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Adaptive Type-II Progressive Hybrid Censoring and Its Impact on Rayleigh Data Overlap Estimation

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Abstract This article employs an adaptive type-II progressive hybrid censoring scheme for estimating the overlap of two Rayleigh distributions with different scale parameters. The estimators for these overlap measures are obtained using this censoring method, and their asymptotic bias and variance are also presented. Furthermore, confidence intervals for these measures are constructed using both the bootstrap method and Taylor approximation. To emphasize the practical significance of our proposed estimators, we analyze real-life data focusing on the impact of mercaptopurine on maintaining remission in patients with acute leukemia.

Keywords Key Words: Bootstrap method; Overlap measures; adaptive type-II progressive hybrid censoring

AMS 2010 subject classifications 62E10, 62N01, 62N02, 62G30

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1. Introduction

Life-testing experiments pose challenges in controlling test duration and conserving experimental units while ensuring efficient estimation. Censoring techniques offer a solution by removing active units and stopping the experiment before all units fail. Progressive censoring is crucial, as it involves removing units at predetermined or random time points during the experiment, accounting for potential losses or removals.

Over the years, progressive censoring has been extensively studied, with models falling into two categories: progressive Type-I censoring, concluding the experiment at predefined times, and progressive Type-II censoring, ending after a predetermined number of failures. Both approaches provide flexibility by allowing unit removal at non-terminal times.

Progressive Type-I censoring involves fixed durations at a specific time, potentially resulting in few or no observed failures for units with long lifetimes. In contrast, progressive Type-II censoring, although flexible, may lead to extended test durations when units have extended lifetimes, which is considered a drawback.

Kundu and Joarder (2006) introduced two progressive hybrid censoring schemes, offering alternatives to traditional progressive Type-II censoring by ending experiments at a certain time T. These schemes adapt to the data, allowing fewer than m observations in Type-I hybrid censoring or extended testing in Type-II hybrid censoring.

During real-life experiments, it is imperative to acknowledge that a fixed censoring scheme may not always be a practical approach. Any intentional or unintentional alternation during the experiment can significantly impact the results. However, Ng et al. (2009) have introduced a new model (depicted in Figure (1) that allows the censoring

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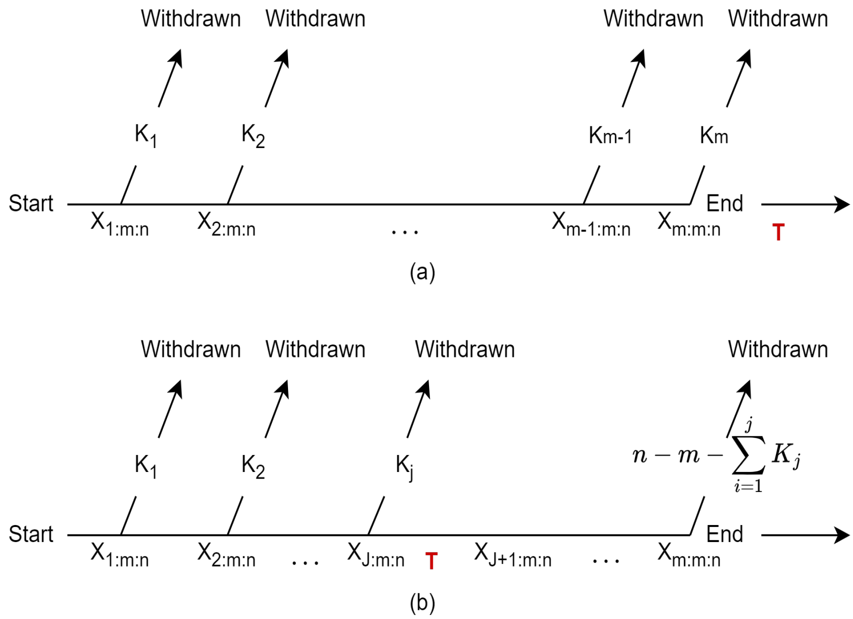

Figure 1. Adaptive type-II progressive hybrid censoring model as proposed by Ng et al. (2009). (a) Experiment ends before time T. (b) Experiment ends after time T.

scheme to be changed as required during the experiment. This model is called adaptive type-II progressive hybrid censoring (Adaptive-IIPH), in which a threshold time T switches between the original and modified schemes.

Assume there are n units in a life-testing experiment, and the effective sample size $m(< n)$ is predetermined, along with the censoring scheme $(K_1, K_2, \dots, K_m)$; however, the values of some of the K_i may change as the experiment progresses. Assuming the experimenter has provided an ideal total test time T. If the m-th failure occurs before time T (see Figure 1(a)), the experiment proceeds similarly to type-II progressive censoring. It halts at time X_m with the pre-fixed censoring scheme $(K_1, K_2, \dots, K_m)$. Otherwise, if the experimental time has passed T, but the number of observed failures has not yet reached m, we do not remove any items from the experiment by setting $K_{j+1} = K_{j+2} = \cdots = K_{m-1} = 0$ and $K_m = n - m - \sum_{i=1}^{j} K_i$. This setting can be seen as a design that guarantees m observed failure times while keeping the total test time not too far away from the ideal test time T (depicted in Figure 1(b)). Note that if we set $T = 0$, we will have a traditional type-II censoring method. However, if $T \to \infty$, the Adaptive-IIPH process becomes a progressive type-II censoring technique.

Adaptive-IIPH significantly impacts real-life applications, as evidenced by its widespread use in literature. Most recently, Alslman and Helu (2023) developed new methods for estimating the stress strength of the inverse Weibull distribution using the Adaptive-IIPH censoring scheme. Asadi et al. (2022) employed Adaptive-IIPH censoring to conduct accelerated life tests on virus-containing microdroplets, monitoring Virus-MD persistence during coughs at different time points. Alotaibi et al. (2022) utilized Adaptive-IIPH censoring for testing sodium sulfur battery lifetimes in a chemical application employing the XLindley distribution. Furthermore, Helu and Samawi (2021) applied Adaptive-IIPH censoring to radar-evaluated rainfall data from 52 cumulus clouds in South Florida, highlighting its versatile utility in various fields.

Estimating the proportion of machines or electronic devices with similar failure time ranges is crucial in reliability analysis, especially when dealing with different sources or stress levels. Various overlap coefficients (OVL), such as Matusia's measure ρ, Morisita's measure λ, and Weitzman's measure Δ, are utilized to achieve this. These coefficients represent the common area between two probability density functions. The depiction of

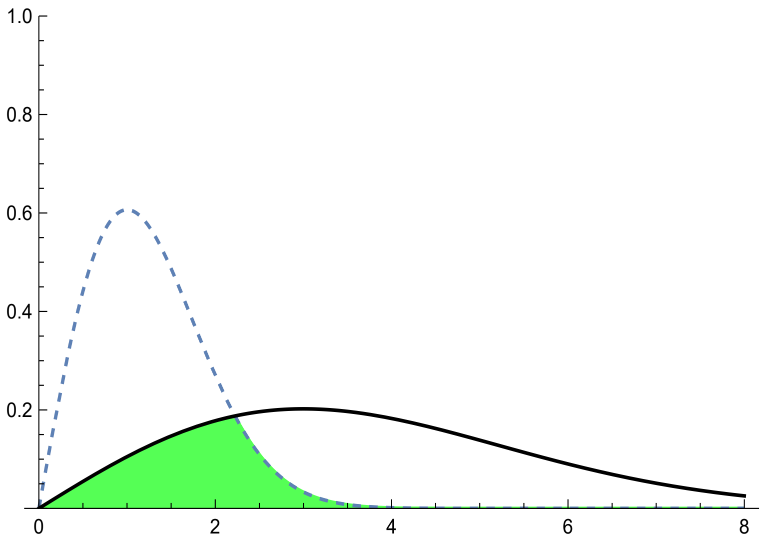

Figure 2. Overlap of two Rayleigh distributions.

OVL for two distributions in Figure 2 displays the natural interpretations of OVL as a fraction of probability mass under either density, represented by the shaded area in 2.

OVL has found widespread use in various practical applications as well. It has been utilized in quantitative ecology, as demonstrated by Gastwirth (1975). Furthermore, OVL has been applied to electromyographic assessment of muscular asymmetry by Ferrario et al. (2000) and in treatment assessment during clinical trials, as discussed by Mizuno et al. (2005).

For a deeper exploration of the various applications of overlap coefficients, interested readers can refer to the works of Wang and Tian (2017), Martinez-Camblor (2022), Helu (2024).

The mathematical form of the OVL measures are as follows: Suppose two samples of observations are drawn from two continuous distributions $f_1(x)$ and $f_2(x)$. Then the overlap measures are defined as follows:

$$\begin{aligned}
\text{Matusita's Measure (1955):} \quad & \rho = \int \sqrt{f_1(x)f_2(x)}\,dx, \\
\text{Morisita's Measure (1959):} \quad & \lambda = \frac{2\int f_1(x)f_2(x)\,dx}{\int [f_1(x)]^2\,dx + \int [f_2(x)]^2\,dx}, \\
\text{Weitzman's Measure (1970):} \quad & \Delta = \int \min(f_1(x), f_2(x))\,dx.
\end{aligned}$$

It is possible to adapt these measures for discrete distributions by using summations. They can also be extended to multivariate distributions. They are quantified on a scale from 0 to 1, with values near 0 indicating significant inequality (or disagreement) and 1 suggesting exact equality (perfect agreement) between density functions.

The mathematical structure of these measures is intricate, and there are no results available on the exact sampling distributions of their estimators. Prior work includes Smith (1982) on discrete Weitzman's measure, Mishra et al. (1986) on sampling properties under homogeneity assumptions, Mulekar and Mishra's (1994) simulations on normal densities, and Lu et al.'s (1989) study of sampling variability. Additionally, Dixon (1993) applied bootstrapping and jackknife techniques, while Mulekar and Mishra (2000) addressed inference problems.

The sampling behavior of a nonparametric estimator of OVL was analyzed by Helu and Samawi (2011). Samawi et al. (2017) conducted a study investigating the similarities and distinctions between the maximum of the Youden index (J) and overlap coefficient (OVL), highlighting the advantages of OVL over J.

In this article, our primary focus lies in making inferences regarding the measure of overlap (OVL) while utilizing Adaptive-IIPH censoring data from two independent Rayleigh distributions with different scale parameters.

Section 2 introduce the Rayleigh distribution and the derivation of the measures. Section 3 introduces the estimators, explores their approximate biases, and establishes confidence intervals via the delta method and bootstrap techniques. Transitioning to Section 4, we present the results of our simulations and engage in a comprehensive discussion. Section 5, on the other hand, spotlights a practical example using real data, culminating in our concluding remarks.

2. The model

The Rayleigh distribution is crucial in life testing experiments because its failure rate is a linear function of time, making it suitable for modeling the aging process of products. Dyer and Whisenand (1973) highlighted its importance in communication engineering, while Polovko (1968) noted its relevance in electrovacuum devices. Introduced by Rayleigh in 1880 and appearing as a special case of the Weibull distribution, the Rayleigh distribution is one of the most commonly used distributions for analyzing lifetime data. It plays a key role in various fields, including project effort loading modeling, survival and reliability analysis, communication theory, physical sciences, technology, diagnostic imaging, applied statistics, and clinical research. This distribution's origin and other aspects are detailed in Siddiqui (1962) and Miller and Sackrowitz (1967), further emphasizing its broad applicability and significance.

A random variable U is said to have a Rayleigh distribution with scale parameter θ (Ray(θ)), if its probability density function (pdf) is given by

$$f\left(u\right)=\frac{2u}{\theta}e^{-u^2/\theta},\theta>0,u>0. \tag{1}$$

The cumulative distribution function (cdf) corresponding to (1) for $u>0$, is

$$F(u)=1-e^{-u^2/\theta}.$$

Let $X=U^2$, thus, X has a one-parameter exponential distribution $(Exp\left(\theta\right))$, with pdf and cdf as follows:

$$g(x)=\frac{1}{\theta}e^{-\frac{x}{\theta}},\quad \text{for } x\geq 0,\theta>0, \tag{2}$$

and

$$G(x)=1-e^{-\frac{x}{\theta}},\quad \text{for } x\geq 0,\theta>0. \tag{3}$$

Let $R=\frac{\theta_1}{\theta_2}$, as in Helu and Samawi (2011), the continuous version of the three proposed overlap measures can be expressed as a function of R as follows:

$$\rho=\frac{2\sqrt{R}}{1+R}, \tag{4}$$

$$\lambda=\frac{4R}{\left(1+R\right)^2}, \tag{5}$$

and

$$\Delta=1-R^{\frac{1}{1-R}}\left|1-\frac{1}{R}\right|,R\neq 1. \tag{6}$$

According to Mulekar and Mishra (2000), ρ, λ, and Δ are not monotone for all $R>0$. However, they exhibit certain properties, such as symmetry in R, meaning that $OVL(R)=OVL(\frac{1}{R})$. They also remain invariance under linear transformations, $Y=aX+b$, $a\neq 0$ and attain the maximum value of 1 at $R=1$.

2.1. Maximum likelihood estimates

The OVL measures ρ, λ and Δ are functions of θ_1 and θ_2. In order to draw any inference about the OVL measures, we need to estimate the unknown parameters, θ_1 and θ_2. In this section we obtain the maximum likelihood estimates ($MLEs$) of the parameters θ_1 and θ_2 based on the Adaptive-IIPH censored samples. Let $\mathbf{U} = U_{1:m_1:n_1} < U_{2:m_1:n_1} < ... < U_{m_1:m_1:n_1}$ be an Adaptive-IIPH censoring sample from Ray(θ_1) under the censoring scheme $\{n_1, m_1, K_1, ..., K_{J_1}, 0, ..., 0, K^* = n_1 - m_1 - \sum_{i=1}^{J_1} K_i\}$ such that $U_{J_1:m_1:n_1} < T_1 < U_{J_1+1:m_1:n_1}$. And, $\mathbf{V} = \{V_{1:m_2:n_2} < V_{2:m_2:n_2} < ... < V_{m_2:m_2:n_2}\}$ be an Adaptive-IIPH censoring sample from Ray(θ_2) under the scheme
$\{n_2, m_2, L_1, ..., L_{J_2}, 0, ..., 0, L^* = n_2 - m_2 - \sum_{i=1}^{J_2} L_i\}$ such that $V_{J_2:m_2:n_2} < T_2 < V_{J_2+1:m_2:n_2}$. For simplicity, let $U_i = U_{i:m_1:n_1}$ and $V_i = V_{i:m_2:n_2}$. Then the joint likelihood function of the Adaptive-IIPH censored sample (see Balakrishnan and Cramer, 2014) can be written as

$$\begin{aligned} L(\theta_1, \theta_2|\mathbf{X}, \mathbf{Y}) = C_1 C_2 [1 - F_1(u_{m_1})]^{K^*} \prod_{i=1}^{m_1} f_1(u_i) \prod_{i=1}^{J_1} [1 - F_1(u_i)]^{K_i} \\ [1 - F_2(v_{m_2})]^{L^*} \prod_{i=1}^{m_2} f_2(v_i) \prod_{i=1}^{J_2} [1 - F_2(v_i)]^{L_i}, \end{aligned} \tag{7}$$

where,

$$C_1 = n_1(n_1 - K_1 - 1)(n_1 - K_1 - K_2 - 2)...(n_1 - K_1 - K_2 - ... - K_{m_1-1} - m_1 + 1),$$

$$C_2 = n_2(n_2 -_1 -1)(n_2 -_1 -L_2 - 2)...(n_2 - L_1 - L_2 - ... - L_{m_2-1} - m_2 + 1),$$

$$f_1(u) = \frac{2u}{\theta_1} e^{-u^2/\theta_1}, \qquad F_1(u) = 1 - e^{-u^2/\theta_1}, \quad \text{for } u > 0, \tag{8}$$

$$f_2(v) = \frac{2v}{\theta_2} e^{-v^2/\theta_2}, \qquad F_2(v) = 1 - e^{-v^2/\theta_2}, \quad \text{for } v > 0. \tag{9}$$

After substituting equations 8 and 9 into equation 7, and then taking the log-likelihood function, we obtain the following:

$$\begin{aligned} l \quad \propto \quad & -m_1 \log(\theta_1) - \frac{\left(K^* u_{m_1}^2 + \sum_{i=1}^{m_1} u_i^2 + \sum_{i=1}^{J_1} K_i u_i^2\right)}{\theta_1} \\ & -m_2 \log(\theta_2) - \frac{\left(L^* v_{m_2}^2 + \sum_{i=1}^{m_2} v_i^2 + \sum_{i=1}^{J_2} L_i v_i^2\right)}{\theta_2}. \end{aligned} \tag{10}$$

Using the transformation $X = U^2$ and $Y = V^2$, Eq. 10 becomes

$$\begin{aligned} l \quad \propto \quad & -m_1 \log(\theta_1) - \frac{\left(K^* x_{m_1} + \sum_{i=1}^{m_1} x_i + \sum_{i=1}^{J_1} K_i x_i\right)}{\theta_1} \\ & -m_2 \log(\theta_2) - \frac{\left(L^* y_{m_2} + \sum_{i=1}^{m_2} y_i + \sum_{i=1}^{J_2} L_i y_i\right)}{\theta_2}. \end{aligned} \tag{11}$$

The MLEs of the parameters θ_1 and θ_2 can be obtained by taking the first derivative of Eq.11 with respect to θ_1 and θ_2 and equating the normal equations to 0 to get

$$\hat{\theta}_1 = \frac{K^* x_{m_1} + \sum_{i=1}^{m_1} x_i + \sum_{i=1}^{J_1} K_i x_i}{m_1}, \tag{12}$$

$$\hat{\theta}_2 = \frac{L^* y_{m_2} + \sum_{i=1}^{m_2} y_i + \sum_{i=1}^{J_2} L_i y_i}{m_2}. \tag{13}$$

Viveros and Balakrishnan (1994; page 88) showed that when the underlying distribution is an exponential with unknown mean θ, and when data $W_{1:m:n} < W_{2:m:n} < \cdots < W_{m:m:n}$ are based on progressively type-II censored sample with censoring scheme $\mathbf{K} = (K_1, K_2, \ldots, K_m)$, $\hat{\theta} = \frac{\sum_{i=1}^{m}(K_i+1)w_i}{m}$ is the MLE of θ, and $\hat{\theta} \sim Gamma(m, \frac{\theta}{m})$ in which $Gamma\,(.,.)$ denote the Gamma distribution. Cramer and Iliopolous (2010; Theorems 5 and 7) showed that the MLE when data are based on Adaptive-IIPH coincide with the MLE in deterministic progressive type-II censoring schemes. Thus, the distribution of this particular random variable is invariant with respect to random (fixed) progressive type-II censoring procedure. Thus, we obtain $\hat{\theta}_i \sim G(m_i, \frac{\theta_i}{m_i}); i = 1, 2$. Consequently, the means and variances of the MLEs in (12) and (13) are

$$E(\hat{\theta}_1) = \theta_1, \qquad E(\hat{\theta}_2) = \theta_2, \tag{14}$$

$$Var(\hat{\theta}_1) = \frac{\theta_1^2}{m_1}, \qquad Var(\hat{\theta}_2) = \frac{\theta_2^2}{m_2}, \tag{15}$$

therefore, the MLE of R is $\hat{R} = \frac{\hat{\theta}_1}{\hat{\theta}_2}$. Hence, $\frac{\theta_2}{\theta_1}\hat{R}$ has F-distribution with $2m_1$ and $2m_2$ degrees of freedom $(F_{2m_1,2m_2})$. Thus, the variance of $\hat{R}$ can be approximated by:

$$Var(\hat{R}) = \frac{m_2^2(m_1 + m_2 - 1)}{m_1(m_2-1)^2(m_2-2)} R^2. \tag{16}$$

Clearly, an unbiased estimator of R is given by $\hat{R}^* = \frac{(m_2-1)}{m_2}\hat{R}$ with variance $Var(\hat{R}^*) = \frac{(m_1+m_2-1)}{m_1(m_2-2)}R^2$ and hence $Var(\hat{R}^*) < Var(\hat{R})$. Since the OVL measures are functions of R, therefore, based on the MLE estimate of R, the OVL measures can be estimated by

$$\hat{\rho} = \frac{2\sqrt{\hat{R}^*}}{1 + \hat{R}^*}, \tag{17}$$

$$\hat{\lambda} = \frac{4\hat{R}^*}{\left(1 + \hat{R}^*\right)^2}, \tag{18}$$

and,

$$\hat{\Delta} = 1 - \hat{R}^{*\frac{1}{1-\hat{R}^*}} \left| 1 - \frac{1}{\hat{R}^*} \right|, \hat{R}^* \neq 1. \tag{19}$$

3. Asymptotic properties of OVL

Using the delta method, the asymptotic variance and bias for OVL measures are as follows: Let $OVL = g(\hat{R}^*)$, then the asymptotic variance are given by

$$Var(\hat{\rho}) = \sigma^2_{\hat{\rho}} \cong \frac{(m_1 + m_2 - 1)}{m_1(m_2 - 2)} \frac{R(1-R)^2}{(1+R)^4}, \tag{20}$$

$$Var(\hat{\lambda}) = \sigma^2_{\hat{\lambda}} \cong \frac{16(m_1 + m_2 - 1)}{m_1(m_2 - 2)} \frac{R^2(1-R)^2}{(1+R)^6}, \tag{21}$$

$$Var(\hat{\Delta}) = \sigma^2_{\hat{\Delta}} \cong \frac{(m_1 + m_2 - 1)}{m_1(m_2 - 2)} \frac{R^{\frac{2}{1-R}}(\ln R)^2}{(1-R)^2}. \tag{22}$$

with the asymptotic bias

$$Bias(\hat{\rho}) \cong \frac{(m_1 + m_2 - 1)}{m_1(m_2 - 2)} \times \frac{\sqrt{R}(3R^2 - 6R - 1)}{2(1+R)^3}, \tag{23}$$

$$Bias(\hat{\lambda}) \cong \frac{(m_1 + m_2 - 1)}{m_1(m_2 - 2)} \times \frac{4R^2(R-2)}{(1+R)^4}, \tag{24}$$

and,

$$Bias(\hat{\Delta}) \cong \left\{ \begin{array}{ll} H(R)\frac{(m_1+m_2-1)}{m_1(m_2-2)}, & R > 1 \\ -H(R)\frac{(m_1+m_2-1)}{m_1(m_2-2)}, & R < 1 \end{array} \right\}, \tag{25}$$

where, $H(R) = R^2 \left[\frac{R^{\frac{2R-1}{1-R}} R\{2R - \ln R - 2\} \ln R - (R-1)^2}{(R-1)^3} \right]$.

Consistent estimators for the above variances and biases can be obtained by substituting R by $\hat{R}^*$ in the above formulas.

3.1. Interval estimation

Two types of interval estimation for the OVL measure are considered, namely the asymptotic confidence interval and the bootstrap confidence interval that were introduced by Efron (1992). For a large sample, normal approximation to the sampling distribution using the delta-method, works fairly well. Therefore, the asymptotic $100(1-\alpha)\%$ confidence interval for the OVL measures is given by:
$\left\{ \widehat{OVL} \mp \hat{\sigma}_{\widehat{OVL}} Z_{\alpha/2} \right\}$, where $Z_{\alpha/2}$ is the $\frac{\alpha}{2}$ upper quantile of the standard normal distribution.

There is an obvious bias involved in all OVL measure estimates, however, for large samples, they work fairly well. Thus, the bias corrected interval can be computed as follows:

$$\left(\widehat{OVL} - Bias(\widehat{OVL}) \right) \pm \hat{\sigma}_{\widehat{OVL}} Z_{\alpha/2}. \tag{26}$$

However, uniform bootstrap resampling approach for estimating bootstrap confidence intervals as described by Efron (1992), is designed for one sample case. For a two-sample case, the uniform resampling rules will apply to each sample separately and independently (see Helu and Samawi, 2011).
Let $\mathbf{X} = (X_1, X_2, \cdots, X_{m_1})$ and $\mathbf{Y} = (Y_1, Y_2, \cdots, Y_{m_2})$ be two independent
Adaptive-IIPH samples drawn from $f_1(x)$ and $f_2(y)$ respectively. Assume that the parameter of interest is the OVL coefficient. Let S be an estimate of OVL based on the mentioned two random samples. For B uniform re-samples, say $(X^*_{i1}, X^*_{i2}, ..., X^*_{im_1})$ and $(Y^*_{i1}, Y^*_{i2}, ..., Y^*_{im_2}), i = 1, 2, ..., B$, let $S^*_1, S^*_2, ..., S^*_B$ be the re-sampling realization of S. Then, the uniform re-sampling approximation to the $100(1-\alpha)\%$ bootstrap confidence limits can be obtained as follows: Let $S^*_{(1)}, S^*_{(2)}, ..., S^*_{(B)}$ be the order statistics of $S^*_1, S^*_2, ..., S^*_B$. Define $\omega_1 = int(B(\alpha))$ and $\omega_2 = int(B(1-\alpha))$. Then the uniform re-sampling approximation of the $100(1-\alpha)\%$ confidence interval is $\left(\frac{S^*_{(\omega_1)} + S^*_{(\omega_1+1)}}{2}, \frac{S^*_{(\omega_2)} + S^*_{(\omega_2+1)}}{2} \right)$.

4. Simulation Study

This simulation study aims to rigorously compare the performance of maximum likelihood estimators for the measures of overlap. These estimators are derived from diverse sets of Adaptive-IIHP censoring samples, as described by Ng et al. (2009), generated from two independent Rayleigh distributions. The algorithm proceeds as follows:

1. Generate two independent progressive type-II censored samples, denoted as $U_1, U_2, ..., U_{m_1}$ and $V_1, V_2, ..., V_{m_2}$ from Ray(θ_1) and Ray(θ_2), respectively. Use censoring schemes $\mathbf{K} = (K_1, K_2, \ldots, K_{m_1})$ and $\mathbf{L} = (L_1, L_2, \ldots, L_{m_2})$ as proposed by Balakrishnan and Cramer(2014).
2. Determine the values of J_1 and J_2, such that $U_{J_1} < T_1 < U_{J_1+1}$ and $V_{J_2} < T_2 < V_{J_2+1}$. Then, remove $U_{J_1+2}, \ldots, U_{m_1}$ and $V_{J_2+2}, \ldots, V_{m_2}$.
3. Generate the first $m_1 - j_1 - 1$ order statistics from the truncated distribution $\frac{f_1(u)}{1-F_1(u_{J_1+1})}$ as $U_{J_1+2}, \ldots, U_{m_1}$, and adjust the censoring scheme to $\mathbf{K} = (K_1, \ldots, K_{J_1}, 0, \ldots, 0, K^* = n_1 - m_1 - \sum_{i=1}^{J_1} K_i)$. Similarly, generate the first $m_2 - j_2 - 1$ order statistics from the truncated distribution $\frac{f_2(v)}{1-F_2(v_{J_2+1})}$ as $V_{J_2+2}, \ldots, V_{m_2}$, and update the censoring scheme to $\mathbf{L} = (L_1, \ldots, L_{J_2}, 0, \ldots, 0, L^* = n_2 - m_2 - \sum_{i=1}^{J_2} L_i)$. Use the transformation $X = U^2$ and $Y = V^2$.
4. Calculate $\hat{\theta}_1$ and $\hat{\theta}_2$, and subsequently obtain the estimates of the measures of overlap $\hat{\rho}$, $\hat{\lambda}$, and $\hat{\Delta}$.

In this study, we executed a total of 10,000 simulations, each corresponding to one of four distinct values of R. Specifically:
1. When $R = 0.003$, the resulting parameter values are as follows: $\rho = 0.1, \lambda = 0.01$, and $\Delta = 0.02$.
2. For $R = 0.03$, we observed $\rho = 0.34, \lambda = 0.11,$, and $\Delta = 0.13$.
3. When $R = 0.2$, the associated parameter values are $\rho = 0.70, \lambda = 0.50$, and $\Delta = 0.42$.
4. Lastly, $R = 0.8$ yielded parameter values of $\rho = 0.98, \lambda = 0.96$, and $\Delta = 0.85$.

These simulations are conducted based on four distinct sets of population parameters: $(\theta_1, \theta_2) = (3, 1000), (3, 100), (0.1, 0.5)$, and $(4, 5)$. This comprehensive range of parameter combinations allowed us to explore varying degrees of similarity between the two Rayleigh distributions. Additionally, three primary stopping times are considered: $T_1 = X_{\lfloor \frac{m}{4} \rfloor}$, $T_2 = X_{\lfloor \frac{4m}{5} \rfloor}$, and $T_3 = X_m + 2$.

We then computed the associated approximate 95% confidence intervals, bias ($|Bias|$), mean squared error (MSE), length of the confidence intervals (L) and coverage probability (Cov) using Taylor and bootstrap approximation techniques. The bootstrap approximation is based on $B = 1000$ resamples. For illustrative purposes we generated the censoring samples using $n = n_1 = n_2 = 20, 30, m = m_1 = m_2 = 5, 10, 20,$ and set $\mathbf{K} = \mathbf{L}$, employing three censoring schemes:

- Scheme-I: $(n - m, 0^{*(m-1)})$, known as scheme-I, where $n - m$ units are removed just after the first failure.
- Scheme-II: $\left(0^{*(m-1)}, n - m\right)$, known as scheme-II, where $n - m$ units are removed after the last failure.
- Scheme-III: $\left(\frac{n-m}{2}, 0^{*(m-2)}, \frac{n-m}{2}\right)$, known as scheme-III, where $\frac{n-m}{2}$ units are removed after the first and last failures. For brevity, we use the notation 0^{*p} to denote p successive zeros. Thus, the scheme $(9, 0, 0, 0, 0, 0)$ is denoted by $(9, 0^{*5})$.

4.1. Data analysis and comparison study

In this study, we investigate the behavior of overlap estimators when applied to samples drawn from two Rayleigh distributions of varying degrees of similarity based on Adaptive-IIPH censored data. Our research has shed light on the crucial relationship between the similarity between two distributions and the accuracy of the estimators.

Most favorable estimators tend to have minimal $|Bias|$, smallest MSE, and shortest L. These desirable properties manifest prominently when a substantial disagreement exists between the two Rayleigh density distributions, when $\rho = 0.1, \lambda = 0.01$ and $\Delta = 0.02$ as depicted in Tables 6 and 10. Conversely, as the similarity between the source distributions increases, we consistently observe an escalation in $|Bias|$, MSE, and L across all

OVL estimators. Based on this pattern, these estimators appear less accurate and precise as the source distributions become more congruent (see Tables 6 to 9).

Interestingly, a notable inverse relationship surfaces concerning coverage probability. As the source distributions become more alike, the coverage probability, Cov, decreases for the estimators $\hat{\rho}$ and $\hat{\lambda}$. However, this trend diverges for $\hat{\Delta}$, where the Cov improves with increasing similarity between the two densities.

Furthermore, as the values of $\hat{\rho}$ and $\hat{\lambda}$ approach 1, signifying strong agreement between the source distributions, we observe a similar behavior pattern: $|Bias|$ increases while MSE, L, and Cov decrease. Intriguingly, the $\hat{\Delta}$ estimator deviates from this pattern. As $\hat{\Delta}$ approaches 1, indicating maximum similarity, only the $|Bias|$ of the $\hat{\Delta}$ estimator declines, while MSE and L increase. This suggests that the $\hat{\Delta}$ possesses unique characteristics, performing optimally when the source distributions completely agree or disagree.

Moreover, it is crucial to underscore the consistent behavior of the $\hat{\Delta}$ regarding coverage. Specifically, as $\hat{\Delta}$ approaches the extremes of 0 or 1, there is a consistent increase in Cov values. This highlights the remarkable stability of the $\hat{\Delta}$ estimator in scenarios where the source distributions either fully align or diverge.

It is noteworthy that when there exists a substantial disagreement between the two Rayleigh densities, there are minimal differences between the three stopping times. Additionally, when the ratio of m/n is large ($\geq 2/3$), $|Bias|$, MSE, L, and coverage probability show noticeable improvement.

Shifting our focus to the bootstrap method, results presented in Tables 10 to 13 align with the observations made in Tables 6 to 9, except for instances when the two densities have perfect agreement. In such cases, all three OVL estimates exhibit similar behavior: $|Bias|$ and Cov increase while MSE and L values decrease. Furthermore, the bootstrap results indicate no significant impact from varying censoring schemes or OVL values, except for the consistent coverage values, which remain stable regardless of the source distributions aligning or diverging.

5. Real life data

In this section, we utilize real-life data to demonstrate and validate our proposed method in practical scenarios. The dataset we employ was initially reported by Freireich (1963) and has since been utilized by Gehan (1965) and, more recently, Zhou (2020).

This study aimed to compare the efficacy of 6-mercaptopurine (6-MP), a medication, to placebo regarding sustaining remission in patients with acute leukemia. Table 1 presents the remission durations for two separate cohorts, each comprising 21 patients. One cohort received the placebo, while the other received the drug 6-MP.

Table 1. The duration of the remission time (in weeks) for two groups of leukemia patients

6-MP	6	6	6	6	7	9	10	10	11	13	16
	17	19	20	22	23	25	32	32	34	35	
placebo	1	1	2	2	3	4	4	5	5	8	8
	8	8	11	11	12	12	15	17	22	23	

The legitimacy of the Rayleigh model is checked for 6-MP group and placebo group based on $\theta_1 = 13.64$ and $\theta_2 = 6.92$, respectively, using the Kolmogorov-Smirnov (K-S), Anderson-Darling (A-D), and chi-square tests. The results, presented in Table 2, demonstrate that the Rayleigh model effectively fits both sets of data.

Table 2. Test statistic and p-value associated with each test for 6-MP and placebo

Data	K-S(p-value)	A-D(p-value)	chi-squared(p-value)
6-MP	0.1509 (0.6701)	0.693 (0.2502)	1.911 (0.3846)
placebo	0.1985 (0.3341)	1.652 (0.3312)	2.245 (0.3255)

Additionally, the fitted pdfs and Q-Q plots for the 6-MP and placebo datasets are depicted in Figures 3 to 6, confirming that the Rayleigh model is a good fit for both data sets.

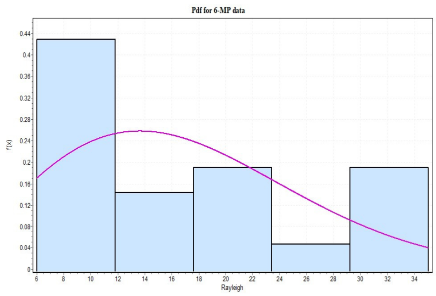

Figure 3. Estimated pdf of the 6-MP data

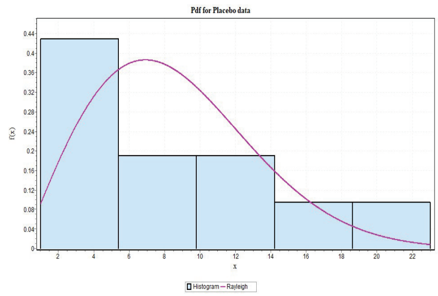

Figure 4. Estimated pdf of the placebo data

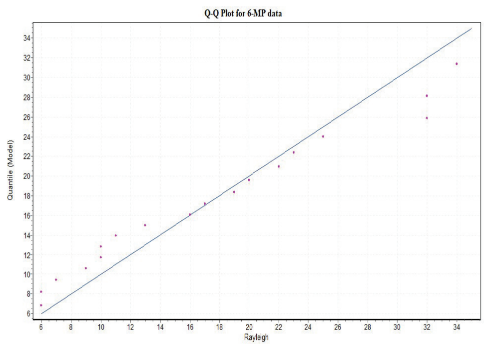

Figure 5. Q-Q plot for the 6-MP data

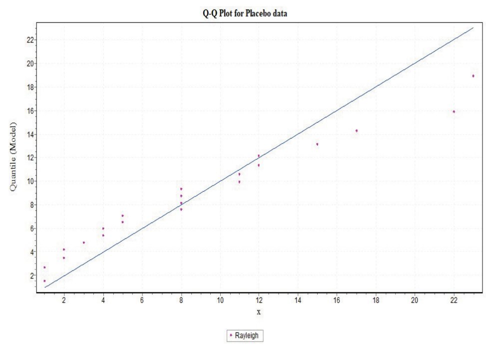

Figure 6. Q-Q plot for the placebo data

Three different artificial Adaptive-IIHP censored data are created for both sets using the same censoring schemes as those in Section 4. The associated stopping time for each scheme and the generated censored samples are given in Table 3 to 6.

The estimates of the OVLs are calculated based on $m_1 = 11$, $m_2 = 11$. The corresponding MLEs, $|Bias|$, asymptotic variance and 95% confidence intervals for OVLs, using Taylor approximation and bootstrap methods, are reported in Table 5, which reveal that the estimates based on Scheme-III, are the closest to those of the complete data set.

Table 3. Artificial Adaptive-IIHP censored samples for 6-MP group

scheme	T	censored data for 6-MP										
I	T_1	9	17	19	20	22	23	25	32	32	34	35
II	T_2	6	6	6	7	10	11	13	17	19	22	23
III	T_3	9	10	10	11	13	16	17	19	20	22	23

Table 4. Artificial Adaptive-IIHP censored samples for the placebo group

scheme	T	censored data for placebo										
I	T_1	1	1	2	5	3	8	11	12	15	17	22
II	T_2	1	1	2	2	3	4	4	5	8	8	11
III	T_3	4	4	5	5	8	8	8	8	11	11	12

Table 5. Results based on the real data of Efron (1988)

				Asymptotic Inference 95% confidence	Bootstrap Inference 95% confidence		
Scheme	**Coeff**	**MLEs (—Bias—)**	**Asymptotic variance**	**Lower**	**Upper**	**Lower**	**Upper**
Complete	ρ	0.9622(0.0038)	0.0018	0.8837	1	0.8750	0.9939
	λ	0.9258(0.0207)	0.0133	0.5540	1	0.6628	0.9244
	Δ	0.7971(0.0055)	0.0066	0.7730	1	0.8041	0.9895
1	ρ	0.9537(0.0055)	0.0044	0.8298	1	0.8758	0.9971
	λ	0.9096(0.0429)	0.0261	0.4159	1	0.6297	0.9441
	Δ	0.7753(0.0061)	0.0159	0.6689	1	0.7671	0.9942
2	ρ	0.9421(0.0026)	0.0053	0.8020	1	0.7379	0.9963
	λ	0.8875(0.0431)	0.0254	0.3927	1	0.4571	0.9365
	Δ	0.7484(0.0040)	0.0188	0.6184	1	0.5446	0.9926
3	ρ	0.9636(0.0082)	0.0035	0.8556	1	0.7981	0.9911
	λ	0.9285(0.0428)	0.0266	0.4383	1	0.5251	0.9016
	Δ	0.8009(0.0124)	0.0131	0.7168	1	0.6368	0.9822

Table 6. Taylor approximation: absolute value of bias ($|Bias|$), length L, mean squared error (MSE) & coverage probability (Cov), when $R = 0.003, \rho = 0.1, \lambda = 0.01$ and $\Delta = 0.02$

(n,m)	Scheme		$T_1 = X_{[\frac{m}{4}]}$				$T_2 = X_{[\frac{4\times m}{5}]}$				$T_3 = X_{[m]}+2$			
		Estimate	Bias	MSE	L	Cov	Bias	MSE	L	Cov	Bias	MSE	L	Cov
(20,6)	I	ρ	0.0060420	0.0013632	0.1370793	0.9694	0.0060822	0.0013836	0.1379369	0.9688	0.0060681	0.0013837	0.1375773	0.9610
		λ	0.0000437	0.0000857	0.0309033	0.9314	0.0000454	0.0000890	0.0313709	0.9288	0.0000469	0.0000916	0.0313801	0.9225
		Δ	0.0003581	0.0001541	0.0436748	0.9433	0.0003639	0.0001586	0.0441820	0.9417	0.0003643	0.0001606	0.0441039	0.9324
	II	ρ	0.0061776	0.0014654	0.1396942	0.9397	0.0064261	0.0016554	0.1443622	0.8810	0.0060681	0.0013837	0.1375773	0.9610
		λ	0.0000591	0.0001147	0.0332784	0.8990	0.0000950	0.0001812	0.0377267	0.8434	0.0000469	0.0000916	0.0313801	0.9225
		Δ	0.0003887	0.0001860	0.0458336	0.9112	0.0004467	0.0002508	0.0497573	0.8579	0.0003643	0.0001606	0.0441039	0.9324
	III	ρ	0.0061647	0.0014545	0.1394582	0.9456	0.0063227	0.0015726	0.1424907	0.9058	0.0060699	0.0013851	0.1376121	0.9601
		λ	0.0000570	0.0001110	0.0330225	0.8998	0.0000760	0.0001466	0.0357717	0.8629	0.0000470	0.0000919	0.0314113	0.9216
		Δ	0.0003853	0.0001822	0.0456132	0.9139	0.0004208	0.0002198	0.0481029	0.8755	0.0003647	0.0001610	0.0441332	0.9321
(20,12)	I	ρ	0.0025772	0.0005621	0.0905628	0.9598	0.0025948	0.0005699	0.0911549	0.9621	0.0025915	0.0005692	0.0910362	0.9568
		λ	0.0000155	0.0000306	0.0200225	0.9342	0.0000160	0.0000316	0.0203013	0.9363	0.0000161	0.0000317	0.0202772	0.9320
		Δ	0.0001493	0.0000597	0.0286885	0.9415	0.0001515	0.0000612	0.0290106	0.9457	0.0001513	0.0000612	0.0289680	0.9430
	II	ρ	0.0025957	0.0005789	0.0910966	0.9152	0.0026421	0.0006078	0.0925539	0.8812	0.0025910	0.0005691	0.0910178	0.9565
		λ	0.0000177	0.0000348	0.0206357	0.8901	0.0000211	0.0000413	0.0216871	0.8684	0.0000161	0.0000317	0.0202760	0.9316
		Δ	0.0001544	0.0000649	0.0292291	0.8980	0.0001630	0.0000729	0.0302602	0.8738	0.0001513	0.0000612	0.0289632	0.9413
	III	ρ	0.0025862	0.0005710	0.0908168	0.9336	0.0026162	0.0005875	0.0917872	0.9221	0.0025910	0.0005692	0.0910177	0.9556
		λ	0.0000167	0.0000329	0.0203466	0.9078	0.0000183	0.0000359	0.0209424	0.9000	0.0000161	0.0000317	0.0202798	0.9310
		Δ	0.0001520	0.0000625	0.0289690	0.9165	0.0001568	0.0000665	0.0295882	0.9096	0.0001514	0.0000613	0.0289658	0.9402
(30,10)	I	ρ	0.0031920	0.0007036	0.1007145	0.9607	0.0032022	0.0007083	0.1010205	0.9608	0.0031977	0.0007070	0.1008776	0.9619
		λ	0.0000203	0.0000400	0.0224566	0.9314	0.0000206	0.0000405	0.0226069	0.9299	0.0000207	0.0000407	0.0225653	0.9357
		Δ	0.0001867	0.0000763	0.0320296	0.9415	0.0001880	0.0000772	0.0322024	0.9395	0.0001877	0.0000772	0.0321388	0.9441
	II	ρ	0.0032593	0.0007565	0.1025033	0.8898	0.0033437	0.0008199	0.1047500	0.8207	0.0031980	0.0007072	0.1008855	0.9620
		λ	0.0000278	0.0000543	0.0241877	0.8701	0.0000377	0.0000730	0.0262678	0.8255	0.0000207	0.0000407	0.0225715	0.9348
		Δ	0.0002027	0.0000928	0.0335925	0.8791	0.0002220	0.0001132	0.0354647	0.8280	0.0001877	0.0000772	0.0321448	0.9438
	III	ρ	0.0032456	0.0007457	0.1021431	0.9031	0.0033084	0.0007910	0.1038460	0.8506	0.0031986	0.0007076	0.1009013	0.9608
		λ	0.0000261	0.0000511	0.0238336	0.8811	0.0000324	0.0000631	0.0253146	0.8457	0.0000208	0.0000409	0.0225861	0.9345
		Δ	0.0001994	0.0000893	0.0332769	0.8933	0.0002130	0.0001031	0.0346436	0.8524	0.0001879	0.0000774	0.0321582	0.9431
(30,20)	I	ρ	0.0014774	0.0003203	0.0690805	0.9556	0.0014808	0.0003217	0.0692359	0.9578	0.0014806	0.0003218	0.0692212	0.9542
		λ	8.4339E-6	0.0000167	0.0152506	0.9412	8.5130E-6	0.0000168	0.0153194	0.9409	8.5554E-6	0.0000169	0.0153266	0.9374
		Δ	0.0000854	0.0000334	0.0219223	0.9459	0.0000858	0.0000336	0.0220038	0.9446	0.0000858	0.0000337	0.0220054	0.9438
	II	ρ	0.0014913	0.0003307	0.0696436	0.8878	0.0015025	0.0003383	0.0701038	0.8458	0.0014812	0.0003222	0.0692489	0.9529
		λ	9.6420E-6	0.0000190	0.0157599	0.8844	0.0000105	0.0000206	0.0161265	0.8532	8.5830E-6	0.0000169	0.0153430	0.9376
		Δ	0.0000885	0.0000364	0.0223969	0.8856	0.0000908	0.0000385	0.0227504	0.8508	0.0000859	0.0000338	0.0220228	0.9427
	III	ρ	0.0014850	0.0003259	0.0693905	0.9176	0.0014911	0.0003295	0.0696480	0.9067	0.0014814	0.0003223	0.0692567	0.9525
		λ	9.066E-6	0.0000179	0.0155254	0.9109	9.3906E-6	0.0000185	0.0156975	0.9028	8.5925E-6	0.0000170	0.0153481	0.9372
		Δ	0.0000871	0.0000350	0.0221803	0.9131	0.0000881	0.0000359	0.0223560	0.9049	0.0000860	0.0000338	0.0220281	0.9426

Table 7. Taylor approximation: absolute value of bias ($|Bias|$), length L, mean squared error (MSE) & coverage probability (Cov), when $R = 0.03, \rho = 0.34, \lambda = 0.11$ and $\Delta = 0.13$

(n,m)	Scheme		$T_1 = X_{[\frac{m}{4}]}$				$T_2 = X_{[\frac{4\times m}{5}]}$				$T_3 = X_{[m]+2}$			
		Estimate	Bias	MSE	L	Cov	Bias	MSE	L	Cov	Bias	MSE	L	Cov
(20,6)	I	ρ	0.0204771	0.0111874	0.3955369	0.9645	0.0206316	0.0113058	0.3973138	0.9626	0.0205933	0.0112605	0.3958114	0.9553
		λ	0.0035217	0.005917	0.2681589	0.9359	0.0036393	0.0060885	0.2713501	0.9331	0.0037193	0.0061538	0.2706366	0.9257
		Δ	0.0036213	0.0044013	0.2423341	0.9513	0.003673	0.0044912	0.2443893	0.9490	0.0036738	0.004508	0.2436084	0.9409
	II	ρ	0.0210332	0.0115481	0.3977913	0.9251	0.0220021	0.0120970	0.4009120	0.8550	0.0205943	0.0112613	0.3958068	0.9551
		λ	0.0043745	0.0069695	0.2799135	0.9007	0.0060105	0.0087409	0.2989461	0.8365	0.0037214	0.0061568	0.2706639	0.9253
		Δ	0.0038586	0.0048750	0.2486130	0.9160	0.0042817	0.0057103	0.2594475	0.8547	0.0036744	0.0045092	0.2436179	0.9404
	III	ρ	0.0209807	0.0115173	0.3977284	0.9321	0.0216161	0.0119158	0.4003236	0.8847	0.0205925	0.0112645	0.3959309	0.9538
		λ	0.0042777	0.0068583	0.2788200	0.9025	0.0052663	0.0080231	0.2919459	0.8606	0.0037034	0.0061412	0.2706679	0.9240
		Δ	0.0038342	0.0048250	0.2480514	0.9204	0.0041052	0.0053606	0.2552895	0.8784	0.0036721	0.0045035	0.2436402	0.9390
(20,12)	I	ρ	0.0087177	0.0047014	0.2629534	0.9581	0.0087840	0.0047528	0.2643301	0.9586	0.0087725	0.0047429	0.2639170	0.9528
		λ	0.0013225	0.0023063	0.1771143	0.9366	0.0013595	0.0023659	0.1792302	0.9378	0.0013638	0.0023688	0.1789095	0.9349
		Δ	0.0015284	0.0018131	0.1610290	0.9468	0.0015491	0.0018471	0.1624516	0.9501	0.0015469	0.0018457	0.1621582	0.9448
	II	ρ	0.0088030	0.0047607	0.2630584	0.9066	0.0089899	0.0048874	0.2652340	0.8671	0.0087714	0.0047423	0.2638882	0.9510
		λ	0.0014689	0.0025097	0.1803549	0.8916	0.0016846	0.0027998	0.1865002	0.8635	0.0013631	0.0023682	0.1788853	0.9341
		Δ	0.0015693	0.0019000	0.1624282	0.9019	0.0016409	0.0020359	0.1659389	0.8681	0.0015467	0.0018453	0.1621351	0.9460
	III	ρ	0.0087607	0.0047318	0.2629068	0.9296	0.0088787	0.0048168	0.2647038	0.9138	0.0087717	0.0047425	0.263876	0.9502
		λ	0.0014015	0.0024170	0.1788053	0.9101	0.0015093	0.0025711	0.1826563	0.8998	0.0013643	0.0023698	0.1788991	0.9339
		Δ	0.0015500	0.0018598	0.1617125	0.9212	0.0015919	0.0019359	0.1640500	0.9104	0.0015469	0.0018459	0.1621361	0.9456
(30,10)	I	ρ	0.0108072	0.0058486	0.2918114	0.9573	0.0108460	0.0058805	0.2925129	0.9566	0.0108278	0.0058635	0.292003	0.9534
		λ	0.0017069	0.0029477	0.1975957	0.9351	0.0017285	0.0029846	0.1987612	0.9328	0.0017314	0.0029823	0.198237	0.9339
		Δ	0.0019047	0.0022784	0.1790819	0.9464	0.0019173	0.0022996	0.1798359	0.9454	0.001913	0.0022944	0.1794276	0.9449
	II	ρ	0.0110940	0.0060344	0.2930754	0.8743	0.0114405	0.0062340	0.2948357	0.8025	0.0108311	0.0058652	0.2920933	0.9569
		λ	0.0021555	0.0035330	0.2065013	0.8667	0.0027091	0.0041899	0.2164828	0.8059	0.0017322	0.0029829	0.1982940	0.9377
		Δ	0.0020313	0.0025364	0.1834888	0.8755	0.0021795	0.0028319	0.1888169	0.8111	0.0019135	0.0022950	0.1794895	0.9487
	III	ρ	0.0110366	0.0059998	0.2928498	0.8918	0.0112961	0.0061617	0.2945406	0.8355	0.0108336	0.0058669	0.292107	0.9558
		λ	0.0020618	0.0034161	0.2047811	0.8808	0.0024410	0.0038930	0.2124531	0.8344	0.0017359	0.0029879	0.198373	0.9359
		Δ	0.0020060	0.0024849	0.1826192	0.8909	0.0021141	0.0026990	0.1867192	0.8405	0.0019146	0.0022972	0.1795289	0.9471
(30,20)	I	ρ	0.0049963	0.0026944	0.2008537	0.9530	0.0050091	0.0027043	0.2012285	0.9568	0.0049993	0.0026963	0.2008613	0.9517
		λ	0.0007305	0.0012907	0.1356726	0.9418	0.0007368	0.0013009	0.1362098	0.9412	0.0007356	0.0012978	0.1358169	0.9398
		Δ	0.0008773	0.0010347	0.1233741	0.9486	0.0008812	0.0010409	0.1237461	0.9479	0.0008787	0.0010376	0.1234365	0.9469
	II	ρ	0.0050569	0.002737	0.2013936	0.8813	0.0051037	0.0027703	0.2019908	0.8383	0.0050112	0.0027055	0.2012031	0.9505
		λ	0.0008146	0.0014107	0.1385952	0.8796	0.0008716	0.0014925	0.1407890	0.8438	0.0007416	0.0013075	0.1363206	0.9386
		Δ	0.0009035	0.0010879	0.1247962	0.8821	0.0009225	0.0011253	0.1259452	0.8422	0.0008823	0.0010435	0.1237812	0.9449
	III	ρ	0.0050294	0.0027181	0.2011708	0.9112	0.0050539	0.0027364	0.2016124	0.9020	0.0050119	0.0027061	0.2012168	0.9501
		λ	0.0007751	0.0013552	0.1372771	0.9078	0.0007988	0.0013906	0.1384018	0.8998	0.0007423	0.0013086	0.1363543	0.9370
		Δ	0.0008915	0.0010634	0.1241592	0.9121	0.0009007	0.0010807	0.1247977	0.9019	0.0008826	0.001044	0.1238005	0.9444

Table 8. Taylor approximation: absolute value of bias ($|Bias|$), length L, mean squared error (MSE) & coverage probability (Cov), when $R = 0.2, \rho = 0.7, \lambda = 0.5$ and $\Delta = 0.42$

(n,m)	Scheme	Estimate	$T_1 = X_{[\frac{m}{4}]}$ Bias	MSE	L	Cov	$T_2 = X_{[\frac{4\times m}{5}]}$ Bias	MSE	L	Cov	$T_3 = X_{[m]+2}$ Bias	MSE	L	Cov
(20,6)	I	ρ	0.0561336	0.0280601	0.6115175	0.9062	0.0563402	0.0279526	0.6095261	0.8962	0.0560135	0.0277758	0.6073368	0.8934
		λ	0.0573248	0.0534306	0.8504819	0.9002	0.0580894	0.0535296	0.8498966	0.8917	0.0576224	0.0529338	0.8437433	0.8841
		Δ	0.0176496	0.0302162	0.6617900	0.9563	0.0177717	0.0304703	0.6642145	0.9554	0.0176330	0.0302843	0.6613138	0.9464
	II	ρ	0.055609	0.0267589	0.5913676	0.8470	0.0545089	0.0248601	0.5614885	0.7559	0.0560201	0.0277748	0.6073331	0.8928
		λ	0.0587323	0.0510490	0.8177863	0.8284	0.0601761	0.0478430	0.7711955	0.7311	0.0576422	0.0529451	0.8438204	0.8842
		Δ	0.0176000	0.0306001	0.6607184	0.9166	0.0169592	0.0311928	0.6587618	0.854	0.0176267	0.0302890	0.6613547	0.9472
	III	ρ	0.0557011	0.0269050	0.5936547	0.8540	0.0551381	0.0255823	0.5725892	0.7877	0.0560166	0.0277519	0.6070221	0.8929
		λ	0.0586537	0.0513384	0.8217347	0.8398	0.0600976	0.0491827	0.7894182	0.7664	0.0576815	0.0529139	0.8433706	0.8825
		Δ	0.0176341	0.0305805	0.6611364	0.9234	0.0174013	0.0310488	0.6604199	0.8783	0.0176403	0.0302964	0.6613315	0.9460
(20,12)	I	ρ	0.0245337	0.0116356	0.4103973	0.9293	0.0246633	0.0116055	0.4096383	0.9210	0.0246138	0.0115810	0.4091421	0.9186
		λ	0.0251655	0.023698	0.5891432	0.9230	0.0255470	0.0237927	0.5901636	0.9184	0.0254732	0.0236937	0.5886158	0.9131
		Δ	0.0076864	0.0131518	0.4432914	0.9515	0.0077600	0.0132766	0.4453584	0.9547	0.0077443	0.0132440	0.4446240	0.9474
	II	ρ	0.0243795	0.0112616	0.4023571	0.8647	0.0243939	0.0109262	0.3943589	0.8155	0.0246044	0.0115744	0.4090123	0.9179
		λ	0.0255312	0.0228468	0.5747403	0.8553	0.0262467	0.0222394	0.5633911	0.8008	0.0254632	0.0236713	0.5882696	0.9124
		Δ	0.0077342	0.0131776	0.4415097	0.8997	0.007842	0.0133772	0.4431762	0.8579	0.0077382	0.0132384	0.4444832	0.9461
	III	ρ	0.0244384	0.0114191	0.4058208	0.8945	0.0245334	0.0112636	0.4021534	0.8660	0.0246014	0.0115707	0.4089352	0.9178
		λ	0.0253610	0.0231984	0.5807898	0.8815	0.0259084	0.0230039	0.5769389	0.8573	0.0254633	0.0236610	0.5881063	0.9121
		Δ	0.0077099	0.0131575	0.4420965	0.9221	0.0078031	0.0133203	0.4441580	0.9064	0.0077380	0.0132375	0.4444441	0.9455
(30,10)	I	ρ	0.0301952	0.0143751	0.4526093	0.9191	0.0302716	0.0143413	0.4519526	0.9124	0.0302023	0.0143384	0.4518647	0.9126
		λ	0.0311222	0.0289369	0.6464249	0.9123	0.0313821	0.0289823	0.6466351	0.9069	0.0312122	0.0288592	0.6453685	0.9107
		Δ	0.0094986	0.0162334	0.4909534	0.9516	0.0095446	0.0163083	0.4919347	0.9493	0.0095077	0.0162590	0.4912166	0.9524
	II	ρ	0.0298487	0.0135034	0.4346217	0.8204	0.0294606	0.0127279	0.4174843	0.7289	0.030202	0.0143361	0.4518109	0.9127
		λ	0.0320305	0.0270155	0.6153402	0.8050	0.0326978	0.0253604	0.5867516	0.7122	0.0312165	0.0288556	0.6452906	0.9090
		Δ	0.0095784	0.0163886	0.4887660	0.8659	0.0094726	0.0165870	0.4875954	0.7924	0.0095086	0.0162602	0.4912212	0.9509
	III	ρ	0.0299264	0.0136619	0.4380814	0.8369	0.0297202	0.0130783	0.4252073	0.7689	0.0301996	0.0143287	0.4516575	0.9129
		λ	0.0318978	0.0273708	0.6213760	0.8227	0.0325932	0.0261755	0.6004719	0.7486	0.0312256	0.0288402	0.6450327	0.9073
		Δ	0.0095903	0.0163613	0.4891926	0.8832	0.0095955	0.0165474	0.4889469	0.8248	0.0095146	0.0162618	0.4912072	0.9485
(30,20)	I	ρ	0.0141883	0.0065927	0.3129512	0.9349	0.0142166	0.0065896	0.312850	0.9329	0.0142053	0.0065797	0.3125814	0.9281
		λ	0.0146809	0.0139374	0.4569044	0.9299	0.0147562	0.0139656	0.4573885	0.9296	0.0147478	0.0139288	0.4567110	0.9257
		Δ	0.0044467	0.0076292	0.3395444	0.9500	0.0044607	0.0076547	0.3401326	0.9525	0.0044586	0.0076491	0.3399362	0.9480
	II	ρ	0.0141395	0.0063857	0.3072105	0.8484	0.0141288	0.0062578	0.3035824	0.8027	0.0142081	0.0065758	0.3124737	0.9268
		λ	0.0149681	0.0134595	0.4471119	0.8438	0.0151948	0.0131940	0.4414508	0.7905	0.0147625	0.0139253	0.4566162	0.9230
		Δ	0.0044860	0.0076699	0.3390108	0.8762	0.0045205	0.0077137	0.3390990	0.8316	0.0044610	0.0076530	0.3399978	0.9466
	III	ρ	0.0141656	0.0064795	0.3098565	0.8851	0.0141802	0.0064280	0.3084448	0.8652	0.0142086	0.0065743	0.3124328	0.9262
		λ	0.0148472	0.0136805	0.4516897	0.8789	0.0149845	0.0135981	0.4498979	0.8624	0.0147667	0.0139230	0.4565661	0.9227
		Δ	0.0044688	0.0076532	0.3392947	0.9072	0.0044898	0.0076848	0.3396659	0.8969	0.0044616	0.0076540	0.3400100	0.9462

Table 9. Taylor approximation: absolute value of bias ($|Bias|$), length L, mean squared error (MSE) & coverage probability (Cov), when $R = 0.8, \rho = 0.98, \lambda = 0.96$ and $\Delta = 0.85$

(n,m)	Scheme		$T_1 = X_{[\frac{m}{4}]}$				$T_2 = X_{[\frac{4\times m}{5}]}$				$T_3 = X_{[m]}+2$			
		Estimate	Bias	MSE	L	Cov	Bias	MSE	L	Cov	Bias	MSE	L	Cov
(20,6)	I	ρ	0.0628599	0.0132117	0.3163052	0.7579	0.0622613	0.0131673	0.3153199	0.7513	0.0620232	0.0134395	0.3218799	0.7623
		λ	0.1099254	0.0417299	0.5767550	0.7556	0.1087742	0.0415122	0.5742898	0.7483	0.1077835	0.0420299	0.5840120	0.7579
		Δ	0.0149958	0.0569762	0.9219693	0.9796	0.0141448	0.0569572	0.9217030	0.9773	0.0139956	0.0566481	0.9189326	0.9734
	II	ρ	0.0590036	0.0142943	0.3427446	0.7781	0.0529279	0.0156013	0.3736531	0.7893	0.0620124	0.0134358	0.3217695	0.7613
		λ	0.1001853	0.0434441	0.6129014	0.7702	0.0857503	0.0446047	0.6464507	0.7615	0.1077610	0.0420120	0.5837770	0.7559
		Δ	0.0111262	0.0555355	0.9087240	0.9542	0.0057311	0.0532156	0.8862410	0.9001	0.0139709	0.0566482	0.9189291	0.9728
	III	ρ	0.0594170	0.0141703	0.3401905	0.7814	0.0552299	0.0150335	0.3604461	0.7826	0.0619422	0.0134522	0.3221124	0.7609
		λ	0.1012420	0.0432653	0.6098925	0.7732	0.0912217	0.0439973	0.6316049	0.7596	0.1075778	0.0420270	0.5841566	0.7562
		Δ	0.0113792	0.0557139	0.9103978	0.9581	0.0076706	0.0541806	0.8956163	0.9214	0.0138572	0.0566228	0.9186953	0.9720
(20,12)	I	ρ	0.0274156	0.0032256	0.1626660	0.7719	0.0272011	0.0031841	0.1617597	0.7752	0.0271755	0.0032359	0.1627422	0.7655
		λ	0.0505603	0.0111200	0.3080009	0.7708	0.0502064	0.0110021	0.3066167	0.7747	0.0500570	0.0111324	0.3078386	0.7647
		Δ	0.0068623	0.0246272	0.6114814	0.9769	0.0074449	0.0246558	0.6118408	0.9780	0.0074233	0.0246113	0.6112521	0.9744
	II	ρ	0.0264531	0.0036779	0.1785982	0.7887	0.0253864	0.0039738	0.1893665	0.7982	0.0271650	0.0032481	0.1631904	0.7647
		λ	0.0477822	0.0122304	0.3330539	0.7829	0.0451727	0.0129317	0.3496698	0.7876	0.0500126	0.0111626	0.3085644	0.7641
		Δ	0.0062699	0.0242145	0.6059324	0.9451	0.0047812	0.0239243	0.6019526	0.9229	0.0073837	0.0246009	0.6111155	0.9729
	III	ρ	0.0268702	0.0034961	0.1720771	0.7796	0.0263060	0.0035932	0.1764825	0.7905	0.0271569	0.0032536	0.1633889	0.7644
		λ	0.0489531	0.0117937	0.3228657	0.7767	0.0476755	0.0120354	0.3301333	0.7859	0.0499862	0.0111769	0.3088877	0.7639
		Δ	0.0068694	0.0243852	0.6082447	0.9602	0.0061624	0.0242910	0.6069576	0.9520	0.0073839	0.0245963	0.6110542	0.9727
(30,10)	I	ρ	0.0335102	0.0045755	0.1924975	0.7734	0.0333348	0.0045632	0.1919662	0.7677	0.0333986	0.0046140	0.1942514	0.7739
		λ	0.0610859	0.0154592	0.3615524	0.7728	0.0607331	0.0153957	0.3603460	0.7662	0.0608079	0.0155952	0.3648422	0.7725
		Δ	0.0090110	0.0302981	0.6772082	0.9751	0.0086157	0.0302944	0.6771284	0.9748	0.0087616	0.0302733	0.6769275	0.9763
	II	ρ	0.0311298	0.0055325	0.2223317	0.7963	0.0288354	0.0062805	0.2453153	0.8032	0.0333909	0.0046157	0.1942493	0.7756
		λ	0.0546173	0.0176528	0.4068264	0.7855	0.0488433	0.0191104	0.4385988	0.7823	0.0607893	0.0155965	0.3647908	0.7738
		Δ	0.0057684	0.0293545	0.6656364	0.9297	0.0036912	0.0284750	0.6543745	0.8711	0.0087961	0.0302709	0.6768965	0.9762
	III	ρ	0.0315705	0.0053644	0.2173951	0.7944	0.0298371	0.0059134	0.2341346	0.8021	0.0333701	0.0046236	0.1944414	0.773
		λ	0.0557841	0.0172952	0.3997688	0.7862	0.0514101	0.0183925	0.4232141	0.7837	0.0607329	0.0156134	0.3650473	0.7715
		Δ	0.0062017	0.0295326	0.6678581	0.9380	0.0045549	0.0289090	0.6599558	0.8994	0.0087616	0.0302630	0.6767998	0.9756
(30,20)	I	ρ	0.0156684	0.0012409	0.1027082	0.7702	0.0156328	0.0012284	0.1019473	0.7635	0.0156163	0.0012471	0.1032183	0.7707
		λ	0.0296295	0.0044721	0.1978643	0.7696	0.0295816	0.0044371	0.1965359	0.7637	0.0295166	0.0044971	0.1988627	0.7705
		Δ	0.0040049	0.0141436	0.4646368	0.9767	0.0050797	0.0141531	0.4648012	0.9771	0.0050690	0.0141403	0.4645835	0.9760
	II	ρ	0.0151228	0.0015051	0.1161561	0.7863	0.0147613	0.0016487	0.1233158	0.7941	0.0156023	0.0012498	0.1032498	0.7697
		λ	0.0280672	0.0052412	0.2207643	0.7813	0.0270887	0.0056358	0.2326061	0.7862	0.0294835	0.0045034	0.1988692	0.7695
		Δ	0.0039334	0.0139402	0.4611026	0.9363	0.0031630	0.0138211	0.4589962	0.9158	0.0050644	0.0141379	0.4645403	0.9748
	III	ρ	0.0153690	0.0013854	0.1101935	0.7794	0.0152084	0.0014355	0.1128121	0.7860	0.0155975	0.0012512	0.1033053	0.7694
		λ	0.0287705	0.0048985	0.2107182	0.7772	0.0283568	0.0050407	0.2151519	0.7816	0.0294714	0.0045073	0.1989563	0.7692
		Δ	0.0044002	0.0140340	0.4627395	0.9533	0.0039987	0.0139938	0.4620353	0.9470	0.0050458	0.0141367	0.4645204	0.9747

Table 10. Bootstrap: absolute value of bias ($|Bias|$), length L, mean squared error (MSE) & coverage probability (Cov), when $R = 0.003, \rho = 0.1, \lambda = 0.01$ and $\Delta = 0.02$

(n,m)	Scheme		$T_1 = X_{[\frac{m}{4}]}$				$T_2 = X_{[\frac{4\times m}{5}]}$				$T_3 = X_{[m]+2}$			
		Estimate	Bias	MSE	L	Cov	Bias	MSE	L	Cov	Bias	MSE	L	Cov
(20,6)	I	ρ	0.0046081	0.0013357	0.1347898	0.937	0.0048502	0.0014364	0.1386855	0.938	0.0050730	0.0015668	0.1410920	0.950
		λ	0.0024443	0.0001274	0.0388790	0.937	0.0026011	0.0001456	0.0406242	0.938	0.0027037	0.0002213	0.0415232	0.950
		Δ	0.0026371	0.0001966	0.0494241	0.937	0.0027893	0.0002179	0.0512655	0.938	0.0029017	0.0002768	0.0521720	0.950
	II	ρ	0.0066111	0.0019649	0.1648653	0.945	0.0106885	0.0038318	0.2182345	0.947	0.0050844	0.0015695	0.1412640	0.949
		λ	0.0037282	0.0002337	0.0521711	0.945	0.0068181	0.0007349	0.0832756	0.947	0.0027126	0.0002202	0.0415943	0.949
		Δ	0.0038863	0.0003249	0.0633862	0.945	0.0066419	0.000823	0.0924787	0.947	0.0029099	0.0002767	0.0522521	0.949
	III	ρ	0.0063645	0.0018742	0.1615085	0.944	0.0089978	0.0030300	0.1966780	0.939	0.0051217	0.0015745	0.1420107	0.948
		λ	0.0035624	0.0002147	0.0506244	0.944	0.0054750	0.0004896	0.0698516	0.939	0.002739	0.0002168	0.0419165	0.948
		Δ	0.0037297	0.0003040	0.0617960	0.944	0.005468	0.0005939	0.0802618	0.939	0.0029349	0.0002756	0.0526018	0.948
(20,12)	I	ρ	0.0022029	0.0005905	0.0902794	0.952	0.0023185	0.0006114	0.0911821	0.945	0.0023881	0.0006159	0.0938341	0.942
		λ	0.0010644	0.0000420	0.0227563	0.952	0.0011107	0.0000466	0.0230267	0.945	0.0011548	0.0000464	0.0238634	0.942
		Δ	0.0012076	0.0000739	0.0308529	0.952	0.0012622	0.0000794	0.0311767	0.945	0.0013090	0.0000797	0.0322213	0.942
	II	ρ	0.0032377	0.0009558	0.1110832	0.934	0.0043640	0.0012504	0.1293809	0.948	0.0024064	0.0006184	0.0941866	0.941
		λ	0.0016392	0.0000820	0.0297519	0.934	0.0022711	0.0001257	0.0365059	0.948	0.0011642	0.0000464	0.0239735	0.941
		Δ	0.0018166	0.0001325	0.0391451	0.934	0.0024721	0.0001886	0.0468364	0.948	0.0013196	0.0000799	0.0323546	0.941
	III	ρ	0.0027804	0.0007984	0.1022750	0.933	0.0032695	0.0009230	0.1105440	0.940	0.0024179	0.0006211	0.0944156	0.943
		λ	0.0013812	0.0000633	0.0266740	0.933	0.0016383	0.0000806	0.0295029	0.940	0.0011702	0.0000467	0.0240477	0.943
		Δ	0.0015463	0.0001060	0.0355571	0.933	0.0018210	0.0001292	0.0388831	0.940	0.0013262	0.0000803	0.0324445	0.943
(30,10)	I	ρ	0.0027715	0.0007079	0.1005795	0.946	0.0028034	0.0006851	0.1018061	0.960	0.0028967	0.0007269	0.1038537	0.955
		λ	0.0013514	0.0000527	0.0259418	0.946	0.0013681	0.0000543	0.0262638	0.960	0.0014200	0.0000556	0.0270144	0.955
		Δ	0.0015242	0.0000907	0.0347748	0.946	0.0015427	0.0000911	0.0351963	0.960	0.0015975	0.0000948	0.0360749	0.955
	II	ρ	0.0049823	0.0013278	0.1395441	0.954	0.0073802	0.0022008	0.1753712	0.956	0.0029007	0.0007259	0.1039106	0.955
		λ	0.0026255	0.0001328	0.0405729	0.954	0.0042404	0.0002984	0.0572204	0.956	0.0014219	0.0000555	0.0270205	0.955
		Δ	0.002837	0.0002005	0.0513863	0.954	0.0043752	0.0003918	0.0684057	0.956	0.0015999	0.0000946	0.0360859	0.955
	III	ρ	0.0045327	0.0011971	0.1321595	0.950	0.0061363	0.0017654	0.1576577	0.954	0.0029140	0.0007261	0.1041405	0.955
		λ	0.0023509	0.0001127	0.0375232	0.950	0.0033952	0.0002120	0.0484930	0.954	0.0014292	0.0000555	0.0270846	0.955
		Δ	0.0025624	0.0001747	0.0480636	0.950	0.0035796	0.0002942	0.0596582	0.954	0.0016078	0.0000946	0.0361662	0.955
(30,20)	I	ρ	0.0013903	0.0002981	0.0685802	0.954	0.0013630	0.0002931	0.0692894	0.955	0.0013926	0.0003145	0.0701542	0.949
		λ	0.0006215	0.0000176	0.0162309	0.954	0.0006247	0.0000176	0.0165015	0.955	0.0006387	0.0000187	0.0167115	0.949
		Δ	0.0007261	0.0000335	0.0226553	0.954	0.0007248	0.0000334	0.0229861	0.955	0.0007412	0.0000356	0.0232667	0.949
	II	ρ	0.0022546	0.0005192	0.0888002	0.946	0.0028454	0.0007552	0.1031100	0.945	0.0014108	0.0003169	0.0705761	0.952
		λ	0.0010560	0.0000355	0.0220238	0.946	0.0014135	0.0000603	0.0269832	0.945	0.0006473	0.0000190	0.0168242	0.952
		Δ	0.0012118	0.0000633	0.0300585	0.946	0.0015836	0.0001008	0.0359506	0.945	0.0007513	0.0000360	0.0234161	0.952
	III	ρ	0.0018502	0.0004112	0.0797334	0.944	0.0020501	0.0005065	0.0866655	0.951	0.0014168	0.0003184	0.0707356	0.951
		λ	0.0008484	0.0000263	0.0193178	0.944	0.0009850	0.0000350	0.0216141	0.951	0.0006503	0.0000191	0.0168677	0.951
		Δ	0.0009824	0.0000483	0.0266586	0.944	0.0011233	0.0000624	0.0294629	0.951	0.0007547	0.0000362	0.0234731	0.951

Table 11. Bootstrap: absolute value of bias ($|Bias|$), length L, mean squared error (MSE) & coverage probability (Cov), when $R = 0.03, \rho = 0.1, \lambda = 0.01$ and $\Delta = 0.02$

(n,m)	Scheme		$T_1 = X_{[\frac{m}{4}]}$				$T_2 = X_{[\frac{4\times m}{5}]}$				$T_3 = X_{[m]+2}$			
		Estimate	Bias	MSE	L	Cov	Bias	MSE	L	Cov	Bias	MSE	L	Cov
(20,6)	I	ρ	0.0097409	0.0096547	0.3643612	0.937	0.0100960	0.0101931	0.3729073	0.938	0.0107014	0.0101712	0.3786757	0.950
		λ	0.0157744	0.0071669	0.2956564	0.937	0.0165164	0.0077681	0.3049557	0.938	0.0171821	0.0082555	0.3075351	0.950
		Δ	0.0107146	0.0047671	0.2481810	0.937	0.0112363	0.0051350	0.2556039	0.938	0.0118111	0.0054922	0.2588454	0.950
	II	ρ	0.0130639	0.0131683	0.4303414	0.945	0.0194634	0.0207306	0.5233829	0.947	0.0107160	0.0101948	0.3790738	0.949
		λ	0.0218570	0.0108950	0.3652007	0.945	0.0325285	0.0209561	0.4751116	0.947	0.0172182	0.0083005	0.3080081	0.949
		Δ	0.0150728	0.0070880	0.3040826	0.945	0.0230899	0.013537	0.3977712	0.947	0.0118327	0.0055154	0.2592159	0.949
	III	ρ	0.0126663	0.0127265	0.4231908	0.944	0.0169307	0.0177783	0.4882753	0.939	0.0107747	0.0102624	0.3807421	0.948
		λ	0.0211553	0.0103708	0.3576306	0.944	0.0280569	0.0170965	0.4327783	0.939	0.0173395	0.0083791	0.3099261	0.948
		Δ	0.0145537	0.0067556	0.2979062	0.944	0.0198414	0.0109869	0.3611974	0.939	0.0119055	0.0055579	0.2607006	0.948
(20,12)	I	ρ	0.0050751	0.0046399	0.2545171	0.952	0.0053370	0.0047259	0.2568724	0.945	0.0055021	0.0047718	0.2638985	0.942
		λ	0.0077441	0.0028468	0.1902737	0.952	0.0080666	0.0030179	0.1919660	0.945	0.0083638	0.0030523	0.1984353	0.942
		Δ	0.0052270	0.0020369	0.1647237	0.952	0.0054769	0.0021262	0.1662718	0.945	0.0056569	0.0021467	0.1713866	0.942
	II	ρ	0.0071497	0.0071456	0.3075101	0.934	0.0093346	0.0089167	0.3521897	0.948	0.0055424	0.0047947	0.2648175	0.941
		λ	0.0112822	0.0049409	0.2381870	0.934	0.0149129	0.0067326	0.2809415	0.948	0.0084294	0.0030595	0.1992133	0.941
		Δ	0.0076362	0.0033675	0.2032795	0.934	0.0101643	0.0044713	0.2374336	0.948	0.0057024	0.0021539	0.1720209	0.941
	III	ρ	0.0062459	0.0061019	0.2854485	0.933	0.0072450	0.0068705	0.3063514	0.940	0.0055660	0.0048139	0.2654118	0.943
		λ	0.0097434	0.0040170	0.2177584	0.933	0.0113276	0.0048034	0.2367405	0.940	0.0084690	0.0030744	0.1997453	0.943
		Δ	0.0065842	0.0027889	0.1869087	0.933	0.0076879	0.0032635	0.2022491	0.940	0.0057294	0.0021637	0.1724481	0.943
(30,10)	I	ρ	0.0062923	0.0054996	0.2816658	0.946	0.0063826	0.0052381	0.2850578	0.960	0.0065560	0.0056060	0.2901491	0.955
		λ	0.0096741	0.0034699	0.2136271	0.946	0.0097855	0.0034603	0.2160206	0.960	0.0101101	0.0036185	0.2213451	0.955
		Δ	0.0065480	0.0024496	0.1837223	0.946	0.0066299	0.0024024	0.1859244	0.960	0.0068403	0.0025314	0.1899111	0.955
	II	ρ	0.0105237	0.0094612	0.3762042	0.954	0.0145124	0.0139783	0.4518406	0.956	0.0065638	0.0055991	0.2903254	0.955
		λ	0.0169646	0.0072106	0.3066002	0.954	0.0240152	0.0124375	0.3862158	0.956	0.0101250	0.0036119	0.2213976	0.955
		Δ	0.0115521	0.0047639	0.2571019	0.954	0.0167405	0.0080101	0.3226124	0.956	0.0068511	0.0025272	0.1899854	0.955
	III	ρ	0.0097041	0.0087018	0.3592105	0.950	0.0124666	0.0117858	0.4159667	0.954	0.0065894	0.0056006	0.2909445	0.955
		λ	0.0155328	0.0063887	0.2889073	0.950	0.0204876	0.0098838	0.3465038	0.954	0.0101738	0.0036124	0.2218581	0.955
		Δ	0.0105519	0.0042661	0.2431543	0.950	0.0141183	0.0064061	0.2900977	0.954	0.0068846	0.0025276	0.1903848	0.955
(30,20)	I	ρ	0.0033985	0.0024416	0.1964889	0.954	0.0033022	0.0023922	0.1982941	0.955	0.0033627	0.0025711	0.2007221	0.949
		λ	0.0047609	0.0013002	0.1404614	0.954	0.0047548	0.0012991	0.1426166	0.955	0.0048595	0.0013826	0.1442947	0.949
		Δ	0.0032634	0.0009929	0.1242022	0.954	0.0032296	0.0009845	0.1257738	0.955	0.0033015	0.0010514	0.1272548	0.949
	II	ρ	0.0052820	0.0041207	0.2513358	0.946	0.0063673	0.0057614	0.2876432	0.945	0.0034032	0.0025882	0.2018945	0.952
		λ	0.0078083	0.0024337	0.1855100	0.946	0.0100001	0.0038241	0.2204385	0.945	0.0049241	0.0013972	0.1452183	0.952
		Δ	0.0053209	0.0017678	0.1614796	0.946	0.0067428	0.0026481	0.1888532	0.945	0.0033466	0.0010606	0.1280348	0.952
	III	ρ	0.0044118	0.003312	0.2270663	0.944	0.0047611	0.0040067	0.2450832	0.951	0.0034171	0.0025996	0.2023344	0.951
		λ	0.0063848	0.0018689	0.1649277	0.944	0.0072477	0.0024095	0.1820473	0.951	0.0049460	0.0014049	0.1455689	0.951
		Δ	0.0043622	0.0013869	0.1446639	0.944	0.0048885	0.0017393	0.1580778	0.951	0.0033615	0.0010659	0.1283288	0.951

Table 12. Bootstrap results: absolute value of bias ($|Bias|$), length L, mean squared error (MSE) & coverage probability (Cov), when $R = 0.2, \rho = 0.7, \lambda = 0.5$ and $\Delta = 0.42$

(n,m)	Scheme		$T_1 = X_{[\frac{m}{4}]}$				$T_2 = X_{[\frac{4\times m}{5}]}$				$T_3 = X_{[m]+2}$			
			Bias	MSE	L	Cov	Bias	MSE	L	Cov	Bias	MSE	L	Cov
(20,6)	I	ρ	0.0141394	0.0173213	0.4851841	0.936	0.0145958	0.0168466	0.4907228	0.945	0.0154351	0.0168592	0.5030400	0.960
		λ	0.0248150	0.0327084	0.6737593	0.936	0.0252386	0.0316914	0.6822360	0.945	0.0258206	0.0312920	0.6968440	0.960
		Δ	0.0206772	0.0239120	0.5957980	0.936	0.0208736	0.0232406	0.6052197	0.945	0.0205663	0.0227013	0.6204231	0.960
	II	ρ	0.0212542	0.0205052	0.5397494	0.975	0.0317404	0.0240698	0.5926571	0.976	0.0154326	0.0168694	0.5032963	0.961
		λ	0.0375534	0.0363193	0.7306063	0.975	0.0545288	0.0388452	0.7811766	0.976	0.0258696	0.0313106	0.6969537	0.961
		Δ	0.0302423	0.0266497	0.6580472	0.975	0.0461813	0.0284703	0.7165494	0.976	0.0206154	0.0227084	0.6202409	0.961
	III	ρ	0.0167805	0.0186198	0.499659	0.951	0.0285407	0.022935	0.5770888	0.975	0.0155263	0.0169781	0.5040291	0.964
		λ	0.0294620	0.0347652	0.6898956	0.951	0.0491613	0.0380757	0.7677236	0.975	0.0261829	0.0314877	0.6974452	0.964
		Δ	0.0229306	0.0254761	0.6168704	0.951	0.0410265	0.0279932	0.7015736	0.975	0.0208371	0.0228436	0.6205804	0.964
(20,12)	I	ρ	0.0062522	0.0100786	0.3784570	0.947	0.0064308	0.0091113	0.3839179	0.957	0.0067700	0.0102060	0.3896721	0.955
		λ	0.0111692	0.0211436	0.5469973	0.947	0.0115208	0.0195308	0.5550169	0.957	0.0121018	0.0214703	0.5625002	0.955
		Δ	0.0093952	0.0142180	0.4550504	0.947	0.0098415	0.0133135	0.4608642	0.957	0.0101267	0.0146796	0.4707832	0.955
	II	ρ	0.0102894	0.0132533	0.4431201	0.952	0.0141285	0.0153531	0.4867802	0.958	0.0067968	0.0102199	0.3905714	0.955
		λ	0.0182029	0.0269432	0.6297215	0.952	0.0247680	0.0296295	0.6781844	0.958	0.0121632	0.0214814	0.5634463	0.955
		Δ	0.0147740	0.0193593	0.5468504	0.952	0.0201619	0.0216598	0.5976450	0.958	0.0101707	0.0146867	0.4715560	0.955
	III	ρ	0.0070016	0.0109277	0.3837849	0.933	0.0106113	0.0128473	0.4498006	0.956	0.0068356	0.0102462	0.3913009	0.955
		λ	0.0120188	0.0228443	0.5528494	0.933	0.0185282	0.0258794	0.6361695	0.956	0.0122382	0.0215189	0.5642711	0.955
		Δ	0.0099236	0.0157058	0.4621617	0.933	0.0153227	0.0185765	0.549465	0.956	0.0102173	0.0147184	0.4723755	0.955
(30,10)	I	ρ	0.0063859	0.0102970	0.3811191	0.946	0.0065258	0.0092576	0.3864415	0.960	0.0067700	0.0102060	0.3896721	0.955
		λ	0.0114118	0.0215415	0.5503783	0.946	0.0117229	0.0198060	0.5582560	0.960	0.0121018	0.0214703	0.5625002	0.955
		Δ	0.0095755	0.0145333	0.4587801	0.946	0.0099746	0.0135339	0.4643317	0.960	0.0101267	0.0146796	0.4707832	0.955
	II	ρ	0.0121409	0.0145562	0.4640182	0.954	0.0193504	0.0183464	0.5289440	0.975	0.0067787	0.0102024	0.3901201	0.955
		λ	0.0214772	0.0290525	0.6541411	0.954	0.0337041	0.0333715	0.7223986	0.975	0.0121252	0.0214542	0.5629440	0.955
		Δ	0.0174493	0.0212052	0.5749933	0.954	0.0271128	0.0245698	0.6482635	0.975	0.0101429	0.0146647	0.4710832	0.955
	III	ρ	0.0109696	0.0139877	0.4505301	0.950	0.0159464	0.0165252	0.5018451	0.962	0.0068184	0.0102117	0.3909943	0.955
		λ	0.0194377	0.0281156	0.6378461	0.950	0.0278609	0.0312857	0.6940647	0.962	0.0121964	0.0214546	0.5639524	0.955
		Δ	0.0156113	0.0203021	0.5563984	0.950	0.0225508	0.0229354	0.6162352	0.962	0.0101831	0.0146654	0.4719769	0.955
(30,20)	I	ρ	0.0032899	0.0053282	0.2867375	0.953	0.0033912	0.0050746	0.2872892	0.955	0.0035233	0.0055372	0.2909235	0.949
		λ	0.0058104	0.0114351	0.4219052	0.953	0.0055839	0.0111278	0.4246590	0.955	0.0059573	0.0119694	0.4288250	0.949
		Δ	0.0053432	0.0068492	0.3288784	0.953	0.0051606	0.0067049	0.3318073	0.955	0.0052665	0.0072067	0.3357459	0.949
	II	ρ	0.0050797	0.0082811	0.3523053	0.946	0.0067238	0.0101398	0.3842908	0.945	0.0035242	0.0055508	0.2924221	0.952
		λ	0.0091046	0.0173277	0.5117599	0.946	0.0117173	0.0215014	0.5568929	0.945	0.0059851	0.0120183	0.4310227	0.952
		Δ	0.0081745	0.0111856	0.4155083	0.946	0.0100717	0.0148207	0.4668933	0.945	0.0053283	0.0072512	0.3376696	0.952
	III	ρ	0.0041932	0.0069043	0.3248072	0.944	0.0049234	0.0077694	0.3408670	0.951	0.0035335	0.0055702	0.2929728	0.951
		λ	0.0075741	0.0146285	0.4743576	0.944	0.0081987	0.0167407	0.4991944	0.951	0.0060020	0.0120601	0.4318050	0.951
		Δ	0.0068491	0.0091507	0.3778483	0.944	0.0072995	0.0108885	0.4052772	0.951	0.0053489	0.0072820	0.3383823	0.951

Table 13. Bootstrap results: absolute value of bias ($|Bias|$), length L, mean squared error (MSE) & coverage probability (Cov), when $R = 0.8, \rho = 0.98, \lambda = 0.96$ and $\Delta = 0.85$

(n,m)	Scheme		$T_1 = X_{[\frac{m}{4}]}$				$T_2 = X_{[\frac{4\times m}{5}]}$				$T_3 = X_{[m]}+2$			
			Bias	MSE	L	Cov	Bias	MSE	L	Cov	Bias	MSE	L	Cov
(20,6)	I	ρ	0.0256331	0.0092335	0.3190778	0.972	0.0268439	0.0101879	0.3246501	0.963	0.0275662	0.0101138	0.3362773	0.975
		λ	0.0418709	0.0234674	0.5198999	0.972	0.0441155	0.0250764	0.5267932	0.963	0.0446518	0.0252833	0.5428017	0.975
		Δ	0.0401992	0.0170452	0.5551702	0.972	0.0418672	0.0178359	0.5612405	0.963	0.0438236	0.0178230	0.5725074	0.975
	II	ρ	0.0345148	0.0137594	0.3814834	0.968	0.0484404	0.0208063	0.4699428	0.978	0.0276257	0.0100530	0.3363941	0.974
		λ	0.0560001	0.0326848	0.5982563	0.968	0.0776273	0.0457035	0.6994106	0.978	0.0447455	0.0251461	0.5430846	0.974
		Δ	0.0538555	0.0215869	0.6141477	0.968	0.0745706	0.0281688	0.6929044	0.978	0.0439522	0.0177296	0.5728792	0.974
	III	ρ	0.0335026	0.0132024	0.3743334	0.966	0.0428451	0.0170800	0.4347689	0.974	0.0278334	0.0100581	0.3370045	0.973
		λ	0.0545443	0.0315727	0.5895938	0.966	0.0689110	0.0386669	0.6618134	0.974	0.0450844	0.0251302	0.5438824	0.973
		Δ	0.0528126	0.0210208	0.6077113	0.966	0.0663987	0.0241874	0.6648531	0.974	0.0442911	0.0176947	0.5736903	0.973
(20,12)	I	ρ	0.0137872	0.0042226	0.2223206	0.980	0.0166589	0.0044808	0.2488951	0.983	0.0172531	0.0052693	0.2538862	0.979
		λ	0.0225551	0.0121702	0.3842989	0.980	0.0269685	0.0127342	0.4248585	0.983	0.0279873	0.0146244	0.4311325	0.979
		Δ	0.0198903	0.0118252	0.4445881	0.980	0.0234981	0.0114650	0.4802916	0.983	0.0247607	0.0126795	0.4849411	0.979
	II	ρ	0.0192864	0.0067740	0.2735534	0.967	0.0303645	0.0102381	0.3531977	0.981	0.0173158	0.0052887	0.2549676	0.981
		λ	0.0314269	0.0182382	0.4581346	0.967	0.0488903	0.0253840	0.5656754	0.981	0.0280648	0.0146780	0.4327294	0.981
		Δ	0.0283435	0.0146430	0.5049036	0.967	0.0459228	0.0172494	0.5919784	0.981	0.0248221	0.0126908	0.4861427	0.981
	III	ρ	0.0169406	0.0057145	0.2525194	0.968	0.0245736	0.0076393	0.3120757	0.985	0.0173914	0.0053153	0.2557385	0.980
		λ	0.0276082	0.0157983	0.4282597	0.968	0.0397132	0.0199333	0.5125891	0.985	0.0281731	0.0147411	0.4338447	0.980
		Δ	0.0245232	0.0135349	0.4806442	0.968	0.0361879	0.0146676	0.5504559	0.985	0.0248841	0.0127107	0.4870471	0.980
(30,10)	I	ρ	0.0165870	0.0053597	0.2477558	0.972	0.0169271	0.0045830	0.2511586	0.985	0.0172531	0.0052693	0.2538862	0.979
		λ	0.0268764	0.0149466	0.4214594	0.972	0.0273922	0.0129815	0.4281182	0.985	0.0279873	0.0146244	0.4311325	0.979
		Δ	0.0237060	0.0131300	0.4755201	0.972	0.0238892	0.0115883	0.4830445	0.985	0.0247607	0.0126795	0.4849411	0.979
	II	ρ	0.0273474	0.0088989	0.3270972	0.982	0.0379352	0.0141299	0.4061639	0.980	0.0172769	0.0052730	0.2544561	0.981
		λ	0.0445708	0.0225453	0.5316133	0.982	0.0607604	0.0332310	0.6297802	0.980	0.0280079	0.0146385	0.4319803	0.981
		Δ	0.0421065	0.0161027	0.5659168	0.982	0.0571185	0.0212488	0.6403437	0.980	0.0247848	0.0126762	0.4854938	0.981
	III	ρ	0.0252627	0.0083255	0.3126206	0.978	0.0330044	0.0115788	0.3721305	0.985	0.0173590	0.0052915	0.2553454	0.981
		λ	0.0412146	0.0213851	0.5120882	0.978	0.0530268	0.0282407	0.589123	0.985	0.0281162	0.0146826	0.4332893	0.981
		Δ	0.0389276	0.0157217	0.5501774	0.978	0.0496806	0.0187254	0.609433	0.985	0.0248733	0.0126816	0.4867085	0.981
(30,20)	I	ρ	0.0084865	0.0025045	0.1770101	0.966	0.0088860	0.0022436	0.1745364	0.973	0.0089539	0.002556	0.1781497	0.973
		λ	0.0138249	0.0076836	0.3149329	0.966	0.0145944	0.0069664	0.3113451	0.973	0.0146635	0.0078201	0.3166548	0.973
		Δ	0.0086141	0.0087570	0.3795276	0.966	0.0091197	0.0083647	0.3792337	0.973	0.0098799	0.009058	0.3827891	0.973
	II	ρ	0.0134685	0.0042626	0.2243817	0.971	0.0173353	0.0051381	0.2477537	0.975	0.0090524	0.0025514	0.1790548	0.974
		λ	0.0216885	0.0122635	0.3876439	0.971	0.0282564	0.0142235	0.4218728	0.975	0.0148138	0.0078103	0.318179	0.974
		Δ	0.0174228	0.0115499	0.4464439	0.971	0.0251954	0.0124804	0.4788383	0.975	0.0098893	0.0090471	0.3843949	0.974
	III	ρ	0.0111650	0.0034057	0.2039765	0.970	0.0130339	0.0036929	0.2122456	0.975	0.0090894	0.0025609	0.1794043	0.974
		λ	0.0180013	0.0100853	0.3569524	0.970	0.0213336	0.0107525	0.3694064	0.975	0.0148699	0.0078367	0.3187352	0.974
		Δ	0.0132306	0.0102860	0.4190811	0.970	0.0166728	0.0108394	0.4340187	0.975	0.0099098	0.0090628	0.3849257	0.974

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