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Estimating Kappa Distribution Parameters: A Comparative Study of Maximum Likelihood and LQ-Moment Approaches

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Abstract The Kappa distribution, pioneered by researchers such as Hosking, is a widely applied continuous model in diverse scientific fields. This study delves into its practical utility, with a specific focus on amalgamating Gamma and Log-Normal distributions. The vital distributional parameters (α, β, θ) are subject to estimation through both Maximum Likelihood (MLE) and LQ-moment methods. Across a spectrum of sample sizes (25, 50,100, and 150), the LQ-moment method consistently exhibits superior performance compared to MLE. Furthermore, the research introduces two essential reliability metrics: Mean inactivity time (MIT) and stress strength reliability (SSR). MIT, influenced by the distribution parameters, provides insights into the temporal behavior of the random variable. The SSR evaluates the reliability of the system by accounting for the probability of component failure under stress conditions. The paper concludes with a comparative analysis of parameter estimation methods, emphasizing the enhanced accuracy of the LQ-moment approach, particularly noticeable in smaller sample sizes (50 and 100).

Keywords Kappa distributions, maximum likelihood, LQ - momente.

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1. Introduction

*I*n many significant real-life applications, extreme values of a random sample are of substantial importance. For example, in ecological studies, the occurrence of high levels of pollutants such as ozone, acid rain, or sulfur dioxide in the air is of critical concern [1]. Accurate modeling of such extreme events is essential for effective environmental monitoring and decision making. The Kappa distribution, introduced by Mielke and Johnson, has emerged as a powerful tool to represent data with positive skewness, particularly in cases where traditional distributions may fall short [2]. This distribution belongs to a broader family of probability distributions that have been developed to capture the complexities of asymmetric data in various fields.

The Kappa distribution is characterized by its flexibility in modeling skewed data, making it applicable in various domains, including hydrology, environmental science, and finance. Despite its potential, the Kappa distribution remains less familiar to those outside specialized areas of statistics. Therefore, this study seeks to provide a comprehensive introduction to the Kappa distribution, exploring its properties, estimation methods, and applications, thereby bridging the gap for a broader audience.

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2. Literature Review

The development and refinement of probability distributions have been central to statistical modeling, particularly in capturing the behavior of extreme values. The Kappa distribution is part of a lineage of distributions that have been extended and generalized to better accommodate data complexities. A significant advancement in this area was made by Gupta et al. [3], who generalized the standard exponential distribution by introducing an exponentiation parameter, giving rise to the exponentiated exponential family. This innovation paved the way for further extensions of well-known distributions.

Building on this foundation, Nadarajah (2005) [4] applied the generalization to the Gumbel distribution, creating an exponentiated version that enhances its flexibility in modeling extreme values. Similarly, Nadarajah and Gupta (2007) [5]extended the gamma distribution by exponentiation, demonstrating the versatility of this approach across different types of data. These developments underscore the importance of generalizing existing distributions to create more adaptable models for various applications. In 2006, Ani and Jemain[6] introduced the LQ-moments method, which has shown promise in outperforming traditional estimation methods, such as L-moments and maximum likelihood estimation (MLE), particularly in the context of small sample sizes. Their simulation studies demonstrated that the LQ-moments method consistently provides more accurate parameter estimates.

Research on parameter estimation for the Kappa distribution has been extensive. Samir et al. [7] investigated the use of MLE for the three-parameter Kappa distribution in Type II censored samples, deriving the asymptotic variance-covariance matrix. Their work highlighted the challenges and intricacies involved in parameter estimation for complex distributions. The practical utility of the Kappa distribution and its estimation methods has also been explored in hydrological studies. For example,Wan Zin et al. (2009) [8]applied both L-moment and LQ-moment methods to determine the best-fitting distribution for annual maximum rainfall data in Peninsular Malaysia. Their findings underscore the value of the Kappa distribution in real-world scenarios where accurate modeling of extreme events is essential. Further contributions to the estimation of Kappa distribution parameters were made by Hassun et al.(2014) [9],who introduced three approaches: maximum likelihood, maximum entropy, and L-moments. These methods offer different advantages depending on the nature of the data and the specific requirements of the analysis.

Phaphan and Ibrahim (2023) [10]developed two maximum likelihood estimation methods: the expectation maximization (EM) algorithm and the simulated annealing algorithm to estimate the three parameters of the weighted mixture generalized gamma distribution (WMGG), with the aim of improving the precision of parameter estimation for this distribution. The goal of the present study is to further advance the understanding and application of the Kappa distribution by developing two estimation methods, maximum likelihood and LQ moments, specifically for the three-parameter Kappa distribution. Furthermore, this study derives the rth central moment about the origin of the Kappa distribution, providing new information on its distributional properties. These methods are then applied to analyze rainfall data in Duhok, with the aim of accurately estimating the parameters of the three-parameter Kappa distribution (α, β, θ). The findings of this study contribute to the broader field of probability distributions, offering practical tools for researchers and practitioners dealing with asymmetric and extreme data.

3. Methodology

The Kappa distribution is characterized by its advantageous features, including flexibility, skewness, and a heavy tail. These attributes make it a valuable tool, particularly due to its explicit formulations for percentiles and moments. The distribution function $F(x)$ of the Kappa distribution is defined as follows:

$$F(x) = \left\{ \begin{array}{rcc} \left[\frac{(\frac{x}{\beta})^{\alpha\theta}}{\alpha+(\frac{x}{\beta})^{\alpha\theta}}\right]^{\frac{1}{\alpha}} & if & X > 0 \\ 0 & if & X \leq 0 \end{array} \right\} \tag{1}$$

Accompanying this, the corresponding quantile function $(X(F))$ is given by:

$$X(F) = \beta \left(\frac{\alpha F(x)^{\alpha}}{1 - F(x)^{\alpha}} \right)^{\frac{1}{\alpha\theta}} \tag{2}$$

Furthermore, the probability density function $f(x)$ takes the form:

$$f(x) = \left\{ \begin{array}{rcl} \frac{\alpha\theta}{\beta}(\frac{x}{\beta})^{\theta-1}\left(\alpha + (\frac{x}{\beta})^{\alpha\theta}\right)^{-(\frac{\alpha+1}{\alpha})} & if & X > 0 \\ 0 & if & X \leq 0 \end{array} \right\} \tag{3}$$

In this context, the shape parameters θ and α exclusively influence the right tail, while β impacts both tails. Figure (1) visually represents various shapes of the Kappa density function for different parameter choices, illustrating its versatility in capturing diverse distributional forms.

3.1. Interpretation of Parameters

The three-parameter Kappa distribution parameter θ, α and β have critical implications in various real-world scenarios.

Shape parameter α: Controls the distribution's tail behavior, particularly its skewness and the weight of extreme values. A higher α enhances the prominence of the right tail, making it suitable for modeling extreme phenomena such as rare but severe weather events, catastrophic insurance losses, or financial risks. In contrast, a lower α results in a more symmetric distribution.

Scale parameter β: Influences the spread or dispersion of the data. A larger β results in a broader range of values, providing flexibility in modeling data that spans a wide range, such as environmental pollutants or market price changes. Conversely, a smaller β concentrates values around a central point, making it suitable for tightly clustered data.

Shape parameter θ: Adjusts the decay rate of the tail, impacting the likelihood of extreme values. In contexts such as risk management or environmental studies, this parameter dictates how rapidly the probability of extreme occurrences diminishes. A lower θ suggests a slower decay rate, implying a higher risk or frequency of extreme events.

3.2. Analysis of Reliability

The Survival Function, also known as the complementary cumulative distribution function, for the Kappa distribution can be expressed as follows: The Survival Function, denoted as $S(x)$, represents the likelihood that a random variable X from the Kappa distribution is greater than a specific value x. Figure (2) depicts the reliability function for various (α,β,θ) values.

$$S(x) = 1 - F(x) \tag{4}$$

Here, $F(x)$ is the cumulative distribution function as given in Equation 1.

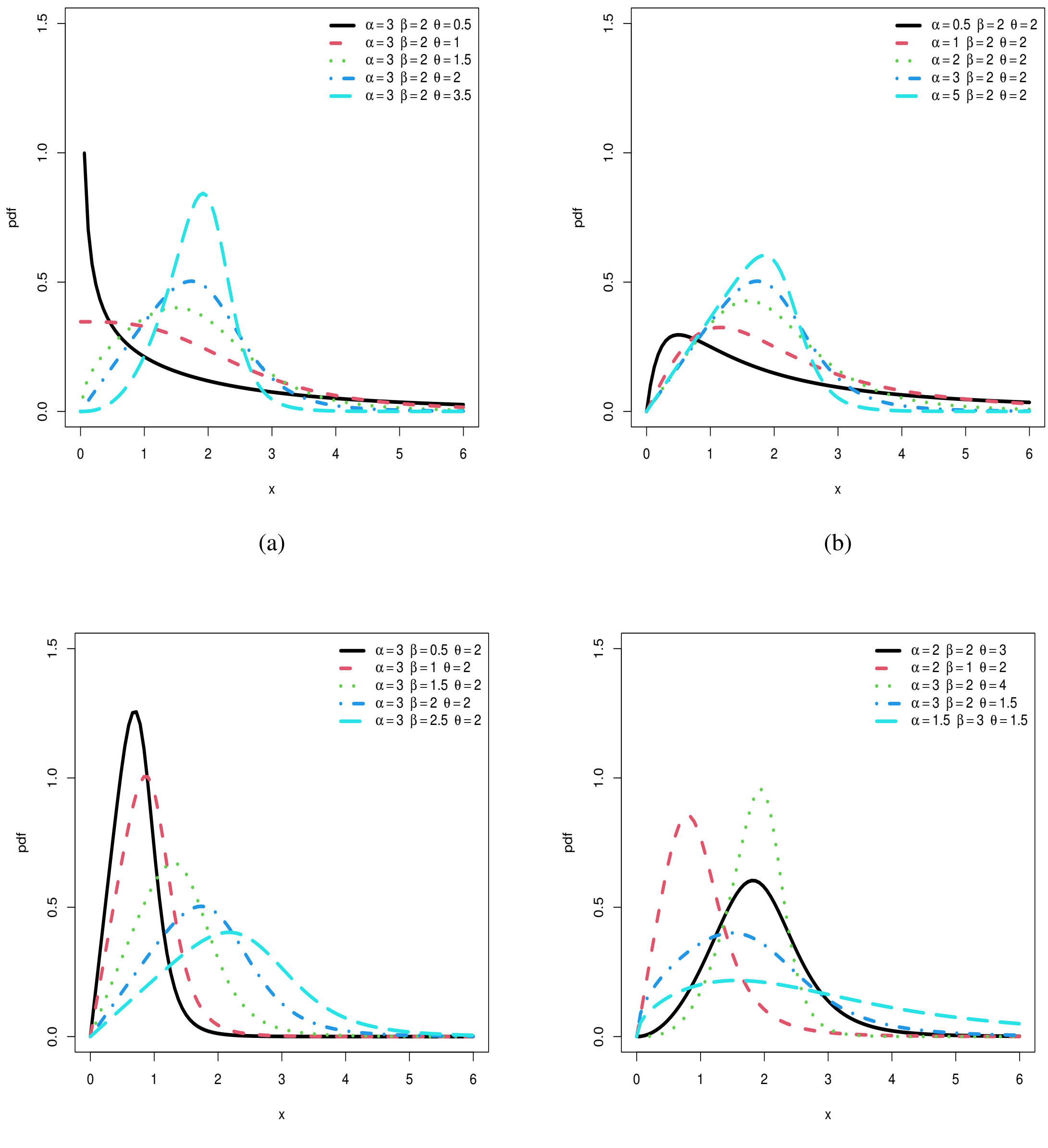
3.3. Hazard Rate Function

The Hazard Rate Function for the Kappa distribution, represented as $h(x)$, is defined as the instantaneous rate of failure at time x. It is calculated as the ratio of the probability density function to the survival function:

$$h(x) = \frac{f(x)}{S(x)} \tag{5}$$

In this equation, $f(x)$ is the probability density function as provided in Equation 3. Now, let's formulate the equations for the Survival Function and Hazard Function of the Kappa distribution: Survival Function:

$$S(x) = 1 - \left[\frac{(\frac{x}{\beta})^{\alpha\theta}}{\alpha + (\frac{x}{\beta})^{\alpha\theta}} \right]^{\frac{1}{\alpha}} \tag{6}$$



(a) (b) (c) (d)

Figure 1. various parameter choices and the shape of the Kappa density function.

Hazard Function:

$$h(x) = \frac{\frac{\alpha\theta}{\beta}(\frac{x}{\beta})^{\theta-1}\left(\alpha + (\frac{x}{\beta})^{\alpha\theta}\right)^{-(\frac{\alpha+1}{\alpha})}}{1 - \left[\frac{(\frac{x}{\beta})^{\alpha\theta}}{\alpha+(\frac{x}{\beta})^{\alpha\theta}}\right]^{\frac{1}{\alpha}}} \tag{7}$$

Figure (3) llustrates the Hazard Rate Function of the Kappa distribution with specified values for the parameters θ, β, and α.

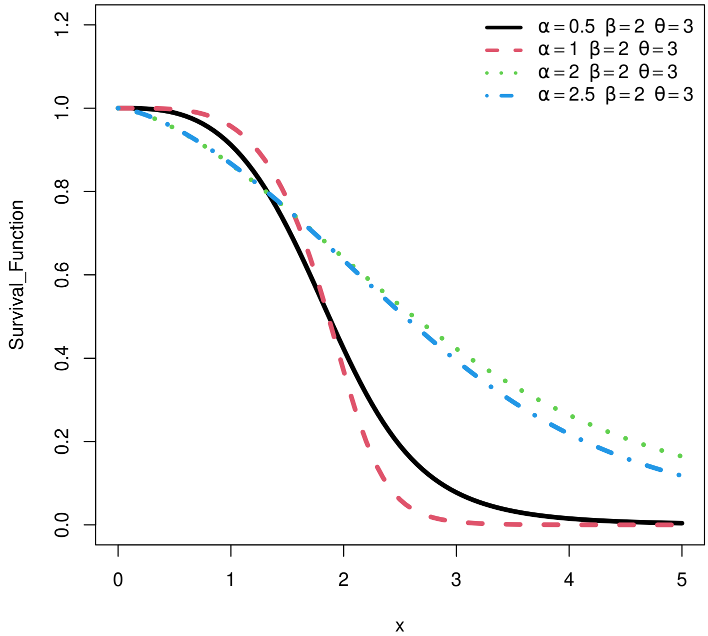


Figure 2. The survival analysis (Reliability Function) of the Kappa Distribution

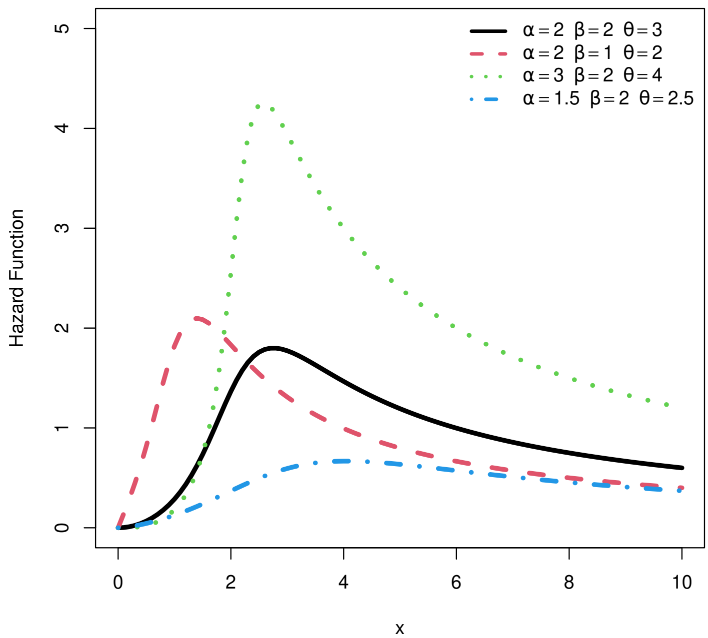


Figure 3. The Hazard Rate Function of the Kappa Distribution

3.4. *Reversed Hazard Rate Function*

The reversed hazard rate function (RHRF) of the Kappa distribution, denoted as $\lambda^*(x)$, can be expressed as the reciprocal of the distribution function's derivative with respect to x. Mathematically, it is defined as [11]:

$$\lambda^*(x) = \frac{f(x)}{F(x)}$$

where $f(x)$ is the probability density function (PDF) and $F(x)$ is the distribution function of the Kappa distribution provided in equations (1) and (3). Therefore, substituting the expressions from equations (1) and (3) into the reversed hazard rate function, we get:

$$\lambda^*(x) = \frac{\frac{\alpha\theta}{\beta}\left(\frac{x}{\beta}\right)^{\theta-1}\left(\alpha + \left(\frac{x}{\beta}\right)^{\alpha\theta}\right)^{-\left(\frac{\alpha+1}{\alpha}\right)}}{\left[\frac{\left(\frac{x}{\beta}\right)^{\alpha\theta}}{\alpha+\left(\frac{x}{\beta}\right)^{\alpha\theta}}\right]^{\frac{1}{\alpha}}} \tag{8}$$

The Reversed Hazard Rate Function, illustrated in Figure (4), varies based on the values of the parameters (α,β,θ)

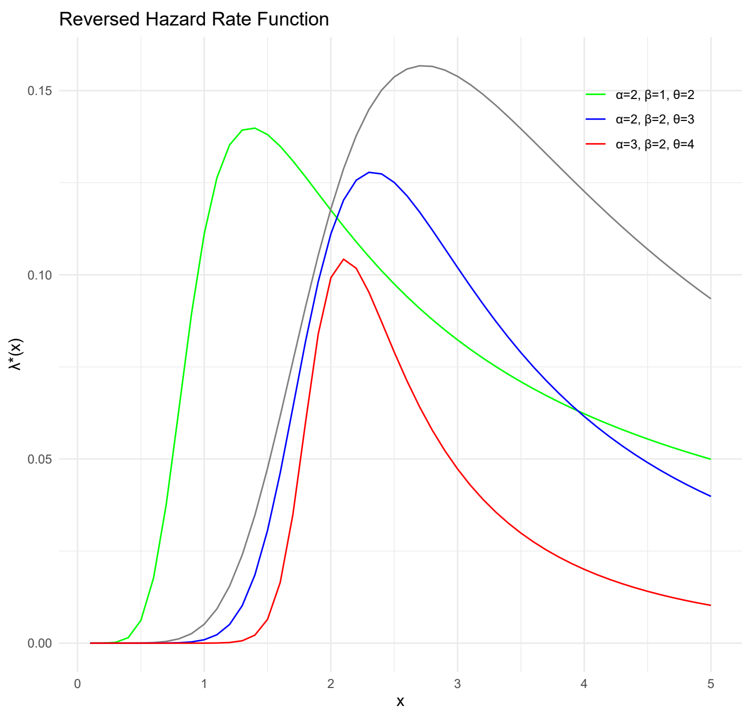


Figure 4. The Reversed Hazard Rate Function for the Kappa Distribution varies with the parameters (α,β,θ)

3.5. *Mean Residual Life (MRL)*

The Mean Residual Life (MRL) for the Kappa distribution is a crucial measure in survival analysis and reliability theory. It is defined as follows[12]:

$$MRL(t) = E(X - t \,|\, X > t)$$

The analytical expression for the Mean Residual Life, denoted as $MRL(t, \theta, \alpha, \beta)$, is given by:

$$MRL(t, \theta, \alpha, \beta) = \frac{1}{R(t)} \int_t^{\infty} x f(x)\, dx - t \tag{9}$$

where:

- $t > 0$ is a specific time point for which the MRL is calculated.
- θ, α, β are the parameters of the Kappa distribution.
- $R(t)$ is the survival function, representing the probability that X is greater than or equal to t with the given parameters.
- x is a variable representing time to failure.

The MRL provides valuable insights into the expected remaining lifetime of a system or process at a given time, considering its survival history. It is a fundamental concept in reliability engineering for assessing the longevity and performance of systems modeled by the Kappa distribution.The Mean Residual Life (MRL), illustrated in Figure (5), varies based on the values of the parameters (α,β,θ)

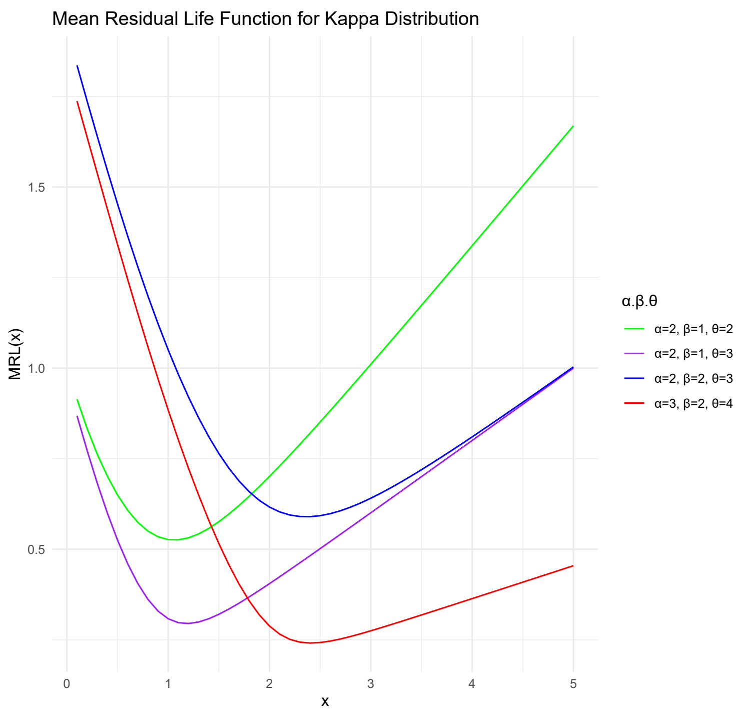


Figure 5. Mean Residual Life (MRL) for the Kappa Distribution varies with the parameters (α,β,θ)

3.6. *Mean Inactivity time*

The Mean Inactivity Time (MIT) for a lifetime random variable x is defined as[13]:

$$MIT(x) = t - (\frac{1}{F(t)} \int_t^{\infty} x f(x)) \, dx \quad t > 0 \tag{10}$$

where F(x) is the distribution function of the Kappa distribution. The Mean Inactivity time, illustrated in Figure (6), varies based on the values of the parameters (α,β,θ)

3.7. *Stress-Strength Reliability*

Based on the given distribution functions, the equation for Stress-Strength Reliability (SSR) can be expressed as[14, 15]:

$$SSR = \int_0^{\infty} f_1(t) \cdot [F_2(t)] \, dt \tag{11}$$

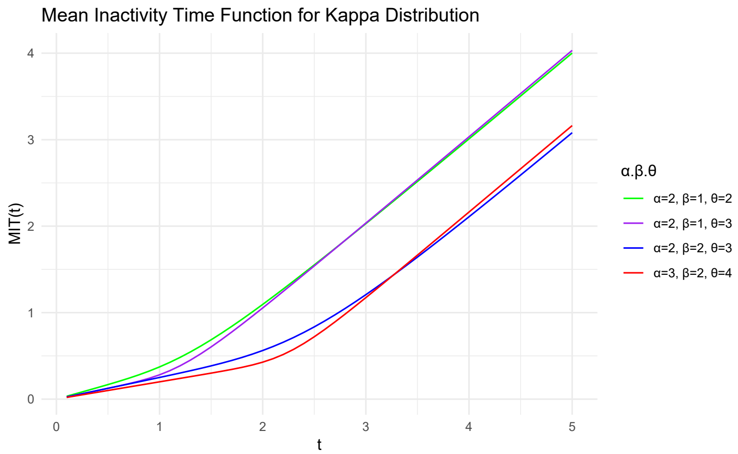


Figure 6. Mean Inactivity Time for the Kappa Distribution varies with the parameters (α,β,θ)

where:

$f_1(t)$ is the probability density function (PDF) of X with parameters $(\alpha_1, \beta_1, \theta_1)$.

$F_2(t)$ is the cumulative distribution function (CDF) of Z with parameters $(\alpha_2, \beta_2, \theta_2)$.

This integral represents the probability that the strength X is greater than the stress Z, indicating the reliability of the component. It considers the failure occurring when X $>$ Z.

$$L(\alpha_1, \beta_1, \theta_1, \alpha_2, \beta_2, \theta_2) = \prod_{i=1}^{n} f_1(x_i; \alpha_1, \beta_1, \theta_1) \cdot [F_2(x_i; \alpha_2, \beta_2, \theta_2)] \tag{12}$$

To maximize the likelihood, we typically take the logarithm (log-likelihood) and solve for the parameters:

$$\log L(\alpha_1, \beta_1, \theta_1, \alpha_2, \beta_2, \theta_2) = \sum_{i=1}^{n} \log f_1(x_i; \alpha_1, \beta_1, \theta_1) + \log[F_2(x_i; \alpha_2, \beta_2, \theta_2)] \tag{13}$$

To find the maximum likelihood estimates of the parameters, we take the derivative of the log-likelihood function with respect to each parameter, set them equal to zero, and solve for the parameters. This usually involves numerical methods as the solution often does not have a closed-form.

$$\frac{\partial}{\partial \alpha_1} \log L = 0, \quad , \frac{\partial}{\partial \beta_1} \log L = 0, \quad , \frac{\partial}{\partial \theta_1} \log L = 0, \quad , \frac{\partial}{\partial \alpha_2} \log L = 0, \quad , \frac{\partial}{\partial \beta_2} \log L = 0,$$
$$\frac{\partial}{\partial \theta_2} \log L = 0$$

These equations represent the conditions for finding the maximum likelihood estimates (MLE) of the parameters. The derivatives are taken with respect to each parameter, and then each derivative is set to zero to find the critical points. Due to the complexity of the distribution functions, numerical methods such as optimization algorithms are often used to find the MLE.

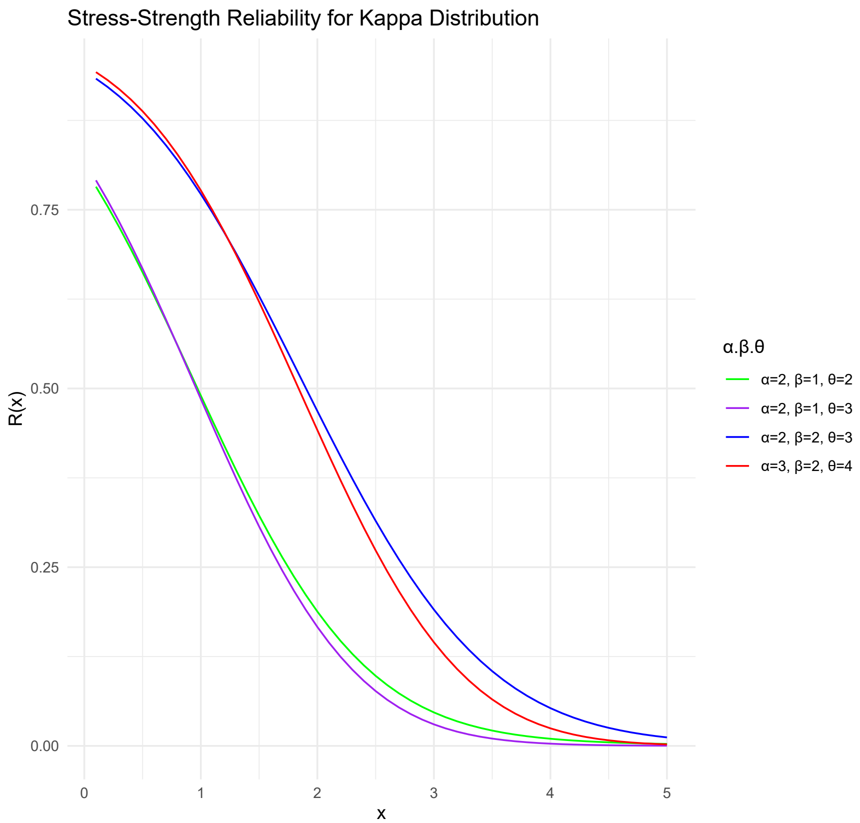


Figure 7. Stress-Strength Reliability Kappa Distribution varies with the parameters (α,β,θ)

the Table (1) compares the Mean Squared Error (MSE) of parameter estimates from two methods, LQM and MLE, at sample sizes 50 and 100.The MLE method appears to have a higher Mean Squared Error (MSE) than the LQM method for both sample sizes (50 and 100), suggesting that the LQM method provides more accurate estimates.

The Kappa distribution's rth central moment has the following formula:

$$E(x)^r = \acute{\mu}_r = \int_0^\infty x^r (\frac{\alpha\theta}{\beta})(\frac{x}{\beta})^{\theta-1}\left(\alpha + (\frac{x}{\beta})^{\alpha\theta}\right)^{-(\frac{\alpha+1}{\alpha})} dx \tag{14}$$

$$let\ u = \frac{x}{\beta} \Longrightarrow X = u\beta \Longrightarrow dx = \beta du$$

$$= \int_0^\infty (u\beta)^r (\frac{\alpha\theta}{\beta}) u^{\theta-1}\left(\alpha + u^{\alpha\theta}\right)^{-(\frac{\alpha+1}{\alpha})} \beta\, du$$

let $Z = u^{\alpha\theta} \Longrightarrow u = Z^{\frac{1}{\alpha\theta}}$

$$= \beta^r \int_0^\infty Z^{\frac{r+\theta}{\alpha\theta}-\frac{1}{\alpha\theta}} \alpha\theta\, (\alpha + Z)^{-(\frac{\alpha+1}{\alpha})} \frac{1}{\alpha\theta} Z^{\frac{1}{\alpha\theta}-1}\, dz$$

$$= \beta^r \alpha^{-(\frac{\alpha+1}{\alpha})} \int_0^\infty Z^{\frac{r}{\alpha\theta}+\frac{1}{\alpha}-1}\left(1 + \frac{Z}{\alpha}\right)^{-(\frac{\alpha+1}{\alpha})} dz$$

Table 1. MSE of the parameter estimations and a comparison of the two methods of estimation at the sample sizes (50,100) For the initial value set ($\alpha_1 = 1, \beta_1 = 1, \theta_1 = 1, \alpha_2 = 1, \beta_2 = 1, \theta_2 = 1$).

methods	Sample size	Parametes	estimate	AIC	BIC	MSE
LQM	50	α_1	1.00268	12.23520	23.70733	0.075250
		β_1	0.26348			
		θ_1	0.74396			
		α_2	0.85766			
		β_2	1.90169			
		θ_2	1.43216			
MLE	50	α_1	1.32350	-210.976	-199.504	7.338912
		β_1	2.09971			
		θ_1	1.45640			
		α_2	0.44394			
		β_2	0.00034			
		θ_2	0.26265			
LQM	100	α_1	0.95941	12.16747	27.79849	0.068774
		β_1	0.51882			
		θ_1	0.71163			
		α_2	0.87835			
		β_2	1.74434			
		θ_2	1.33885			
MLE	100	α_1	1.12350	-208.796	-192.414	7.016811
		β_1	1.9971			
		θ_1	1.05640			
		α_2	0.34394			
		β_2	0.00144			
		θ_2	0.17143			

let $y = \frac{Z}{\alpha} \Longrightarrow Z = \alpha y \Longrightarrow dz = \alpha dy$

$$= \beta^r \alpha^{-\left(\frac{\alpha+1}{\alpha}\right)} \int_0^\infty (\alpha y)^{\frac{r}{\alpha\theta}+\frac{1}{\alpha}-1} (1+y)^{-\left(\frac{\alpha+1}{\alpha}\right)} \alpha dy$$

$$= \beta^r \alpha^{\frac{r}{\alpha\theta}-1} \int_0^\infty \frac{(y)^{\frac{r}{\alpha\theta}+\frac{1}{\alpha}-1}}{(1+y)^{-\left(\frac{\alpha+1}{\alpha}\right)}} dy$$

comparison with the second Beta distribution formula

$$\beta(\alpha,\beta) = \int_0^\infty \frac{x^{\alpha-1}}{(1+x)^{\alpha+\beta}} dx = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

The two parameters' values are as follows.

$$\alpha = \frac{r}{\alpha\theta} + \frac{1}{\alpha}$$

$$\beta = (\alpha+\beta) - \alpha\beta = \frac{r}{\alpha\theta} + \frac{1}{\alpha} - \frac{1}{\alpha\theta}$$

After replacing (α, β) values, we obtain r^{th} Centeral Moment about Origin for Kappa distribution

$$E(x)^r = \beta^r \alpha^{\frac{r}{\alpha\theta}-1} \frac{\Gamma\left(\frac{r}{\alpha\theta} + \frac{1}{\alpha}\right)\Gamma\left(1 - \frac{r}{\alpha\theta}\right)}{\Gamma\left(1+\frac{1}{\alpha}\right)} \tag{15}$$

When solving equation (15) for the mean, we assume that $(r = 1)$.

$$E(x) = \mu = \beta\alpha^{\frac{1}{\alpha\theta}-1}\frac{\Gamma\left(\frac{1}{\alpha\theta}+\frac{1}{\alpha}\right)\Gamma\left(1-\frac{1}{\alpha\theta}\right)}{\Gamma\left(1+\frac{1}{\alpha}\right)} \tag{16}$$

Furthermore, the variance, skewness and kurtosis are obtained as follows:

$$E(x-\mu)^r = \int_0^\infty (x-\mu)^r f(x)\,dx \tag{17}$$

$$E(x-\mu)^r = \int_0^\infty (x-\mu)^r (\frac{\alpha\theta}{\beta})(\frac{x}{\beta})^{\theta-1}\left(\alpha+(\frac{x}{\beta})^{\alpha\theta}\right)^{-(\frac{\alpha+1}{\alpha})} dx$$

$$let\ u = \frac{x}{\beta} \Longrightarrow X = u\beta \Longrightarrow dx = \beta du$$

$$= \int_0^\infty (u\beta-\mu)^r(\frac{\alpha\theta}{\beta})u^{\theta-1}\left(\alpha+u^{\alpha\theta}\right)^{-(\frac{\alpha+1}{\alpha})}\beta\,du$$

similar to what was stated earlier, we obtain

$$E(x-\mu)^r = \sum_{j=0}^{r} C_j^r \beta^j \alpha^{\frac{j}{\alpha\theta}-1}(-\mu)^{r-j}\left(\frac{\Gamma(\frac{j}{\alpha\theta}+\frac{1}{\alpha})\Gamma(1-\frac{j}{\alpha\theta})}{\Gamma(1+\frac{1}{\alpha})}\right) \tag{18}$$

The variance is also obtained in the manner described below:

$$\begin{aligned} var(x) = \Bigg\{ &\beta^2\alpha^{-3+(\frac{2}{\alpha\theta})}\left(\frac{\Gamma(\frac{1+\theta}{\alpha\theta})\Gamma(\frac{\alpha\theta-1}{\alpha\theta})}{\Gamma(\frac{\alpha+1}{\alpha})}\right)^2\left(\frac{\Gamma(\frac{1}{\alpha})}{\Gamma(1+\frac{1}{\alpha})}\right) - 2\beta^2\alpha^{-2+(\frac{2}{\alpha\theta})}\left(\frac{\Gamma^2(\frac{1+\theta}{\alpha\theta})\Gamma^2(\frac{\alpha\theta-1}{\alpha\theta})}{\Gamma(\frac{\alpha+1}{\alpha})}\right) \\ &+ \beta^2\alpha^{-1+(\frac{2}{\alpha\theta})}\left(\frac{\Gamma(\frac{2}{\alpha\theta}+\frac{1}{\alpha})\Gamma(1-\frac{2}{\alpha\theta})}{\Gamma(1+\frac{1}{\alpha})}\right)\Bigg\} \end{aligned} \tag{19}$$

The following is the coefficient of skewness.

$$C.S = \frac{E(x-\mu)^3}{\sigma^3} \tag{20}$$

where

$$E(x-\mu)^3 = \beta^j\alpha^{\frac{j}{\alpha\theta}-1}\sum_{j=0}^{3} C_j^3(-\mu)^{3-j}\left(\frac{\Gamma(\frac{j}{\alpha\theta}+\frac{1}{\alpha})\Gamma(1-\frac{j}{\alpha\theta})}{\Gamma(1+\frac{1}{\alpha})}\right)$$

and

$$\begin{aligned} \sigma^3 = \Bigg\{ &\beta^2\alpha^{-3+(\frac{2}{\alpha\theta})}\left(\frac{\Gamma(\frac{1+\theta}{\alpha\theta})\Gamma(\frac{\alpha\theta-1}{\alpha\theta})}{\Gamma(\frac{\alpha+1}{\alpha})}\right)^2\left(\frac{\Gamma(\frac{1}{\alpha})}{\Gamma(1+\frac{1}{\alpha})}\right) - 2\beta^2\alpha^{-2+(\frac{2}{\alpha\theta})}\left(\frac{\Gamma^2(\frac{1+\theta}{\alpha\theta})\Gamma^2(\frac{\alpha\theta-1}{\alpha\theta})}{\Gamma(\frac{\alpha+1}{\alpha})}\right) \\ &+ \beta^2\alpha^{-1+(\frac{2}{\alpha\theta})}\left(\frac{\Gamma(\frac{2}{\alpha\theta}+\frac{1}{\alpha})\Gamma(1-\frac{2}{\alpha\theta}\alpha\theta)}{\Gamma(1+\frac{1}{\alpha})}\right)\Bigg\}^3 \end{aligned} \tag{21}$$

The kurtosis coefficient is as follows.

$$C.k = \frac{E(x-\mu)^4}{\sigma^4} \tag{22}$$

Equation (18) is used to obtain $E(x-\mu)^4$ by substituting r=4 and squaring equation (21) to obtain σ^4.

The coefficient of variation is one of the measures of relative dispersion. It is defined as the ratio between The standard deviation of a given distribution to the mean of that distribution can be calculated according to following formula:

$$C.v = \frac{\sqrt{\sigma^2}}{E(x)} \times 100 \tag{23}$$

The value of the coefficient of variation is the product of dividing the variance extracted in equation (19) On the expectation extracted in equation (16)

4. Various Estimation Techniques

Numerous estimating methods within the classical paradigm are documented in the statistical literature. However, we will only provide two of these techniques here: maximum likelihood and LQ-moment.

4.1. Maximum Likelihood Estimator

This section explains how to derive MLEs from a Kappa (α, β, θ) distribution's unknown parameters. Consider a sample of size n from a Kappa$(\alpha, \beta, theta)$ distribution as $X = (X_1, X_2, ..., X_n)$ [16, 17, 18]. The likelihood function can be given as follows based on the observation:

$$Lf(x_i, \alpha, \beta, \theta) = \prod_{i=1}^{n} \frac{\alpha\theta}{\beta}(\frac{x}{\beta})^{\theta-1} \left(\alpha + (\frac{x}{\beta})^{\alpha\theta}\right)^{-(\frac{\alpha+1}{\alpha})} \tag{24}$$

The log-likelihood function in formula (24) can then be expressed as follows:

$$\begin{aligned} \log Lf(x_i, \alpha, \beta, \theta) =& \{n \log\alpha + n\log\theta - n\log\beta + (\theta-1)\sum_{i=1}^{n} \log(\frac{x_i}{\beta}) \\ & - (\frac{\alpha+1}{\alpha})\sum_{i=1}^{n}\log(\alpha + (\frac{x_i}{\beta})^{\alpha\theta})\} \end{aligned} \tag{25}$$

In this case, we suppose that the parameters α, β and θ are unknown . We partially differentiate equation (25) with respect to α, β and θ and equal to zero as follows to obtain the normal equations for the unknown parameters:

$$\frac{\partial Lf(x_i, \alpha, \beta, \theta)}{\partial \alpha} = \left\{ \frac{n}{\alpha} + \frac{1}{\alpha^2}\sum_{i=1}^{n}\log(\alpha + (\frac{x}{\beta})^{\alpha\theta} - \frac{\alpha+1}{\alpha}\sum_{i=1}^{n}\frac{1+\theta(\frac{x}{\beta})^{\alpha\theta}\log(\frac{x}{\beta})}{\alpha + (\frac{x}{\beta})^{\alpha\theta}} \right\} = 0 \tag{26}$$

$$\widehat{\alpha}_{MLE} = \frac{n}{\left\{ -\frac{1}{\alpha^2}\sum_{i=1}^{n}\log(\alpha + (\frac{x}{\beta})^{\alpha\theta} + \frac{\alpha+1}{\alpha}\sum_{i=1}^{n}\frac{1+\theta(\frac{x}{\beta})^{\alpha\theta}\log(\frac{x}{\beta})}{\alpha+(\frac{x}{\beta})^{\alpha\theta}} \right\}} \tag{27}$$

$$\frac{\partial Lf(x_i, \alpha, \beta, \theta)}{\partial \beta} = -\frac{n\theta}{\beta} + \frac{(\alpha+1)\theta}{\beta}\sum_{i=1}^{n}\frac{(\frac{x}{\beta})^{\alpha\theta}}{\alpha + (\frac{x}{\beta})^{\alpha\theta}} = 0 \tag{28}$$

$$\widehat{\beta}_{MLE} = \frac{n\theta}{\frac{(\alpha+1)\theta}{\beta}\sum_{i=1}^{n}\frac{(\frac{x}{\beta})^{\alpha\theta}}{\alpha+(\frac{x}{\beta})^{\alpha\theta}}} \tag{29}$$

$$\frac{\partial Lf(x_i, \alpha, \beta, \theta)}{\partial \theta} = \frac{n}{\theta} - n\log\beta + \sum_{i=1}^{n}\log(x) - (\alpha+1)\sum_{i=1}^{n}\frac{(\frac{x}{\beta})^{\alpha\theta}\log(\frac{x}{\beta})}{\alpha + (\frac{x}{\beta})^{\alpha\theta}} = 0 \tag{30}$$

$$\widehat{\theta}_{MLE} = \frac{n}{n\log\beta - \sum_{i=1}^{n}\log(x) + (\alpha+1)\sum_{i=1}^{n}\frac{(\frac{x}{\beta})^{\alpha\theta}\log(\frac{x}{\beta})}{\alpha+(\frac{x}{\beta})^{\alpha\theta}}} \tag{31}$$

It has been noted that closed-form solutions for these estimations are not possible, hence the nonlinear equations must be solved numerically. Although the Newton-Raphson approach works well for calculating maximum likelihoods, it has problems with convergence, especially when dealing with complex likelihood functions like the Kappa distribution's. Due to its great sensitivity to initial values, if the initial values are not chosen correctly, this approach may result in sluggish convergence or divergence. Additionally, it could get stuck in saddle points or local minima. Strategies to improve convergence include regularisation to handle singular Hessians, global optimisation approaches, and well-chosen initial values. Furthermore, high-precision arithmetic and the use of line search techniques to modify step sizes can enhance numerical stability and accuracy in parameter estimation.

4.2. Linear Quantile Moment method (L-Q)

Let $X_1, X_2, ..., X_n$ be a random sample from a continuous distribution function F(x) with quantile [19] function

$$Q(F) = \beta\left(\frac{\alpha F(x)^{\alpha}}{1 - F(x)^{\alpha}}\right)^{\frac{1}{\alpha\theta}} \tag{32}$$

and let $X_{(1:n)} \leq .X_{(2:n)} \leq, ..., X_{(n:n)} \leq$ denote the order statistics. where (ϵ_r) represents the Linear Quantile Moment of the random variable τ with two parameters (p,m) specified by Muolkar and Hutson [20] . Suppose that the Linear Quantile Moment for a sample of size n from a Kappa (α, β, θ) distribution is as follows[14]:

$$\widehat{\epsilon}_r = \frac{1}{r}\sum_{k=0}^{r-1}(-1)^k \begin{pmatrix} r-1 \\ k \end{pmatrix} \widehat{\tau}_{p,m}(X_{r-k:r}), r = 1, 2, .. \tag{33}$$

if

$$\widehat{\tau}_{p,m}(X_{r-k:r}) = p\widehat{Q}_{r-k:r}(m) + (1-2p)\widehat{Q}_{r-k:r}\left(\frac{1}{2}\right) + p\widehat{Q}_{r-k:r}(1-m) \tag{34}$$

$$= p\widehat{Q}[B^{-1}_{r-k:r}(m)] + (1-2p)\widehat{Q}[B^{-1}_{r-k:r}\left(\frac{1}{2}\right)] + p\widehat{Q}[B^{-1}_{r-k:r}(1-m)] \tag{35}$$

Assume that the sample is aware of $\widehat{Q}(u)$ as follows.

$$\widehat{Q}(u) = \sum_{i=1}^{n}\left[(n)^{-1}k_h\left[\sum_{j=1}^{i} w_{j,n} - u\right]\right](X_{i,n}), 0 < u < \infty \tag{36}$$

when k is a random variable for parameter h at the estimator (.)

$$w_{i,n} = \left\{ \begin{array}{cc} \frac{1}{2}(1 - \left[\frac{n-2}{\sqrt{n(n-1)}}\right]) & i = 1, n \\ \frac{1}{\sqrt{n(n-1)}} & i = 1, 2, ..., n-1 \end{array} \right\} \tag{37}$$

$$k(t) = (2\pi)^{-\frac{1}{2}} exp(-\frac{t^2}{2})$$

$$h = (\frac{uv}{n})^{\frac{1}{2}}$$

$$v = 1 - u$$

For the sample, the first four quantitative moments of the Moment-LQ method are specified as follows:

$$\hat{\epsilon}_1 = \hat{\tau}_{p,m}(X) \tag{38}$$

$$\hat{\epsilon}_2 = \frac{1}{2}\left[\hat{\tau}_{p,m}(X_{2:2}) - \hat{\tau}_{p,m}(X_{1:2}]\right. \tag{39}$$

$$\hat{\epsilon}_3 = \frac{1}{3}\left[\hat{\tau}_{p,m}(X_{3:3}) - 2\hat{\tau}_{p,m}(X_{2:3} + \hat{\tau}_{p,m}(X_{1:3})\right] \tag{40}$$

$$\hat{\epsilon}_4 = \frac{1}{4}\left[\hat{\tau}_{p,m}(X_{4:4}) - 3\hat{\tau}_{p,m}(X_{3:4} + 3\hat{\tau}_{p,m}(X_{2:4}) + \hat{\tau}_{p,m}(X_{1:4})\right] \tag{41}$$

LQ- Skewnes and LQ- Kurtosis of a sample are defined:

$$LQ - Skewnes = \hat{\eta}_3 = \frac{\hat{\epsilon}_3}{\hat{\epsilon}_2}$$

$$LQ - Kurtosis = \hat{\eta}_4 = \frac{\hat{\epsilon}_4}{\hat{\epsilon}_2}$$

LQ technique is used to produce estimates with the desired accuracy. To determine the values of the three parameters, implicit equations are solved using a simple iteration in the R software program.

5. Application and Analysis

In this section, we evaluate the correctness and dependability of the Kappa distribution model in describing the phenomenon under study using an experimental technique. Well-designed experiments are frequently necessary for scientific advancement, and simulation is a popular technique for producing precise and reliable results. This study uses simulation as a reasonable and useful approach to assess the Kappa distribution model's performance.

Simulation Settings: The simulation experiment was conducted using six distinct sample sizes:(10, 25, 50, 100, 150, and 500). Data for random variables following the three-parameter Kappa distribution were generated using formula (3). The process began by producing random values from a continuous uniform distribution defined over the interval (0,1). These uniform random values were then transformed using the cumulative distribution function (CDF) of the Kappa model, as defined by equation (1). Subsequently, the inverse of this CDF was used to generate random variables corresponding to the Kappa distribution. The parameters were systematically varied within the following ranges:α-[2,2,3], β=[2,1,2], and θ=[2,2,4] . These ranges were selected to represent different distribution shapes and scales, including variations in skewness and kurtosis, commonly observed in environmental and reliability studies.

Simulation Procedure: Each simulation scenario was repeated 1000 times for every combination of sample size and parameter set to ensure statistical robustness. The simulation was performed using the [R version 4.3.3], which provided the necessary tools for random number generation, parameter estimation, and goodness-of-fit testing.

Performance Evaluation: The performance of the maximum likelihood estimators (MLEs) and LQ-moments estimators was evaluated by calculating the mean squared error (MSE) for each scenario. Lower MSE values indicate better estimator performance. The results, summarized in Tables 2, 3, and 4, demonstrate that the LQ-moments estimator generally achieved a lower MSE compared to the MLE, particularly for smaller sample sizes. This suggests that the LQ-moments estimator provides greater robustness in estimating the parameters of the Kappa distribution.The non-linear equations for the maximum likelihood estimation (MLE) of the Kappa distribution parameters were solved using the Newton-Raphson method. Although this approach is popular because, in ideal circumstances, it converges quickly, it is dependent on the initial values selected. The selection of initial values in this study was done with great care, taking into account plausible estimates from the data and exploratory analysis. However, when the likelihood surface is flat close to the optimum or when the starting points are excessively remote from the genuine parameter values, convergence problems may occur.

Table (2) shows that the (LQM) method is the best for estimating the general model of distribution. When the three parameters are ($\hat{\alpha} = 2.552177, \hat{\beta} = 3.045866, \hat{\theta} = 1.542476$) and the sample size is (n=25), the mean squared error was (MSE =4.76E-05).

Table 2. MSE of the parameter estimations and a comparison of the two methods of estimation at the sample sizes (10,25,50,100,150,500) For the initial value set ($\alpha = 2, \beta = 2, \theta = 3$).

MSE					
Sample size	Parametes	methods		Performance	
		MLE	LQ-moment	MLE	LQ-moment
10	α	133.8760	0.1026561	0.0053	4.97E-05
	β	0.8962	0.0040		
	θ	0.7421	0.0032		
25	α	131.1876	0.0035	0.0027	4.76E-05
	β	0.6296	0.0029		
	θ	0.6041	0.0028		
50	α	5.4511	0.0032	0.0012	6.37E-05
	β	0.3192	0.0022		
	θ	0.1072	0.0040		
100	α	0.6119	0.0027	0.0006	5.36E-05
	β	0.1055	0.0031		
	θ	0.0886	0.0032		
150	α	0.3439	0.0029	0.00027	5.45E-05
	β	0.0513	0.0032		
	θ	0.0474	0.0033		
500	α	0.07589	0.004811	0.000264	5.05E-05
	β	0.002420	0.00481		
	θ	0.0440898	1.121779		

Table 3. MSE of the parameter estimations and a comparison of the two methods of estimation at the sample sizes (10,25,50,100,150,500) For the initial value set ($\alpha = 2, \beta = 1, \theta = 2$).

MSE					
Sample size	Parametes	methods		Performance	
		MLE	LQ-moment	MLE	LQ-moment
10	α	48.20388	3.43402720	0.876588	1.804576
	β	0.1063615	0.9496205		
	θ	25.78638	3.43402720		
25	α	41.5401	0.0034	0.0034	3.65E-05
	β	0.6248	0.0035		
	θ	5.4029	0.0035		
50	α	6.8975	0.0026	0.0012	3.90E-05
	β	0.2552	0.0037		
	θ	0.1691	0.0035		
100	α	1.9765	0.0039	0.0004	3.70E-05
	β	0.0700	0.0033		
	θ	0.067	0.0032		
150	α	1.2172	0.0037	0.0004	3.70E-0
	β	0.0779	0.0039		
	θ	0.0566	0.0029		
500	α	0.0545205	0.0061150	0.0001266398	9834E-4
	β	0.0008436005	0.0062		
	θ	0.02185435	0.03819039		

Table (3) shows that the (LQM) method is the best for estimating the general model of distribution. When the three parameters are ($\hat{\alpha} = 3.047708, \hat{\beta} = 4.036641, \hat{\theta} = 2.05011$) and the sample size is (n=25), the mean squared error was (MSE =3.65E-05).

Table 4. MSE of the parameter estimations and a comparison of the two methods of estimation at the sample sizes (10,25,50,100,150,500) For the initial value set ($\alpha = 3, \beta = 2, \theta = 4$).

MSE					
Sample size	Parametes	methods		Performance	
		MLE	LQ-moment	MLE	LQ-moment
10	α	21.34852	7.7732061	0.3190029	4.86E-05
	β	0.0669413	3.813840		
	θ	9.566965	3.7325726		
25	α	139.9068	0.0032	0.0031	4.66E-05
	β	0.2850	0.0029		
	θ	1.5244	0.0036		
50	α	29.5004	0.0041	0.0016	3.68E-05
	β	0.1857	0.0026		
	θ	0.3887	0.0028		
100	α	1.0751	0.3037	0.00061	4.43E-05
	β	0.0657	0.0039		
	θ	0.0672	0.0026		
150	α	1.0912	0.0035	0.00041	4.70E-05
	β	0.04298	0.0038		
	θ	0.0496	0.0030		
500	α	0.0545205	0.4124429	0.0003974	3.642217E-05
	β	0.0008436	2.696877		
	θ	0.02185	0.1280085		

Table (4) shows that the (LQM) method is the best for estimating the general model of distribution. When the three parameters are ($\hat{\alpha} = 4.057929, \hat{\beta} = 4.043308, \hat{\theta} = 2.045325$) and the sample size is (n=50), the mean squared error was (MSE =3.68E-05).

The second case data set represents the monthly average of rainfall in Duhok meteorological station -Iraq, from 1993 to 2006. The null hypothesis (H_0) for the Chi-Square test aims to assess the goodness-of-fit of the Kappa distribution to the observed rainfall data. In this context, the null hypothesis typically states that there is no significant difference between the observed rainfall data and the Kappa distribution with specified parameters ($\alpha = 2, \beta = 2, \theta = 3$). The alternative hypothesis (H_a) would suggest that there is a significant difference. Table (5) presents the results of estimating the three parameters using the Chi-Square criterion. The methods employed for parameter estimation include Maximum Likelihood Estimate (MLE) and the Method of Quantile L-Moments (MQL). The estimated parameters (α, β, θ) for both methods are reported, along with the corresponding Chi-Square statistic, p-value, and Mean Squared Error (MSE).

For MLE, the estimated parameters are $\alpha = 1.625983$, $\beta = 0.144314$, and $\theta = 5.01906$. The associated Chi-Square statistic is 163.68 with a p-value of 0.004161, indicating a rejection of the null hypothesis at a significance level of 0.05. The MSE is reported as 2.498371.

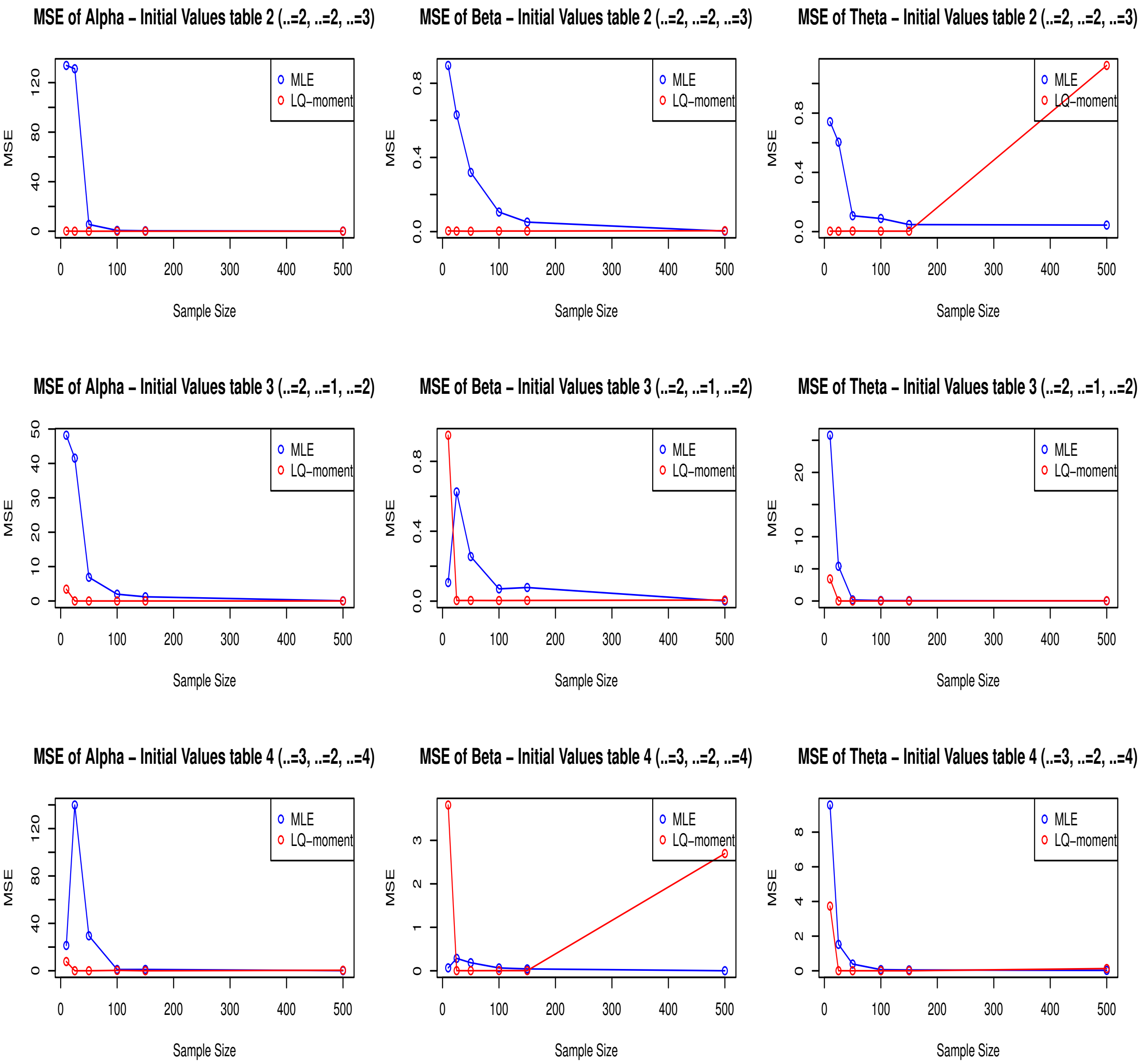


Figure 8. mse of (α, β, θ) for table 3,table 4 and table 5 .

For MQL, the estimated parameters are $\alpha = 1.687656$, $\beta = 1.084304$, and $\theta = 1.616303$. The Chi-Square statistic is 173.44 with a p-value of 0.0008383, again leading to the rejection of the null hypothesis. The MSE for MQL is reported as 1.130276. These results suggest that both MLE and MQL provide parameter estimates that significantly differ from the assumed Kappa distribution parameters based on the Chi-Square criterion.

The cumulative distribution function for rainfall data is shown in Figure (9).

Table(5) displays the results of best estimating the three parameters of real data using the Square-Chi criterion.

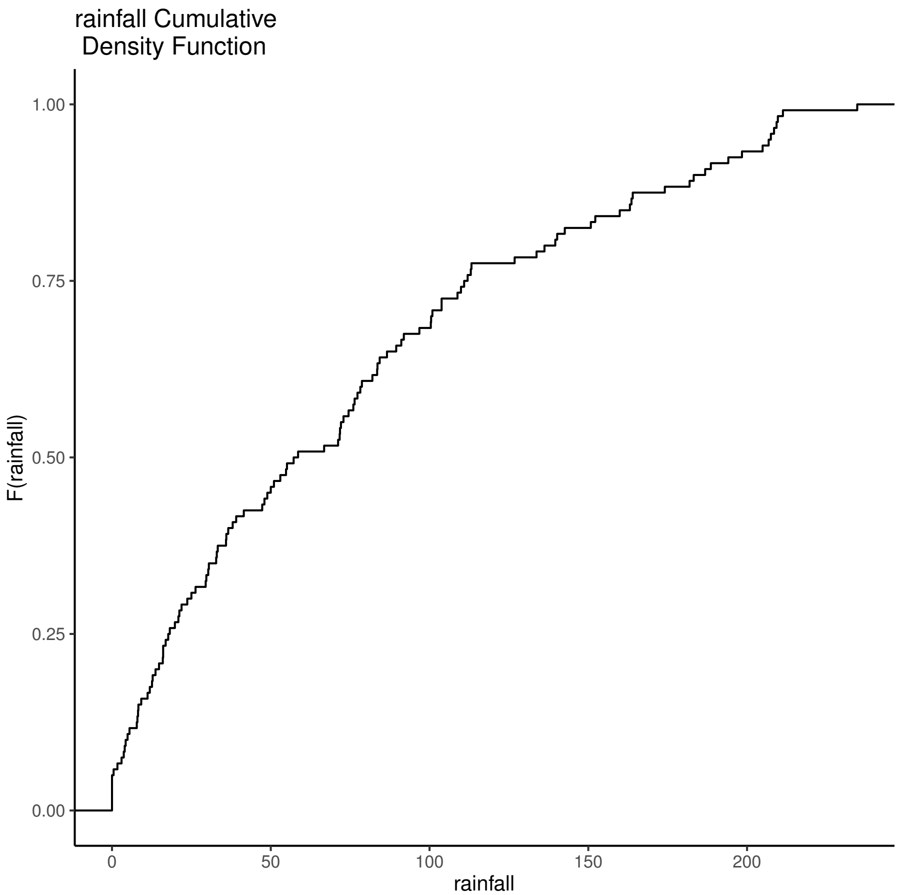


Figure 9. cumulative distribution function for the rainfall data.

Table 5. results of best estimating the three parameters (α, β, θ) of rainfall data using the Chi-Square.

Method	parameters			chi-square		MSE
	α	β	θ	static	p-value	
MLE	1.625983	0.144314	5.01906	163.68	0.004161	2.498371
MQL	1.687656	1.084304	1.616303	173.44	0.0008383	1.130276

The use of LQ-moments, a skewness measure known as the LQ-skewnes. The two measures were calculated for LQ-skewnes and LQ-kurtosis, and the results were 0.2135066 and -2.961426, respectively. Figure (10) shows the probability distribution function for the rainfall data using LQM.

6. Conclusion

The maximum likelihood and LQ-moments methods were used to derive estimators of the three-parameter Kappa distribution's parameters and quantiles. The results obtained using maximum likelihood are compared to those obtained using the LQ-moments method. Because of the importance of its applications, its parameters must be evaluated precisely, accurately, and efficiently. We investigate the performance of these estimators using two data sets from the simulation case and from the monthly average of rainfall in Duhok, Iraq. The results demonstrate

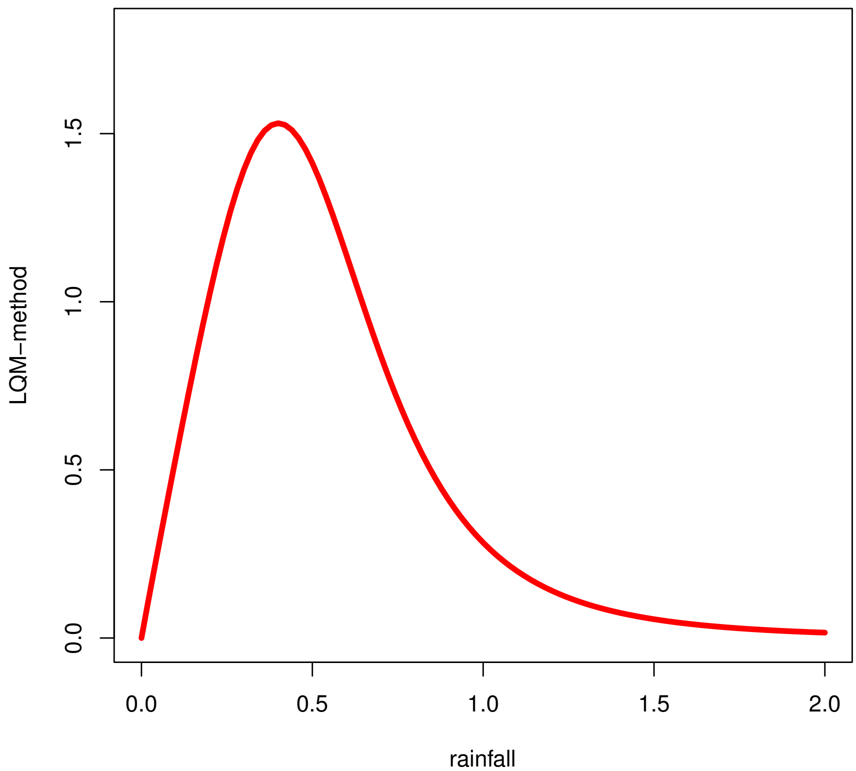


Figure 10. probability distribution function for the rainfall data using LQM method.

that the LQ-moments and maximum likelihood methods are comparable and that the three-parameter kappa distribution fits the data well. For the three simulated cases using the two methods, the LQ-moments performed better than the MLE for all cases in simulation for Estimating the general model parameters according to the MSE criterion and selecting sample sizes at all default values. As a result, the researcher used them for estimating the real data. When comparing the p-value, For the criterion χ^2 with 0.05 then the fit of Goodness test is used on the real data according to the null hypothesis, i.e. the real data is distributed the kappa distribution. Future studies should investigate ways to improve convergence reliability, such as employing hybrid approaches that combine grid search with Newton-Raphson for starting values or alternative optimisation algorithms like the Broyden-Fletcher-Goldfarb-Shanno (BFGS) technique. Furthermore, robustness may be increased by using multi-start approaches, which evaluate various initial values, particularly in situations with small sample sizes or highly skewed data.

Availability of Data
The datasets that support the paper's results are included in the paper.
Author Contributions Both authors contributed to and approved the final version of the work.
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Conflicts of Interest
According to the writers, they have no conflicts of interest.

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