in various scenarios. Most control charts are based on the assumption that quality criteria follow a normal distribution. However, in many applications, the distribution of the relevant quality feature is uncertain or does not follow a normal distribution. Stoumbos and Reynolds [14] investigated the EWMA chart's resistance to non-normal distributions of the quality parameter and found that it is resistant to various distributions [1]. Stoumbos and Sullivan [15] suggested alternative distributions for non-normal data. EWMA control charts for non-normal distributions have been developed by numerous researchers, including [16, 17, 18].

Li et al. [19] and Lin & Chou [20] employed the beta distribution to represent non-normal skewed or symmetric populations to study control chart robustness to non-normality and proposed the EWMA control chart alternative. The EWMA function processes these distributions before plotting on the control chart. The EWMA control chart's main issue is data imbalance around UCL and LCL. Processing with the EWMA function mitigates data skewness, however asymmetry data may remain. The chart should have half the data above the upper control limit (UCL) and half below the lower control limit (LCL), but symmetric data does not.

The difference between data above UCL and below LCL of the regulating process called data asymmetry. The Average Run Length (ARL) is typically utilized to determine whether the process is under control or if there has been a shift in the data mean. However, ARL is only reliable for symmetric data, as the aforementioned proportions can differ for asymmetric data. A small shift towards the side with a smaller proportion can increase ARL, rather than decrease it. Regrettably, the issue of asymmetry has not been adequately addressed in the existing literature. The EWMA function is capable of reducing data skewness and mitigating the issue of asymmetry. However, in cases of severe skewness or when the smoothing factor is relatively large, the EWMA function may not decrease the skewness to an acceptable degree.

EWMA can identify tiny process mean shifts better than the Shewhart chart. The EWMA chart, created by Roberts in 1959, gives recent data more weight by decreasing historical observation weights exponentially. This lets EWMA respond faster to modest, steady process changes that Shewhart charts may miss due to their sensitivity to bigger, more abrupt alterations. Shewhart charts are simple and excellent at highlighting substantial deviations, but their reliance on the most recent data point can make them less useful for ongoing process monitoring when minor adjustments are crucial. In comparison, the EWMA chart uses all past data points to create a smoother, more steady signal that better represents the process. EWMA is useful in semiconductor manufacturing, healthcare monitoring, and financial risk management, where early detection of slight deviations can prevent major ones. Modern process control relies on EWMA's blend of responsiveness and stability to precisely control sensitive processes. This study generalizes the EWMA chart to make it more useful in situations when data distributions vary from normality, expanding its use in numerous domains.

In general, skewness, skewness coefficient, or asymmetry coefficient are indicators utilized in descriptive statistics and probability theory to assess the extent and direction of asymmetry in the probability distribution function of authentic random variates. It represents an important formal characteristic of the probability distribution, together with the kurtosis coefficient, central tendency, and statistical dispersion parameters, which aid in understanding the structure of variables and statistical data. If a distribution is skewed to the right, it is positively skewed. This means that the tail on the right side of the distribution is longer or fatter than the left side. In such a distribution, the mean is typically greater than the median. This is because the outliers on the high end of the distribution pull the mean upward. The greater the absolute value of the skewness measure, the more asymmetrically distributed the distribution is. The distribution skewed to the left is negatively skewed. Symmetric distributions have a skewness value close to zero or zero [21].

This study concentrates on evaluating the level of skewness reduction attained by utilizing the EWMA function at varying smoothing factors. The beta distribution was chosen to represent a range from a highly skewed to an approximately unskewed probability density function, as it is capable of approximating a broad range of distributions encountered in real-world situations. The MATLAB software was utilized to generate random data that followed a beta distribution, to apply the EWMA function to this data, and to gauge the resulting skewness and kurtosis. The study examines and discusses the relationship between the skewness and kurtosis of EWMA statistics and their associated smoothing parameters. This research is particularly valuable if the process data distribution is known. Most recommendations in the literature deal with unknown distributions. This is as a result of the fact that the majority of distributions are for well-established, long-running processes. The data distribution will eventually be known, even if the process has undergone major changes, and the control chart settings must be changed accordingly. The results support the recommendations of EWMA control chart researchers and practitioners in selecting the optimal parameters.

1.1. EWMA Control Chart

Control charts are widely used in various industries and applications and are considered one of the most effective tools for Statistical Process Control (SPC) [22, 23, 24]. The EWMA control chart was developed by Crowder [24]. This chart is particularly effective in detecting minor shifts in process mean. The EWMA is a type of moving average, where the average is calculated by considering all the measurements and giving greater weight to recent observations and exponentially decreasing weight to older ones. The statistics for the EWMA control chart are as follows:

$$Z_i = \lambda X_i + (1-\lambda) Z_{i-1}, \quad i = 1, 2, \ldots, n; \quad 0 \leq \lambda \leq 1 \tag{1}$$

Where λ is the smoothing constant; $0 < \lambda < 1$.

The EWMA statistic (Z_i) is based on the current measurement (X_i) and the EWMA statistic from the previous measurement (Z_{i-1}). The weight given to the most recent measurement is determined by the weighting factor (lambda λ), represented by the symbol λ, which impacts the contribution of all earlier measurements based on their distance from the current measurement. For example, if $\lambda = 0.6$ and the current measurement is the fourth statistic taken $(i = 4)$, the weight of the current measurement is 0.6, the weight of the previous measurement $(i = 3)$ is 0.24, and the weight of the measurement at $(i = 2)$ is 0.096, while the remaining weight of 0.064 is for the initial measurement $(i = 1)$. As the number of measurements increases, the impact of old measurements becomes minimal, especially if λ is large, and when $\lambda = 1$, the EWMA only uses the most recent measurement.

UCL and LCL in EWMA are dependent on the number of previous measurements, represented by i, as shown in Equations (2) and (3).

$$UCL = \mu_0 + L\sigma \sqrt{\frac{\lambda}{(2-\lambda)} \left[1 - (1-\lambda)^{2i}\right]} \tag{2}$$

$$LCL = \mu_0 - L\sigma \sqrt{\frac{\lambda}{(2-\lambda)} \left[1 - (1-\lambda)^{2i}\right]} \tag{3}$$

Where L is a scale factor. In this methodology, the random variable is Z_i, and it is applicable to the distribution of any continuous random variable. However, when the value of i is very large, the component $(1-\lambda)^{2i}$ becomes very small, and as a result, the UCL and LCL equations can be reformulated as shown in Equations (4) and (5).

$$UCL = \mu_0 + L\sigma\sqrt{\frac{\lambda}{(2-\lambda)}} \tag{4}$$

$$LCL = \mu_0 - L\sigma\sqrt{\frac{\lambda}{(2-\lambda)}} \tag{5}$$

A smaller λ value leads to a more gradual trend, which also reduces the importance of the current measurement. According to Hunter's [26] recommendation, λ should be within the range of 0.2 to 0.3. To ensure false signals are detected and the average run length (ARL) is controlled, L can be selected with ARL set to either 370 or 500. Borror et al. [27] determined the appropriate value of L for an ARL of 370, while Lucas and Saccucci [28] determined the corresponding value of L for an ARL of 500.

1.2. The normal and non-normal Distribution

The robustness of control charts to non-normality has been investigated by several authors such as [19, 20, 27, 29], where the gamma and Student's t-distributions were used to model various non-normal skewed or symmetric populations. The normal distribution, also known as the Gaussian distribution, is a continuous probability distribution for a real-valued random variable, with the general form of its probability density function. The normal distribution is denoted by $\mathcal{N}(\mu, \sigma^2)$.

The normal distribution is a standard probability distribution that is commonly referred to as a "bell curve." It is utilized to describe the distribution of a variable's values, and many common datasets, such as adult human heights, test results from large classes, and measurement errors, frequently follow a normal distribution. The distribution is symmetrical around its mean, which is also the median and mode. The standard deviation is a statistic that measures the degree of dispersion in a set of data with a normally distributed distribution. It indicates how closely each instance in a set of data is clustered around the mean. The mean and standard deviation decide how the normal distribution will look. The standard deviation drops as the bell curve steepens, and if the examples are widely scattered, the bell curve will be flatter, signifying a high standard deviation. The normal distribution has a central peak where most observations are concentrated, and probabilities for values that are further from the mean decrease equally in both directions. The distribution's two tails contain rare extreme values.

1.3. Beta distribution

The beta distribution is a continuous probability distribution family that is widely used in statistics and probabilities. It is defined by two positive parameters, alpha (α) and beta (β), which act as exponents of the random variable and determine the shape of the distribution. The beta distribution has been widely used in many fields to model random variables constrained to finite intervals. For example, see [30, 31, 32]. It is a useful model for describing the behavior of proportions and percentages. If a random variable X has a beta distribution with the first shape parameter (shape1) $\alpha > 0$ and second shape parameter (shape2) $\beta > 0$, it is denoted as $X \sim \mathrm{B}(\alpha, \beta)$, and its probability density function is given by Equation 6.

$$f(x;\alpha,\beta) = \frac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha,\beta)} = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}x^{\alpha-1}(1-x)^{\beta-1}, \quad 0 < x < 1 \tag{6}$$

Where $\Gamma(z)$ is the gamma function. The beta function, B, is a normalization constant to ensure that the total probability is 1. In the above equations x is a realization of an observed value that occurred of a random process X. Where $B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$ denotes the beta function $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt$, for $x > 0$. Gamma function has the following property: $\Gamma(x+1) = x\Gamma(x)$ and $\Gamma(1) = 1$.

The Cumulative distribution function is addressed in Equation 7.

$$F(x;\alpha,\beta) = \frac{B(x;\alpha,\beta)}{B(\alpha,\beta)} = I_x(\alpha,\beta) \tag{7}$$

In the discussion regarding the beta distribution, we encounter two significant functions: $B(\alpha, \beta)$ and $I_x(\alpha, \beta)$. $B(\alpha, \beta)$ refers to the incomplete beta function, which is crucial in calculating probabilities that involve the beta distribution. The regularized incomplete beta function, $I_x(\alpha, \beta)$ is used to express the cumulative distribution function of the beta distribution, effectively normalizing the incomplete beta function by dividing it by $B(\alpha, \beta)$ to scale the result between 0 and 1. This method of representation is detailed in Berger's 2013 work, highlighting its importance in statistical applications. Further refining our understanding, the beta distribution can be reparametrized in terms of its mean μ and the sum of its shape parameters, denoted as $\nu = \alpha + \beta$, where ν must be greater than zero. This reparametrization not just a mathematical convenience; it provides a more intuitive understanding of the distribution by directly relating the parameters to the mean and the dispersion (variance). Kruschke's 2011 exploration provides an excellent foundation for this approach, emphasizing how the mean μ (which ranges between 0 and 1) and the parameter ν, determine the shape of the beta distribution. This parameterization is particularly useful in Bayesian statistics, where the beta distribution often serves as a prior distribution for binomial proportions, allowing for a straightforward interpretation and integration of prior knowledge into the statistical analysis. The expected value (mean) and the variance of the beta distribution is expressed in Equation 8 and 9, respectively.

$$\mathrm{E}(T) = \frac{\alpha}{\alpha + \beta} \tag{8}$$

$$\mathrm{Var}(T) = \frac{\alpha\beta}{(\alpha + \beta)^2(\alpha + \beta + 1)} \tag{9}$$

The beta distribution density curve can take on different shapes depending on the values of its shape parameters. When the distribution is symmetric, with equal shape parameters, the curve is balanced, exhibiting equal probabilities on both sides of the peak. In contrast, an asymmetric distribution occurs when the shape parameters differ, resulting in an uneven distribution of probabilities favoring one side over the other. A bell-shaped curve is observed when both shape parameters are greater than 1, presenting a smooth and symmetric curve with a single peak. On the other hand, a U-shaped curve appears when both shape parameters are less than 1, indicating a higher probability of extreme values near the boundaries of the interval. As the shape parameters deviate from equality, the bell shape can start to skew to the right or left. Skewing to the right implies a stretched distribution toward higher values, while skewing to the left represents a higher probability of smaller values. Lastly, when both shape parameters are equal to 1, the beta distribution takes on a uniform shape, resulting in a flat curve with equal probabilities across the entire interval. In summary, the beta distribution density curve showcases a range of shapes, including symmetric and asymmetric distributions, bell-shaped curves, U-shaped curves, skewed curves to the right or left, and uniform distributions, determined by the values of the shape parameters. Figure 1 provided examples about above mentioned cases of beta distribution. The shape of the Beta distribution can change dramatically with changes in the parameters, as explained below.

1.4. The effect of data skewness on the control chart performance

This section explores the impact of skewness on the performance of the control chart. Typically, UCL and LCL are placed at a distance of $L\sigma$ above and below the process mean, respectively, to create the standard control chart. If the input distribution to the control chart is symmetric about the mean, where the likelihood of receiving a point higher than the UCL is the same as receiving a point lower than the LCL, the performance of the control chart will be perfect. For this reason, the Shewhart control chart uses subgroups as imputes to solve the problem of asymmetry. However, in many cases, the control chart must be very sensitive to small shift such as temperature signal indicating of machine failure, and Shewhart is not an option. In these cases, Shewhart with subgroups cannot be used, and the problem of asymmetry must be tracked and solved in a different way. To maintain the control chart's strong performance, the symmetry around the mean of Z, or the output data from the EWMA, is crucial.

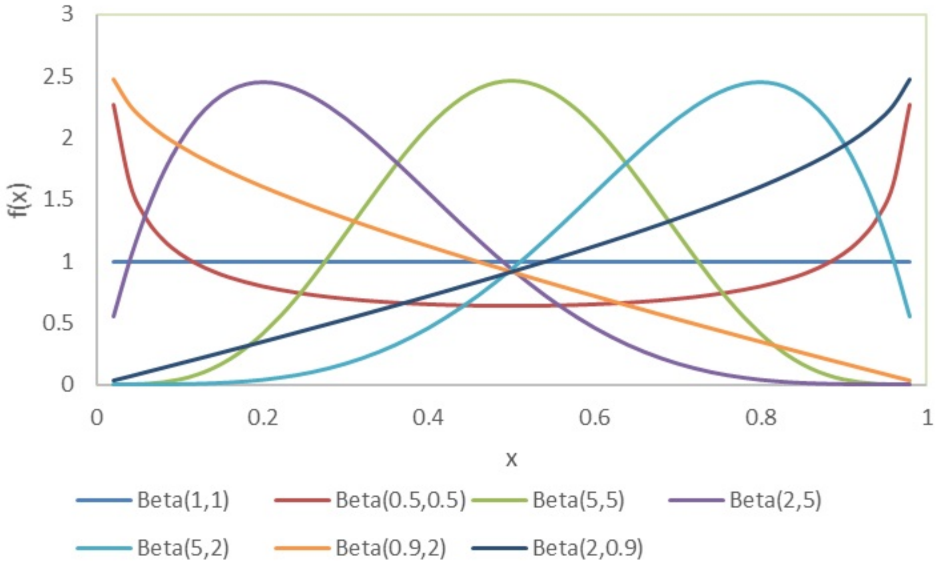

Figure 1. Some shape of Beta distribution.

2. Experimental Methodology

The experimental methodology was implemented using two distinct methods, both of which yielded closely aligned results. The first method generated beta-distributed random data using MATLAB, applied the EWMA function to this data, then measured the skewness and kurtosis of the resulting data. To enhance accuracy, one million randomly generated data points are used to estimate skewness or kurtosis. The MATLAB code is presented in Appendix 1. All data presented in this paper is produced in this method. The second method is used to validate the first method. In the second method, random variates were generated from selected beta distributions using MINITAB. These variates were then copied and pasted in a column within Excel. The first random number generated was presumed to be the initial EWMA statistic. Subsequently, the series of random variates was used as defined in Equation 1 to determine the complete series of EWMA statistics, located in the column adjacent to the column of random variates. For each beta distribution, a million EWMA statistics were generated at a time, with skewness and kurtosis being estimated for all the simulated EWMA statistics. Both approaches are predicated on generating random variates from a specific beta distribution, and using them to calculate EWMA statistics. By doing so, a real process employing EWMA statistics is simulated, assuming the original process data are beta distributed.

3. Results and Discussion

The discussion of the experiment involving the effect of the EWMA smoothing factor, λ, on the shape of the Beta distribution reveals a progressive transition towards normality as λ decreases. Initially, at $\lambda = 1.0$, the shape of the distribution is identical to the original Beta (0.4, 0.4), with its distinctive bimodal characteristics. See Figure 2. As λ is reduced, the twin peaks of the Beta distribution begin to merge, and the distribution starts to exhibit a unimodal form. The valley that initially dips between the peaks at higher λ values becomes less pronounced, indicating the beginning of the smoothing effect.

With further reduction in λ, the skewness and kurtosis, which are indicative of the deviation from normality, are noticeably lessened. The distribution's tails, initially heavy and contributing to the bimodality, retract and the central peak becomes more pronounced and rounded. This trend continues until the distribution adopts the bell shape of a normal distribution, particularly evident when λ reaches values as low as 0.1. The culmination

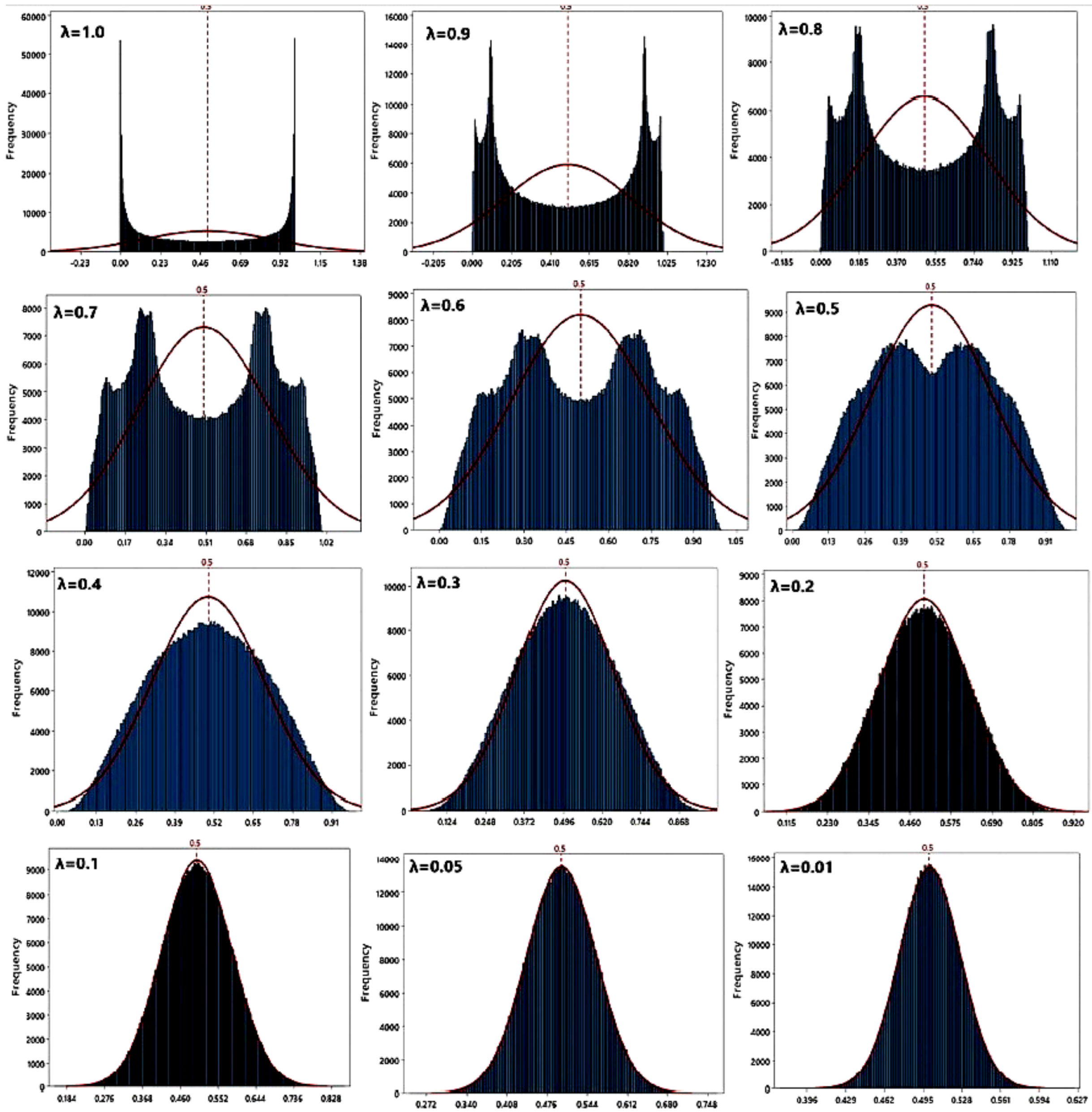

Figure 2. Distribution of beta-based random data after processing with the EWMA function at various λ values.

of this smoothing process is observed at $\lambda = 0.01$, where the distribution closely approximates the normal curve, showcasing the potency of the EWMA method in normalizing distributions that are initially non-normal. This experiment thus illustrates the power of the EWMA smoothing factor in transforming a distribution and underscores its potential utility in areas where normality is a critical assumption.

In the exploration of the EWMA smoothing factor's influence on Beta distribution characteristics, six distinct scenarios are analyzed to ascertain the impact on skewness and kurtosis: $\alpha > \beta > 1$, $\beta > \alpha > 1$, $1 > \beta > \alpha$, $1 > \alpha > \beta$, $\alpha > 1 > \beta$, and $\beta > 1 > \alpha$. These scenarios facilitate a comprehensive understanding of the skewness behavior in response to varying EWMA smoothing factors.

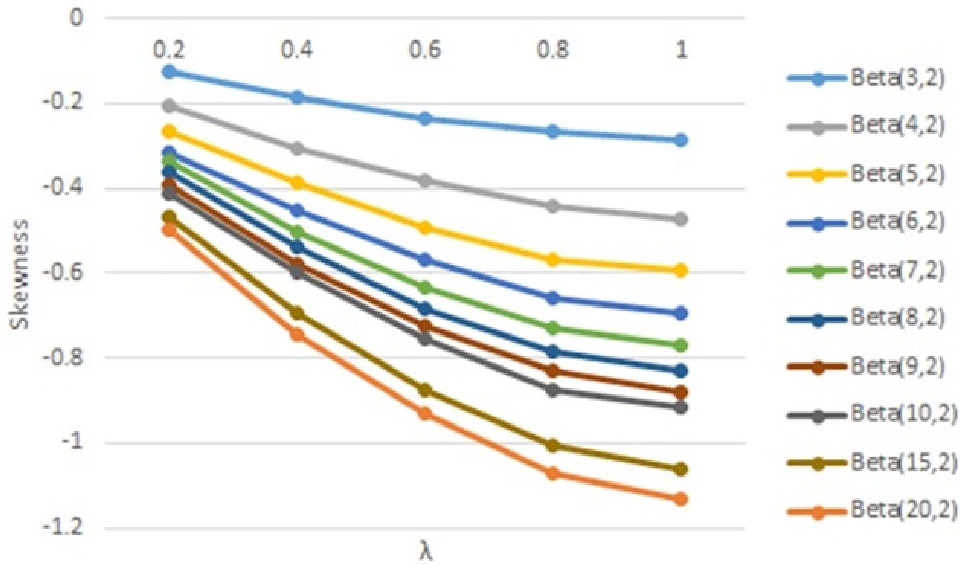

Figure 3. The effect of the parameter λ on the skewness of the beta distribution when $\alpha > \beta > 1$.

For distributions where $\alpha > \beta > 1$, an inherent leftward skew is observed, attributed to the higher β values pulling the distribution's peak towards the right. This behavior is exemplified in the curve labeled Beta(5,2) in Figure 1. Subsequent analysis across Figures 3 to 8 showcases the skewness of various Beta distributions after EWMA application, plotting skewness against different λ values on the y-axis and x-axis, respectively. Each figure corresponds to one of the six cases, with $\lambda = 1$ representing the baseline skewness in the absence of EWMA modification.

Upon examining the skewness trends as λ approaches 0, and through comparison of different curves within each figure, notable patterns emerge. In the first case ($\alpha > \beta > 1$), a decrease in λ from 1 leads to an increase in the negative skewness of each curve towards zero, indicating a diminution of the leftward skew, particularly evident in Figure 3. For curves with a constant β factor, such as Beta(3,2) versus Beta(4,2), the latter, with a higher α factor, exhibits less negative skewness, suggesting that an increase in α, with β held constant, reduces the distribution's skewness. Conversely, the second case ($\beta > \alpha > 1$) exhibits a rightward skew, with higher β values shifting the peak towards the left, as demonstrated by Beta(2,5) in Figure 1. Figure 4, acting as a negative mirror to Figure 3, illustrates the skewness response to EWMA function intervention in this scenario. As λ decreases from 1, the positive skewness of each curve lessens, moving closer to zero, hence indicating a reduction in rightward skew. When comparing curves with a constant α factor, such as Beta(2,3) and Beta(2,4), the curve with the higher β factor demonstrates greater skewness, implying that an increase in α, with β constant, amplifies the skewness of the distribution. The third case ($\beta < \alpha < 1$) and the fourth case ($\alpha < \beta < 1$) both feature U-shaped distributions but with different skewness directions influenced by their respective α and β values. Figures 5 and 6, respectively, illustrate the adjustments in skewness as λ decreases, with each showing a trend towards a reduction in skewness, either positive or negative. The fifth case ($\alpha < 1 < \beta$) suggests a right-skewed distribution with a left-side peak, as could be inferred from the curve Beta(0.9,2) in Figure 1. Figure 7 portrays this scenario, where a decrease in λ leads to a reduction in skewness towards zero. Curves with higher β factors, for a given α, display greater skewness, highlighting the EWMA process's accentuation of the original distribution's skewness. Lastly, the sixth scenario ($\beta < 1 < \alpha$) presents a distribution peaking on the right, indicative of a leftward skew, exemplified by the curve Beta(2,0.9). Figure 8 addresses this case, revealing that as λ decreases, the negative skewness increases towards zero. This trend implies that within this specific context, comparing curves like Beta(2,0.1) and Beta(2,0.2), an increase in the β factor results in less negative skewness, suggesting that a higher β factor, when α is constant, mitigates the skewness effect after EWMA processing.

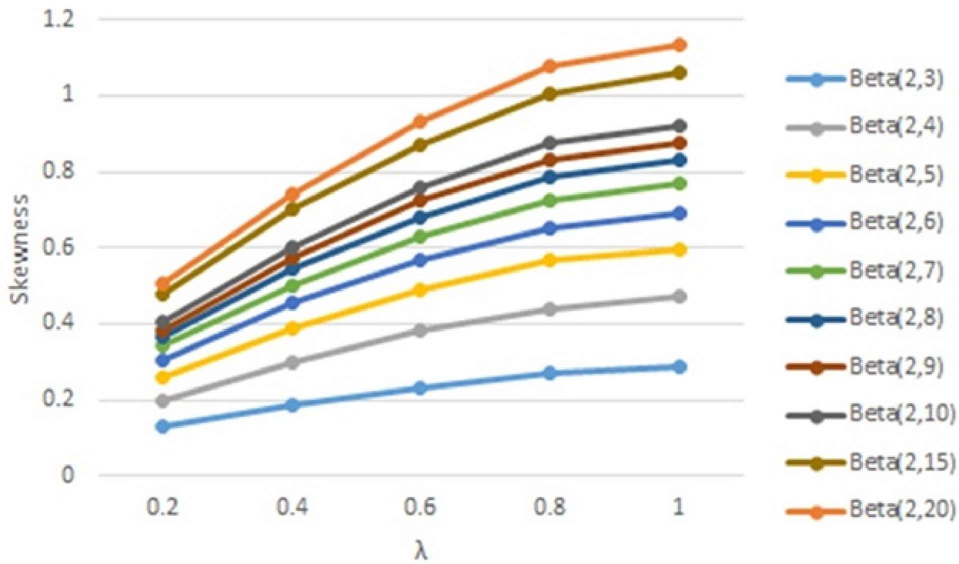

Figure 4. The effect of the parameter λ on the skewness of the beta distribution when $\beta > \alpha > 1$.

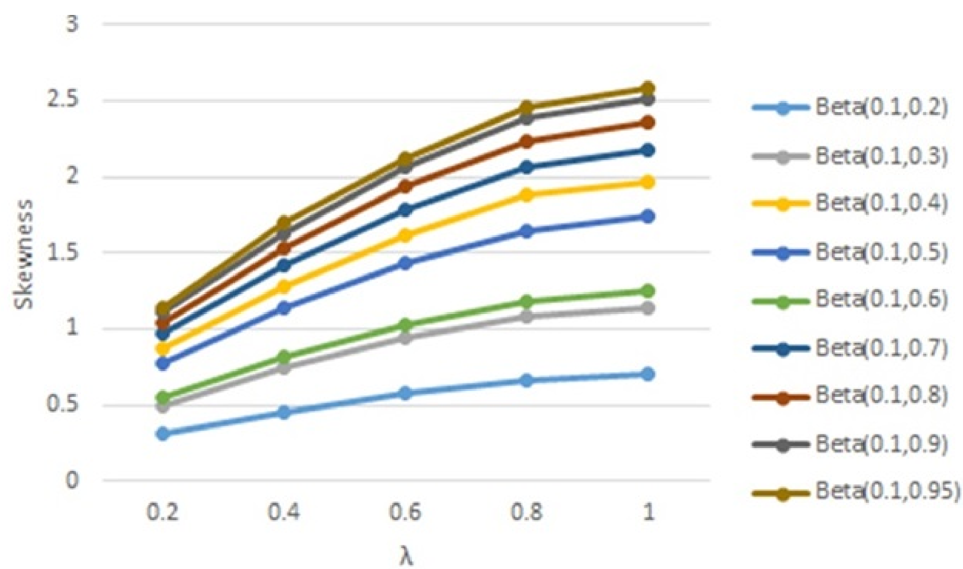

Figure 5. The effect of the parameter λ on the skewness of the beta distribution when $1 > \beta > \alpha$.

The data illustrated in Figures 9 through 14 elucidate the effects of applying the Exponentially Weighted Moving Average (EWMA) function on the kurtosis of various Beta distributions across differing values of λ, paralleling the analysis of skewness previously conducted. These figures sequentially represent distinct cases, with Figure 9 corresponding to case 1, Figure 10 to case 2, and so forth. Depending on the chosen parameters α and β, each case exhibits either negative or positive original kurtosis. A prevalent observation is the convergence of all distributions' kurtosis towards zero as λ decreases, indicative of a shift towards a normal-like kurtosis, or mesokurtosis. Further analysis reveals that, for a constant β, an increase in α results in a reduction of kurtosis, and conversely, a decrease in α leads to its augmentation. Thus, the EWMA process significantly affects not only the skewness, as previously

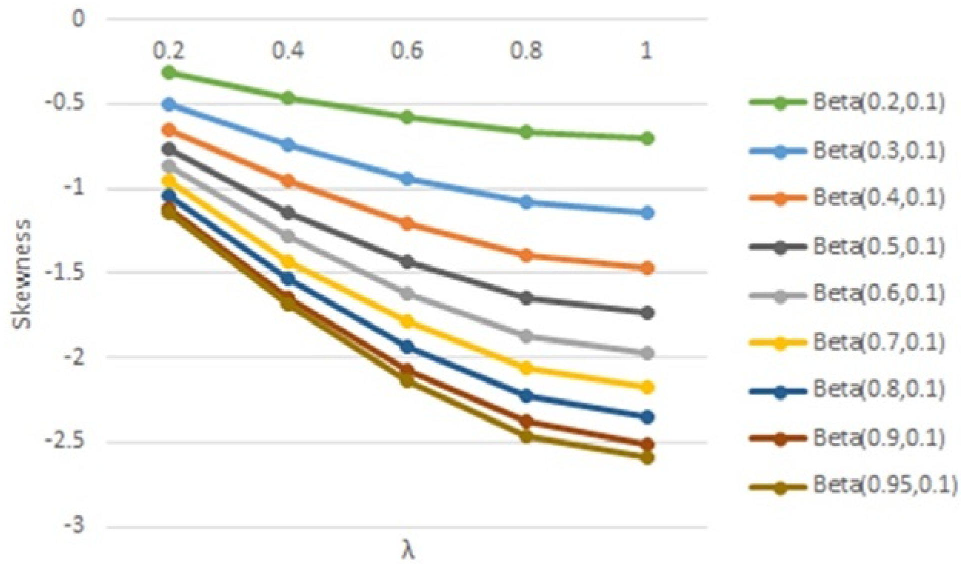

Figure 6. The effect of the parameter λ on the skewness of the beta distribution when $1 > \alpha > \beta$.

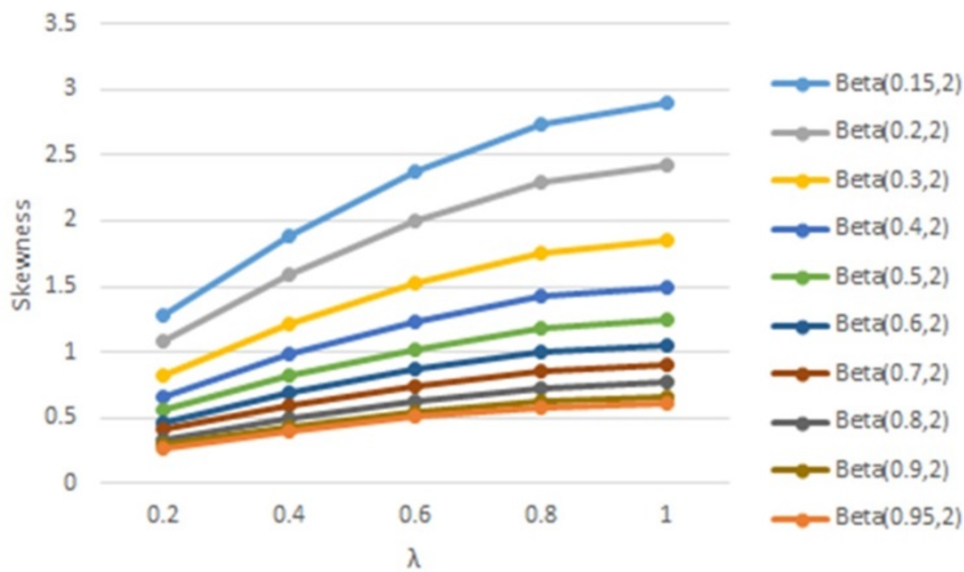

Figure 7. The effect of the parameter λ on the skewness of the beta distribution when $\alpha > 1 > \beta$.

demonstrated, but also markedly influences the kurtosis of the Beta distributions, altering their peakedness and the weight of their tails.

Figure 9 (Case 1: $\beta > \alpha > 1$) shows kurtosis decreasing as λ decreases from 1. All curves start with positive kurtosis (leptokurtic) and approach mesokurtic (kurtosis closer to 0). Comparing curves such as Beta(3,2) and Beta(4,2), with a constant β factor, the higher α factor in Beta(4,2) results in lower kurtosis. Thus, an increase in α with constant β reduces the kurtosis under EWMA. Figure 10 (Case 2: $\beta < \alpha < 1$) also displays a decrease in kurtosis with decreasing λ. Here, the curves start near zero or with slight negative kurtosis and move towards zero. Comparing curves like Beta(2,3) and Beta(2,4), an increase in the β factor results in a decrease in kurtosis

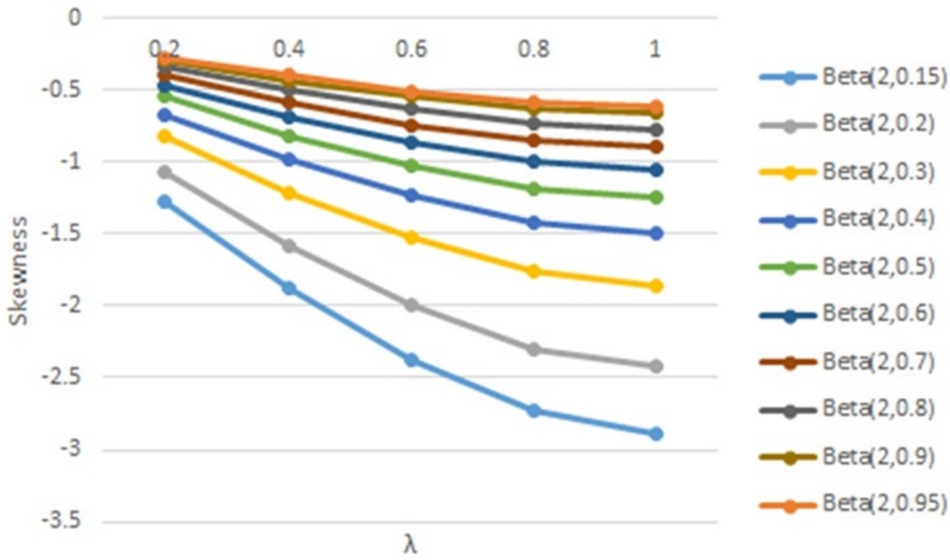

Figure 8. The effect of the parameter λ on the skewness of the beta distribution when $\beta > 1 > \alpha$.

when α is constant. Figure 11 ($\alpha < \beta < 1$) would indicate a similar trend to Figure 12, assuming this represents the corrected condition $\alpha < 1 < \beta$. The kurtosis increases for lower λ values, moving towards zero, suggesting that the distributions become less platykurtic. Curves with higher β factors at a constant α, such as Beta(0.1,0.7) vs. Beta(0.1,0.8), show greater kurtosis, indicating that higher β values increase kurtosis under EWMA. Figure 12 ($\beta < 1 < \alpha$) displays an increase in kurtosis as λ decreases. The curves start with negative kurtosis and move towards zero. For constant α, increasing β, like comparing Beta(2,0.9) and Beta(2,0.8), results in higher kurtosis, showing that higher β factors raise kurtosis when α is held constant. Figure 13 ($\alpha < 1 < \beta$) shows a trend where kurtosis decreases with decreasing λ. The curves are initially positive and decrease towards zero, indicating that the distributions are becoming less leptokurtic. For curves with a constant α, such as Beta(0.1,0.8) and Beta(0.1,0.9), an increase in β results in lower kurtosis. Lastly, Figure 14 ($\beta < 1 < \alpha$) shows an increasing trend in kurtosis as λ decreases, with curves starting from high positive kurtosis values and remaining positive but moving closer to zero. Comparing Beta(2,0.3) with Beta(2,0.4), where α is constant, the kurtosis is lower for the higher β factor, indicating that an increase in β reduces kurtosis.

In summary, across these figures, the general trend is that the kurtosis of all distributions approaches zero as λ decreases, moving the distributions towards a normal-like kurtosis (mesokurtic). Comparing curves with a constant β factor, an increase in the α factor leads to lower kurtosis, and vice versa. The EWMA process thus not only impacts the skewness, as seen in the previous figures but also significantly influences the kurtosis of the Beta distributions, affecting their peakedness and tail heaviness.

This study has major implications for real optimization problems, especially in process control and quality assurance. The Exponentially Weighted Moving Average (EWMA) function can optimize control chart settings in manufacturing, healthcare, and finance by reducing skewness and normalizing data distributions. In manufacturing, the results can be used to fine-tune control chart settings to monitor production lines more accurately, minimizing waste and improving product quality. EWMA normalizes skewed data in patient monitoring systems to improve early warning indications, resulting in faster interventions and better patient outcomes. These discoveries may also improve asset price monitoring in financial markets, enabling more precise risk management and decision-making. This study shows how adjusting the smoothing factor λ can normalize a distribution, enabling practitioners to alter control charts dynamically based on data properties. Real-world data distributions are often non-normal

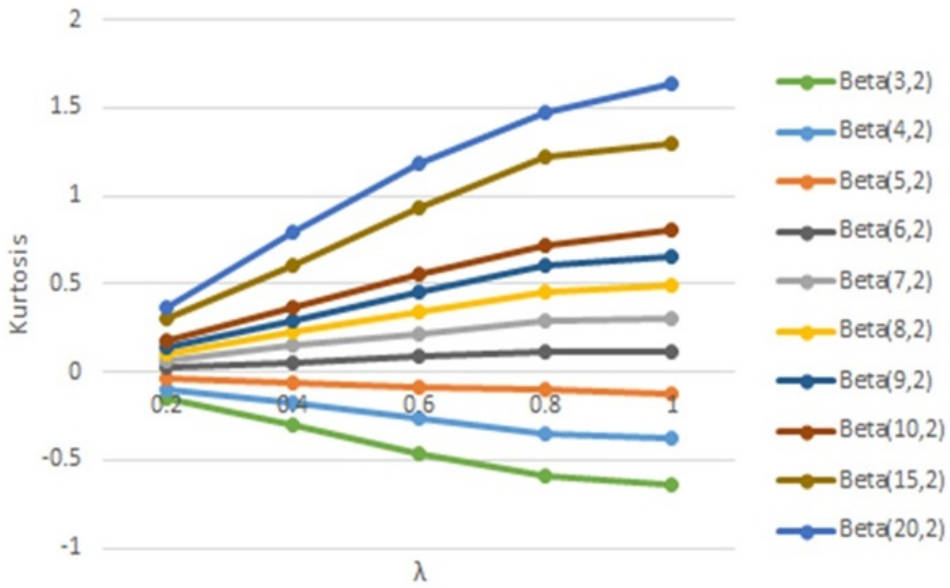

Figure 9. The effect of the parameter λ on the kurtosis of the beta distribution when $\alpha > \beta > 1$.

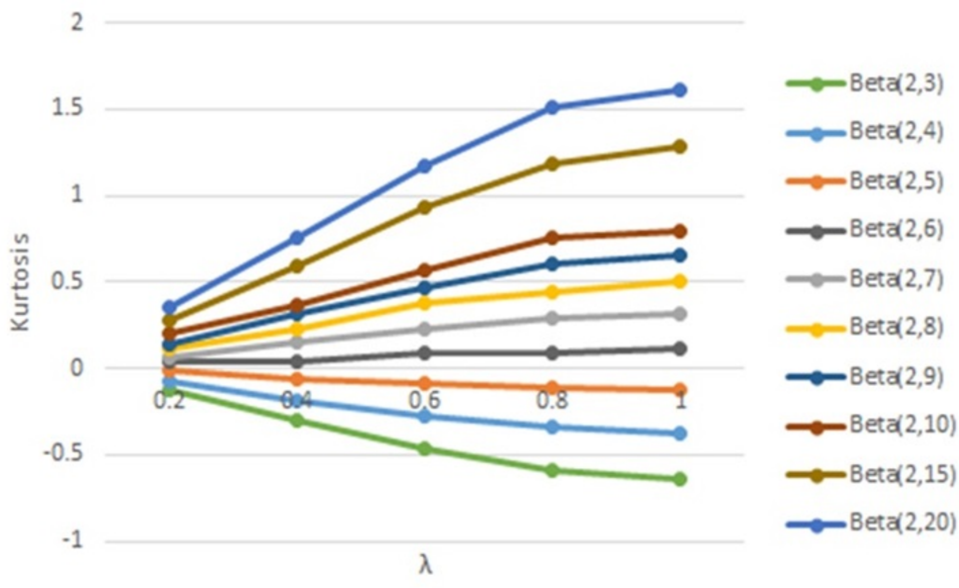

Figure 10. The effect of the parameter λ on the kurtosis of the beta distribution when $\beta > \alpha > 1$.

and change often, making adaptation essential. This research advances EWMA in statistical process control and provides practical solutions for process stability and optimization in various applications.

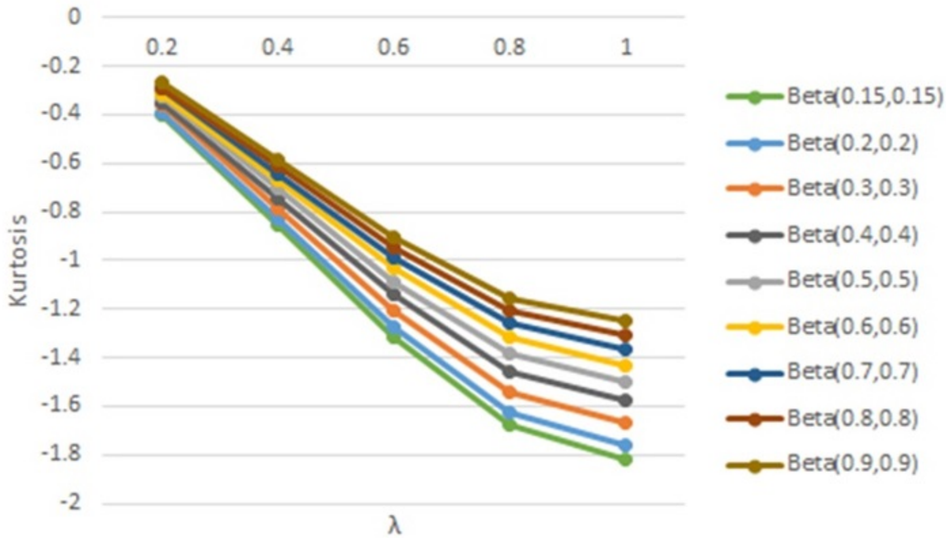

Figure 11. The effect of the parameter λ on the kurtosis of the beta distribution when $1 > \beta > \alpha$.

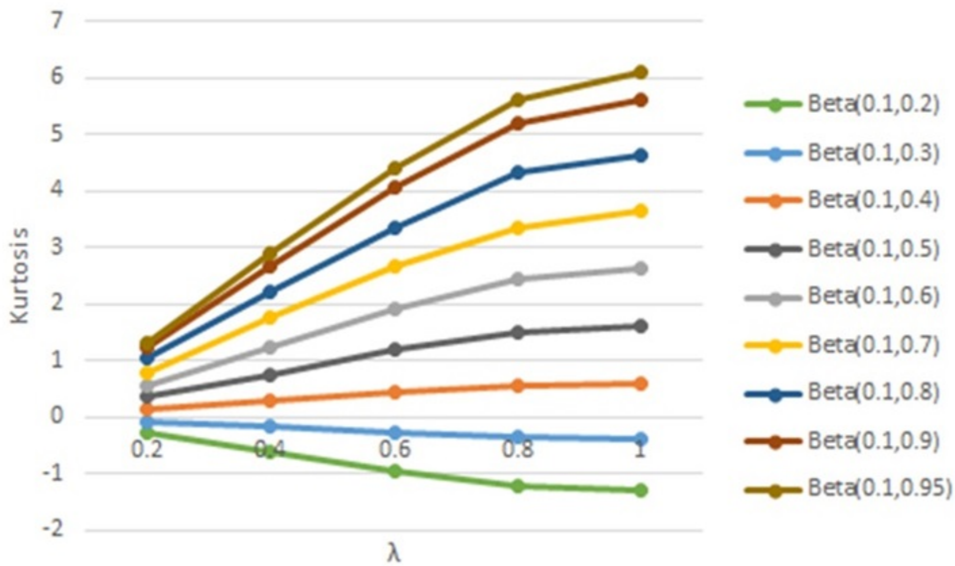

Figure 12. The effect of the parameter λ on the kurtosis of the beta distribution when $1 > \alpha > \beta$.

4. Conclusion

In conclusion, the comprehensive empirical analysis conducted in this research manuscript on "Exploring the Symmetry Shift Phenomenon in EWMA Statistics Generated from Beta Data Entry" offers significant insights into the impact of the Exponential Weighted Moving Average (EWMA) smoothing factor on the reduction of skewness and normalization of data distributions. Through meticulous experimentation and analysis, the study illustrates the capability of the EWMA function to significantly mitigate asymmetry and skewness in data derived from beta distributions, which are emblematic of a wide range of real-world scenarios. The findings underscore

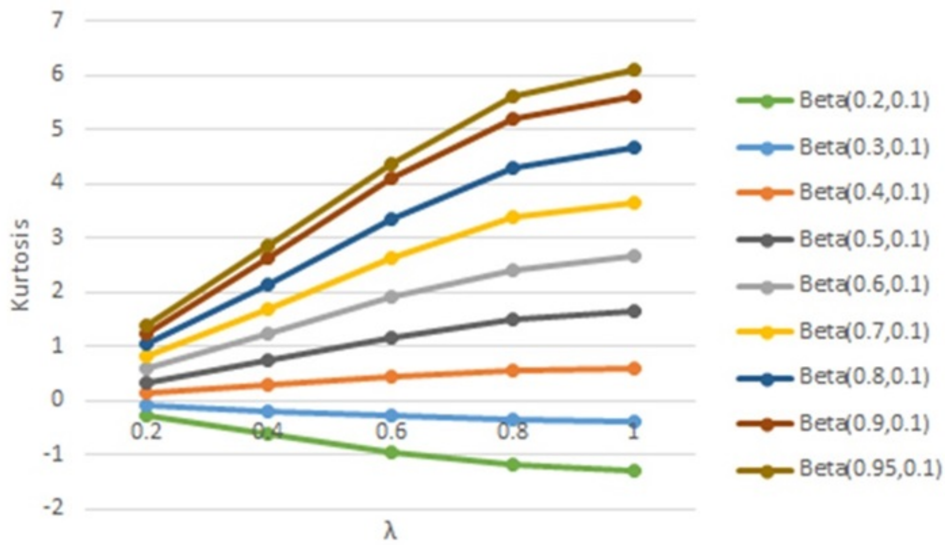

Figure 13. The effect of the parameter λ on the kurtosis of the beta distribution when $\alpha > 1 > \beta$.

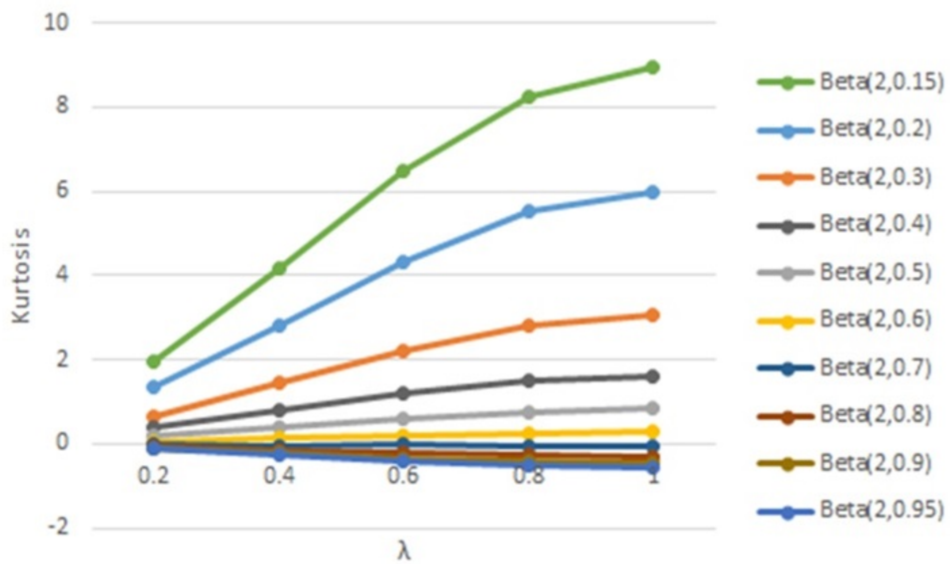

Figure 14. The effect of the parameter λ on the kurtosis of the beta distribution when $\beta > 1 > \alpha$.

the importance of selecting appropriate λ values to achieve the desired level of symmetry in quality control charts, particularly in contexts where the distribution of process data is non-normal. The research demonstrates that as λ decreases, there is a marked transition towards a more symmetrical and normal-like distribution, showcasing the EWMA method's efficacy in normalizing distributions with varying degrees of skewness and kurtosis.

Moreover, this study contributes to the body of knowledge by highlighting the dynamic interplay between the smoothing factor λ and the beta distribution parameters α and β. It provides a nuanced understanding of how different combinations of these parameters influence the resulting distribution's skewness and kurtosis, thereby offering a pathway to more precise and effective application of EWMA control charts in statistical process control.

The practical implications of this research are profound for quality engineers and professionals in fields where accurate monitoring and control of process measures are crucial to operational excellence. By offering a deeper understanding of how to manipulate EWMA parameters to counteract the effects of data skewness and achieve more stable and predictable process behaviors, this study lays the foundation for improved optimization and quality assurance across various industries. In light of these findings, future research could explore the application of the findings in more complex or multi-dimensional data scenarios, potentially extending the utility of EWMA control charts beyond the current realms. Additionally, further investigation into the optimization of λ values based on specific process requirements or industry standards could provide even more targeted guidelines for practitioners.

Overall, the research significantly advances our understanding of the EWMA control chart's functionality and its critical role in managing process variability, especially in the presence of non-normal data distributions. The insights gleaned from this study not only validate the theoretical underpinnings of EWMA but also underscore its practical relevance and adaptability to a broad spectrum of statistical process control applications.

In future work, our aim is to extend our analysis by incorporating models where Z_i is explicitly beta-distributed and by introducing beta-distributed white noise into the EWMA equation. These enhancements will allow us to assess the robustness of EWMA control charts in more complex and realistic scenarios.

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