STATISTICS, OPTIMIZATION AND INFORMATION COMPUTING
Stat., Optim. Inf. Comput., Vol. **13**, March 2025, pp 977–992.
Published online in International Academic Press (www.IAPress.org)

Inverse Multi-Objective Optimization for Portfolio Allocation in Commercial Banks

Nagwa M. Albehery*, Marwa A. Helal, Amal F. Ghania

Department of Mathematics, Applied Statistics and Insurance,
Faculty of Commerce and Business Administration, Helwan University, Egypt

Abstract Optimal portfolio allocation in commercial banks is a critical decision for financial institutions. This paper proposes a multi-objective linear programming model to address this challenge. We employ a generalized inverse optimization approach to ensure the model's feasibility and efficiency, replacing regular optimality with Pareto optimality. The proposed models were applied to 2020/2021 fiscal year data from two leading Egyptian banks, Banque Misr and the national Bank of Egypt. A sensitivity analysis was subsequently conducted to evaluate the robustness of the optimal solutions to variations in input parameters. The multi-objective model and the sensitivity analysis were solved using LINGO 19, while the inverse multi-objective model was solved using R programming. Our analysis of the results provides valuable insights into optimal portfolio distribution for commercial banks.

Keywords Linear programming, Multi-objective linear programming, Sensitivity analysis, Inverse optimization, Efficiency, Pareto optimality, Inverse multi-objective linear programming.

DOI: 10.19139/soic-2310-5070-2188

1. Introduction

Multi-objective linear programming (MOLP) is a powerful mathematical optimization technique designed to handle decision-making problems with multiple, often conflicting, objectives. Unlike traditional linear programming, which aims to optimize a single objective, MOLP seeks optimal solutions that balance these competing objectives. By considering a diverse range of criteria, such as maximizing returns, minimizing risk, and adhering to regulatory constraints, MOLP provides a more comprehensive and realistic approach to decision-making. MOLP has various applications in various fields, including finance, engineering, and medicine [8, 28, 51].

Inverse optimization flips the traditional optimization paradigm. Instead of determining optimal decisions given a fixed set of constraints and objectives, IO seeks to identify the underlying parameters (constraints or objectives) that would make a given set of decisions optimal [17]. This involves finding the parameters of the original "forward" optimization problem by formulating an "inverse" optimization problem. Early research in IO, often referred to as "classical" IO, assumed perfect alignment between the model and real-world data. These models are valuable for introducing new reformulation techniques and applications where precise parameter identification is crucial. However, real-world scenarios often involve model misspecification and noisy data. To address these challenges, "data-driven" IO has emerged. Data-driven IO builds upon classical techniques but incorporates additional considerations, such as model uncertainty and data noise. It leverages loss functions to measure the degree of suboptimality and guide the parameter estimation process. inverse optimization techniques have been explored and utilized in various studies such as [1, 2, 3, 6, 16, 30, 31, 33, 34, 40, 54, 55]. Recently, inverse

*Correspondence to: Nagwa M. Albehery (Email: nagwa.el-behairy@commerce.helwan.edu.eg). Department of Mathematics, Applied Statistics and Insurance, Faculty of Commerce and Business Administration, Helwan University. Ain Helwan, Cairo, Egypt (11795).

ISSN 2310-5070 (online) ISSN 2311-004X (print)
Copyright © 2025 International Academic Press

optimization has seen increased interest and has been applied in diverse fields, including vehicle routing (e.g., [21]), transportation systems (e.g., [44]), portfolio optimization (e.g., [35, 53]), power systems (e.g., [12, 24, 46, 47]), electric vehicle charging (e.g.,[25]), network design (e.g.,[23], healthcare (e.g., [9]), and controller design ([5]). A comprehensive overview of IO applications can be found in [17].

Recently, a novel approach has been introduced in e.g.,[18, 19] where the optimal solution can be an interior point of the feasible region. In this case, the objective function coefficients and the solution are modified to achieve optimality. Inverse optimization has been investigated for a specific class of multi-objective problems, demonstrating how to minimally modify the criteria matrix to transform a set of feasible points into Pareto optimal points [45]. This method has been applied to radiotherapy cancer treatment, focusing on turning infeasible or interior points into near-optimal solutions [20]. Naghavi et al [43] generalized inverse linear programming (ILP) to inverse multi-objective linear programming (IMOLP), addressing the non-convexity of the problem and introducing necessary tools for efficient convex optimization.

Portfolio optimization has been a cornerstone of financial decision-making for decades. Traditional approaches, such as the mean-variance model proposed in e.g.,[38] provided a foundation for portfolio allocation. Moreover, Modern Portfolio Theory (MPT) posits that the assessment of portfolio risk is based on the variability of anticipated returns. This framework assumes that investors' decisions are primarily driven by these two objectives: maximizing expected return and minimizing risk and the trade-off between these objectives is often quantified by a risk tolerance parameter.

In recent years, multi-objective optimization has emerged as a powerful tool for addressing the multi-faceted nature of portfolio allocation. By considering multiple objectives simultaneously, such as risk, return, liquidity, and regulatory constraints, multi-objective optimization techniques can provide more comprehensive and robust solutions. Studies demonstrated the effectiveness of multi-objective optimization in generating efficient frontiers that capture the trade-offs between different objectives [7, 10, 13, 27, 29, 37, 39, 41, 52]. Inverse optimization offers a novel approach to portfolio optimization by determining the parameters of an optimization problem that leads to a desired outcome. By leveraging inverse optimization, we can identify the optimal portfolio weights that align with specific investment goals and risk tolerances. Recent studies, e.g.,[11, 53] have explored the application of inverse optimization in various financial contexts, including portfolio management. While existing research has made significant contributions to portfolio optimization, more advanced techniques are needed to address the complexities of real-world portfolio allocation in commercial banks. This paper addresses this gap by proposing a novel inverse multi-objective optimization framework that combines the strengths of multi-objective optimization and inverse optimization.

In this research paper, a multi-objective linear programming model will be employed to optimize portfolio distribution in commercial banks. A sensitivity analysis will assess the impact of parameter variations in the objective functions on the model's reliability and will inform decision-making under different scenarios. Additionally, we will utilize inverse optimization for multi-objective linear programming to transform a feasible point into a weakly efficient solution, aligning with the decision-maker's preferences for portfolio allocation problems in commercial banks. This paper is organized as follows: Section 2 presents a suggested multi-objective linear programming model for optimizing portfolio distribution in commercial banks. Section 3 outlines and demonstrates the inverse optimization approach for multi-objective linear programming, drawing on the study by Naghavi et al [43]. Section 4 applies the proposed model, sensitivity analysis, and inverse optimization approach for multi-objective linear programming to 2020/2021 fiscal year data from Banque Misr and the National Bank of Egypt (NBE). Section 5 presents the conclusions.

2. A proposed Multi-Objective Linear Programming Model for Optimal Portfolio Distribution.

Several studies in the literature have proposed goal programming models to address portfolio optimization problems. For example, Albehery et.al [7] introduced the model to achieve the best compromise distribution for portfolios in commercial banks using the chance-constrained goal programming model. This model maximized the following objectives: capital adequacy ratio, stock profit value, loan returns, total balance sheet, and minimized credit risk ratio according to their priorities. A probabilistic model is constructed when the capital adequacy ratio, stock profit value, and credit risk ratio are random variables. The model has been applied to the real data of the Egyptian Bank (Bank Misr) during the period (2009/2010)-(2018/2019). Mohammedi et.al [41] presented a model to optimize bank liquidity management. This model integrated two techniques: the goal programming approach and the Fuzzy Analytic Hierarchy Process (FAHP). FAHP is a multivariate decision-making method that prioritizes the model's objectives. It considers three elements: the overall goal (improved liquidity management), the decision-making criteria (e.g., capital adequacy liquidity ratio), and the available alternatives. This model maximizes capital adequacy, liquidity ratio, total asset growth, investment portfolio, and fixed assets and minimizes liquidity risk, claims from other banks, and the ratio of consumption to resources. This model was applied to one of the Iranian banks, Persian Bank to improve bank liquidity management. Halim et.al [27] presented a goal programming model for managing the financial statements of a Malaysian commercial bank, Maybank. This model was solved by combining the weighting and the preemptive methods.

This section introduces a multi-objective linear programming model for optimizing portfolio distribution.

First. we introduce the following notation:

• **Decision variables:** The model includes 10 decision variables: x_1: Cash with the central bank, x_2: Due from banks, x_3: Loans to banks and customers, x_4: Total financial investments, x_5: Investments in subsidiaries and associates, x_6: Intangible and other assets, x_7: Fixed assets, x_8: Tier 1 capital, x_9: Tier 2 capital, x_{10}: Total risk-weighted assets,

• **Parameters:** We have 19 parameters, as follows:

Objective function parameters: I: Interest rate on loans, CAR: Capital adequacy ratio

Left-hand side parameter: P: Ratio of retained earnings to total assets

Right-hand side parameters: b_1: Ratio of cash to deposits, b_2: Ratio of due from banks to deposits, b_3: Ratio of loans and advances to banks and customers to deposits, b_4: Ratio of total financial investments to deposits, b_5: Ratio of investments in subsidiaries and associates to deposits, b_6: Ratio of intangible and other assets to shareholder's equity, b_7: Ratio of fixed assets to shareholder's equity, b_8: Ratio of tier 1 capital to shareholder's equity, b_9: Ratio of tier 2 capital to shareholder's equity, b_{10}: Ratio of total risk-weighted assets to deposits,

Additional parameters: DB: Due to banks, CD: Total customers' deposits, LP: Liabilities and provisions, PC: Paid-in capital, R: Reserves, RE: Retained earnings, These parameters are defined and calculated from the balance sheet (liabilities and shareholder's equity).

• **Objectives and Priorities: :** The proposed model includes three objectives, arranged in order of priority:

1. Maximize total balance sheet: $\mathrm{f}_1(x) = \sum_{j=1}^{7}(x_j)$

2. Maximize loan returns: $\mathrm{f}_2(x) = I \times (x_3)$

3. Maximize capital adequacy ratio: $\mathrm{f}_3(x) = (x_8 + x_9 - CAR \times x_{10})$

Second. based on the above, we propose the following multi-objective linear programming model: to determine the optimal values of the decision variables (x_j) where $(j = 1, 2, \ldots, 10)$ which maximizes (Z):

$$Lexic.Max.z = (\sum_{j=1}^{7}(x_j), I \times (x_3), x_8 + x_9 - CAR \times (x_{10})), \tag{1}$$

s.t.

$$x_j \geq b_i \times CD, (i, j = 1, 2, ..., 5), \tag{2}$$

$$x_j \geq b_i \times (PC + R + RE), (i, j = 6, 7, ..., 9), \tag{3}$$

$$x_{10} \leq b_{10} \times CD, \tag{4}$$

$$x_8 - x_9 \geq 0, \tag{5}$$

$$x_{10} \geq x_8 + x_9, \tag{6}$$

$$P \times \sum_{j=1}^{7} x_i \leq RE, \tag{7}$$

$$(x_j \geq 0, (j, i = 1, 2, ..., 10) \tag{8}$$

The assumptions of the proposed model in (1)-(8) are as follows:

1. The assets side of the balance sheet consists of seven items: $x_1, x_2, x_3, x_4, x_5, x_6$ and x_7,

2. The interest rate on loans (I) is constant for loans and advances to customers and banks,

3. The suggested model is deterministic; the selected parameter values are from the real Bank data,

4. Tier 2 capital (x_9) does not exceed 100% of Tier 1 capital (x_8),

5. The ratio of retained earnings to total assets (P) is a measure of the bank's profitability used to assess the bank's performance in investing its assets.

6. Total risk-weighted assets (x_{10}) are greater than or equal to total capital (Tier 1 capital + Tier 2 capital).

The proposed model could incorporate additional objectives and structural Constraints, including those related to liquidity, risk management, and regulatory compliance. However, data availability is limited due to restrictions imposed by the Central Bank of Egypt on publicly accessible information from the bank's websites. Consequently, the model's current objectives and structural Constraints are based on the available financial data. The proposed multi-objective model will be used to apply the generalized inverse optimization approach in the following section.

3. A Generalized Inverse Optimization Approach.

In this section, we will introduce the notation used throughout the paper in Subsection 3.1. Next, in Subsections 3.2 and 3.3, we will discuss inverse single-objective linear programming, as well as explore inverse multi-objective linear programming based on the study by Naghavi et al [43] and provide our commentary.

3.1. Notations

Let R^n and $R^{m \times n}$ denote the sets of real n vectors and $m \times n$ matrices, respectively. We denote the i-th row of matrix A as a_i. For $x \in$R^n, $||.||_p$ denotes the $p-norm$ of the vector x.

Consider the following two problems:

min $(cx|x \in \mathcal{S}), (LP(c))$,

min $(Cx = (c_1x, ..., c_kx)|x \in \mathcal{S})$,(MOLP(C)),

where$A \in \mathcal{R}^{k \times n}$,$C \in \mathcal{R}^{k \times n}$, $c \in \mathcal{R}^n$,$b \in$ R^m, and $S = (x \in \ R^n)|Ax \geq b)$ is the feasible region. A feasible point $x^0 \in \mathcal{S}$ is weakly efficient or (weak Pareto optimal) for MOLP(C) if there exists no $x \in \mathcal{S}$ such that $Cx < Cx^0$ (component-wise). The set of all weakly efficient points of MOLP(C) is denoted by $S_{we}(C)$ and the set of all optimal solutions of LP(c) is denoted by S$_0(c)$.

3.2. Inverse single-objective linear programming (ILP)

This section investigates the inverse problem of single-objective linear programming problem under the p-norm, focusing on modifications to the cost vector. Suppose there exists a feasible solution $x^0 \in \mathcal{S}$ and the decision-maker desires to make x^0 the optimal solution while maintaining the current optimal objective function value. One way to formulate an inverse optimization problem is by adjusting the coefficient parameters of the decision variables in the objective function. The goal is to find new coefficients that make (x^0, z) the optimal solution to the following $(ILP(c, x^0))$ problem:

$$Min||c - \hat{c}||_p,$$

$$\text{s.t. } x^0 \in \mathcal{S}_0(\hat{c}), \hat{c} \in \mathcal{R}^n,$$

(ILP) problems under the p-norm have been extensively studied in [55, 55, 56]. Ahuja and Orlin [1] have also investigated weighted norms for such problems. Naghavi et al. [43] reformulated $ILP(c, x^0)$ using the following lemma into a more convenient form involving the conic hull$(\hat{K})$ of the active constraints, where $\hat{K}$ is defined as:

$\hat{K} = cone(a_i|i \in I(x^0)) = (x \in \mathcal{R}^n|x) = \sum_{i\in(I(x^0)} \beta_i a_i, \beta_i \geq 0)$,$\mathrm{I}(\mathrm{x}^0) = (i|a_i x^0 = b_i)$ be the set of all active constraints at x^0.

Lemma I Let $x^0 \in \mathcal{S}$ be a feasible point of the inverse linear programming $ILP(c, x^0)$, then $x^0 \in \mathcal{S}_0(\hat{c})$ if and only if $\hat{c} \in \hat{K}$.

Using Lemma 1, $ILP(c, x^0)$ can be reformulated based on the conic hull concept. The constraint $x^0 \in S_0(\hat{c})$ in $ILP(c, x^0)$ is replaced with $\hat{c} \in \hat{K}$, resulting in the following equivalent problem:

$$Min||c - \hat{c}||_p,$$

$$\text{s.t. } \hat{c} \in \hat{K},\ \ \hat{c} \in \mathcal{R}^n,$$

This reformulation (using Lemma 1) provides an elegant and insightful approach to checking a point's optimality. By stating that a point x^0 is optimal if and only if its corresponding cost vector lies within the conic hull of the active constraints at x^0, the authors have introduced a valuable criterion for assessing optimality. This reformulation simplifies the process of verifying whether a solution is indeed the best possible.

3.3. Inverse multi-objective linear programming (IMOLP)

By replacing the cost vector with a criteria matrix ($\hat{C}$) and the concept of Pareto optimality, $IMOLP(C, x^0)$ extends the scope of inverse optimization to multi-objective problems, seeking to make a given feasible solution x^0 weakly efficient, unlike $(ILP(c, x^0))$ where the aim is to make x^0 optimal for a single objective. $IMOLP(C, x^0)$ as follows:

$$Min||C - \hat{C}||_p,$$

$$\text{s.t.}$$

$$x^0 \in \mathcal{S}_{we}(\hat{C}),$$

$$\hat{C} \in \mathcal{R}^{k\times n},$$

Naghavi et al. in [43] utilized the weighted-sum theorem to establish a connection between inverse linear programming (ILP) and inverse multi-objective linear programming (IMOLP), building upon the work of Ehrgott in [22, 36, 50]. This theorem serves as a bridge between single-objective and multi-objective optimization, allowing insights from ILP to be applied to the more complex world of IMOLP. Notably, $IMOLP(C, x^0)$ reduces to $ILP(C, x^0)$ when only one objective function is considered.

Theorem I: $x^0 \in \mathcal{S}$ to be a weakly efficient solution of MOLP if it is both necessary and sufficient that there exists a weighted vector $w \in \mathcal{W}$ where $W = (w \in \mathcal{R}^k | \sum_{i=1}^k w_i = 1, w_i \geq 0)$ such that x^0 is an optimal solution of the weighted-sum (LP): min $(w\hat{C}x|x \in \mathcal{S})$.

Based on Lemma I and Theorem I, they realized that the following statements are equivalent:

1. $x^0 \in \mathcal{S}_{we}(\hat{C})$.

2. There exists $w \in \mathcal{W}$ can be found such that $w\hat{C} \in \hat{K}$.

3. d (conv $(\hat{C}, \hat{K})$ = 0.

Where conv(C) denotes the convex hull of the rows of the matrix C, and d is the distance between two nonempty sets A and B, defined as:

$$d(A, B) = inf(||x - y||_p | x \in \mathcal{A}, y \in \mathcal{B})$$

Based on the above, Naghavi et al [43] expressed $IMOLP(C, x^0)$ in the following equivalent form:

$$Min||C - \hat{C}||, \tag{11}$$

$$\text{s.t.}$$

$$w\hat{C} \in \hat{K}, \tag{12}$$

$$w \in \mathcal{W}, \tag{13}$$

$$\hat{C} \in \mathcal{R}^{k\times n}, \tag{14}$$

In the case of x^0 is not a weakly efficient solution, IMOLP aims to minimally modify the criteria matrix C so that a convex combination of its rows falls within $\hat{K}$, the convex cone generated by the active constraints at x^0. They rewrote the constraint in problem (15)-(18) using the actual active constraints at the point x^0 as follows:

$$Min.q = \sum_{i=1}^{k} ||c_i - \hat{c}_i||_p, \tag{15}$$

s.t.

$$\sum_{i=1}^{k} w_i\hat{c}_i - \sum_{r\in I(x^0)} \beta_r a_r = 0, \tag{16}$$

$$\sum_{i=1}^{k} w_i = 1, \tag{17}$$

$$w_i \geq 0, i = 1, 2, ..., k, \tag{18}$$

$$\beta_r \geq 0, r \in \ I(x^0), \tag{19}$$

$$\hat{c}_j \in R^n \tag{20}$$

This makes the problem easier to work with because we're now dealing with explicit linear inequalities. Unfortunately, the presence of product terms like $w_i\hat{c}_i$ in the IMOLP in (19)-(24) makes it a non-convex problem. This is a challenge because non-convex problems are generally harder to solve than convex ones, with multiple local optima and potential difficulty in finding the global optimum (the absolute best solution). The authors addressed the non-convexity issue by proving that under certain conditions (when the convex hull of the original criteria matrix is sufficiently far from the conic hull of active constraints), an optimal solution for IMOLP can always be found by modifying just one of the objective functions. This significant simplification allows us to focus on changing objective functions one by one instead of all at once. Moreover, they provided a lower bound on the necessary modification to the criteria matrix. This bound is valuable because it enables us to stop searching for an optimal solution early if a solution already close to the lower bound is found. These findings are presented in the following theorem.

Theorem II: For $IMOLP(C, x^0)$ as defined in (19)-(24), if the distance between $conv(C)$ and $\hat{K})$ is positive $d(conv(C), \hat{K}) > 0$, then,

1. There exists an optimal solution for $IMOLP(C, x^0)$ such that $\hat{c}_i^* = c_i, i \in (1, 2, .., k), i \neq j.$

2. $d(conv(C), \hat{K})$ is a lower bound on the optimal value of $IMOLP(C, x^0)$,

3. Thus $\hat{q}^* \geq d(conv(C), \hat{K})$.

Assuming only the jth objective function in I$MOLP(C, x^0)$ (19)-(24) is modified while others remain unchanged, the authors reformulated the model as the following P_j problem:

$$Min.||c_j - \hat{c}_j||_p, \tag{21}$$

s.t.

$$w_j\hat{c}_j + \sum_{i=1,i\neq j}^{k} w_i c_i - \sum_{r\in I(x^0))} \beta_r a_r = 0, \tag{22}$$

$$\sum_{i=1}^{k} w_i = 1, \tag{23}$$

$$w_i \geq 0, i = 1, 2, ..., k, \tag{24}$$

$$\beta_r \geq 0, r \in I(x^0), \tag{25}$$

$$\hat{c}_j \in \mathcal{R}^n, i = 1, 2, ..., k, \tag{26}$$

The proof for part (1) of Theorem II is constructive, demonstrating a transformation process that modifies an initial optimal solution of the IMOLP by altering only a single objective function to yield a new, at least equally good solution. This process begins with an assumed optimal solution and identifies the objective function with the highest weight within that solution. This function is then modified to create a new solution. The proof subsequently verifies that this modified solution remains feasible, satisfying all constraints, and maintains or improves upon the original optimality. This confirms that an optimal solution can be achieved through a single objective function modification. In contrast, the proof for part (2) uses properties of distances, convex hulls, and conic hulls to establish a lower bound. It shows that the distance between the convex hull of the original criteria matrix and the conic hull of the active constraints defines the minimum modification required to achieve optimality. The approach involves solving this problem (P_j) in (25)-(30) for each objective function and selecting the solution with the smallest overall modification. However, there's a caveat: the problem (P_j) in (25)-(30) is still non-convex. Naghavi et.al [43] introduced the following theorem, which is considered a game-changer because it reveals that under certain conditions, the non-convex problem (P_j) in (25)-(30) can be reformulated into an equivalent convex problem.

Theorem III: If $d(conv(C), \hat{K}) > 0$, then P_j in (25)-(30) will be equivalent to the following convex optimization model Q_j:

$$q_j^* = Min.||c_j - \hat{c}_j||_p, \tag{27}$$

s.t.

$$\hat{c}_j + \sum_{i=1, i\neq j}^{k} \gamma_i c_i - \sum_{r \in I(x^0)} \theta_r a_r = 0, \tag{28}$$

$$\gamma_i \geq 0, i = 1, 2, ..., k, i \neq j, \tag{29}$$

$$\theta_r \geq 0, r \in I(x^0), \tag{30}$$

$$\hat{c}_j \in \mathcal{R}^n, j = 1, 2, ..., k, \tag{31}$$

The theorem above determines the condition that the convex hull of the original criteria matrix (conv(C)) must be sufficiently distant from the conic hull of the active constraints $\hat{K}$. In other words, the original objectives and constraints must be somewhat "separated" for this technique to work. The theorem states that under the condition $d(conv(C), \hat{K}) > 0$, the optimization problem P_j in(25)-(30) is equivalent to the convex optimization problem Q_j in (31)-(35). The proof of the theorem demonstrates this equivalence by establishing a one-to-one correspondence between the feasible regions of P_j and Q_j. It does so by showing that for every feasible point in P_j, there exists a corresponding feasible point in Q_j with the same objective function value, and vice versa. For interested readers, the proofs of theorems II and III can be found in Naghavi et al [43]. Based on Theorem III, Naghavi et al [43] proposed an algorithm to address the IMOLP problem when the initial point (x^0) might not be weakly efficient. The goal is to find the minimal modification to the objective functions that make x^0 weakly efficient. In the next section, we will apply this algorithm to portfolio allocation in commercial banks.

4. Application

In this section, the proposed multi-objective linear programming model from section 2 and IMOLP-based algorithm described in section 3 will be applied to real-world data from two leading Egyptian commercial financial institutions: Banque Misr and the National Bank of Egypt (NBE). The data, covering the fiscal year 2020/2021, was obtained from the publicly available financial statements on the bank's official websites (www.banquemisr.com and www.nbe.com.eg). We have specified the technical names of the financial statements used for each bank. Subsections 4.1 and 4.2 present the model's applications, the sensitivity analysis and the inverse optimization. Subsection 4.3 addresses the algorithm's computational efficiency.

4.1. Banque Misr case study

The data for this study originates from the following sources within Banque Misr's Separate Financial reporting for the Period concluding June 30, 2021:

1. The separate Statements of Financial Position as of June 30, 2021.
2. The separate Income Statements for the Financial Period from July 1, 2020, to June 30, 2021.
3. The summarized notes to the Separate Financial Statements for the Financial Period Ended June 30, 2021 (Capital Management).

First: The proposed multi-objective linear programming model defined in equations (1)-(8) will be applied and solved using LINGO package version 19 as follows:

1- The values of the constant parameters derived from the real - world data for Banque Misr during the 2020/2021 fiscal year were calculated and are shown in Table 1 below.

Table 1. Constant parameters values: Banque Misr

Symbol	parameter values
I	0.2118
CAR	0.153115
P	0.0074
b_1	0.0558
b_2	0.2201
b_3	0.5105
b_4	0.4470
b_5	0.0226
b_6	0.4566
b_7	0.0724
b_8	0.922
b_9	0.2119
b_{10}	0.6388
DB	122,206,054
CD	1,120,349,800
LP	119,103,527
PC	15,000,000
R	70,836,820
RE	10,800,574

2- The mathematical form of the multi-objective linear programming problem takes the following form: Determine the optimal values of the decision variables (x_j) where (j=1,2, ...,10) that yields the optimal solution which maximizes (Z):

$$Lexic.Max.z = (\sum_{j=1}^{7}(x_j), 0.2118(x_3), (x_8 + x_9 - 0.153115x_{10})), \tag{32}$$

s.t.

$$x_1 \geq 0.0558CD, \tag{33}$$

$$x_2 \geq 0.2201CD, \tag{34}$$

$$x_3 \geq 0.5105CD, \tag{35}$$

$$x_4 \geq 0.4470CD, \tag{36}$$

$$x_5 \geq 0.0226CD, \tag{37}$$

$$x_6 \geq 0.4566(PC + R + RE), \tag{38}$$

$$x_7 \geq 0.0724(PC + R + RE), \tag{39}$$

$$x_8 \geq 0.922(PC + R + RE), \tag{40}$$

$$x_9 \geq 0.2119(PC + R + RE), \tag{41}$$

$$x_{10} \leq 0.6388CD, \tag{42}$$

$$x_8 - x_9 \geq 0, \tag{43}$$

$$x_{10} \geq x_8 + x_9, \tag{44}$$

$$0.0074 \sum_{j=1}^{7} (x_j \leq RE, \tag{45}$$

$$(x_j \geq 0, (j = 1, 2, ..., 10) \tag{46}$$

3- The model defined by equations (36)-(50) was solved using LINGO version 19. The resulting optimal values of the decision variables are presented in Table 2 below.

Table 2. The optimal values for the decision variables (expressed in thousands of pounds): Banque Misr

Variable	The derived values of the decision variables
x_1	62,515,520
x_2	246,589,000
x_3	573,195,000
x_4	500,796,400
x_5	25,319,910
x_6	44,124,630
x_7	6,996,595
x_8	357,839,700
x_9	357,839,700
x_{10}	715,679,500

In Table 2, we can observe that total risk-weighted assets (x_{10}) constitute the largest portion of the balance sheet, while fixed assets (x_7) are the smallest. The optimal solution for the objective function Z in equation (36) is 606,098,200 Egyptian pounds.

Second, a sensitivity analysis was conducted to determine the range within which the loan interest rate (I) and the capital adequacy ratio (CAR) could vary, the results are presented in Table 3 below.

Table 3. Sensitivity analysis of objective function coefficients: allowable changes in interest rate (I) and capital adequacy ratio (CAR): Banque Misr

Variable	Current Coefficient	Allowable Increase	Allowable Decrease
x_3	0.2118	Infinity	0.2118
x_{10}	- 0.153115	Infinity	- 0.846885

Table 3 shows that the interest rate on loans can be decreased by up to 0.2118, and the capital adequacy ratio can be decreased by up to 0.844885. Both can be increased without limit without affecting the optimal solution of the objective functions. **Third,** if the decision-maker wishes to set specific values for certain decision variables, such as loans to banks and customers (x_3), total financial investments (x_4), investments in subsidiaries and associates (x_5), fixed assets (x_7) and total risk-weighted assets (x_{10}) while maintaining the optimal objective function value of 606,098,200 Egyptian pounds as shown in Table 4 below.

Table 4. The different (feasible) values x^0 of the decision variables (Values in thousands of pounds): Banque Misr

Variable	The derived values of the decision variables
x_3^0	573191595
x_4^0	500,796,390
x_5^0	25,319,900
x_7^0	7,000,000
x_{10}^0	715,679,400

Then, we apply the IMOLP-based algorithm described in Section 3 to the suggested multi-objective linear programming model defined in equations (36)-(50). We will use R programming to make the feasible values x^0 align with the decision-maker's preference for a weakly efficient solution. We start by identifying the binding constraints: The binding constraints are (47) and (48)

$x_8 - x_9 \geq 0,$

$x_{10} \geq x_8 + x_9,$

Accordingly, the set of active constraints $I(x^0)$ is $I(x^0) = (12, 13)$ and $\hat{K} = (x \in \mathcal{R}^{10} | x = \beta_1 a_1 + \beta_2 a_2, \beta_1, \beta_2 \geq 0), where a_1 = (0,0,0,0,0,0,0,1,-1,0)$, $a_2 = (0,0,0,0,0,0,0,-1,-1,1)$.

Then $IMOLP(C, x^0)$ is formulated as

Min. q =$||c_1 - \hat{c}_1||_p + ||c_2 - \hat{c}_2||_p + ||c_3 - \hat{c}_3||_p,$

s.t.

$w_1\hat{c}_1 + w_2\hat{c}_2 + w_3\hat{c}_3 - \beta_1 a_1 - \beta_2 a_2 = 0,$

$w_1 + w_2 + w_3 = 1,$

$w_1, w_2, w_3, \beta_1, \beta_2 \geq 0,$

$\hat{c}_1, \hat{c}_2, \hat{c}_3 \in \mathcal{R}^{10},$

The algorithm is employed for the case where p = 2, aiming for an exact solution by setting the threshold $\epsilon = 0$.

Phase I: The solving of the following convex problem $d(conv(C), \hat{K}) = (min.\ ||x - y||_p | x \in conv(C), y \in \hat{K})$ yields distance $d^* = 0.20254$. Since $d^* > 0$, this indicates that x^0 is not a weakly efficient solution. Therefore, we set j=1 and proceed to step 1.

Phase II: Step 1: We solve Q_1 as follows:

Min. q$_1$ $= |||c_1 - \hat{c}_1||_2,$

s.t.

$\hat{c}_1 + \gamma_2 c_2 + \gamma_3 c_3 - \theta_1 a_1 - \theta_2 a_2 = 0,$

$\gamma_2, \gamma_3, \theta_1, \theta_2 \geq 0,$

$\hat{c}_1 \in \mathcal{R}^{10},$

This problem finds the minimum distance between c_1 and $K_1 = cone(a_1, a_2, -c_2, -c_3) = cone(a_1, -c_2)$. The optimal objective value is $q_1^* = 2.64588$. Since $q_1^* - d^* > 0$, we will proceed to solve Q_2 as follows:

Min. q$_2$ $= |||c_2 - \hat{c}_2||_2,$

s.t.

$\hat{c}_2 + \gamma_1 c_1 + \gamma_3 c_3 - \theta_1 a_1 - \theta_2 a_2 = 0,$

$\gamma_1, \gamma_3, \theta_1, \theta_2 \geq 0,$

$\hat{c}_2 \in \mathcal{R}^{10},$

The optimal solution corresponds to the shortest distance between c_2 and $K_2 = cone(a_1, a_2, -c_1, -c_3) = cone(a_1, -c_1)$. The optimal objective value is $q_2^* = 0.69148$. Since $q_2^* - d^* > 0$, we will proceed to solve Q_3 as follows:

Min. $\mathrm{q}_3 = |||c_3 - \hat{c}_3||_2$,
s.t.
$\hat{c}_3 + \gamma_1 c_1 + \gamma_2 c_2 - \theta_1 a_1 - \theta_2 a_2 = 0$,
$\gamma_1, \gamma_2, \theta_1, \theta_2 \geq 0$,
$\hat{c}_3 \in \mathcal{R}^{10}$,

The optimal objective value is $q_3^* = 0.21181$. As Q_3 yields the smallest objective value compared to all the problems, j is determined to be 3, $j^* = 3$. The optimal solution of Q_3 is $\eta_3^* = (\hat{c}_3^*, \gamma_1^*, \gamma_2^*, \theta_1^*, \theta_2^*) = ((0, 0, 0, 0, 0, 0, 0, 0.71771, 0.71771, -0.71771), 0, 0, 0, 0)$. Therefore, the optimal solution of $IMOLP(C, x^0)$ is $\eta^* = (\hat{c}_1^*, \hat{c}_2^*, \hat{c}_3^*, w_1^*, w_2^*, w_3^*, \beta_1^*, \beta_2^*) = (c_1, c_2, \hat{c}_3^*, 0, 0, 1, 0, 0.71771)$.

The new criteria matrix C^*, obtained by moving c_3 to $\hat{c}_3^*$, indicates that $conv(C^*)$ intersects $\hat{K}$. This implies that x^0 is weakly efficient for the new criteria matrix.

4.2. National Bank of Egypt (NBE) case study

The National Bank of Egypt's data is derived from its Summarized Separate Financial Statements for the financial period spanning July 1, 2020, to the end of 2021. The General Assembly approved on October 13, 2020, an amendment to Article 25 of the Bank's Articles of Association. This amendment transitions the fiscal year to coincide with the calendar year beginning on January 1, 2022. The current period (July 1, 2020 - December 31, 2021) is considered a transitional 18-month transitional period. The data is based on:

1. The National Bank of Egypt's Separate Statements of Financial Position as of December 31, 2021.
2. The National Bank of Egypt's Separate Income Statements for the Financial Period from July 1, 2020, to December 31, 2021.
3. Summarized notes to the Separate Financial Statements for the Financial Period Ended December 31, 2021 (Capital Management).

First The proposed multi-objective linear programming model defined in equations (1)-(8) will be applied and solved using LINGO package version 19 as follows:

1- Real-world data from NBE's 2020/2021 fiscal year was used to derive constant parameter values presented in Table 5 below.

2- The mathematical form of the multi-objective linear programming problem takes the following form: Determine the values of the decision variables (x_j) where (j=1, 2, ...,10) that yield the optimal solution which maximizes (Z):

$$Lexic.Max.z = (\sum_{j=1}^{7}(x_j), 0.3121(x_3), (x_8 + x_9 - 0.2139x_{10})), \tag{47}$$

s.t.

$$x_1 \geq 0.0298CD, \tag{48}$$

$$x_2 \geq 0.2225CD, \tag{49}$$

$$x_3 \geq 0.4736CD, \tag{50}$$

$$x_4 \geq 0.568CD, \tag{51}$$

$$x_5 \geq 0.0035CD, \tag{52}$$

$$x_6 \geq 0.706(PC + R + RE), \tag{53}$$

$$x_7 \geq 0.0465(PC + R + RE), \tag{54}$$

Table 5. Constant parameters values: National Bank of Egypt

Symbol	parameter values
I	0.3121
CAR	0.2139
P	0.0097
b_1	0.0298
b_2	0.2225
b_3	0.4736
b_4	0.568
b_5	0.0035
b_6	0.706
b_7	0.0465
b_8	0.95991
b_9	0.3802
b_{10}	0.4783
DB	342,817,000,000
CD	2,386,450,000,000
LP	322,328,000,000
PC	50,000,000,000
R	100,836,000,000
RE	31,332,000,000

$$x_8 \geq 0.95991(PC + R + RE), \tag{55}$$

$$x_9 \geq 0.3802(PC + R + RE), \tag{56}$$

$$x_{10} \leq 0.4783CD, \tag{57}$$

$$x_8 - x_9 \geq 0, \tag{58}$$

$$x_{10} \geq x_8 + x_9, \tag{59}$$

$$0.0097 * \sum_{j=1}^{7}(x_j \leq RE, \tag{60}$$

$$(x_j \geq 0, (j = 1, 2, ..., 10) \tag{61}$$

3- The model defined by equations (51)-(65) was solved using LINGO version 19. The optimal values of the decision variables are shown in Table 6 below.

As shown in table 6, total risk-weighted assets x_{10} represent the largest component of the balance sheet, while fixed assets x_7 constitute the smallest. The optimal solution for the objective function Z in equation (51) is 606,098,200000 Egyptian pounds.

Second: a sensitivity analysis was conducted, to determine the range within which the loan interest rate (I) and the capital adequacy ratio (CAR) can vary. The results are shown in Table 7.

Table 7 shows that the interest rate on loans can be decreased by up to 0.3121, and the capital adequacy ratio can be decreased by up to 0.7861. Both can be increased without limit, without affecting the optimal solution of the objective functions.

Third: if the decision-maker wishes to set specific values for certain decision variables, such as loans to banks and customers x_3, total financial investments x_4, investments in subsidiaries, associates x_5, intangible and other assets x_6, and fixed assets x_7 while maintaining the optimal objective function value of 897,285,200,000 Egyptian Pounds, as shown in Table 8 below.

Table 6. The optimal values of the decision variables (Values in thousands of pounds): National Bank of Egypt

Variable	The derived values of the decision variables
x_1	71,116,210
x_2	530,985,100
x_3	1,135,535,000
x_4	1,355,504,000
x_5	8,352,575
x_6	128,610,600
x_7	117.5014
x_8	570,719,500
x_9	570,719,500
x_{10}	1,141,439,000

Table 7. Sensitivity analysis of objective function coefficients: allowable changes in interest rate (I) and capital adequacy ratio (CAR): National Bank of Egypt

Variable	Current Coefficient	Allowable Increase	Allowable Decrease
x_3	0.3121	Infinity	0.3121
x_{10}	- 0.2139	Infinity	- 0.7861

Table 8. The different (feasible) values x^0 of the decision variables (Values in millions of pounds): National Bank of Egypt

Variable	The derived values of the decision variables
x_3^0	1130300
x_4^0	1355503
x_5^0	13588.575
x_6^0	120000
x_7^0	9785.614

Then we apply the IMOLP algorithm described in Section 3 to the proposed multi-objective linear programming model defined in equations (51)-(65). This will enable us to find the minimal modification of the objective functions that makes the decision-makers's preference a weakly efficient solution. We will apply the algorithm using R programming. We start by identifying the binding constraints: The binding constraints are (52), (62) and (63)

$x_1 \geq 0.0298 * CD,$

$x_8 - x_9 \geq 0,$

$x_{10} \geq x_8 + x_9$

Accordingly, the set of active constraints $I(x^0)$ is $I(x^0) = (1, 11, 12)$ and $\hat{K} = (x \in \mathcal{R}^{10}/x = \beta_1 a_1 + \beta_2 a_2 + \beta_3 a_3, \beta_1, \beta_2, \beta_3 \geq 0)$, where $a_1 = (1, 0, 0, 0, 0, 0, 0, 0, 0, 0)$, $a_2 = (0, 0, 0, 0, 0, 0, 0, 1, -1, 0)$, and $a_3 = (1, 0, 0, 0, 0, 0, 0, -1, -1, 1)$.

Then $IMOLP(C, x^0)$ is formulated as

Min. q $=||c_1 - \hat{c}_1||_p + ||c_2 - \hat{c}_2||_p + ||c_3 - \hat{c}_3||_p,$

s.t.

$w_1\hat{c}_1 + w_2\hat{c}_2 + w_3\hat{c}_3 - \beta_1 a_1 - \beta_2 a_2 - \beta_3 a_3 = 0,$

$w_1 + w_2 + w_3 = 1$

$w_1, w_2, w_3, \beta_1, \beta_2, \beta_3 \geq 0,$
$\hat{c}_1, \hat{c}_2, \hat{c}_3 \in \mathcal{R}^{10},$

The algorithm is employed for the case when p = 2, aiming for an exact solution by setting the threshold $\epsilon = 0$.

Phase I: The solving of the following convex problem $d(conv(C), \hat{K}) = (min.(||x - y||_p | x \in conv(C), y \in \hat{K})$ yields distance $d^* = 0.280646$. Since $d^* > 0$, this indicates that x^0 is not a weakly efficient solution. Therefore, we set j=1 and proceed to step 1.

Phase II: Step 1: We solve Q_1 as follows:
$Min.q_1 = |||c_1 - \hat{c}_1||_2,$
s.t.
$\hat{c}_1 + \gamma_2 c_2 + \gamma_3 c_3 - \theta_1 a_1 - \theta_2 a_2 - \theta_3 a_3 = 0,$
$\gamma_2, \gamma_3, \theta_1, \theta_2, \theta_3 \geq 0$
$\hat{c}_1 \in \mathcal{R}^{10},$

The optimal objective value is $q_1^* = 2.64575$. Since $q_1^* - d^* > 0$, we will proceed to solve Q_2 as follows:
$Min.q_2 = |||c_2 - \hat{c}_2||_2,$
s.t.
$\hat{c}_2 + \gamma_1 c_1 + \gamma_3 c_3 - \theta_1 a_1 - \theta_2 a_2 - \theta_3 a_3 = 0,$
$\gamma_1, \gamma_3, \theta_1, \theta_2, \theta_3 \geq 0,$
$\hat{c}_2 \in \mathcal{R}^{10},$

The optimal objective value is $q_2^* = 0.64169$. Since $q_2^* - d^* > 0$, we will proceed to solve Q_3 as follows:
$Min.q_3 = |||c_3 - \hat{c}_3||_2,$
s.t.
$\hat{c}_3 + \gamma_1 c_1 + \gamma_2 c_2 - \theta_1 a_1 - \theta_2 a_2 - \theta_3 a_3 = 0,$
$\gamma_1, \gamma_2, \theta_1, \theta_2, \theta_3 \geq 0,$
$\hat{c}_3 \in \mathcal{R}^{10},$

The optimal objective value is $q_3^* = 0.3121$. As Q_3 yields the smallest objective value compared to all other problems, j is determined to be 3, $j^* = 3$. The optimal solution of q_3 is $\eta_3^* = (\hat{c}_3^*, \gamma_1^*, \gamma_2^*, \theta_1^*, \theta_2^*) = ((0, 0, 0, 0, 0, 0, 0, 0.73801, 0.73801, -0.73801), 0, 0, 0, 0)$. Therefore, the optimal solution of $IMOLP(C, x^0)$ is $\eta^* = (\hat{c}_1^*, \hat{c}_2^*, \hat{c}_3^*, w_1^*, w_2^*, w_3^*, \beta_1^*, \beta_2^*) = (c_1, c_2, \hat{c}_3^*, 0, 0, 1, 0, 0.73801)$.

The new criteria matrix C^*, obtained by moving c_3 to $\hat{c}_3^*$, indicates that $conv(C^*)$ intersects $\hat{K}$. This implies that x^0 is weakly efficient for the new criteria matrix.

4.3. *Computational efficiency of the algorithm*

In this subsection, we provide a detailed discussion of the computational efficiency of the algorithm, including runtime and scalability. To mitigate the computational challenges associated with non-convexity, we have implemented the following strategies:

1. We have explored problem-specific reformulations to transform the non-convex IMOLP into a more tractable form. By exploiting the specific structure of our problem, the algorithm has been able to reduce the computational complexity and improve the convergence properties of the optimization algorithms.

2. We used CVXR, a powerful R package for convex optimization, and the Splitting Conic Solver (SCS), a numerical optimization package designed to efficiently solve large-scale convex cone problems.

5. Conclusion

The paper's key contribution is the development and successful implementation of a multi-objective linear programming model to optimize portfolio distribution in commercial banks especially Banque Misr and the National Bank of Egypt during the period 2020/2021. The findings indicate that the total risk-weighted assets represent the largest portion of the balance sheet, while fixed assets constitute the smallest. The optimal solution for the objective function has been determined to be 606,098,200,000 Egyptian pounds for Banque Misr and 897,285,200,000 for the National Bank of Egypt. The sensitivity analysis was performed to assess the impact of input parameter variations on the robustness of the optimal solutions. Furthermore, we applied inverse multi-objective linear programming (IMOLP) to the suggested model to explore its flexibility and practical application. Our analysis of both case studies using IMOLP demonstrates that decision-makers can achieve Pareto optimal portfolio allocations by adjusting the coefficients of the third objective, which is related to Tier 1 capital (x_8), Tier 2 capital (x_9), and Total risk-weighted assets (x_{10}). This findings provide valuable insights for commercial banks seeking to optimize their portfolio allocation and enhance their financial performance.

REFERENCES

1. R. K Ahuja, and J. B Orlin, *Inverse optimization, part I: linear programming and general problems*, working paper, Indian Institute of Technology, pp. 183–202, 2009.
2. R. K Ahuja, and J. B Orlin, *Inverse optimization*, Operations Research, vol. 49, No. 5, pp. 771-783, 2001.
3. R. K Ahuja, and J. B Orlin, *Inverse optimization: an application-based survey Nitin Arya*, International Journal of Creative Research Thoughts (IJCRT), Vol. 8, No. 5, pp 2320-2882, 2020.
4. Z. Akbari and M. R. Peyghami, *An interior-point algorithm for solving inverse linear optimization problem*, Optimization, Vol. 61, No. 4, pp 373-386, 2012.
5. S. A Akhtar, A. S Kolarijani and P. M Esfahani, *Learning for control: An inverse optimization approach*, IEEE Control Systems Letters, 6:187–192, Vol. 6, pp 187-192, 2021.
6. N. M Albehery, *Optimum value of poverty measure using inverse optimization programming problems*, International Journal of Contemporary Mathematical Science, Vol. 14, pp 31- 42, 2019.
7. N. M Albehery, A. A EL-Dash and M. A Helal, *A suggested chance-constrained goal programming model to solve the portfolio problem*, Far East Journal of Mathematical Sciences (FJMS), Vol. 137, pp 11-25, 2022.
8. J. E Annetts and E. Audsley, *Multiple objective linear programming for environmental farm planning*, Journal of the Operational. Research Society, Vol. 53, Issue. 9, pp 933–943, 2002.
9. T. Ayer, *Inverse optimization for assessing emerging technologies in breast cancer screening*, Journal of Annals of operations research, Vol.230, pp 57–85, 2015.
10. A. Battal, W. Al-Anezi and F. Alsubaihil, *Determining the optimal investment portfolio for the Dubai Financial Market under the policy od diversification*, Journal of the Islamic University of Economic and Administrative Studies, College of Commerce, University of Anbar, Vol. 24, No. 3, pp 47-61, 2016.
11. D. Bertsimas, V. Gupta and I. C Paschalidis, *Inverse optimization: A new perspective on the Black-Litterman model*, Operation Research. Vol.60, No. 6, pp 1389–1403, 2012.
12. J. R Birge, A. H Su and J. M Pavlin, *Inverse optimization for the recovery of market structure from market outcomes: An application to the miso electricity market*, Operations Research, Vol. 65, No. 4, pp 837–855, 2017.
13. F. Black and R. Litterman, *Global portfolio optimization*, Financ Anal Journal, Vol. 48, No. 5, pp 28-43, 1992.
14. G. G Booth and G. H Dash, *Alternate programming structures for bank portfolios*, Journal of Banking and Finance, Vol. 3, No. 1, PP 67 - 82, 1979.
15. S. Boyd and L. Vandenberghe, *Convex Optimization*, Cambridge University Press, Cambridge, 2004.
16. D. Burton and P. L Toint, *On an instance of the inverse shortest paths problem*, Mathematical Programming, Vol. 53, pp 45-61, 1992.
17. T. C. Chan, Y. Mahmood and Zhu, *Inverse Optimization. Theory and Applications. Operations Research*, Articles in Advance, pp. 1–29, 2023.
18. T. C. Chan and T. Lee, *Trade-off preservation in inverse multi-objective convex optimization*, European Journal of Operational Research, Vol. 270, No. 1, pp 25–39, 2018.
19. T. C. Chan, T. Lee and D. Terekhov, *Inverse optimization: closed-form solutions, geometry, and goodness of fit*, Mathematics and Optimization and Control, Vol. 2, pp 1– 21, 2018.
20. T. C Chan, T. Craig, T. Lee and M. B. Sharpe, *Generalized inverse multi-objective optimization with application to cancer therapy*, Operations Research, Vol. 62, No. 3, pp 680–695, 2014.
21. L. Chen, Y. Chen and A. Langevin, *An inverse optimization approach for a capacitated vehicle routing problem*, Eurpian Journal of Operation Research, Vol. 295. No. 3, pp 1087–1098, 2021.
22. M. Ehrgott, *Multicriteria Optimization*, Springer, Berlin, 2006.
23. A. S. Farag, R. Szentesi and B. Szviatovszki, *Inverse optimization in high-speed networks*, Discrete Applied Mathematics, Vol. 129, No. 1, pp 83–98, 2003.
24. R. Fern´andez-Blanco, JM. Morales and S. Pineda, *Forecasting the price-response of a pool of buildings via homothetic inverse optimization*, Applied Energy, Vol. 290, pp 116791, 2021.
25. R. Fern´andez-Blanco, JM. Morales and S. Pineda, *Inverse optimization with kernel regression: Application to the power forecasting and bidding of a fleet of electric vehicles*, Computer and Operation Research, Vol. 134, pp 105405, 2021.
26. A. Ghate, *Inverse optimization in countably infinite linear programs*, Operations Research Letters, Vol. 43, No. 3, pp 231–235, 2015.
27. B.A. Halim et. al., *Bank Financial Statement Management using a Goal Programming Model*, 2nd Global Conference on Business and Social Science, Bali, Indonesia, Vol. 211, PP. 498-504, 2015.
28. H. W Hamacher and K. H. Küfer, *Inverse radiation therapy planning: a multiple objective optimization approach*, Discrete Applied Mathematics, Vol. 118, pp 145–161, 2002.
29. G. He and R. B. Litterman, *The intuition behind Black-Litterman model portfolios*, Cap. Mark.: Asset Pricing Valuat, 2002.

30. C. Heuberger, *Inverse combinatorial optimization: a survey on problems, methods, and results*, Journal of combinatorial optimization, Vol. 8, No. 3, pp 329–361, 2004.
31. D. S. Hochbaum, *Inverse problems and efficient convex optimization algorithms*, Technical report. University of California. Berkely. CA, 2008.
32. G. Iyengar and W. Kang, *Inverse conic programming with applications*, Operations Research Letters,Vol. 33, No. 3, pp 319–330, 2005.
33. S. Jain and N. Arya, *An Inverse Capacitated Transportation Problem*, IOSR Journal of Mathematics, Vol. 5, pp 24-27, 2013.
34. Y. Jiang, X. Xiao, L. Zhang and J. Zhang, *A perturbation approach for a type of inverse linear programming problems*, International Journal of Computer Mathematics, Vol. 88, pp 508–516, 2013.
35. Li JY-M, *Inverse optimization of convex risk functions*, Management Sci. Vol. 67, No. 11, pp 7113–7141, 2021.
36. D. T. Luc, *Multiobjective Linear Programming*, Springer, Berlin, 2016.
37. A.L. M. Mansi, *Evaluation of Market Pricing and the Pricing Model of Capital Assets in the Egyptian Stock Market*, Journal of Business Research, Faculty of Commerce, Zagazig University, Vol. 23, No. 1, pp 217-262, 2001.
38. H. Markowitz, *Portfolio Selection*, The Journal of Finance, Vol.7, No.1, pp 77-91, 1952.
39. M. J. Mohagheghnia and M. Shirgholami, *Optimal Bank Loan Portfolio in Iranian's Banks (Based Linear Programming Modelling)*, Research Journal of Finance and Accounting, Vol. 4, No.2, PP 2222–2847, 2013.
40. H. Mohammed and N. M. Albehery, *Applications of inverse optimization for convex nonlinear Problems*, Institute of Statistical Studies and Research, Cairo University, pp 72-82, 2016.
41. R. Mohamammadi and M. Sherafati, *Optimization of Bank Liquidity Management using Goal Programming and Fuzzy AHP*, Research Journal of Recent Sciences, Vol. 4, No. 6, pp. 53 -61, 2015.
42. A. Mostafaee, M. Hladík and M. Cerny, *Inverse linear programming with interval coefficients*, Journal of Computational and Applied Mathematics, Vol. 292, pp 591–608, 2016.
43. M. Naghavi, A. Foroughi and M. Zarepisheh, *Inverse optimization for multi-objective linear programming*, Optimization Letters, Vol. 13, pp 281-294, 2019.
44. M.Patriksson, *The traffic assignment problem: models and methods*, Courier Dover Publications, 2015.
45. J. Roland, Y. D.Smet and J. R. Figueira, *Inverse multi-objective combinatorial optimization*, Discrete Applied Mathematics, Vol. 161, No. 16, pp 2764–2771, 2013.
46. J. Saez-Gallego and JM. Morales, *Short-term forecasting of price-responsive loads using inverse optimization*, IEEE Trans. Smart Grid, Vol. 9, No.5, pp 4805–4814, 2017.
47. J. Saez-Gallego, JM. Morales, M. Zugno and H. Madsen, *A data-driven bidding model for a cluster of price-responsive consumers of electricity*, IEEE Trans. Power Systems Vol. 31. No. 6, pp 5001–5011, 2016.
48. A. J. Schaefer, *Inverse integer programming*, Optimization Letters, Vol. 3, No. 4, pp 483–489, 2009.
49. Z. Shahmoradi and T. Lee, *Quantile inverse optimization: Improving stability in inverse linear programming*, Operations research, Vol. 70, No. 4, pp 2538–2562, 2022.
50. R. E. Steuer, *Multiple Criteria Optimization: Theory, Computation and Applications*, Wiley, London, 1986.
51. R. E. Steuer and P. Na, *Multiple criteria decision making combined with finance: a categorized bibliographic study*, European Journal of Operational Research Eur, Vol. 150, No. 3, pp 496–515, 2003.
52. H. Togo, K. Asanuma, T. Nishi and Z. Liu, *Machine learning and inverse optimization for estimation of weighting factors in multi-objective production scheduling problems*, Applied Sciences, Vol. 12, No. 19, 2022.
53. S. Yu, H. Wang and C. Dong, *Learning risk preferences from investment portfolios using inverse optimization*, Res. Internat. Bus. Finance. Vol. 64, pp 101879, 2023.
54. J. Zhang and Z. Liu, *Calculating some inverse linear programming problems*, Journal of Computational and Applied Mathematics, Vol. 72, No. 2, pp 261–273, 1996.
55. J. Zhang and Z. Liu, *A further study on inverse linear programming problems*, Journal of Computational and Applied Mathematics, Vol. 106, No. 2, pp 345–359, 1999.
56. J. Zhang, *Inverse optimization for linearly constrained convex separable programming problems*, European Journal of Operational Research, Vol. 200, pp 671–679, 2010.