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Uncertain Portfolio Optimization Problems: Systematic Review

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Abstract This paper presents a literature review and analysis of Uncertain Portfolio Optimization Problems (UPOP), where security returns are described by uncertain variables due to a lack of historical data. To identify the major gaps in literature and offer perspectives for future research to address these limitations, this paper reviews more than 80 works that have shaped the field among foundational works and recent advancements in UPOP until 2024. We have presented the definitions and some comparisons between various mathematical risk measures to allow the decision-maker to choose which one is appropriate for their situation. In addition, some real features that marked literature are introduced. This has provided a number of enhancements that have been suggested as, artificial intelligence utilization, considering environmental constraints and, using other techniques to model asset returns as the uncertain random variables and employing dynamic and multi-period optimization methods.

Keywords Uncertainty theory, Statistical risk measure, Portfolio optimization, Risk measure, Real features.

AMS 2010 subject classifications 62P05, 91G70, 91G10

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1. Introduction

Since Markowitz's seminal work in 1952 [1], portfolio optimization has been a significant research area, taking attention of numerous scholars. According to Markovitz approach, optimal portfolio is obtained by minimizing variance return, that could be viewed as the investment risk, for a minimum level of return θ.

Although variance has been frequently used to measure risk investment, it has notable shortcomings [2]. For instance, it gives low and high returns equal consideration, failing to distinguish between gains and losses. To address these inconveniences, scholars have considered alternative risk measures, such as downside risk measures as risk index, semi-absolute deviation, semi-variance, risk curve and others [3–5]. This approach aids in making more informed decisions for portfolio selection. Additionally, JP Morgan proposed Value at Risk (VaR) as risk measure [6]. Numerous researchers have studied the VaR measure and proposed various models for portfolio optimization [7–9].

Even though VaR is an intuitive risk measure, but it is not a good since it doesn't verify coherence axioms. To address this shortcoming, Tail Value at Risk (TVaR) is often preferred, as it quantifies risk beyond value-at-risk.

In these articles and others, researchers modelled security returns as random variables using historical data. When the real frequency distributions are close enough to the past data, they use probability theory. However, unexpected events, such as company-specific surprises or insufficient data on newly listed stocks, can lead scholars to doubt that past data accurately reflects future returns, thereby undermining confidence in probability theory.

Lotfi Zadeh has developed fuzzy set theory in 1965 [10] and several contributions on fuzzy portfolio optimization have since appeared [11–23]. However, using fuzzy set theory presents certain limitations, one of which is that the

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possibility measure is not self-dual. This means that two fuzzy events with different probabilities of occurrence may have the same possibility value.
For instance, a security return characterized as a fuzzy number with a triangular membership function $\xi = (-0.4, 0.3, 1.0)$ (Figure. 1). Based on this membership function, the events "the return is exactly 0.3 " and "the return is not exactly 0.3" both have the same belief degree of 1 in the possibility measure. Based on the membership function, the two event "the return is exactly 0.3 " and "the return is not exactly 0.3 " have the same belief degree 1 in possibility measure. This result is unacceptable.

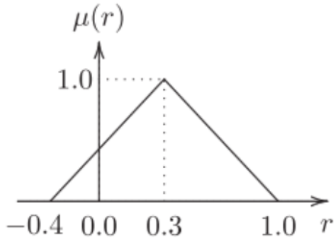

Figure 1. Membership function of a security return ξ

To achieve a more balanced method, Liu [24, 25] introduced uncertain measure that is a self-dual measure using to modelling subjective estimations. This measure forms the core of uncertainty theory, a branch of mathematics dedicated to modelling subjective uncertain events.
The introduction of uncertainty theory has undoubtedly helped overcome several limitations and has led to new solutions for portfolio selection. This development has been a major factor in the growing quantity of research on POP in recent decades. The purpose of this work is twofold. Providing general overview and critical analysis

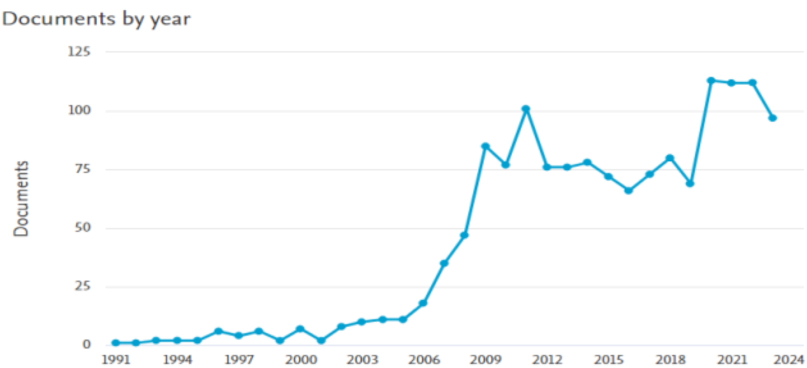

Figure 2. Evolution of POP publications

of both the fundamental studies and the more recent developments in UPOP until 2024 and also sharing with the readers certain mathematical definitions and comparisons of uncertain risk measures and real-features constraint, which allow to facilitate decision-making in the field of investment management and introduced to researchers some recommendations and perspectives to fill the restrictions.
The rest of the paper is structured as follows. In section 2, we will describe the method used for this literature review. To facilitate understanding of the entire paper, we will introduce, in section 3, some elements of uncertainty theory. Section 4 will be dedicated to present fundamentals models and recent approaches addressed in the literature, based on uncertainty theory, and introduce the major advantages and limits of risk measurements. In section 5, some real features employed in UPOP are presented. Finally, some conclusions, gaps and limitations remarked on UPOP and a proposal solutions and recommendations to fill them are presented in the last section.

2. The method for the literature review

The accomplish this review, we used the following method. During the publication collection phase, the foremost task involved selecting bibliographic databases to conduct a thorough search. Publications were sourced from Scopus and Web of Science. A structured keyword search encompassing terms such as ”optimization,” ”selection,” ”evaluation”, ”portfolio”, ”mathematical modeling” and ”uncertainty theory” was executed across various databases and major publisher websites. The objective was to gather a maximal number of pertinent papers to facilitate a comprehensive review and thorough analysis. To achieve this, a scientific approach was employed to pinpoint the most relevant search terms. All the articles examined are edited between 2007 and 2024.
Portfolios optimization problems can be categorized based on two fundamental aspects for analysis.
Framework: This aspect involves understanding the nature of the security returns. Are they characterized as random variables, fuzzy variables, or uncertain variables?
Risk measure: This involves identifying what is considered as investment risk. Is it modelled by the risk curve, variance, or the chance of a specific loss level occurring?
Different individuals may have varying preferences for each aspect, leading to diverse outcomes. However, any POP can be expressed through the combination of these three elements: Framework, Investment Strategy, and Risk. Due to the scarcity of review studies within uncertainty theory, this paper will focus on analyzing uncertain

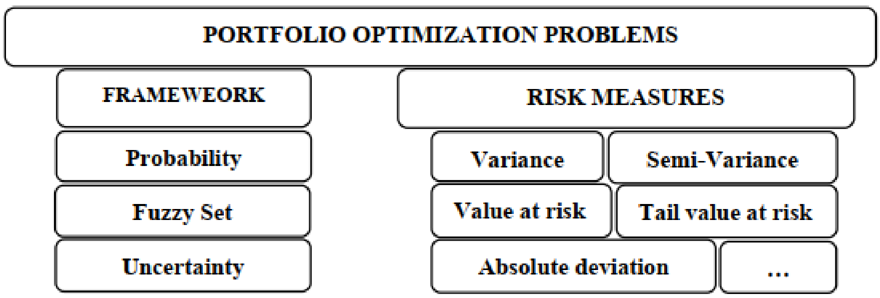

Figure 3. Fundamental aspects of Portfolio Optimization

portfolio optimization problems.

3. Uncertainty theory fundamentals

In the following, we will introduce the core concepts and principles of uncertainty theory:

3.1. Uncertainty Space

The uncertainty space is the foundational element in uncertainty theory. It consists of a set of possible outcomes or events. Formally, an uncertainty space is defined as a triplet ($\Gamma, \mathrm{L}, \mathrm{M}$), where:

- Γ : The sample space, representing all possible outcomes.
- L: A σ-algebra of subsets of Γ, known as the collection of events.
- M: An uncertainty measure, a function that assigns a value between 0 and 1 to each event in L, representing the belief of the event.

3.2. Uncertainty Measure

The uncertainty measure M has similar properties to probability measures but is designed to handle more general situations. These key properties include:
Normality: $\mathrm{M}\{\Gamma\} = 1$.
Duality: For any event Λ : $\quad \mathrm{M}\{\Lambda\} + \mathrm{M}\left\{\Lambda^c\right\} = 1$.
Subadditivity: For every countable sequence of events Λ_i, we have:

$$\mathrm{M}\left\{\mathrm{U}_{\mathrm{i}=1}^{\infty} \Lambda_i\right\} \leq \sum_{i=1}^{\infty} \mathrm{M}\left\{\Lambda_i\right\}$$

3.3. Uncertain Variables

An uncertain variable is a measurable function from ($\Gamma, \mathrm{L}, \mathrm{M}$) to IR. The uncertainty distribution of the uncertain variable ξ is Φ : IR $\rightarrow[0,1]$ defined by

$$\Phi(\mathrm{x})=\mathrm{M}(\xi \leq \mathrm{x})$$

for any real number r .

3.4. Uncertain Expected Value

Let ξ be an uncertain variable. The expected value of ξ is a measure of the central tendency of its distribution defined as:

$$\mathrm{E}(\xi)=\int_0^{+\infty}(1-\mathrm{M}(\xi \leq \mathrm{x})) d x-\int_{-\infty}^0 \mathrm{M}(\xi \leq \mathrm{x}) d x$$

By understanding and applying these fundamental concepts, practitioners can better handle and model the inherent uncertainty present in many real-world problems. For more details about uncertain theory, consult [25].

4. Notable Uncertain problem for portfolio optimization

4.1. Return portfolio

At the beginning of a period, investor assigns a weight to each security by allocating its capital among the selected securities. During the investment period, each security generates a rate of return. At the end of the period, this involves in a change of capital invested observed.
For investors, all decisions are based on individual securities returns. The return rate can be defined as follows:

$$\text{Return rate } =\frac{\text{Current Price-Purchase Price }+\text{ Dividends}}{\text{Purchase Price}}$$

This formula captures the profitability of an investment by considering the capital gain or loss (the difference between the current price and the purchase price) and any dividends received, all relative to the initial investment cost.
Let ξ_i be an uncertain variable that represents the return of the ith security and x_i the proportion of capital in security i , then, the portfolio return is $\sum_{i=1}^n \mathrm{x}_i \xi_i$. Therefore, portfolio mean return is expressed as $\mathrm{E}\left(\sum_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{x}_{\mathrm{i}} \xi_{\mathrm{i}}\right)=\sum_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{x}_{\mathrm{i}} \mathrm{E}\left(\xi_{\mathrm{i}}\right)$

4.2. Uncertain problem for portfolio optimization

Financial risk refers to the possibility of losing money on an investment. It is a broad term that encompasses various types of risks associated with financial activities. It is an important factor in POP and a major subject for researchers.
Within the framework of uncertainty theory, general risk measures and downside risk measure are used. Unlike more general risk measures that examine the entire distribution of returns, downside risk measures specifically focus on potential losses or negative returns.
These measures are important because many investors are more sensitive to losses than gains, and capital preservation is often a priority. They allow investors to understand potential scenarios where losses could be greater than expected and adjust their strategies accordingly.

In the following, let ξ be an uncertain variable with finite expected value e and uncertainty distribution $\Phi.\mathrm{r_f}$ is the risk-free interest rate. For accordance, let:

$$(\xi-\mathrm{e})^{-}=\min(\xi-\mathrm{e};0) \text{ and } (\xi-\mathrm{e})^{+}=\max(\xi-\mathrm{e};0)$$

Uncertain Mean-Variance problem for portfolio optimization

In 1952, Markowitz [1] used variance of a portfolio returns as investment risk measurement. In 2009, Liu [26] extended Markovitz idea and in uncertain environment and used it to characterize investment risk. Liu defined the variance of the uncertain variable ξ by:

$$V(\xi)=E\left[(\xi-e)^2\right]=\int_0^{+\infty} M\left\{(\xi-e)^2\geq t\right\}dt$$
$$=\int_0^{+\infty} M\{(\xi\geq e+\sqrt{t})\cup(\xi\leq e-\sqrt{t})\}dt=2\int_e^{+\infty}(t-e)(1-\Phi(t)+2\Phi(2e-t))dt$$

with e is the finite expected value of ξ.
The goal of mean-variance for portfolio optimization, which was first proposed by Markovitz and extended by Huang [27] within uncertain environment, is to create an investment portfolio that is as efficient as possible. Finding the best asset allocation within a portfolio to maximize total return subject to maximum risk θ is the main objective of MVO.

$$\mathrm{Max}\, E\left(\sum_{j=1}^{n} x_i\xi_i\right)$$
$$\text{S.c.}\begin{cases} V\left(\sum_{j=1}^{n} x_i\xi_i\right)\leq\theta \\ \sum_{i=1}^{n} x_i=1 \\ x_i\geqslant 0, \quad i=1\ldots,n \end{cases}$$

which can be expressed as the following UPOP.

$$\mathrm{Max}\, E\left(\sum_{j=1}^{n} x_i\xi_i\right)$$
$$\text{S.c.}\begin{cases} 2\int_e^{+\infty}(t-e)(1-\Phi(t)+2\Phi(2e-t))dt\leq\theta \\ \sum_{\mathrm{i=1}}^{\mathrm{n x_i=1}} \\ \mathrm{x_i}\geqslant 0, \quad \mathrm{i}=1\ldots,\mathrm{n} \end{cases}$$

In addition, Zhai and Bai [36] introduced a mean-variance model for portfolio optimization that takes background risk, liquidity, and transaction costs into account. There is a discussion of how liquidity constraints and background risk affect portfolio choices. In the other hand, Sajedi and Yari [46] defined order v cross entropy and order v entropy of uncertain variables. Moreover, order v entropy and order v cross entropy of uncertain variables were applied to mean-variance portfolio selection model. They demonstrated that there is no distinction between the two models. Additionally, it was demonstrated that choosing a more diversified portfolio is achieved by reducing the values of the v parameter in the order v cross entropy.

Uncertain Mean-Semi-variance problem for portfolio optimization

When Huang initially adopted uncertain variance as a risk measure, it suggested that the uncertain security returns were assumed to be symmetrical. This assumption stems from the idea that when variance is disregarded, deviations in returns-both high and low-are treated equally in relation to the expected return. However, it's crucial to note that high return deviations signal the potential for significant investment gains, which investors typically favor. Consequently, in cases where security returns exhibit asymmetry, relying solely on variance as a risk measure becomes impractical.

For this reason, Huang [27] defined the semi-variance for uncertain variable ξ, that measure only the low deviations from the expected return, as follows:

$$SV[\xi] = E\left[\left((\xi - e)^{-}\right)^{2}\right] = \int_{0}^{+\infty} M\left\{\left((\xi - \boldsymbol{e})^{-}\right)^{2} \geq t\right\} dt$$
$$= \int_{0}^{+\infty} \Phi(e - \sqrt{t})dt = \int_{0}^{\beta} \left(\Phi^{-1}(\alpha) - e\right)^{2} dt$$

where β is the root of $\Phi^{-1}(\beta) = e$. Using semi-variance as risk measure, the corresponding mean semi-variance model is:

$$\text{MaxE}\left(\sum_{j=1}^{n} x_i \xi_i\right)$$
$$\text{S.c.}\left\{\begin{array}{l} \text{SV}\left(\sum_{j=1}^{n} \quad x_i \xi_i\right) \leq \theta \\ \sum_{i=1}^{n} x_i = 1 \\ x_i \geqslant 0, \quad i = 1 \ldots, n \end{array}\right.$$

Then, the determinist UPOP is as follow.

$$\text{Max} \sum_{j=1}^{n} E\left(\xi_i\right) x_i$$
$$\text{S.c.}\left\{\begin{array}{l} \int_0^\beta \left(\sum_{\mathrm{j}=1}^{\mathrm{n}} \mathrm{x_i} \phi_i^{-1}(\alpha) - e\right)^2 dt \leq \theta \\ \sum_{\mathrm{j}=1}^{\mathrm{n}} \mathrm{x_i} \Phi_i^{-1}(\beta) = e \\ \sum_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{x_i} = 1 \\ \mathrm{x_i} \geqslant 0, \quad \mathrm{i} = 1 \ldots, \mathrm{n} \end{array}\right.$$

where β is the root of $\Phi^{-1}(\beta) = e$
While the semi variance is approximated by Huang numerically, Chen et al. [41] introduced a method to calculate it under specific distribution types. They present a multi-period mean semi-variance portfolio model with real-world constraints that considers bounding restrictions, cardinality, and transaction costs. Semi-variance is calculated. The accompanying optimization model is meant to be solved by a modified imperialist competitive algorithm. The experimental findings show that the best investment plan is greatly influenced by real-world restrictions, and the suggested model can be solved well by the developed MICA algorithm.

Uncertain Mean-Chance problem for portfolio optimization

In practical scenarios, investors often worry about the chance of their portfolio's return falling below a certain threshold. To address this concern, Huang [28] suggested employing the chance of the portfolio return being less than a specified threshold k , as a statistical measure for investment risk assessment. Investors can then stipulate that the chance of this unfavorable event (the portfolio return is lower than the threshold return H) must be sufficiently low than a predetermined tolerance level denoted as β.

$$\mathrm{M}\{\xi \leq \mathrm{k}\} \leq \beta$$

One major concern in investment risk management is the possibility that the return on the portfolio will drop below a predetermined threshold H. Huang suggested calculating investment risk using this probability. Using this method, investors can stipulate that the likelihood of the portfolio return falling below the threshold H must be less than a predetermined acceptable level β. The portfolio with the highest projected return is deemed optimal once this risk control criterion has been met.

By assisting investors in striking a balance between risk and return and making sure that prospective losses stay within reasonable bounds, this risk management strategy helps them achieve optimal performance. This approach is reflected in the mean-chance model that follows.

$$\mathrm{MaxE}\left(\sum_{\mathrm{j=1}}^{\mathrm{n}} \mathrm{x_i}\xi_{\mathrm{i}}\right)$$

$$\text{S.c.}\begin{cases} \mathrm{M}\left(\sum_{\mathrm{j=1}}^{\mathrm{n}} \quad \mathrm{x_i}\xi_{\mathrm{i}} \leq \mathrm{H}\right) \leq \beta \\ \sum_{\mathrm{i=1}} \mathrm{x_i} = 1 \\ \mathrm{x_i} \geqslant 0, \quad \mathrm{i} = 1\ldots,\mathrm{n} \end{cases}$$

which is corresponding to:

$$\mathrm{MaxE}\left(\sum_{\mathrm{j=1}}^{\mathrm{n}} \mathrm{x_i}\xi_{\mathrm{i}}\right)$$

$$\text{S.c.}\begin{cases} \sum_{\mathrm{j=1}}^{\mathrm{n}} \mathrm{x_i}\Phi_i^{-1}(\beta) \geq H \\ \sum_{\mathrm{i=1}}^{\mathrm{h}} \mathrm{x_i} = 1 \\ \mathrm{x_i} \geqslant 0, \quad \mathrm{i} = 1\ldots,\mathrm{n} \end{cases}$$

where ϕ_i^{-1} is the inverse uncertainty distributions of the security return $\mathrm{i}, \mathrm{i} = 1,\ldots,\mathrm{n}$.
Using chance measure, a tentative mean-chance model for an international UPOP is put forth [66]. Additionally, by contrasting the foreign portfolio investment that is unhedged by forward contracts with that is hedged by forward contracts. The risk reduction brought about by the forward contracts also eliminates the possibility of a significant return when the acceptable level of risk increases. In contrast to this work, Chang et al.'s [82] article used the Chance measure as a risk metric. This is a summary of the paper's primary contribution. Firstly, by taking into consideration practical limitations and mental accounts, an uncertain nonlinear multi-period portfolio selection model is generated. Second, this work offers a practical method for converting a nonlinear model to a linear model.

Uncertain Mean- Absolute deviation problem for portfolio optimization

Absolute deviation is a measure of the dispersion or spread of a set of values around the mean. It quantifies how much individual values in a dataset differ from a central value, regardless of whether those differences are positive or negative. The absolute deviation of ξ is defined by Zhang [29] as:

$$\mathrm{AD}(\xi) = \mathrm{E}[|\xi - \mathrm{e}|] = \int_0^{+\infty} M\{|\xi - \mathrm{e}| \geq t\}dt$$
$$= \int_e^{+\infty} (1 - \Phi(r))dr + \int_{-\infty}^{e} \Phi(r)dr$$

Using absolute deviation to risk measure, the mean absolute deviation approach is expressed as follows.

$$\mathrm{Max}\, E\left(\sum_{\mathrm{j=1}}^{\mathrm{n}} \mathrm{x_i}\xi_{\mathrm{i}}\right)$$

$$\text{S.c.}\begin{cases} \mathrm{AD}\left(\sum_{\mathrm{j=1}}^{\mathrm{n}} \mathrm{x_i}\xi_{\mathrm{i}}\right) \leq \beta \\ \sum_{\mathrm{i=1}}^{\mathrm{n}} \mathrm{x_i} = 1 \\ \mathrm{x_i} \geqslant 0, \quad \mathrm{i} = 1\ldots,\mathrm{n} \end{cases}$$

This is corresponding to:

$$\mathrm{Max}\, E\left(\sum_{\mathrm{j=1}}^{\mathrm{n}} \mathrm{x_i}\xi_{\mathrm{i}}\right)$$

$$\text{S.c.}\begin{cases} \int_e^{+\infty}(1 - \Phi(r))dr + \int_{-\infty}^{e} \Phi(r)dr \leq \beta \\ \sum_{\mathrm{i=1}}^{\mathrm{n}} \mathrm{x_i} = 1 \\ \mathrm{x_i} \geqslant 0, \quad \mathrm{i} = 1\ldots,\mathrm{n} \end{cases}$$

Zhang [58] used absolute deviation in the framework of multiperiod UPOP and offers models that account for uncertain mean-absolute deviation while balancing risk and return on investment. The suggested models are transformed into deterministic versions using a variety of uncertainty distributions of the security returns. The efficiency of the suggested models and the impact of transaction cost on portfolio selection are finally exemplified with numerical examples featuring synthetic uncertain returns.

Uncertain Mean- Semi-absolute deviation problem for portfolio optimization

Semi-absolute deviation is another linear uncertain downside risk measurement proposed by Liu et Qin. (2012) [30]. It's defined by:

$$\mathrm{Sa}[\xi]=\mathrm{E}\left[\left|(\xi-\mathrm{e})^{-}\right|\right]=\int_{0}^{+\infty} M\left\{\left|(\xi-e)^{-}\right| \geq t\right\} dt=\int_{-\infty}^{e} \Phi(t) dt$$

Then, the equivalent mean-Semi-absolute deviation UPOP is:

$$\operatorname{Max} E\left(\sum_{\mathrm{j}=1}^{\mathrm{n}} \mathrm{x}_{\mathrm{i}} \xi_{\mathrm{i}}\right)$$

$$\text{S.c.}\left\{\begin{array}{c}\int_{-\infty}^{e} \Phi(t) dt \leq \delta \\ \sum_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{x}_{\mathrm{i}}=1 \\ \mathrm{x}_{\mathrm{i}} \geqslant 0, \quad \mathrm{i}=1 \ldots, \mathrm{n}\end{array}\right.$$

Yang and Huang [83] proposed a non-linear UPOP using a semi-absolute deviation risk metric. The suggested model's analytical solution is provided when returns are represented by linear uncertainty distributions. In addition, the tracking portfolio frontier's form has been provided, who have discovered that the frontier is a continuous curve made up of n-1 line segments. Furthermore, Yang and Huang showed the tracking error tolerance level and benchmark return distribution affected the return and risk of the ideal tracking portfolio.

Uncertain Mean-Risk index problem for portfolio optimization

Investing in risk-free assets will ensure that investors receive a risk-free interest rate, r_f. As a result, a return that is less than the risk-free interest rate will be deemed negative. The term "risk index" refers to the average return of the portfolio below the risk-free interest rate. Then, according to [31], the portfolio's risk index is as follows:

$$\mathrm{RI}(\xi)=\mathrm{E}\left[\left(\mathrm{r}_{\mathrm{f}}-\xi\right)^{+}\right)]=\int_{0}^{+\infty} M\left\{r_{f}-\xi \geq t\right\} dt=\int_{-\infty}^{\mathrm{r}_{\mathrm{f}}} \Phi(r) dr$$

Then, $\mathrm{RI}\left(\sum_{\mathrm{j}=1}^{\mathrm{n}} \mathrm{x}_{\mathrm{i}} \xi_{\mathrm{i}}\right)=\beta \mathrm{r}_{\mathrm{f}}-\int_{0}^{\beta}\left(\sum_{\mathrm{j}=1}^{\mathrm{n}} \mathrm{x}_{\mathrm{i}} \phi_{i}^{-1}(\alpha)\right) d\alpha$

Thus, Huang [31] proposed the following mean-risk index model to seek the maximum expected return:

$$\operatorname{MaxE}\left(\sum_{j=1}^{n} \mathrm{x}_{\mathrm{i}} \xi_{\mathrm{i}}\right)$$

$$\text{S.c.}\left\{\begin{array}{l}\beta \mathrm{r}_{\mathrm{f}}-\int_{0}^{\beta}\left(\sum_{\mathrm{j}=1}^{\mathrm{n}} \mathrm{x}_{\mathrm{i}} \phi_{i}^{-1}(\alpha)\right) d\alpha \leq \delta \\ \sum_{\mathrm{j}=1}^{\mathrm{n}} \mathrm{x}_{\mathrm{i}} \phi_{i}^{-1}(\beta)=\mathrm{r}_{\mathrm{f}} \\ \sum_{\mathrm{i}=1}^{\mathrm{n} \mathrm{x}_{\mathrm{i}}=1} \\ \mathrm{x}_{\mathrm{i}} \geqslant 0, \quad \mathrm{i}=1 \ldots, \mathrm{n}\end{array}\right.$$

where δ denote the maximum mean loss level that the investors can accept below r_{f} and β is the root of the equation $\phi^{-1}(\beta)=\mathrm{r}_{\mathrm{f}}$

A new risk index nonlinear model incorporating background risk is created by Huang et al. [84]. Discussion is had regarding background risk's impact on investment choices. A comparison between Risk index and chance UPOP with background risk is done. To resolve the suggested model, a genetic algorithm is applied. In addition, Choe [85] presents a mean-risk index model for portfolio selection that takes inflation into account within an uncertain environment. To find the optimal solution for the model, the authors introduce a chaos adaptive genetic algorithm (CAGA). Comparisons with other genetic algorithms demonstrate its superior effectiveness.

Uncertain Mean- Risk curve problem for portfolio optimization

If $\mathrm{r_f} - \xi \geq 0$, this quantity means a loss. Then, $\forall \mathrm{r} \geq 0 : \mathrm{r_f} - \xi \geq \mathrm{r}$ describes all the likely losses an investor might face with the security. Huang [32] provided a risk measure which allows to give information on all the possible losses and their occurrence chances, a risk curve is expressed as follows:

$$\forall \mathrm{r} \geq 0, \quad \mathrm{R(r)} = \mathrm{M}\left\{\mathrm{r_f} - \xi \geq \mathrm{r}\right\} = \mathrm{M}\left\{\mathrm{r_f} - \mathrm{r} \geq \xi\right\} = \Phi\left(\mathrm{r_f} - \mathrm{r}\right)$$

A portfolio is considered safe if none of the risk curve's segments are over the investors' confidence curve, $\alpha(r)$, which represents their maximum tolerance for each possible loss. The mean-risk curve model presented below expresses this selection concept:

$$\operatorname{Max} E\left(\sum_{j=1}^{n} x_i \xi_i\right)$$

$$\text{S.c.} \begin{cases} M\left\{r_f - \sum_{j=1}^{n} \quad x_i \xi_i \geq r\right\} \leq \alpha(r), \quad \forall r \geq 0 \\ \sum_{i=1}^{n} x_i = 1 \\ x_i \geqslant 0, \quad i = 1 \ldots, n \end{cases}$$

which can be transformed into the following UPOP.

$$\operatorname{MaxE}\left(\sum_{j=1}^{n} x_i \xi_i\right)$$

$$\text{S.c.} \begin{cases} \sum_{j=1}^{n} x_i \phi_i^{-1}(\alpha(r)) \geq r_f - r, \quad \forall r \geq 0 \\ \sum_{i=1}^{n} x_i = 1 \\ x_i \geqslant 0, \quad \mathrm{i} = 1 \ldots, n \end{cases}$$

Instead of considering only general background risks and a few mental accounts, Deng and Huang [86] introduce multiple background risks by combining various asset risks. They also consider nine types of mental accounts and, for the first time, use a risk curve to measure risk across these accounts. They propose a novel mean-entropy portfolio model that accounts for multiple background risks, mental accounts, transaction fees, and other constraints. The text highlights that neglecting multiple background risks can lead to underestimating potential risks, and as the return threshold increases, so does the risk that the risk curve exceeds the confidence curve.

Uncertain Mean- Value at Risk problem for portfolio optimization

Value at Risk (VaR) is a statistical downside risk measure, originating in the insurance sector but popularized in the financial field in the 1980s. It was introduced in the United States and widely adopted by institutions financial and regulated by standards such as the Basel II Committee for banks and Solvency II for insurance. VaR is a measure that quantify the maximum loss that could affect a portfolio for a certain level of confiance α over a given period t. Peng [33] extends this indicator into uncertainty theory and defined it as:

$$\mathrm{VaR}_\alpha(\xi) = \inf(x \mid M(\xi \leq x) \geq \alpha)$$

when, α is the level of risk confidence. Therefore, for a risk confidence level α, we have: $\mathrm{VaR}_\alpha(\xi) = \Phi^{-1}(\alpha)$ Although VaR has been widely used, it has some limitations. First, VaR is a non-convex function, meaning that merging two portfolios does not necessarily guarantee a reduction in risk, thus making the measurement inconsistent in some cases. Another limitation of VaR is that it only provides an estimate of the maximum loss at a certain confidence level and over a given time horizon. It does not provide any information on potential losses beyond this threshold.

Yan [34] employed VaR definition proposed by peng to introduce a mean VaR approach for POP. If $\sum_{j=1}^{n} x_i\xi_i$ is the portfolio return then, the loss, being its negative value, then, portfolio VaR is given by:

$$\mathrm{VaR}_\alpha(\xi) = -\Phi^{-1}(1-\alpha)$$

where Φ is the distribution function of $\sum_{j=1}^{n} x_i\xi_i$ If an investor wants to maximize the projected return on a portfolio and stipulates that the investment risk cannot exceed a predetermined risk level δ, which is determined based on the investor's preferences. The selection idea's mathematical expression can then be summed up as follows.

$$\operatorname{Max} E\left(\sum_{j=1}^{n} x_i\xi_i\right)$$

$$\text{S.c.} \begin{cases} \mathrm{VaR}_\alpha(\xi) \leq \delta \\ \sum_{i=1}^{n} x_i = 1 \\ x_i \geqslant 0, \quad i = 1\ldots,n \end{cases}$$

Moreover, belabbes et al. [48] introduced international portfolio models with real-world constraints. They considered the future security prices and the foreign exchange rates as uncertain variables. The impact of realworld constraints is demonstraed using actuarial applications.

Uncertain Tail Value at Risk problem for portfolio optimization

Tail Value at Risk (TVaR), also known by other names such as Conditional Value-at-Risk (CVaR) or Expected Shortfall (ES), represents the average loss that exceed the threshold defined by VaR, thus providing a more comprehensive measure of risk. Specifically, TVaR is the quantile corresponding to the potential loss that a security or portfolio may experience due to adverse market price movements, with a certain level of confidence, knowing that this loss already exceeds VaR. Uncertain VaR is defined by Peng [33] as:

$$\mathrm{TVaR}_{\alpha,\mathrm{t}} = \frac{1}{1-\alpha}\int_\alpha^1 \mathrm{VaR}_{\beta,\mathrm{t}} \cdot \mathrm{d}\beta = \frac{1}{1-\alpha}\int_\alpha^1 \Phi^{-\mathbf{1}}(\boldsymbol{\alpha}) \cdot \mathrm{d}\beta$$

Yufu et al. [35] used TVaR definition and proposed the following UPOP.

$$\operatorname{Max} E\left(\sum_{j=1}^{n} x_i\xi_i\right)$$

$$\text{S.c.} \begin{cases} \frac{1}{1-\alpha}\int_\alpha^1 \Phi^{-1}(\boldsymbol{\alpha}) \cdot \mathrm{d}\beta \leq \delta \\ \sum_{\mathrm{i=1}}^{\mathrm{n}} \mathrm{x_i} = 1 \\ \mathrm{x_i} \geqslant 0, \quad \mathrm{i} = 1\ldots,\mathrm{n} \end{cases}$$

Belabbes et al. [49] presented a multi-period for UPOP in 2023. Liquidity, diversification, and transaction costs are taken into consideration when modeling the security returns and turnover rates utilizing uncertain variables. The standard deviation, expected return, and turnover rate values of the specified securities are estimated using the modified Delphi technique. Recently, Wang et al. [81] employs a mean-CVaR model that takes investor preferences and transaction costs into account when selecting a portfolio in an unpredictable market. To address tail risk, the authors additionally make use of the Arbitrage Pricing Theory (APT) and uncertainty theory. A hybrid method is suggested by the study to optimize the portfolio model in these circumstances.

Table 1 sums up the major advantages and limits of risk measurement used in portfolio optimization problems.

Table 1. Notable uncertain risk measure

Risk measure	**Advantage**	**Limits**
Variance	Easy to calculate and interpret.	Does not distinguish losses from gains and does not consider extreme events.
Semi-variance	Suitable for investors concerned about losses, rather than overall fluctuations. Provides a more realistic view of risk when losses are the primary concern.	Ignores positive gains, which may not be appropriate in some situations where one seeks to assess both gain and loss risk. More complex to calculate than the classic variance, which may limit its practical use.
VaR	Easy to interpret	Ignores extreme losses
TVaR	Considers extreme losses, making it a more conservative measure than VaR.	More complex to calculate than VaR and requires more data to correctly estimate the tails of the distribution (the extremes).
Absolute deviation	Useful for assessing the risks associated with active management from the mean.	Only considers deviations from the mean, without considering other sources of risk.
Semi Absolute deviation	Concentrates on downside deviations from the mean.	More difficult to calculate, less commonly used, ignores positive gains.
Risk Index	Ease of use and interpret and give a fast comparison to decision-making	Doesn't take into account important details.
Risk Curve	Provides a detailed, graphical representation of the probability of different loss levels occurring. It allows for the examination of the entire distribution of possible outcomes.	Interpretation is not easy and requires expertise, as it involves understanding uncertainty distributions and their implications. It's difficult to quantify risk as a single value to do comparisons.

Table 2 presents some research addressed in the literature according to the uncertain risk measures used.

Table 2. Notable uncertain risk measures

Risk measures	**Author(s)**
Variance	Yan [34], Qin et al. [15], Huang [27], Zhai and Bai [36], Chen et al. [37], Huang et al. [38], Huang and Jiang [39], Zhang et al. [40], Chen et al. [43], Chen et al. [44], Qin et al. [45], Chen et al. [46], Bhattacharyya et al. [47].
Semi-variance	Huang [27], Chen et al. [37], Chen et al. [41], Huang et al. [38].
Chance Measure	Huang [28], Huang and Zhao [63], Wang and Huang [64], Huang and Ma [65], Huang et al. [66].
Risk Index	Huang [31], Huang and Qiao [59], Huang and Ying [60], Wang and Huang [61], Zhang et al. [62].
Risk Curve	Huang [32].
Entropy/ Cross-entropy	Qian et al. [50], Kar et al. [51], Li et al. [52], Zhang [53].
Absolute deviation	Zhang [29], Zhang [58].
Value at Risk	Yan [34], Mohammadi and Nazemi [42], Belabbes et al. [48].
Tail Value at Risk/	Ning et al. [35], Belabbes et al. [49], Wang et al. [75].

5. Real features

When constructing a portfolio of securities, an investor may consider additional qualities known as real features, or they may be practical limitations imposed by standard financial market conditions. These real features can include transaction lots, transaction costs, securities liquidity, diversification portfolio or the number of securities and other. We will highlight, in this section, the most used real features by researchers to closely mirror the realities of the financial market.

5.1. Transaction costs

A transaction cost is a cost resulting by any economic trade when buying/selling assets. To simplify, many researchers do not consider it when modeling their optimization problems. However, the lack of any transaction costs would imply unrealistic POP (Arnott & Wagner (1990)) [67].
The transaction cost c_i is treated as a V -shaped function between a specified portfolio $x^0 = \left(x_1^0, x_2^0, \ldots, x_n^0\right)$ and a new portfolio $x = (x_1, x_2, \ldots, x_n)$. Therefore, the transaction cost rate of the i th risky asset can be articulated as follows:

$$C_i = c_i \left|x_i - x_i^0\right|$$

Then the total transaction cost rate is $\sum_{i=1}^{n} c_i \left|x_i - x_i^0\right|$. Note that, for a new investor there is no security on hand, it can be assumed that $x_i^0 = 0, i = 1, 2, \ldots, n$.

5.2. Transaction lots

A transaction lot is a normalized group of assets request to invest in an asset. Often, investing in just one asset at a time is not viable because of its real value. In this case, investing in a security must be a multiple of a transaction lot. For instance, in Chinese and American securities markets, the minimum transaction lot required for any type of security is 100 shares. This means that all investors must purchase shares in multiples of 100 for each security.

5.3. Liquidity constraints

Liquidity of a security is the level of the transformation this security into cash without incurring any loss in value. Securities with high liquidity level are the most preferable. For liquidity control within uncertainty environment, several authors are modeled the turnover rate, which is defined as the percentage of traded shares and modeled by uncertain variable $\boldsymbol{\eta}$.

5.4. Background Risk

Background risks is all risks that investors face cannot be insured against or avoided by taking some actions, but they are incurred and considered as a part of their investment. Like Baptista [72], several researchers regarded background assets returns as normal distributed, in uncertainty environment, with null expected value. Some scholars have proved that background risk can affect investments. For instance, within stochastic environment, Huang and young [73] showed that background risk made investor more cautious. The background assets can be expressed by the uncertain normal variable $\boldsymbol{r_b}$, in which $\boldsymbol{r_b} \sim \boldsymbol{N}(\boldsymbol{0}, \boldsymbol{\rho})$.

5.5. Diversification portfolio

In many cases, models can lead to a concentrative solution, which exposes investors to have heavy losses (Mansinia et al. [74]). Contrariwise, if the portfolio is diversified enough, the investor can avoid more risks. For this reason, researchers incorporate a new constraint ensuring portfolio diversification. Chen et al (2016) [37] proposed Shannon entropy measure and used it under uncertain environment. It was introduced within the framework of information theory. It represents the mean quantity of information contained in each symbol of a source information. If each asset in the portfolio is considered as an information source with a probability associated with its weight in the portfolio, then the Shannon entropy of these weights can give an indication of the diversification of the portfolio. If $(x_1, x_2, \ldots, x_n)$ is the weights of the assets in a portfolio, then the Shannon entropy of these weights is:

$$\sum_{i=1}^{n} x_i \cdot \ln\left(\frac{1}{x_i}\right)$$

5.6. *Real securities constraints*

Several securities constraints are also incorporating into uncertain optimization models, like the cardinality constraints which limit the number of securities held in a portfolio, the threshold constraints which controls minimum or maximum capital proportion invested in each security or in each period, the minimum level acceptable of dividend and bankruptcy constraint.

In Table 3, a review of studies that have used reality factors, under uncertainty theory and chance theory, reflecting reality of the market is presented.

Table 3. Notable real features used in UPOP

Risk measures	**Author(s)**
Diversification portfolio	Chen et al. [37], Li et al. [70].
Background Risk	Zhai and Bai [36], Huang and Jiang [39], Zhai et al. [68], Huang and Young[73].
Bankruptcy Risk	Li et al. [52], Zhai et al. [68], Li et al. [69], Lu et al. [71].
Assets liquidity	Zhai and Bai [36], Li et al. [52], Zhai et al. [68], Li et al. [69], Li et al. [70], Lu et al. [71].
Securities constraints	Chen et al. [43], Chen et al. (2018) [41], Huang and Zhao [63], Xue et al. [77], Zhang et al. [78].
Transaction costs	Wang et al. [75], Liu et al. [76], Zhai and Bai [36], Wang et al. [75], Chhatri et al. [79].

6. Conclusion, perspectives and new directions

Since the famous work of Markowitz on POP, many alternative studies have been proposed to deal with some of their gaps. The imprecise of the market parameters and the lack of historical data are an example. The introduction of uncertainty theory has undoubtedly helped overcome several limitations and has led to new solutions for portfolio selection. This development has been a major factor in the growing quantity of research on POP in recent decades. In this work, we reviewed more than 80 works, until 2024, that have addressed the literature among foundational and recent advancements papers in UPOP. The major limitations addressed in literature are identified. In addition, the mathematical definitions of some risk measures and a comparison between them is done. Real features reflecting market reality such as transaction costs, liquidity constraints, diversification constraints and others have been also discussed.

This section reviews recent trends, gaps in the literature, and prospective research directions to provide new insights for future studies:

- The two fundamental types of indeterminacy are uncertainty and randomness. In 2013, Liu [80] provided the uncertain random variable to model hybrid systems that exhibit both uncertainty and randomness. Using uncertain random variables to model future returns could improve the existing methods. Developing models in this area could enrich research and improve risk and portfolio management practices.
- Most portfolio optimization models deal with decisions made at a single point in time or with a single objective, but portfolio management is a multi-period activity or continuous process, where decisions are made over time as new information becomes available. To fill these limitations, we suggest:
  - Using other techniques of optimization such as optimization by interval to estimate parameters by intervals is a solution. This approach is relatively new in the literature and could offer a solution to better manage uncertainties in market parameters. For example, instead of assuming a single expected

return for a stock, one could assume that it lies within a certain interval. This increases the robustness of the model by considering the uncertainty of the estimates.
  - Proposing multi-period optimization problems is not yet well addressed in the literature. Some models in this sense under uncertainty could be a future direction.

- Considering reality factors, such as transaction costs, liquidity constraints, legal restrictions or taxes, portfolio modelling is interesting to reflect more market reality. In a framework of uncertainty and randomness, modeling a portfolio that incorporates these aspects would be more realistic. For example, developing a model that considers long-term fluctuations in interest rates or regulations could give more reliable results that are applicable to real-world situations. Proposing models in this sense of uncertain random framework could be a future work.
- With increasing attention to climate change, investors are increasingly paying attention to climate change risks, such as natural disasters, carbon reduction policies, etc. Optimized portfolios should integrate these environmental risks into their strategies. For example, a portfolio could be adjusted to include companies that are less exposed to climate risks or investments in renewable energy. - Last but not least, employing artificial intelligence (AI) techniques in portfolio optimization can greatly enhance decision-making by identifying patterns, adapting to market changes, and providing more sophisticated risk management. Some of the AI techniques, such as machine learning, neural networks, and genetic algorithms, can be particularly effective in improving portfolio performance.

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