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# A Phase Portrait Stability Analysis for a Reaction Wheel Pendulum Using a Generalized Backstepping Control Approach

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**Abstract** This research presents the stability analysis of a reaction wheel pendulum (RWP) using the phase portrait method. To derive the general control law that stabilizes the RWP system in the upper vertical position, a comprehensive backstepping control design is provided. This control is formulated in a generalized manner for a two-dimensional dynamical system, with the key advantage of ensuring asymptotic stability via Lyapunov's stability theorem. In this approach, the upper vertical position transitions from an unstable saddle point in an open loop to a stable node in a closed loop. The backstepping control design, when applied to the RWP system, incorporates a polynomial-based controller combined with a trigonometric function. A qualitative comparison with existing literature, including passivity- and Lyapunov-based control designs, confirms the generalization capabilities of the proposed backstepping controller. The phase portrait analysis was conducted using MATLAB version 2024$b$.

**Keywords** reaction wheel pendulum, phase portrait analysis, backstepping control design, stability analysis

**AMS 2010 subject classifications** 62J07

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## Nomenclature

### Functions

$\mathcal{H}(\eta, z)$ Second candidate Lyapunov function defined for the state variables $\eta$ and $z$

$\mathcal{V}(\eta)$ First candidate Lyapunov function defined for the state variable $\eta$

$\mathcal{W}(\eta)$ Positive definite function of the state variable $\eta$

$\sin(\varphi)$ Trigonometric function

$\varphi(\eta)$ Auxiliary nonlinear function of the state variable $\eta$

$f(\eta)$ Nonlinear function of the state variable $\eta$

$g(\eta)$ Nonlinear function of the state variable $\eta$

$h(\eta, \zeta)$ Nonlinear function of the state variables $\eta$ and $\zeta$

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$j(\eta, \zeta)$ Nonlinear function of the state variables $\eta$ and $\zeta$, known as the *input function*

**Operators**

$\frac{\partial \mathcal{V}}{\partial \eta}$ Partial derivative applied to the function $\mathcal{V}(\eta)$ with respect to the state variable $\eta$

**Parameters**

$\alpha$ Positive proportional coefficient used for defining the function $\varphi(\eta)$

$\alpha_1$ Proportional control gain used in the control input $u_{PBC}$

$\eta$ Positive integer exponent used for defining the function $\varphi(\eta)$.

$\sigma$ Dynamic friction coefficient

$a$ Constant parameter of the RWP

$b$ Constant parameter of the RWP

$c$ Constant parameter of the RWP

$j_1$ Proportional control gain used in the control input $u_{PBC}$

$k_1$ Proportional control gain used in the control input $u_{LC}$

$k_2$ Proportional control gain used in the control input $u_{LC}$

$k_p$ Proportional control gain

$r_2$ Proportional control gain used in the control input $u_{PBC}$

$x_k^\star$ Generic equilibrium point associated with the state variable $x_k$

**Variables**

$\eta$ State variable for backstepping control

$\omega$ Auxiliary control input

$\theta$ Angular position of the reaction wheel

$\varphi$ Angular position of the pendulum

$\zeta$ State variable for backstepping control (auxiliary control input)

$u$ Control input

$u_{LC}$ Control input obtained through Lyapunov-control theory

$u_{LC}$ Control input obtained through backstepping control theory

$u_{PBC}$ Control input obtained via passivity-based control theory

$z$ Auxiliary state variable

## 1. Introduction

### *1.1. General context*

The analysis of nonlinear systems has been fundamental to the continuous development of the industry in service of humanity [1]. Among others, advances in electrical energy production [2], electric mobility [3], robotics [4], and aeronautics [5] have been possible thanks to robust, reliable, and mathematically well-founded control and optimization methodologies [6].

Dynamical systems are integral to numerous processes in our daily lives, both at commercial and industrial levels. Often complex and nonlinear, these systems demand effective control mechanisms to perform specific tasks [7]. Designing such controls necessitates extensive research and the development of novel strategies that can be applied to both linear and nonlinear dynamical systems. However, before these controls are implemented in real-world applications, they are first evaluated using test systems with similar characteristics [8]. These test or prototype systems are crucial for validating performance. They allow understanding the behavior of dynamical systems, facilitating the simplification and adjustment of controls during implementation [9]. A clear example of this is the application of controls in a pendulum. Despite their small size, pendulums are highly complex systems that can be used to evaluate the effectiveness of a controller. Various types of pendulums, such as reaction wheels [10], pendubots [11], and acrobots [12], among others, provide a means to emulate the challenges faced by real-world complex dynamical systems. These systems include synchronous machines connected to an infinite bus, drones, robots (both walking and mobile), rockets, and aircraft [13, 14].

### *1.2. Motivation*

The application of backstepping control designs to reaction wheel pendulums (RWPs) is motivated by the complex nonlinear dynamics that characterize the system. RWPs serve as an ideal benchmark for testing and validating advanced control strategies, given their inherent instability and the challenges associated with stabilizing their inverted position. By applying backstepping control, which is particularly well-suited for dealing with nonlinear systems, this research aims to achieve a higher level of robustness in controlling pendulums. This approach can systematically handle the nonlinearities and uncertainties of the system, which makes it a promising alternative for enhancing the overall performance and stability of RWPs. Not only does it address the specific challenges posed by RWPs, but it can also be extended to broader applications involving the control of other nonlinear dynamical systems.

Moreover, this research contributes to the theoretical development of backstepping control by exploring its effectiveness in a practical, real-world scenario. The insights gained from this study will deepen the understanding of how backstepping control can be applied to systems with complex dynamics, providing valuable guidance for future implementations in various engineering areas. Additionally, by tackling the representative challenges posed by RWPs, this research offers practical insights into the design process and the tuning of parameters, potentially leading to new advancements in the field of nonlinear control systems. Ultimately, the findings of this study will not only advance the theoretical framework of backstepping control, but also demonstrate its practical viability in addressing the demands of real-world dynamical systems, thereby contributing to the ongoing evolution of control engineering.

### *1.3. Literature review*

In the specialized literature, multiple designs have been proposed to achieve the control objective of an RWP, *i.e.*, stabilizing the system around the top upright position. Some of them are presented and discussed below.

Linear controllers are effective when designed near the equilibrium point through linearization methods such as Taylor series or trigonometric approximations [15]. These methods simplify the nonlinear dynamics of a system into a linear model, making the design and analysis of controllers more straightforward and manageable [16]. While these linearizations work well in the vicinity of the equilibrium point, their effectiveness diminishes as the system deviates further from it. The stability of these controllers is compromised when the RWP experiences disturbances or external events that displace it significantly from the equilibrium, thereby invalidating the linear approximation [17]. Such deviations can lead to poor performance or even instability, which evinces the limitations of linear controllers in handling large disturbances or operating under a broader range of conditions [18]. More robust or adaptive control strategies are required to address these challenges, which should be able to manage nonlinearities and maintain stability across a wider range of operating conditions [10].

To avoid stability issues, researchers have proposed nonlinear controllers that offer a good performance and can operate at points far from the system's equilibrium. These controllers are designed to handle the inherent nonlinearities of the system, maintaining stability and performance across a broader range of conditions. Some notable nonlinear control strategies include exact or partial feedback linearization [19], sliding mode control [20], fuzzy logic control [21], passivity-based control [18], artificial neural network designs [22], the optimal control approach [23], robust control methods [24], and Lyapunov-based control methods [25]. Together, these advanced nonlinear control strategies offer a comprehensive toolkit for addressing the challenges posed by nonlinear systems. They provide improved stability and performance, even when subjected to significant disturbances and while operating away from the equilibrium point. However, some controllers require meticulous tuning of multiple parameters or operating rules, which makes them difficult to implement. Additionally, some drivers must perform real-time optimizations, which can be problematic due to the high hardware and precision requirements involved [17]. Although some of these controllers ensure the stability of closed-loop systems, their implementation can be complex.

### *1.4. Contribution and scope*

Considering the above, this article makes the following contributions:

i. It presents the backstepping controller as a tutorial for analyzing a subclass of nonlinear systems (bi-dimensional dynamical systems), outlining the required design steps to obtain a general control input that ensures asymptotic stability in closed-loop systems in the sense of Lyapunov.
ii. It analyzes the dynamical performance of a RWP in its upright top position via phase portrait plotting when the backstepping controller is applied in open- and closed-loop scenarios.
iii. It provides a qualitative comparison between the proposed backstepping controller and some literature reports using passivity- and Lyapunov-based designs.

It is important to mention that, unlike the Lyapunov-based control design presented by the authors of [25], in this work, the controller corresponds to a generalized design for RWP systems that applies backstepping control theory. Depending on the selection of the control constants, this approach allows obtaining a control law that (i) can be obtained through the exact linearization of the state variables (see Ref. [26]), (ii) is equivalent to the Lyapunov formulation given by the authors of [25], or (iii) belongs to a family of nonlinear controllers that ensure a stable RWP behavior in its upper vertical equilibrium position.

### *1.5. Document structure*

The remainder of this research is structured as follows. Section 2 outlines the backstepping control theory for bi-dimensional dynamics in the form of a tutorial manner, introducing the general control input structure based on the conditions of Lyapunov's stability theorem. Section 3 describes the general dynamical modeling of an RWP and its stability analysis around its upright top and bottom positions, using the phase portrait representation for open-loop operation. Section 4 describes the application of the backstepping controller for the RWP in its upright top position, demonstrating that this open-loop saddle point can be transformed into a stable node during closed-loop operation. In addition, a qualitative analysis with existing passivity- and Lyapunov-based approaches is presented. Finally, section 5 lists the main concluding remarks of this work, as well as some possible future developments.

## 2. Backstepping control design

Backstepping control theory is a systematic and recursive design methodology used in control engineering to stabilize nonlinear systems, particularly those in strict-feedback form. It involves breaking down complex control problems into simpler sub-problems, in addition to incrementally constructing control laws and Lyapunov functions at each step. A Lyapunov function is selected to ensure system stability, with its derivative made negative definite by designing appropriate control inputs at each stage. This method is especially useful for nonlinear systems, but it can be extended to adaptive control scenarios where the system parameters are unknown or varying. This can be done by incorporating parameter estimation techniques into the backstepping framework. Overall, backstepping provides a structured approach to designing stabilizing controllers for various nonlinear systems.

### *2.1. General design procedure*

To design a backstepping controller, consider the general dynamical system defined by (1) and (2).

$$\dot{\eta} = f(\eta) + g(\eta)\zeta, \tag{1}$$

$$\dot{\zeta} = h(\eta,\zeta) + j(\eta,\zeta)u, \tag{2}$$

where $\eta$ and $\zeta$ represent the state variables; $f(\eta)$ and $h(\eta,\zeta)$ are soft nonlinear functions of the state variable $\eta$; and $g(\eta)$ and $j(\eta,\zeta)$ denote the control input matrices relating the state variables and the control inputs.

*Remark 1*
In backstepping control theory, the state variable $\zeta$ defines an auxiliary control input for the first component of the dynamical system described by Equation (1).

To design a backstepping control for the dynamical system given in (1) and (2), a systematic procedure must be followed, as described below.

*2.1.1. Step 1: selection of the $\zeta$ controller* The main idea of this stage is to define the structure of the auxiliary control input $\zeta$, in order to stabilize the dynamical system (1) by fulfilling Lyapunov's stability theorem.

*Theorem 1* (Lyapunov's stability theorem)
A dynamical system $\dot{x} = f(x)$ is asymptotically stable at the origin (*i.e.*) $x = 0$, if there is a candidate Lyapunov function $\mathcal{V}(x)$ such that:

1. $\mathcal{V}(x)$ is a positive definite function, i.e., $\mathcal{V}(x) > 0,\ \forall x \neq 0$ and $\mathcal{V}(0) = 0$, and
2. the derivative function $\dot{\mathcal{V}}(x)$ is negative definite, i.e., $\dot{\mathcal{V}}(x) < 0$

Now, the auxiliary control input $\zeta$ is defined as $\varphi(\eta)$, as well as the candidate Lyapunov function $\mathcal{V}(\eta)$, such that Theorem 1 is fulfilled. Then, the following condition must be ensured:

$$\begin{aligned}\dot{\mathcal{V}}(\eta) &= \frac{\partial \mathcal{V}}{\partial \eta}\dot{\eta} \leq -\mathcal{W}(\eta), \\ &= \frac{\partial \mathcal{V}}{\partial \eta}\left(f(\eta) + g(\eta)\varphi(\eta)\right) \leq -\mathcal{W}(\eta),\end{aligned} \tag{3}$$

where $\mathcal{W}(\eta)$ is a positive definite function.

*Remark 2*
Note that, if condition (3) is fulfilled, the dynamical system defined by Equation (1) is stable. If the lower equal condition is changed for the strictly lower condition, the dynamical system (1) will be asymptotically stable at $\eta = 0$.

*2.1.2. Step 2: equivalent dynamical system* The second step in the backstepping control design consists of transforming the general dynamical system defined by (1) and (2) into an equivalent dynamical one using the original state variable $\eta$ and an auxiliary variable $z$, defined as $z = \zeta - \varphi(\eta)$. Note that, by substituting the definition of $z$ into (1) and (2), the following result is obtained:

$$\dot{\eta} = f(\eta) + g(\eta) z + g(\eta) \varphi(\eta), \tag{4}$$

$$\dot{z} = h(\eta, \zeta) + j(\eta, \zeta) u - \dot{\varphi}(\eta) = \omega, \tag{5}$$

where $\omega$ is a new auxiliary control input.

*Remark 3*
The general control input $u$ can be obtained from (5) if the auxiliary control input $\omega$ is known:

$$u = j^{-1}(\eta, \zeta)(\omega + \dot{\varphi}(\eta) - h(\eta, \zeta)). \tag{6}$$

Here, it is considered that $j(\eta, \zeta)$ is a non-singular function in the $\eta$ domain.

*2.1.3. Step 3: stability analysis of the equivalent dynamical system* To ensure that the equivalent dynamical system defined by (4) and (5) is stable, the auxiliary control input $\omega$ must be adequately selected. This selection is based on the application of Lyapunov's stability theorem 1 to the equivalent system (4) and (5). To this effect, the following candidate Lyapunov function is defined:

$$\mathcal{H}(\eta, z) = \mathcal{V}(\eta) + \frac{1}{2} z^2, \tag{7}$$

which is positive definite and fulfills $\mathcal{H}(0, 0) = 0$. Its time derivative is defined as follows:

$$\dot{\mathcal{H}}(\eta, z) = \dot{\mathcal{V}}(\eta) + z\dot{z}. \tag{8}$$

Now, by using (4) and rearranging some terms, the following result is obtained:

$$\dot{\mathcal{H}}(\eta, z) = \frac{\partial \mathcal{V}}{\partial \eta}(f(\eta) + g(\eta)\varphi(\eta)) + z\left(\frac{\partial \mathcal{V}}{\partial \eta} g(\eta) + \omega\right). \tag{9}$$

Note that (10) considers the existence of the positive definite function $\mathcal{W}(\eta)$, as defined by (3). Thus, (10) can be rewritten as follows:

$$\dot{\mathcal{H}}(\eta, z) \leq \mathcal{W}(\eta) + z\left(\frac{\partial \mathcal{V}}{\partial \eta} g(\eta) + \omega\right), \tag{10}$$

which, with an adequate selection of the auxiliary control input $\omega$, will be asymptotically stable in the sense of Lyapunov.

*Remark 4*
The auxiliary control input $\omega$ can be selected as follows:

$$\omega = -k_p z - \frac{\partial \mathcal{V}}{\partial \eta} g(\eta), \tag{11}$$

which ensures that the time derivative of the candidate Lyapunov function $\mathcal{H}(\eta, z)$ fulfills the second condition of Theorem 1 for $k_p$, a positive constant.

*2.1.4. Step 4: generalized control input calculation* Backstepping control design requires determining the general structure of the original control input $u$ as a function of the original variables $\eta$ and $\zeta$. To determine the structure of $u$, consider the definition in (6) by replacing the structure of the auxiliary control input $\omega$, which yields the following:

$$u = j^{-1}(\eta, \zeta)\left(-k_p z - \frac{\partial \mathcal{V}}{\partial \eta} g(\eta) + \dot{\varphi}(\eta) - h(\eta, \zeta)\right). \tag{12}$$

Now, considering the definition of the auxiliary state variable, $z = \zeta - \varphi(\eta)$, the control input (12) is rewritten as follows:

$$u = j^{-1}(\eta,\zeta)\left(-k_p(\zeta - \varphi(\eta)) - \frac{\partial \mathcal{V}}{\partial \eta} g(\eta) + \dot{\varphi}(\eta) - h(\eta,\zeta)\right). \tag{13}$$

*Remark 5*
The general control input (13) has the time derivative of the function $\varphi(\eta)$, which can be rewritten as $\dot{\varphi}(\eta) = \frac{\partial \varphi}{\partial \eta}\dot{\eta}$. This implies that, if the definition of the original dynamical system in Equation (1) is used, the general control input $u$, via the backstepping design in (13), takes the form presented in (14).

$$u = j^{-1}(\eta,\zeta)\left(-k_p\zeta + k_p\varphi(\eta) - \frac{\partial \mathcal{V}}{\partial \eta} g(\eta) + \frac{\partial \varphi}{\partial \eta}(f(\eta) + g(\eta)\zeta) - h(\eta,\zeta)\right). \tag{14}$$

## 3. RWP modeling and analysis

This section outlines the dynamical model representing the RWP, as well as the open loop analysis via the phase portrait approach.

### 3.1. Dynamical modeling

The schematic representation of the RWP system is presented in Figure 1, where the main variables of this system are presented, *i.e.*, the angular position of the pendulum bar concerning the vertical axis ($\varphi$) and the relative position of the angle of the wheel for the mobile axis associated with the pendulum ($\alpha$). The relative angular position of the reaction wheel is defined as $\theta$ with regard to the fixed reference frame. Note that $\theta = \alpha + \varphi$. Additionally, the effect of the torque ($\tau$) applied by the motor connected to the reaction wheel is shown.

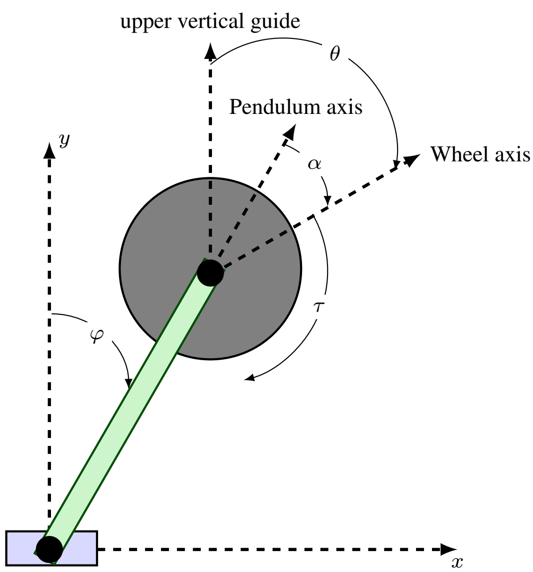


Figure 1. Representation of the RWP in two dimensions

According to the authors of [17], the dynamical model of a RWP is defined according to the system of equations given by (15).

$$\begin{aligned} \ddot{\varphi} &= a\sin(\varphi) - bu - \sigma\dot{\varphi}, \\ \ddot{\theta} &= cu, \end{aligned} \tag{15}$$

where $a$, $b$, and $c$ are constants related to the system's physical parameters, and $\sigma$ defines the dynamic friction coefficient of the pendulum in interaction with the air. For the implementation of the control design, since the behavior of the variable $\theta$ is defined as the double integral of the control input, it is not necessary to include this in the representation of the state variable, as recommended by [25]. To obtain the equivalent representation of the pendulum in terms of the state variables, $\eta = \varphi$ and $\zeta = \dot{\eta}$ are defined. With the above, the first Equation of (15) is represented as follows:

$$\begin{aligned} \dot{\eta} &= \zeta, \\ \dot{\zeta} &= a \sin(\eta) - bu - \sigma\zeta. \end{aligned} \tag{16}$$

Note that (16) corresponds to a second-order nonlinear dynamical system with infinite cyclic equilibrium points when the control input is zero ($u = 0$), which occurs when $\eta = 0$ and $\zeta = 0$, *i.e.*, $x_k^\star = (k\pi, 0)$, $\forall k \in \mathbf{N}$. These balance points are cyclical since the movement of the RWP takes place on a circular plane. Therefore, the balance points occur in the upper vertical and lower vertical part of Figure 1, *i.e.*, $p_1\,(0, 0)$ and $p_2\,(\pi, 0)$, respectively.

### 3.2. Phase portrait analysis

Phase portrait analysis is a two-dimensional methodology for identifying the behavior of a dynamical system's equilibrium point in its vicinity [27]. In this subsection, two equilibrium points ($p_1$ and $p_2$) are analyzed via the phase portrait representation $\eta$ *vs.* $\zeta$.

Figure 2 presents the phase portrait for the equilibrium point $p_1$, which represents the upper vertical position of the RWP. For this simulation, parameters $a$ and $b$ are set as $1.08$ and $198$, and the damping coefficient is defined as 10.

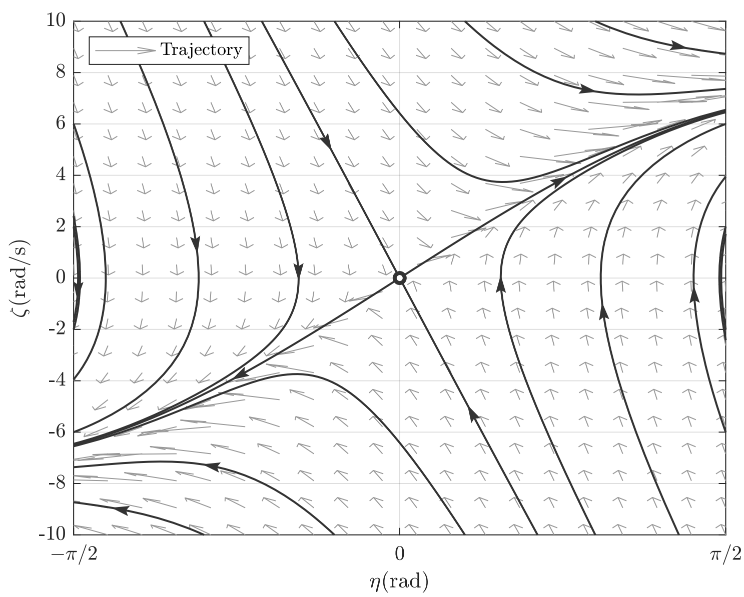


Figure 2. Behavior of the RWP in the vicinity of $p_1$

The behavior of the trajectories around the equilibrium point $p_1$ shows that the RWP has a saddle point in the upper vertical position. This is an expected behavior, given that, in this position, any small perturbation will deviate the pendulum away from $p_1$, carrying it to the equilibrium at $p_2$. The behavior of the RWP at $p_2$ is illustrated in Figure 3.

The trajectories in the vicinity of $p_2$ confirm that, in the lower position, the RWP exhibits a stable focus behavior [27]. This behavior is attributable to the fact that the friction coefficient was considered in the mathematical model, which dissipates the energy of the RWP, showing an asymptotic convergence to $p_2$.

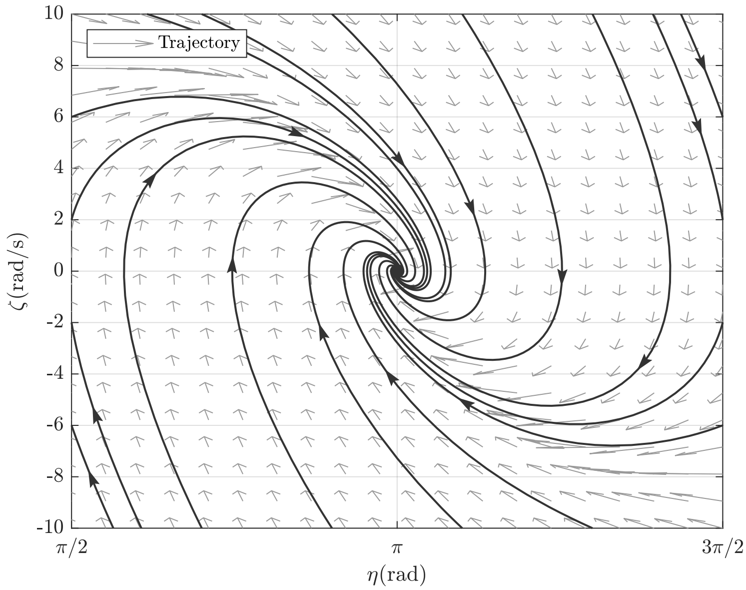


Figure 3. Behavior of the RWP in the vicinity of $p_2$

*Remark 6*

Considering the trajectories around the equilibrium points $p_1$ and $p_2$, the main goal in RWP analysis is to design a controller that allows transforming the saddle point $p_1$ (Figure 2) into a stable focus through the adequate selection of the control input $u$.

## 4. Backstepping control applied to the RWP

With the aim of stabilizing the RWP system in the upper vertical position, this section presents the application of the backstepping control design to the dynamical system described by (16). Note that, in order to compare (16) against the general dynamical system defined by (1) and (2), the following functions are defined:

$$\begin{aligned} f(\eta) &= 0, \\ g(\eta) &= 1, \\ h(\eta,\zeta) &= a\sin(\eta) - \sigma\zeta, \\ j(\eta,\zeta) &= -b. \end{aligned} \tag{17}$$

To illustrate the application of the backstepping controller described in section 2, the next subsection describes the selection of the controller and the stability test carried out for the RWP system.

### *4.1. Selection of the $\zeta$ controller*

To obtain a generalized controller that stabilizes the RWP in the upper vertical position, the auxiliary control input $\zeta = \varphi(\eta)$ is selected as follows:

$$\varphi(\eta) = -\alpha\eta^{2n-1}, \tag{18}$$

where $\alpha$ is a positive definite constant in the real variable domain, and $n$ is a positive integer. To prove that the reduced dynamical system is stable, the candidate Lyapunov function $\mathcal{V}(\eta)$ is selected as a quadratic function, as defined below.

$$\mathcal{V}(\eta) = \frac{1}{2}\eta^2. \tag{19}$$

Thus, as defined in in condition (3), and considering the definitions in (17), the following result is obtained:

$$\dot{\mathcal{V}}(\eta) = \frac{\partial \mathcal{V}}{\partial \eta}\left(f(\eta) + g(\eta)\varphi(\eta)\right) = -\alpha\eta^{2n} \leq 0, \tag{20}$$

which confirms that the first dynamical system is stable in the sense of Lyapunov for the selected $\varphi(\eta)$ function.

### *4.2. General control design*

As defined in section 2, the design of the backstepping controller depends exclusively on the selected functions $\varphi(\eta)$ and $\mathcal{V}(\eta)$, which are defined in the generalized control law (14). To obtain the general control law for the RWP, the partial derivatives of these functions are obtained.

$$\frac{\partial \mathcal{V}}{\partial \eta} = \eta, \tag{21}$$

$$\frac{\partial \varphi}{\partial \eta} = -\alpha(2n-1)\eta^{2n-2}. \tag{22}$$

*Remark 7*
As shown in (14), using the definitions of (21) and (22), and based on the structure of the RWP defined in (17), the general control law for the RWP system using the backstepping control design takes the following form:

$$u = \frac{-1}{b}\left(-k_p\zeta - k_p\alpha\eta^{2n-1} - \eta - \alpha(2n-1)\eta^{2n-2}\zeta - a\sin(\eta) + \sigma\zeta\right). \tag{23}$$

where $\alpha$, $n$, and $k_p$ correspond to control parameters defined by the designer.

### *4.3. Phase portrait analysis*

To illustrate the dynamical behavior reached with the backstepping controller applied to the RWP in its unstable equilibrium point (*i.e.*, $p_1$), Figures 4, 5, and 6 present the phase portrait $\eta$ *vs.* $\zeta$ representation for three different combinations of the control parameters $k_p$, $n$, and $\alpha$.

As depicted in these figures, the behavior of the RWP in the vicinity of the equilibrium point $p_1$ allows stating that:

i. All the combinations of the control gains, selected so that $\alpha$ and $k_p$ are positive real constants and $n$ a positive integer, allow stabilizing the RWP to the operating point $p_1$.
ii. These behaviors can be classified as stable nodes in Figures 4 and 5. In contrast, the behavior illustrated in Figure 6 corresponds to a stable focus [27].
iii. This backstepping design is a global controller since the only stable equilibrium point of the closed-loop system corresponds to $p_1$, which is asymptotically stable in the sense of Lyapunov.

### *4.4. Comments and discussion*

Two literature-reported nonlinear RWP controllers were considered to confirm the effectiveness of our proposal. Note that, in our controller, the damping coefficient $\sigma$ is assumed to be zero.

The first design corresponds to a passivity-based approach, as reported by the authors of [18], which provides the global stabilizing controller with the structure in (24).

$$u_{PBC} = \frac{1}{b}\left(a\sin(\eta) - j_1\alpha_1\eta + r_2\zeta\right), \tag{24}$$

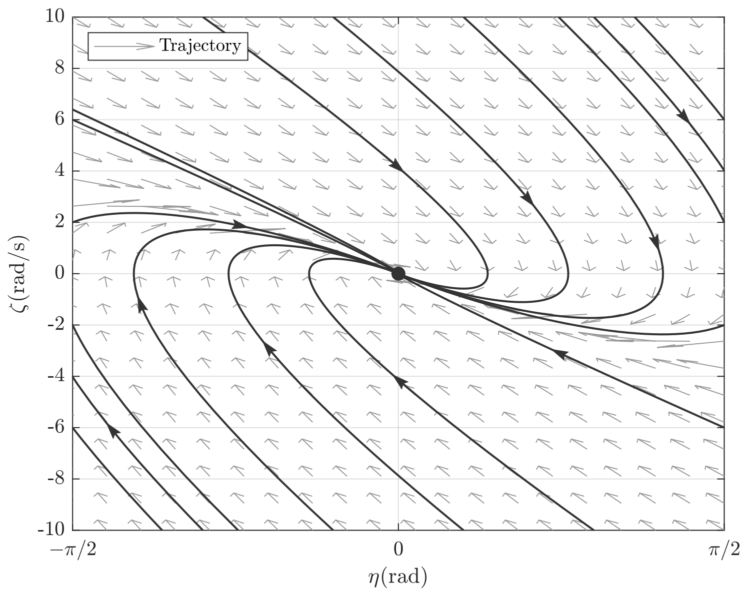


Figure 4. Behavior of the RWP around $p_1$ with control gains $k_p = 3$, $n = 1$, and $\alpha = 4$

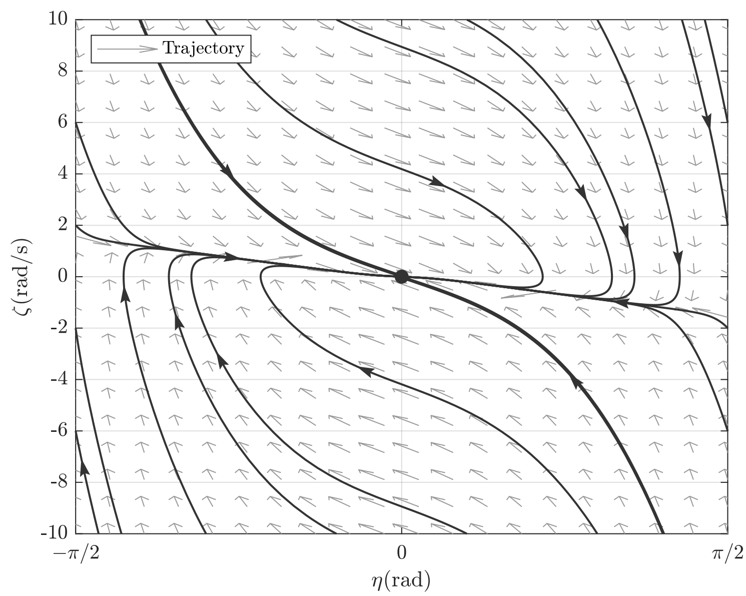


Figure 5. Behavior of the RWP around $p_1$ with control gains $k_p = 3$, $n = 2$, and $\alpha = 4$

where $j_1 = -1$, $\alpha_1 = 3500$, and $r_2 = 135$.

The second controller design employs a Lyapunov-based approach, as proposed by [25]. It is a global, asymptotically stabilizing control with the structure defined in (25).

$$u_{LC} = \frac{1}{b}\left(2a\sin\left(\eta\right) + k_1\eta + k_2\zeta\right), \tag{25}$$

where $k_1$ and $k_2$ are the control gains, set as 3500 and 135, respectively.

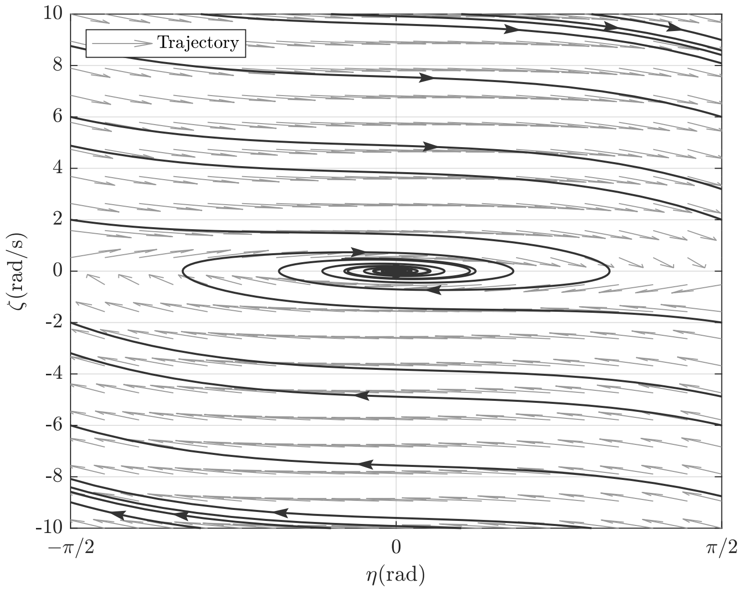


Figure 6. Behavior of the RWP around $p_1$ with control gains $k_p = 0.25$, $n = 2$, and $\alpha = 0.25$

In order make a straightforward comparison between our backstepping design and the literature reports, the proposed control law (23) is rewritten below, simplifying the damping coefficient.

$$u_{BC} = \frac{1}{b}\left(k_p\zeta + k_p\alpha\eta^{2n-1} + \eta + \alpha\left(2n-1\right)\eta^{2n-2}\zeta + a\sin\left(\eta\right)\right). \tag{26}$$

Considering the structure of the proposed design and the controllers selected from the literature, the following facts are observed:

i. The control signal $u_{BC}$ is more general in comparison with $u_{PBC}$, as the proposed structure is a combination of a sinusoidal function with a general polynomial controller design, wherein three control gains (*i.e.*, parameters $\alpha$, $k_p$, and $n$) can be adjusted to obtain the desired performance (as illustrated for various combination of these parameters in the phase portraits of Figures 4, 5, and 6). In the case of the passivity-based control design, two gains in linear form govern the dynamical behavior of the control law defined by (24).
ii. The passivity-based control approach (24) is a particular case of the proposed backstepping controller defined in (26). This can be easily demonstrated if the following parameters are selected: $n = 1$, $1 + k_p\alpha = -j_1\alpha_1$, and $k_p + \alpha = r_2$. For this particular RWP system, $u_{PBC} = U_{BC}$, when $n = 1$, $\alpha = 100.0154$, and $k_p = 34.9846$; or when $n = 1$, $k_p = 100.0154$, and $\alpha = 34.9846$.
iii. The Lyapunov-based design defined by the control law (25) is slightly different from the proposed backstepping control (26) in the case of the trigonometric function, as it is added two times for $u_{LC}$. In contrast, the trigonometric function is added just once in the proposed control law $u_{BC}$. In addition, the linear part $u_{LC}$ is a reduced version of the proposed backstepping controller, obtained by adequately selecting the control gains $n$, $\alpha$, and $k_p$, as demonstrated in the case of the passivity-based control signal.

*Remark 8*
All the three controllers have the general advantage of ensuring asymptotic stability properties in the sense of Lyapunov for the equilibrium point $p_1$ (*i.e.*, the upper vertical position). The most general controller design corresponds to the proposed backstepping approach since it is a general combination of a trigonometric function with a polynomial one.

## 5. Conclusions and future work

This research comprehensively presented the application of a backstepping control design, showing its general formulation for a bi-dimensional dynamical system and providing a general control law structure. An RWP was selected as the nonlinear dynamical system to demonstrate the effectiveness of the backstepping control in stabilizing the upright top position, which exhibited a saddle behavior in open-loop operation and became a stable node in closed-loop operation. Phase portrait analysis was a fundamental part of this research, as it helped to demonstrate the effectiveness of the studied controller.

A qualitative analysis of the proposed backstepping controller for the RWP system was conducted by comparing the general control law against two nonlinear controllers featuring passivity- and Lyapunov-based designs. This analysis revealed that the passivity-based design is a specific instance of the backstepping controller when the control gains are appropriately selected, and that the Lyapunov-based controller also incorporates a combination of linear and trigonometric functions in its control structure. In contrast, the backstepping control design allows including a polynomial structure in the generalized control law.

Future works could explore the following: (i) the application of the proposed backstepping control design to classical second-order DC/DC converters feeding nonlinear loads and (ii) the extension of our proposal to more general dynamical systems with multiple state variables, including applications in electrical systems, motors, and microgrids.

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