STATISTICS, OPTIMIZATION AND INFORMATION COMPUTING
Stat., Optim. Inf. Comput., Vol. **12**, November 2024, pp 1710–1724.
Published online in International Academic Press (www.IAPress.org)

A new family of continuous distributions with applications

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Abstract This article introduces a novel set of optimizing probability distributions known as the Survival Power-G (SP-G) family, which employs a specific approach to introduce an additional parameter with the survival function of the original distributions. The utilization of this family enhances the modeling capabilities of diverse existing continuous distributions. By applying this approach to the single-parameter exponential distribution, a new two-parameter Survival Power-Exponential (SP-E) distribution is generated. The statistical characteristics of this fresh distribution and the maximum likelihood estimator are established, and Monte Carlo simulation is utilized to explore the efficiency of the maximum likelihood estimator of the two parameters under varying sample sizes. Subsequently, the new distribution is used in the analysis of three distinct sets of real data. Compared with alternative distributions on these datasets, it is demonstrated that the new distribution outperforms the other distributions.

Keywords Maximum likelihood estimation, Exponential distribution, Survival function, goodness-of-fit criteria, Monte Carlo simulation.

AMS 2010 subject classifications 60E05, 62E15, 62F10.

DOI: 10.19139/soic-2310-5070-2144

1. Introduction

In the realm of probability distribution theory, the inclination towards a specific probability distribution for representing real-world phenomena might stem from the distribution's tractability or its flexibility. The ease of handling a probability distribution, known as tractability, holds theoretical value as it simplifies operations, particularly in simulating random samples. However, stakeholders and practitioners often find the adaptability of probability distributions intriguing. Opting for probability distributions that align best with the provided data set is more favorable than altering the original data, preserving its integrity. Consequently, recent years have witnessed considerable endeavors to enhance the flexibility and modelling capacity of standard theoretical distributions, ensuring a better fit for real-world data sets. Adding an extra parameter to a probability distribution could improve its flexibility in modelling intricate phenomena. By employing more customized distributions, scholars can adequately accommodate variations in data analysis techniques. Furthermore, the addition of an extra parameter could potentially improve the accuracy of the distribution and result in enhanced predictive capabilities. The incorporation of a new parameter within established categories of continuous distributions has yielded a multitude of unique distribution families in recent years. A technique proposed by A. Mahdavi and D. Kundu (2016) marked the initial introduction of the Alpha Power method for generating innovative distributions [1]. Yilmaz (2017) put forth a novel family of distributions through the incorporation of a modified exponentiated component into the multiplication of two distribution functions, concentrating on the exponential baseline distributions and their characteristics [2]. Nanvapisheh (2019) brought forward the exponentiated Uniform-Pareto distribution (EU-PD), a model with five parameters applicable in economics and biological sciences, to assess its efficacy

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ISSN 2310-5070 (online) ISSN 2311-004X (print)

using actual data [3]. Nanvapisheh et al. (2019) recommended the exponentiated Shanker distribution, a two-parameter lifetime distribution featuring various hazard rate functions, and validated its appropriateness for real datasets through maximum likelihood estimation and simulation studies [4]. Aslam et al. (2020) devised a mechanism for novel continuous lifetime distributions by introducing an additional parameter, demonstrating the model's efficacy through applications to actual datasets and Monte Carlo simulations [5]. X. Xie and J. Shi (2020) introduced a distributed algorithm for the estimation of quantiles in heavy-tailed distributions with extensive datasets [6]. N.J. Hassan et al. (2021) suggested the Weibull Lindley model using a methodology that introduces a new shape parameter to the Weibull distribution [7]. Turner (2021) introduced a new adaptable distribution for discrete data, examining moment estimation and maximum likelihood estimation, and validating the model through real and simulated data [8]. Klakattawi and Khormi (2022) deliberated on the exponentiation of generalized models through the addition of three supplementary parameters, resulting in the exponentiated generaling inverted Kumaraswamy Gompertz distribution, which notably enhances goodness of fit compared to non-exponentiated distributions [9]. Alsolami and Alsulami (2022) introduced the exponentiated Weibull-X (EEW-X) family, particularly the EEWE distribution, which has exhibited exceptional flexibility in fitting intricate data through maximum likelihood estimation [10]. Ünözkan and Yilmaz (2023) introduced a fresh distribution by evaluating the conditional diagonal segment of the bivariate F-GM distribution with exponential marginals, showcasing its effectiveness in modelling lifetime datasets [11]. Amani Alghamdi and Aisha Fouad Fayomi (2023) introduced a new range of distributions utilizing the Dagum distribution as a generator, focusing on characteristics such as hazard rate functions, moments, and Renyi entropy, and applying it to actual datasets to demonstrate its performance [12]. Ibrahim et al. (2023) introduced the Truncated Cauchy Power Kumaraswamy-G family of probability distributions, and certain specialized models within this family have been delineated [13]. Fayomi et al. (2023) unveiled a group of BFGMLG utilizing the Lomax generator and the FGM copula [14]. Jimmy Reyes and Yuri (2023) along with Iriarte (year not provided) unveiled a fresh range of symmetric distributions of heavy-tailed from the ratio of a normal and a Birnbaum-Saunders distribution, capable of capturing high levels of kurtosis and effectively fitting actual datasets [15]. Chamunorwa et al. (2014) developed a new family of distributions called the Marshall-Olkin-Topp-Leone-Gompertz-G family [16]. Mehran et al. (2024) introduced the distribution family (GKM-G) by extending the K-M class to offer flexibility for continuous distributions [17].Oluyede et al. (2014) presented the (MO-TL-Gom-G) Family as the generalized family for creating continuous distributions by adding new two-shape parameters [18].

This research is centred on further enhancing the flexibility of the continuous probability distribution through the introduction of an additional parameter that amalgamates it with the survival function as a Power. The set of distributions presented in this paper is denoted as the survival Power family (SP-G). Specifically, the examination of this family is conducted using the exponential distribution, resulting in the derivation of a novel two-parameter distribution known as survival Power exponential (SP-E) distribution, accompanied by an investigation into the general characteristics of the functions associated with this recent distribution. The resultant Survival Power-Exponential (SP-E) distribution displays many advantageous properties. The manuscript also explores the process of estimating maximum likelihood parameters [19]. Additionally, an evaluation of the new distribution utilizing three real datasets is tested, alongside comparative analyses with similar distributions based on the goodness-of-fit criteria of these datasets, to demonstrate the effectiveness and superiority of the new distribution.

This research is structured as follows. An advantageous extension for the CDF and PDF of the SP-G family is delineated in Section 2. Section 3 introduces the Survival Power-Exponential (SP-E) distribution as a distinctive instance of the SP-G family. Section 4 delves into the statistical characteristics of the SP-E distribution, encompassing the quantile function, moments, and the maximum likelihood approach. In Section 5, The simulation study is applied to assess the maximum likelihood estimators by determining the mean squared error (MSE) and bias for two parameters. Section 6 offers insights into the application of the proposed family through the analysis of real-world data sets.

2. The proposed SP-G family method

This section presents the origin of the suggested approach. In a recent study, the T-X family method was introduced, characterized by the cumulative distribution function CDF.

$$F(x) = \int_{p_1}^{\omega[G(x;\theta)]} h(t)dt, \quad x \in \mathbb{R},\ t \in (p_1, p_2) \tag{1}$$

the conditions under which $\omega[G(x;\theta)\ ]$ is satisfied conditions approach as presented by Alzaatreh et al. [20]. The probability density function $f(x)$ associated with Equation (1) is defined as.

$$f(x) = \left\{ \frac{\partial}{\partial x} \omega[G(x;\theta)] \right\} h\left(\omega[G(x;\theta)]\right), \quad x \in \mathbb{R} \tag{2}$$

Utilizing the T-X methodology, numerous novel families of statistical models have been introduced in scholarly works. If we consider T to follow an exponential distribution with the rate parameter 1, we have the following CDF.

$$H(t) = 1 - e^{-t}, \quad t \ \geq \ 0 \tag{3}$$

In correspondence with Equation (1), the density function of the random variable t is determined.

$$h(t) = e^{-t}, \quad t \ > \ 0 \tag{4}$$

We define the function $\omega[G(x;\theta)]$ as follows.

$$\omega[G(x;\theta)] = -\log\left(-\alpha^{S(x;\theta)} + \alpha S(x;\theta) + 1\right) \tag{5}$$

where $\alpha > 0$ is shape parameter and $S(x;\theta) = 1 - G(x;\theta)$ is the survival function of the baseline distribution function $G(x;\theta)$. To show the function in Equation (5) complies with the methodology of the T-X family approach as presented by Alzaatreh et al. [20], let T be a random variable having PDF of the exponential distribution, that belongs to $(0,\infty)$, and let suppose, X be a random variable having CDF $G(x;\theta)$, then function $\omega[G(x;\theta)]$ in Equation (5) satisfy three conditions of the T-X family method, as under.

1. $\omega[G(x;\theta)] \in (0,\infty)$.
   It is clear because $S(x;\theta) \in (0,1) \Rightarrow \left(-\alpha^{S(x;\theta)} + \alpha S(x;\theta) + 1\right) \in (0,1)$
   and then $-\log\left(-\alpha^{S(x;\theta)} + \alpha S(x;\theta) + 1\right) \in (0,\infty)$.

2. $\omega[G(x;\theta)]$ is differentiable and monotonically increasing function.
   Note that the function $S(x;\theta)$ is differentiable, so $\omega[G(x;\theta)]$ is a differentiable, and also monotonically decreasing function.

3. $x \to -\infty \Rightarrow S(x;\theta) \to 1 \Rightarrow \omega[G(x;\theta)] \to 0$
   and $x \to \infty \Rightarrow S(x;\theta) \to 0 \Rightarrow \omega[G(x;\theta)] \to \infty$

If $h(t)$ follows Equation (4) and setting Equation (5) in Equation (1), then the CDF of the survival power-G (SP-G) family follows as

$$\begin{aligned} F(x;\alpha,\theta) &= 1 - e^{-\omega[G(x;\theta)]} \\ &= 1 - e^{\log\left(-\alpha^{S(x;\theta)} + \alpha S(x;\theta) + 1\right)} \\ &= \alpha^{S(x;\theta)} - \alpha S(x;\theta) \\ &= \alpha^{1-G(x;\theta)} - \alpha\left(1 - G(x;\theta)\right), \quad \alpha > 0, x \in \mathbb{R} \end{aligned} \tag{6}$$

where $G(x;\theta)$ is the CDF of the baseline distribution function which may have the vector of parameters $\theta \in \mathbb{R}$. From Equation (6), we can see if $\alpha = 1$ then $F(x;\alpha,\theta) = G(x;\theta)$.
And when $g(x;\theta)$ is the PDF of baseline distribution, thin the PDF of the SP-G family method associated with Equation (6) is

$$f(x;\alpha,\theta) = g(x;\theta)\left(\alpha - \alpha^{1-G(x;\theta)}\log(\alpha)\right) \tag{7}$$

2.1. *Quantile function of SP-G family*

We employ the cumulative distribution function CDF of SP-G family distributions for the purpose of obtaining the quantile function through the resolution of a non-linear equation.

$$u = \alpha^{1-G(x;\theta)} - \alpha\left(1 - G(x;\theta)\right) \tag{8}$$

Solving for $G(x;\theta)$, then

$$G(x;\theta) = \frac{W\left(-\log(\alpha)\,\alpha^{\frac{-\alpha-u}{\alpha}}\right)\alpha + \log(\alpha)\,(\alpha+u)}{\log(\alpha)\,\alpha} \tag{9}$$

where W is the LambertW function and $0 < u < 1$, the quantile function is

$$\mathrm{Q}(u) = \mathrm{G}^{-1}\left\{\frac{W\left(-\log(\alpha)\,\alpha^{\frac{-\alpha-u}{\alpha}}\right)\alpha + \log(\alpha)\,(\alpha+u)}{\log(\alpha)\,\alpha}\right\} \tag{10}$$

The quantile function serves as a tool for producing random variables to represent the parameters of a specific model, particularly applied in Monte Carlo simulations. Moreover, the quantile function is fundamental in establishing the Median, skewness and kurtosis.

3. The Survival Power-Exponential (SP-E) distribution

Consider the CDF $G(x;\theta) = 1 - e^{-\frac{x}{\beta}}, x \geq 0, \beta > 0$, and PDF $g(x;\theta) = \frac{1}{\beta}e^{-\frac{x}{\beta}}$, the vector of parameters θ is equal to only one-parameter β of the Exponential distribution. Then, the CDF of the new Survival Power-Exponential (SP-E) distribution is defined in the following

$$F(x;\alpha,\beta) = \alpha^{e^{-\frac{x}{\beta}}} - \alpha\, e^{-\frac{x}{\beta}}, \quad x \geq 0, \quad \alpha,\beta > 0 \tag{11}$$

The PDF of the Survival Power-Exponential (SP-E) distribution is

$$f(x;\alpha,\beta) = \frac{e^{-\frac{x}{\beta}}\left(\alpha - \log(\alpha)\,\alpha^{e^{-\frac{x}{\beta}}}\right)}{\beta}, \quad x > 0 \tag{12}$$

Plots for the CDF and PDF of the Survival Power-Exponential (SP-E) distribution are sketched in Figure 1 and Figure 2 respectively.

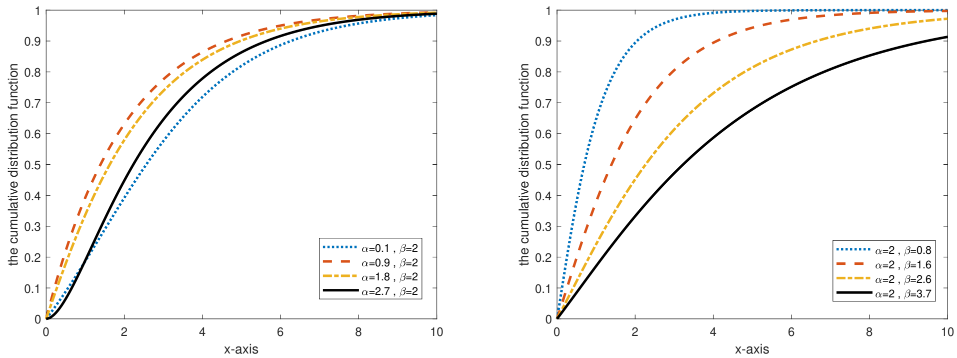

Figure 1. The Plots for the CDF of the SP-E distribution.

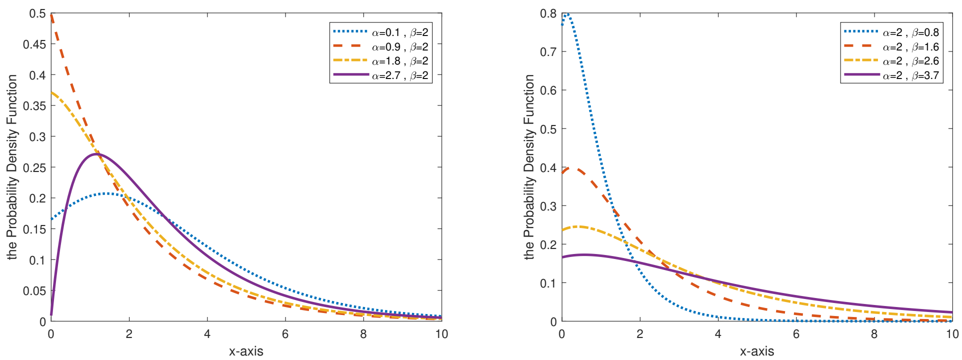

Figure 2. The Plots for the PDF of the SP-E distribution.

4. The Statistical properties of SP-E distribution

In this section, an examination is conducted on various statistical properties of the Survival Power-Exponential (SP-E) distribution, encompassing the quantile function, as well as the moment about the origin and the moment generating function.

4.1. Quantile function

Through the inverse of the function CDF in the Equation (11), we get the Quantile function of the Survival Power-Exponential (SP-E) distribution as following

$$Q_{SPE}(u) = F^{-1}(u) = -\log\left(-\frac{W\left(-\frac{\log(\alpha)e^{-\frac{u\log(\alpha)}{\alpha}}}{\alpha}\right)\alpha + u\log(\alpha)}{\alpha\log(\alpha)}\right)\beta \tag{13}$$

here $0 < u < 1$ and W is LambertW function. So we can use the Quantile Function to find the median of the Survival Power-Exponential (SP-E) distribution by substituting u=0.5 in the Equation (13). One of the initial measures of skewness that was proposed is the Bowley skewness [23], which is defined by as following

$$SK = \frac{Q_{SPE}\left(\frac{3}{4}\right) + Q_{SPE}\left(\frac{1}{4}\right) - 2Q_{SPE}\left(\frac{1}{2}\right)}{Q_{SPE}\left(\frac{3}{4}\right) - Q_{SPE}\left(\frac{1}{4}\right)} \tag{14}$$

and the kurtosis of the Moors calculated based on quantiles is expressed in the following [24]

$$KU = \frac{Q_{SPE}\left(\frac{7}{8}\right) - Q_{SPE}\left(\frac{5}{8}\right) + Q_{SPE}\left(\frac{3}{8}\right) - Q_{SPE}\left(\frac{1}{8}\right)}{Q_{SPE}\left(\frac{6}{8}\right) - Q_{SPE}\left(\frac{2}{8}\right)} \tag{15}$$

Where $Q_{SPE}(.)$ denotes the quantile function of the Survival Power-Exponential (SP-E) distribution is depicted. The metrics SK and KU exhibit reduced sensitivity to outliers and are applicable even in cases of distributions lacking moments.

4.2. Moments

This subsection focuses on the computation of the rth moment of the Survival Power-Exponential (SP-E) distribution. The derivation of the rth moment about the origin point is presented as follows.

$$\mu'_r = \int_0^\infty x^r f(x;\alpha,\beta)dx = \int_0^\infty \frac{x^r e^{-\frac{x}{\beta}}\left(\alpha - \log(\alpha)\,\alpha^{e^{-\frac{x}{\beta}}}\right)}{\beta}dx \tag{16}$$

$$\mu'_r = I = \frac{-\log(\alpha)}{\beta}\int_0^\infty x^r e^{-\frac{x}{\beta}}\alpha^{e^{-\frac{x}{\beta}}}dx + \frac{\alpha}{\beta}\int_0^\infty x^r e^{-\frac{x}{\beta}}dx \tag{17}$$

To solve the integral $I_1 = \int_0^\infty x^r e^{-\frac{x}{\beta}}\alpha^{e^{-\frac{x}{\beta}}}dx$
Substitution: Let $t = e^{-\frac{x}{\beta}}$. Then, $x = -b\log(t)$, and $dx = -\frac{\beta}{t}dt$.

$$I_1 = \int_1^0 (-\beta\log(t))^r t\alpha^t\left(-\frac{\beta}{t}\right)dt = \beta^{n+1}\int_0^1 (\log(t))^n\alpha^t dt \tag{18}$$

Note that the value of the last integral is
$\int_0^1 (\log(t))^r\alpha^t dt = \frac{(-1)^r}{(n+1)^2}\gamma(r+1,\log(\alpha))$
where $\gamma(s,z)$ is the lower incomplete gamma function.

$$I_1 = \beta^{r+1}\frac{(-1)^r}{(r+1)^2}\gamma(r+1,\log(\alpha)) \tag{19}$$

To solve the integral $I_2 = \int_0^\infty x^r e^{-\frac{x}{\beta}}dx$
Using the substitution $u = \frac{x}{\beta}$ (thus $du = \frac{dx}{\beta}$ and $dx = \beta du$):

$$I_2 = \int_0^\infty (\beta u)^r e^{-u}\beta du = b^{r+1}\int_0^\infty u^r e^{-u}du$$

This is a standard Gamma function integral, where

$$\int_0^\infty u^r e^{-u}du = \Gamma(r+1) = r!$$

Thus,

$$I_2 = \beta^{r+1}r! \tag{20}$$

Now, substituting I_1 and I_2 back into the original expression for I :

$$I = \frac{-\log(\alpha)}{\beta}\left(\beta^{r+1}\frac{(-1)^r}{(r+1)^2}\gamma(r+1,\log(\alpha))\right) + \alpha\beta^r r!$$

Where $\gamma(r+1,\log(\alpha))$ is the lower incomplete gamma function. Therefore, the moment is:

$$\mu_r' = \beta^r\left(\alpha r! - \log(\alpha)\frac{(-1)^r}{(r+1)^2}\gamma(r+1,\log(\alpha))\right) \tag{21}$$

4.3. Moment generating function

The moment generating function $M_X(t)$ of the Survival Power-Exponential (SP-E) random variable X, is can derived as

$$M_X(t) = E\left(e^{tx}\right) = \int_0^\infty e^{tx} f_{SPE}(x;\alpha,\beta)dx \tag{22}$$

Taking the Taylor series for the value e^{tx}, as following

$$e^{tx} = \sum_{r=0}^\infty \frac{t^r x^r}{r!} \tag{23}$$

By substituting Equation (23) into (22), then

$$M_X(t) = E(e^{tx}) = \sum_{r=0}^\infty \frac{t^r}{r!}\mu_r' \tag{24}$$

So we get the moment-generating function of the Survival Power-Exponential (SP-E) distribution by using Equation (21) in Equation (24).

4.4. Maximum likelihood estimation

Various approaches have been suggested to estimate the unidentified parameters for the purpose of deriving the predictions. The foremost and extensively utilized technique among these is the maximum likelihood estimation (MLE) method. The estimators acquired through this method exhibit advantageous characteristics and can be employed to formulate the confidence interval and various statistical assessments. The MLEs' normal approximation can be conveniently managed either through numerical computations or analytical techniques. Deeper insights into maximum likelihood estimation can be uncovered in [21]. In this particular section, we opt for the employment of the MLE method to estimate the two parameters of the Survival Power-Exponential (SP-E) distribution. Let us consider $X_1, X_2, \ldots, X_n$ as a randomly selected observed sample from the SP-E distribution. The likelihood function to Equation (2) is

$$L(\alpha,\beta;x_i) = \beta^{-n}\left(\prod_{i=1}^n \mathrm{e}^{-\frac{x_i}{\beta}}\right)\left(\prod_{i=1}^n\left(\alpha - \log(\alpha)\alpha^{e^{-\frac{x_i}{\beta}}}\right)\right). \tag{25}$$

The logarithm of the likelihood function is

$$\ell(\alpha,\beta;x_i) = -n\log(\beta) - \frac{\sum_{i=1}^n x_i}{\beta} + \left(\sum_{i=1}^n \log\left(\alpha - \log(\alpha)\alpha^{e^{-\frac{x_i}{\beta}}}\right)\right) \tag{26}$$

To find out the values of the parameters that make the Likelihood function a maximum value, the expressions for the partial derivatives of equation (26) are provided.

$$\frac{\partial \ell(\alpha,\beta;x_i)}{\partial \alpha} = \sum_{i=1}^{n} \frac{\alpha^{-1+e^{-\frac{x_i}{\beta}}} e^{-\frac{x_i}{\beta}} \log(\alpha) + \alpha^{-1+e^{-\frac{x_i}{\beta}}} - 1}{\log(\alpha)\alpha^{-\frac{x_i}{\beta}} - \alpha} \tag{27}$$

and

$$\frac{\partial \ell(\alpha,\beta;x_i)}{\partial \beta} = -\frac{n}{\beta} + \frac{1}{\beta^2}\sum_{i=1}^{n} x_i + \sum_{i=1}^{n}\left(-\frac{\log(\alpha)^2 \alpha^{e^{-\frac{x_i}{\beta}}} x_i e^{-\frac{x_i}{\beta}}}{\beta^2\left(\alpha - \log(\alpha)\alpha^{e^{-\frac{x_i}{\beta}}}\right)}\right) \tag{28}$$

By setting the nonlinear Equations (27) and (28) to zero and subsequently solving them concurrently, one can obtain the Maximum Likelihood Estimates MLEs to α and β. However, the solution to the machine learning equations is nontrivial, thus requiring the utilization of numerically optimized methods provided by computer software like R or Matlab.

Alternative Bayesian and non-Bayesian methodologies [22]. for estimating the parameters of the SP-E distribution can be extrapolated for prospective investigations, including the estimation via the method of moments, given that the moment concerning the origin has been established in the present study.

5. Simulation study

In this section, an assessment is carried out on the efficacy of the Maximum Likelihood Estimators to the two parameters of the Survival Power-Exponential (SP-E) distribution. The analysis focuses on the utilization of the new distribution as a specialized model for simulation. The simulation methodology entails a sequence of steps, as generated $N = 1000$ samples of sizes $n = 25, 50, 100, 250, 400, 600$ from the Survival Power-Exponential (SP-E) distribution with using the inversion method for four values sets of parameters $(\alpha = 1.5, \beta = 2)$, $(\alpha = 0.5, \beta = 0.5)$, $(\alpha = 1.5, \beta = 0.5)$, $(\alpha = 2, \beta = 3)$, $(\alpha = 0.5, \beta = 1.5)$ and $(\alpha = 3, \beta = 2)$, Compute mean square error (MSE) and average bias of the new distribution parameters with repeating for the sample size n, as the following

$$\begin{aligned} \text{MSE}(\alpha) &= \frac{1}{1000}\sum_{k=1}^{1000}(\hat{\alpha_k} - \alpha)^2 \text{ and } \text{MSE}(\beta) = \frac{1}{1000}\sum_{k=1}^{1000}\left(\hat{\beta_k} - \beta\right)^2 \\ \text{Bias}(\alpha) &= \frac{1}{1000}\sum_{k=1}^{1000}(\hat{\alpha_k} - \alpha) \text{ and } \text{Bias}(\beta) = \frac{1}{1000}\sum_{k=1}^{1000}\left(\hat{\beta_k} - \beta\right) \end{aligned} \tag{29}$$

The presented simulation outcomes can be observed in Table 1 with the Figure (3) of MSE and the Figure (4) of bias. The data displayed in this table and these figures demonstrate the favourable performance of the estimations of the Survival Power-Exponential (SP-E) distribution parameters, exhibiting minimal bias and credible Mean Squared Errors (MSEs) across all analyzed scenarios, thus signifying the reliability and proximity to the true values of these estimations. Additionally, the biases tend towards zero with the growth of sample size, indicating that the estimations function as asymptotically unbiased estimators. Furthermore, the MSEs exhibit a reduction with the increase in sample size, suggesting the consistency of these estimators in estimating the SP-E distribution parameters.

Table 1. Simulation Results for SP-E distribution: MSE and Bias

	$\alpha = 1.5, \beta = 2$				$\alpha = 0.5, \beta = 0.5$			
n	$MSE(\alpha)$	$Bias(\alpha)$	$MSE(\beta)$	$Bias(\beta)$	$MSE(\alpha)$	$Bias(\alpha)$	$MSE(\beta)$	$Bias(\beta)$
25	3.3638	0.5802	0.2653	-0.1722	0.5284	0.2080	0.0132	-0.0602
50	0.2707	0.0561	0.1089	-0.1067	0.4114	0.2177	0.0104	-0.0406
100	0.2429	0.1065	0.0556	-0.0676	0.2546	0.1461	0.0048	-0.0227
250	0.1510	-0.0471	0.0258	-0.0096	0.1053	0.0634	0.0027	-0.0084
400	0.1077	-0.0287	0.0179	-0.0055	0.1377	0.1151	0.0018	-0.0075
600	0.0767	-0.0249	0.0151	-0.0045	0.1003	0.0978	0.0014	0.0009
	$\alpha = 1.5, \beta = 0.5$				$\alpha = 2, \beta = 3$			
n	$MSE(\alpha)$	$Bias(\alpha)$	$MSE(\beta)$	$Bias(\beta)$	$MSE(\alpha)$	$Bias(\alpha)$	$MSE(\beta)$	$Bias(\beta)$
25	2.6551	0.4624	0.0157	-0.0391	3.0126	0.2452	0.5401	-0.1485
50	0.3053	0.0621	0.0071	-0.0111	0.9181	0.0160	0.2429	0.0449
100	0.1907	0.0035	0.0029	-0.0090	0.1423	-0.0259	0.1092	0.0063
250	0.1628	0.0045	0.0015	-0.0107	0.1013	0.0043	0.0530	-0.0098
400	0.1186	-0.0157	0.0013	-0.0028	0.0624	-0.01378	0.03155	-0.01925
600	0.0875	-0.0274	0.0008	-0.0001	0.0260	0.0089	0.02285	-0.0172
	$\alpha = 0.5, \beta = 1.5$				$\alpha = 3, \beta = 2$			
n	$MSE(\alpha)$	$Bias(\alpha)$	$MSE(\beta)$	$Bias(\beta)$	$MSE(\alpha)$	$Bias(\alpha)$	$MSE(\beta)$	$Bias(\beta)$
25	0.2269	-0.0004	0.1707	-0.1607	45.5867	6.5373	0.5580	-0.6538
50	0.3122	0.1782	0.0860	-0.1037	34.9175	5.1521	0.5614	-0.6681
100	0.1396	0.0819	0.0371	-0.0511	35.9211	5.3053	0.7434	-0.8008
250	0.1031	0.1032	0.0188	-0.0303	28.2474	4.3537	0.7013	-0.7766
400	0.0552	0.0456	0.0166	-0.0088	21.0201	3.4354	0.6066	-0.7054
600	0.0720	0.0738	0.0116	-0.0059	8.4658	1.8357	0.3588	-0.5457

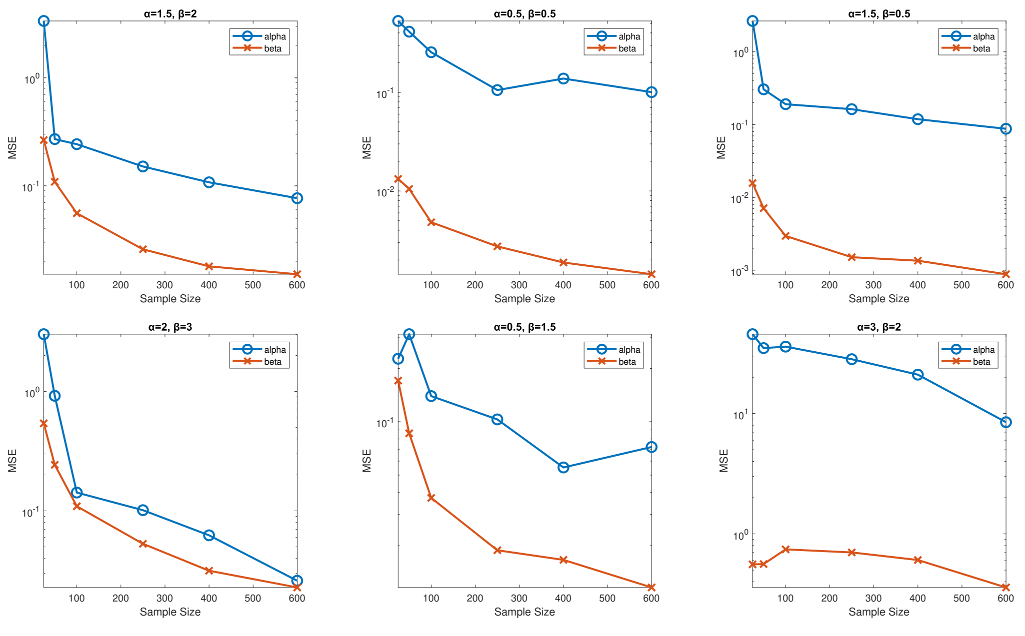

Figure 3. The Plots for the MSE of the SP-E distribution parameters.

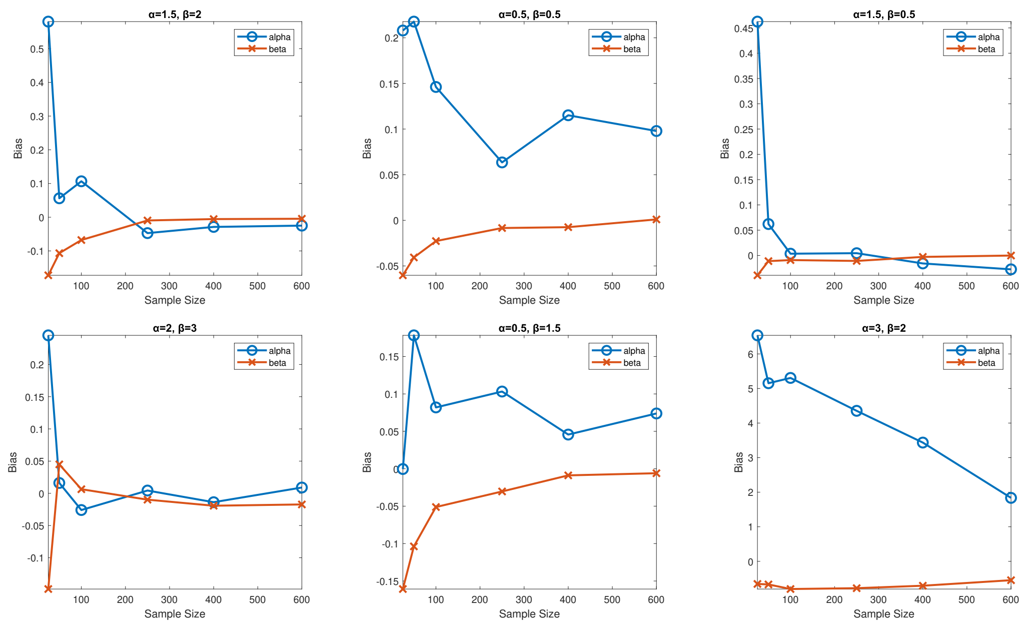

Figure 4. The Plots for the bias of the SP-E distribution parameters.

6. Applications SP-E distribution

In this portion, an examination and explication of three authentic datasets will be carried out to elucidate the advantages associated with the application of the Survival Power-Exponential (SP-E) distribution methodology. The assessment of the model's pertinence entailed the identification of various information criteria.Model selection is commonly performed by assessing a range of information criteria like the Akaike IC (AIC), the consistent Akaike IC (CAIC), the Bayesian IC (BIC), and the Hannan-Quinn IC (HQIC), as well as the Kolmogorov-Smirnov Criterion(K-S). It is essential to underscore that a decrease in the values of goodness-of-fit metrics indicates a more optimal fit of the data. Below we will show the real data to which the new distribution was applied. The profusion and variety of data could pose challenges for researchers in acquiring the most suitable data reflecting the new distribution. Consequently, during the data selection process, alternative distributions were chosen for comparison based on previous research pertaining to the data in question, rather than selecting the distributions themselves for comparison across the three datasets, which will be explained below.

the first dataset pertains to the occurrences of surpassing flood flow rates (measured in cubic meters per second) of the Wheaton River in proximity to Carcross within the Yukon Territory, Canada. There are 72 instances of surpassing measurements spanning from the years 1958 to 1984, the data were examined by various researchers such as Bourguignon et al. [25], S Chhetri et al. [26] and others. The selection of the identical dataset was made to facilitate a comparative analysis of our findings with the alternative models proposed by these and other researchers. Our analysis involves the estimation of parameters for the Survival Power-Exponential (SP-E) model and an evaluation of its suitability in modelling this dataset, along with other models comprising the beta-transmuting Pareto (BTP), Exponentiated Fréchet (EF), beta Pareto distribution (BP), Marshall-Olkin Fréchet (MOF), transmuted Pareto (TP), Beta Exponential Fréchet (BEF), exponentiated Pareto (EP) and Pareto distribution (P) distributions. The first data is given as "1.7, 2.2, 14.4, 1.1, 0.4, 20.6, 5.3, 0.7, 1.9, 13.0, 12.0, 9.3,1.4, 18.7, 8.5, 25.5, 11.6, 14.1, 22.1, 1.1, 2.5, 14.4, 1.7,37.6, 0.6, 2.2, 39.0, 0.3, 15.0, 11.0, 7.3, 22.9, 1.7, 0.1, 1.1, 0.6, 9.0, 1.7, 7.0, 20.1, 0.4, 2.8, 14.1, 9.9, 10.4, 10.7, 30.0, 3.6,5.6, 30.8, 13.3, 4.2, 25.5, 3.4, 11.9, 21.5, 27.6, 36.4, 2.7, 64.0,1.5, 2.5, 27.4, 1.0, 27.1, 20.2, 16.8,5.3, 9.7, 27.5, 2.5, 27.0". The results of the comparison using The goodness-of-fit statistics for the first data are shown in Table 2.

Table 2. The goodness-of-fit statistics for the first dataset

Dis.	$-\ell(.,x)$	AIC	CAIC	BIC	HQIC	KS
SP-E	252.127	508.255	514.809	512.809	510.068	0.142
BTP	256.577	521.154	521.760	530.204	524.753	0.159
EF	255.803	517.606	517.959	524.436	520.325	0.158
BP	283.700	573.400	573.753	580.230	576.119	0.174
MOF	256.605	519.211	519.563	526.041	521.930	0.121
TP	286.201	576.402	576.575	580.954	578.214	0.287
BEF	255.222	521.353	520.444	531.827	524.976	0.976
EP	287.300	578.600	578.774	583.153	580.413	0.198
P	303.100	608.200	608.257	610.477	609.106	0.332

The second dataset comprises 34 instances of vinyl chloride concentrations gathered from remediation gradient groundwater monitoring wells in milligrams per liter. This dataset was gathered from the analysis done by Bhaumik et al. [27]. Our analysis involves the estimation of parameters for the SP-E distribution and an evaluation of its suitability in modelling this dataset, along with the other distributions as comprising Exponentiated generalized modified Weibull distribution (EGMW), inverse Weibull Weibull distribution (IWW), Beta modified Weibull distribution (BMW), Weibull distribution (W), Kumaraswamy Lomax distribution (KL), Weibull Weibull distribution (WW) and Weibull Lomax distribution (WL), these distributions were also selected by Hassan and Abd-Allah [28], and outlined in the Table 3. The second data is given as "5.1, 1.2, 1.3, 0.6, 0.5, 2.4, 0.5, 1.1, 8.0, 0.8, 0.4, 0.6, 0.9, 0.4, 2.0, 0.5, 5.3, 3.2, 2.7, 2.9, 2.5, 2.3, 1.0, 0.2, 0.1, 0.1, 1.8, 0.9, 2.0, 4.0, 6.8, 1.2, 0.4,0.2".

Table 3. The goodness-of-fit statistics for the second dataset

Dis.	$-\ell(.,x)$	AIC	CAIC	BIC	HQIC	KS
SP-E	55.4520	114.905	119.957	117.957	115.946	0.089
EWL	54.3210	118.642	120.642	126.274	121.245	0.106
EGMW	55.4020	120.804	122.949	128.435	123.406	0.121
IWW	54.2351	116.470	117.850	122.576	118.552	0.087
BMW	55.0955	120.191	122.333	127.822	122.793	0.233
W	58.7322	121.253	121.640	124.306	122.294	0.113
KL	64.7635	137.527	138.817	143.632	139.609	0.145
WW	55.5531	119.160	120.539	125.265	121.242	0.094
WL	64.1625	136.325	137.615	142.43	138.407	0.279

The third dataset, presented in Murthy et al. [29], pertains to the time intervals between failures for repairable items. Our study focuses on estimating the parameters of the SP-E distribution and assessing its appropriateness for modelling this specific dataset. This involves considering alternative models such as the Kumaraswamy Weibull exponential (KwWE), the Additive Weibull distribution (AW), the transmuated power function (TPF), the New Modified Weibull distribution (NMW), the exponentiated Weibull exponential (EWE), the traditional Weibull distribution (W), the beta modified Weibull (BMW) and Weibull exponential (WE), these distributions were also selected by Hassan and Nassr [30] and outlined in the Table 3. The dataset is presented as follows, "1.43, 0.11, 0.71, 0.77, 2.63, 1.49, 3.46, 2.46, 0.59, 0.74, 1.23, 0.94, 4.36, 0.40, 1.74, 4.73, 2.23, 0.45, 0.70, 1.06, 1.46, 0.30, 1.82, 2.37, 0.63, 1.23, 1.24, 1.97, 1.86, 1.17". The results of the comparison using The goodness-of-fit statistics for the third data are shown in Table 4.

Table 4. The goodness-of-fit statistics for the third dataset

Dis.	$-\ell(.,x)$	AIC	CAIC	BIC	HQIC	KS
SP-E	43.0050	90.010	94.8131	92.813	90.907	0.184
KWE	53.4332	116.964	119.464	114.350	119.205	0.062
AW	79.8210	167.642	169.242	173.246	169.435	0.283
TPF	45.2279	101.458	102.381	105.662	102.80	0.139
NMW	121.250	250.051	251.651	255.656	251.845	0.942
EWE	56.6513	121.344	122.944	119.252	123.137	0.078
W	46.3755	96.751	97.196	99.554	97.648	0.134
BMW	43.2513	94.406	96.906	101.412	96.647	0.866
EW	55.9511	117.901	118.824	116.332	119.246	0.083

The three distinct data sets were subjected to analysis utilizing the novel SP-E distribution. We derived the mean and variance through the application of moment functions or the moment-generating function; additionally, utilizing the functions inherent in the quantile function, we ascertained the Median, Skewness, and Kurtosis, as shown in Table 5. The extent to which the SP-E distribution matches these three data sets is also shown in Figures 5, 6 and 7.

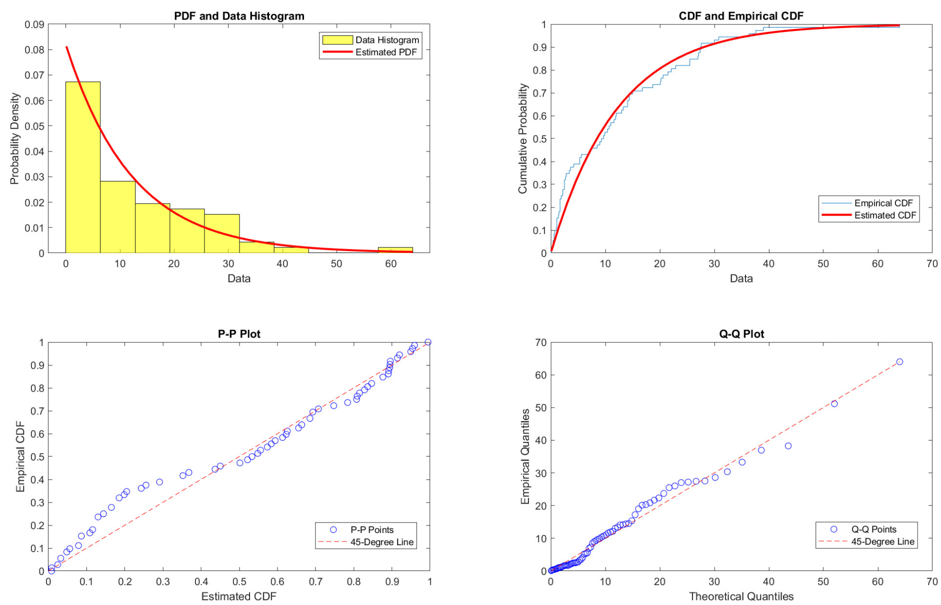

Figure 5. Test results of SP-E distribution for the first dataset

Table 5. Descriptive analyzes for the datasets

Data	Mean	Median	Variance	Skewness	Kurtosis	Min	Max
first	12.20416	9.50000	151.22153	1.472508	5.889549	0.1000	64.000
second	1.879412	1.150000	3.812594	1.603688	5.005408	0.1000	8.0000
third	1.542667	1.235000	1.271675	1.295462	4.319170	0.1100	4.7300

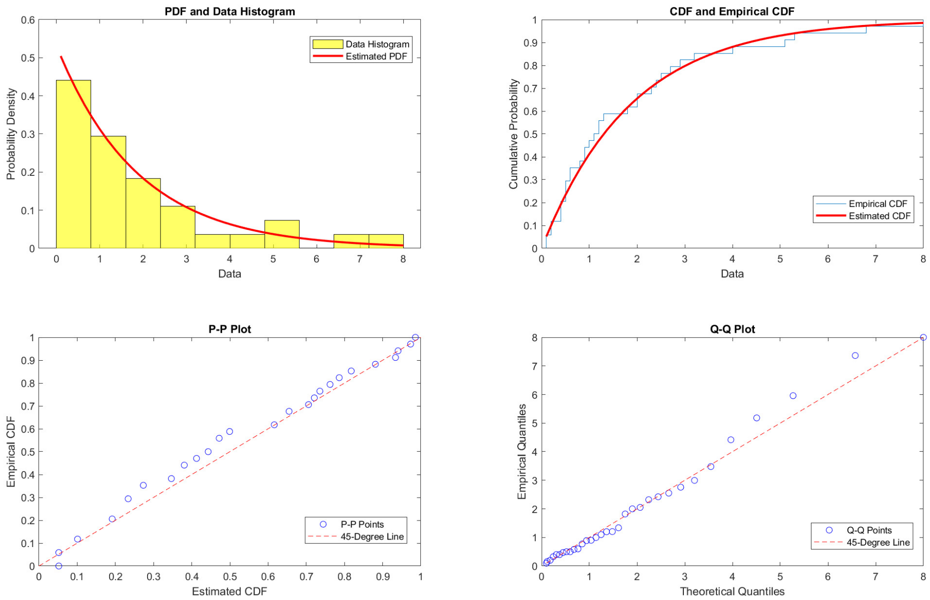

Figure 6. Test results of SP-E distribution for the second dataset

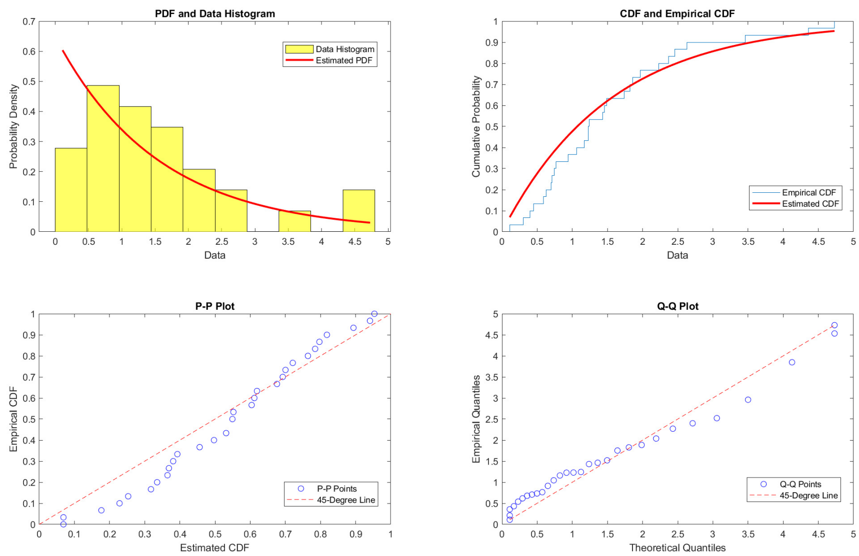

Figure 7. Test results of SP-E distribution for the third dataset

7. Conclusion

This manuscript introduces a novel Survival Power-G (SP-G) family, which involves the incorporation of a singular parameter representing the power function of the survival function in an innovative manner into established continuous distributions. The distinct model for the (SP-G) family were introduced through the selection of the exponential distribution, deemed suitable for the envisaged SP-G family. The SP-E distribution, characterized by two parameters, was formulated. Diverse some mathematical functions and attributes of the SP-E model were studied, with the Maximum Likelihood Estimation (MLE) being calculated for the two parameters, while Monte Carlo simulations were utilized to show the effectiveness of these estimators. The validity of this model was empirically substantiated using three authentic datasets, affirming its superior fitting for selected data when juxtaposed with alternative models.

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