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Reflexive Edge Strength in Certain Graphs with Dominant Vertex

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Abstract Consider a basic, connected graph G with an edge set of $E(G)$ and a vertex set of $V(G)$. The functions f_e and f_v, which take $k = max\{k_e, 2k_v\}$, from the edge set to the first k_e natural number and the non-negative even number up to $2k_v$, respectively, are the components of total k-labeling. An *edge irregular reflexive k labeling* of the graph G is the total k-labeling, if for every two different edges x_1x_2 and $x'_1x'_2$ of G, $wt(x_1x_2) \neq wt(x'_1x'_2)$, where $wt(x_1x_2) = f_v(x_1) + f_e(x_1x_2) + f_v(x_2)$. The reflexive edge strength of graph G is defined as the minimal k for graph G with an edge irregular reflexive k-labeling; it is denoted by $res(G)$. The $res(G)$, where G are the book, triangular book, Jahangir, and helm graphs, was found in this work.

Keywords Edge irregular reflexive $k-$labeling, reflexive edge strength, book graph, triangular book graph, jahangir graph, helm graph.

AMS 2010 subject classifications 05C35, 05C40

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1. Introduction

Graph labeling is the process of assigning labels to elements in a graph to represent certain information [1, 4]. The graph G is a pair of sets $(V(G), E(G))$ where $V(G)$ is a non-empty set and $E(G)$ (possibly empty) is a set of unordered pairs of two elements differs in $V(G)$ [5, 6]. Elements $V(G)$ are called vertices on the graph G and elements $E(G)$ are called edges on the graph G [7, 8]. Labels in graphs are used to describe additional information about vertices or edges in the graph [9]. The history of graph labeling involves the development of concepts and techniques used to label graphs [10, 11]. As technology develops and applications develop, graph labeling continues to develop and is widely used in various fields, such as information modeling, image processing, network analysis, and many more [12]. It becomes a very useful tool for representing information in a form that is easy to understand and analyze [13]. By using labeling, we can turn what may be very complex graphs into representations that are easier to understand and can be used for a variety of purposes, including analysis, visualization, decision making, and more [14].

A total labeling with the range of the first natural number k is called a total k-labeling [15]. An edge irregular total k-labeling was defined by Baca *et.al* [17] as a total k-labeling with the requirement that every two distinct edges have different weights. The total edge irregularity strength of a G is defined as the least k for which G has an edge irregular total k-labeling; it is represented by $tes(G)$ [16]. Additionally, Tanna *et. al* [18] advance the study of erratic labeling. They defined an edge irregular reflexive k-labeling as a generalization of the edge irregular

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total labeling k-labeling of a graph [19]. The Total k-labeling defined the function $f_e : E(G) \longrightarrow \{1, 2, \ldots, k_e\}$ and $f_v : V(G) \longrightarrow \{0, 2, \ldots, 2k_v\}$, where $k = \max\{k_e, 2k_v\}$ [20]. An *edge irregular reflexive k-labeling* of the graph G is the total k-labeling, if for every two different edges x_1x_2 and $x_1'x_2'$ of G, $wt(x_1x_2) \neq wt(x_1'x_2')$, where $wt(x_1x_2) = f_v(x_1) + f_e(x_1x_2) + f_v(x_2)$ [21]. The minimum k for a graph G admitting an edge irregular reflexive k-labelling is called the reflexive edge strength of the graph G, denoted by $res(G)$ [17]. Baca *et al.* also gave the lower bound Lemma of $res(G)$, namely $\left\lceil \frac{|E(G)|}{3} \right\rceil + 1$ for $|E(G)| \equiv 2, 3 (mod\ 6)$ and $\left\lceil \frac{|E(G)|}{3} \right\rceil$ for otherwise [17].

The theory of edge irregular reflexive k-labeling on graphs has been studied by several researchers in previous works. This body of research explores various graphs of edge irregular reflexive k-labeling, contributing to a deeper understanding of their structural properties. The following Table (1) provides a summary of the findings from these studies, presenting key results and insights gathered from the literature.

Table 1. The Previous Results on Edge Irregular Reflexive k Labeling

Previous Research	Edge Irregular Reflexive Graph
Tanna *et al.* [3]	prism (D_n)
Tanna *et al.* [3]	wheel (W_n)
Tanna *et al.* [3]	fan (F_n)
Tanna *et al.* [3]	basket (B_n)
Guirao *et al.* [26]	disjoint union generalized petersen
Baca *et al.* [28]	generalized friendship
Baca *et al.* [23]	Cycle (C_n)
Baca *et al.* [23]	Cartesian product of two cycles $C_n \times C_3$
Baca *et al.* [23]	$(P_n + (2K1), C_n + (2K1))$
Agustin *et al.* [2]	generalized sub-divided star
Agustin *et al.* [2]	broom
Agustin *et al.* [2]	double star
Indriati *et al.* [22]	path
Indriati *et al.* [22]	corona of path
Yoong *et al.* [25]	corona product of two paths
Yoong *et al.* [25]	corona product of a path with isolated vertices
Zhang *et al.* [24]	disjoint wheel-relate
Zhang *et al.* [24]	disjoint specifically gear
Zhang *et al.* [24]	disjoint prism
Ibrahim *et al.* [27]	Star
Ibrahim *et al.* [27]	Double star
Ibrahim *et al.* [27]	Caterpillar
Ke *et al.* [29]	Cartesian product of two paths and two cycles
Agustin *et al.* [30]	ladder (L_n)
Agustin *et al.* [30]	triangular ladder (Tl_n)
Agustin *et al.* [30]	$P_n \times C_3$
Agustin *et al.* [30]	$P_n \odot P_2$
Agustin *et al.* [30]	$P_n \odot C_3$

From several previous research results regarding irregular reflexive labeling on graphs. There are still many open problems related to finding the exact values of res in graph families. Therefore, We want to make new contributions in edge irregular reflexive k labeling. In this article, We determines the exact values of res for graphs that contain dominant vertices. A dominant vertex is a vertex in a graph that is neighbor to most or all other vertices in the graph. A vertex is said to be "dominant" if every vertex that does not belong to the set of dominant vertices has at least one neighbor in the set of dominant vertices. Studying edge irregular reflexive k-labeling on graphs with dominant vertices is important because dominant vertices contribute significantly to the overall edge weights of the graph. This dominance creates a challenge for researchers to achieve sharp results and reach the lower bound of the edge irregular reflexive k-labeling. The influence of dominant vertices complicates the distribution of edge weights, making it crucial to develop precise methods that account for these effects. As such, investigating this topic provides valuable insights into achieving optimal labelings and contributes to the broader understanding of graph theory.

Some of the graphs used in this study that have dominant vertex are book graphs, triangular book graphs, and Jahangir graphs. A book graph is a graph formed from the Cartesian product operation between a path graph with two vertices and a star graph with $n+1$ vertices, namely $B_n = P_2 \times S_n$ [31]. The Jahangir graph is a generalization of the wheel graph by adding one vertex between two adjacent vertices (except the center vertex). Jahangir graph is denoted by J_n (Indriati, 2016). Triangular book is a graph formed from C_3 under edge amalgamation operation. Triangular book of order $n+2$ denoted by $TB_n, n \geq 2$. Helm Graph, denoted by H_n is obtained from wheel graph W_n by adding a single pendant edge at each vertex of cycle C_n. The number of vertices in the helm graph are $2n+1$, and the number of edges are $3n$.

2. Result

Motivated by the motivation showed above, we present four new theorems that add to the understanding of edge irregular reflexive k-labeling in graphs. These theorems enhance and add to the body of knowledge already known in this field. The four theorems, which each address a distinct element of edge irregular reflexive labeling and make significant contributions to the subject, are presented in the following section.

Theorem 1
Let B_n be a book graph. For every natural number $n \geq 3$, $res(B_n) = n+1$.

Proof
Let B_n, for $n \geq 3$, be a book graph with the vertex set $V(B_n) = \{a, b, u_i, v_i; 1 \leq i \leq n\}$ and the edge set $E(B_n) = \{ab, au_i, bv_i, u_iv_i; 1 \leq i \leq n\}$. The cardinality of vertex and edge set of B_n are $2n+2$ and $3n+1$, respectively. Since size of book graph is $3n+1$, it will be equivalent 1 or 4 in modulo 6. Based on lower bound Lemma of $res(G)$, we have $res(B_n) \geq \lceil \frac{|E(G)|}{3} \rceil = \lceil \frac{3n+1}{3} \rceil = n+1$.
Furthermore, we show an upper bound of reflexive edge strength of B_n, by defining the following functions. We divide two cases for odd n and even n as follows.
Case 1. For odd n

$$f_v(a) = f_v(b) = n-1$$

$$f_v(u_i) = 0 : 1 \leq i \leq n-1$$

$$f_v(u_n) = n-1$$

$$f_v(v_i) = 2\left\lceil \frac{i}{2} \right\rceil - 2 : 1 \leq i \leq n-1$$

$$f_v(v_n) = n+1$$

$$f_e(au_i) = i : 1 \leq i \leq n-1$$

$$f_e(ab) = n$$

$$f_e(au_n) = n+1$$

$$f_e(bv_i) = \begin{cases} n, & \text{if } 1 \leq i \leq n-2, \ i \text{ odd} \\ n+1, & \text{if } 2 \leq i \leq n-1, \ i \text{ even} \end{cases}$$

$$f_e(bv_n) = n$$

$$f_e(u_iv_i) = \begin{cases} 1, & \text{if } 1 \leq i \leq n-2, \ i \text{ odd} \\ 2, & \text{if } 2 \leq i \leq n-1, \ i \text{ even} \end{cases}$$

$$f_e(u_nv_n) = n+1$$

The edge weight sets in the following can be obtained using the edge weight from the function above.

$$w(u_iv_i) = i : 1 \leq i \leq n-1$$
$$w(au_i) = n-1+i : 1 \leq i \leq n-1$$
$$w(bv_i) = 2n-2+i : 1 \leq i \leq n-1$$
$$w(ab) = 3n-2, w(au_n) = 3n-1, w(bv_n) = 3n, w(u_nv_n) = 3n+1$$

Case 2. For even n

$$f_v(a) = f_v(b) = n-2$$
$$f_v(u_i) = 0 : 1 \leq i \leq n-2$$
$$f_v(u_i) = n : i \in \{n-1, n\}$$
$$f_v(v_i) = 2\left\lceil \frac{i}{2} \right\rceil - 2 : 1 \leq i \leq n-2$$
$$f_v(v_i) = n : i \in \{n-1, n\}$$
$$f_e(au_i) = i : 1 \leq i \leq n-2$$
$$f_e(ab) = n-1$$
$$f_e(au_{n-1}) = n-2$$
$$f_e(au_n) = n$$

$$f_e(bv_i) = \begin{cases} n-1, & \text{if } 1 \leq i \leq n-3, \ i \text{ odd} \\ n, & \text{if } 2 \leq i \leq n-2, \ i \text{ even} \end{cases}$$

$$f_e(bv_{n-1}) = n-1$$
$$f_e(bv_n) = n+1$$

$$f_e(u_iv_i) = \begin{cases} 1, & \text{if } 1 \leq i \leq n-3, \ i \text{ odd} \\ 2, & \text{if } 2 \leq i \leq n-2, \ i \text{ even} \end{cases}$$

$$f_e(u_{n-1}v_{n-1}) = n$$
$$f_e(u_nv_n) = n+1$$

The edge weight sets in the following can be obtained using the edge weight from the function above.

$$w(u_iv_i) = i : 1 \leq i \leq n-2$$

$$w(au_i) = n - 2 + i : 1 \leq i \leq n - 2$$

$$w(bv_i) = 2n - 4 + i : 1 \leq i \leq n - 2$$

$$w(ab) = 3n - 5$$

$$w(au_{n-1}) = 3n - 4, w(bv_{n-1}) = 3n - 3$$

$$w(au_n) = 3n - 2, w(bv_n) = 3n - 1$$

$$w(u_{n-1}v_{n-1}) = 3n, w(u_n v_n) = 3n + 1$$

We can see that the all edge weights on book graph are distinct. From the labeling on B_n, it gives $res(B_n) \leq n + 1$. Since we have $res(B_n) \geq n + 1$ and $res(B_n) \leq n + 1$. It concludes that $res(B_n) = n + 1$. □

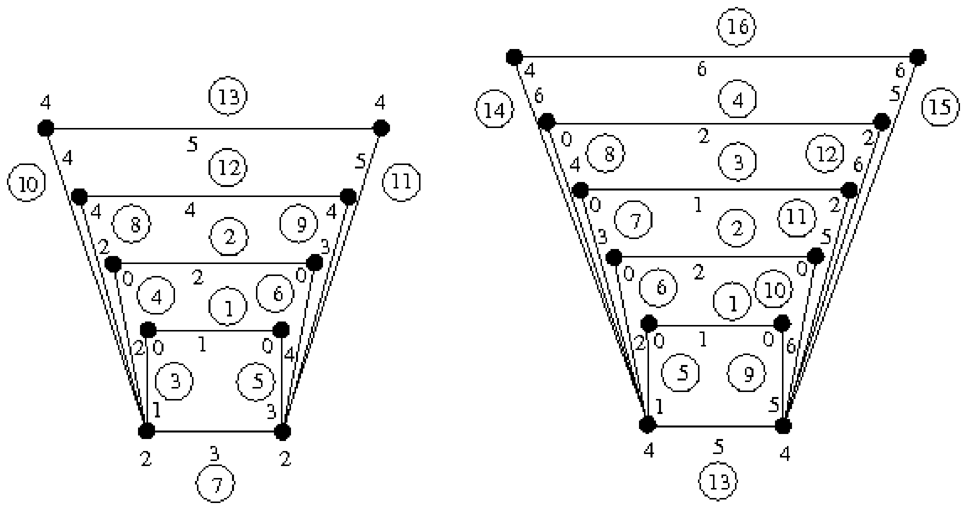

Figure 1. The example of edge irregular reflexive labeling on B_4 and B_5.

Theorem 2
Let TB_n be a triangular book graph. For every natural number $n \geq 2$,

$$res(TB_n) = \begin{cases} \left\lceil \frac{2n+1}{3} \right\rceil + 1, & \text{if } 2n+1 \equiv 3 (\text{mod } 6) \\ \left\lceil \frac{2n+1}{3} \right\rceil, & \text{otherwise} \end{cases}$$

Proof
Let TB_n, for $n \geq 2$, be a triangular book graph with the vertex set $V(B_n) = \{\alpha, \beta, \gamma_i : 1 \leq i \leq n\}$ and the edge set $E(TB_n) = \{\alpha\beta, \alpha\gamma_i, \beta\gamma_i : 1 \leq i \leq n\}$. The order and size of TB_n are $n + 2$ and $2n + 1$, respectively. Based on lower bound Lemma of $res(G)$, we have $res(TB_n) \geq \lceil \frac{|E(G)|}{3} \rceil = \lceil \frac{2n+1}{3} \rceil$. For $n + 1 \equiv 3(mod\ 6)$, it gives $res(TB_n) \geq \lceil \frac{|E(G)|}{3} \rceil + 1 = \lceil \frac{2n+1}{3} \rceil + 1$.
Moreover, by introducing the following functions, we show that k is an upper bound for the edge irregular reflexive k labeling of TB_n.

$$f_v(\alpha) = 0$$

$$f_v(\gamma_i) = \begin{cases} 0, & \text{if } i \in \{1,2\} \\ 2, & \text{if } i \in \{3,4\} \\ 2\left\lceil \frac{i-4}{3} \right\rceil + 2, & \text{if } i \geq 5 \end{cases}$$

$$f_v(\beta) = \begin{cases} k-1, & \text{if } \ k \text{ odd} \\ k, & \text{if } \ k \text{ even} \end{cases}$$

where

$$k = \begin{cases} \left\lceil \frac{2n+1}{3} \right\rceil + 1, & \text{if } 2n+1 \equiv 3 (\text{mod } 6) \\ \left\lceil \frac{2n+1}{3} \right\rceil, & \text{otherwise} \end{cases}$$

$$f_e(\alpha\gamma_i) = \begin{cases} 1, & \text{if } i \in \{1,3\} \\ 2, & \text{if } i \in \{2,4\} \\ \left\lceil \frac{i-4}{3} \right\rceil, & \text{if } i \equiv 2 (\text{mod } 3) \\ \left\lceil \frac{i-4}{3} \right\rceil + 1, & \text{if } i \equiv 0 (\text{mod } 3) \\ \left\lceil \frac{i-4}{3} \right\rceil + 2, & \text{if } i \equiv 1 (\text{mod } 3) \end{cases}$$

$$f_e(\alpha\beta) = n + 1 - f_v(\beta)$$

$$f_e(\beta\gamma_i) = \begin{cases} f_e(\alpha\beta) + 1, & \text{if } i \in \{1,3\} \\ f_e(\alpha\beta) + 2, & \text{if } i \in \{2,4\} \\ f_e(\alpha\beta) + \left\lceil \frac{i-4}{3} \right\rceil, & \text{if } i \equiv 2 (\text{mod } 3) \\ f_e(\alpha\beta)\left\lceil \frac{i-4}{3} \right\rceil + 1, & \text{if } i \equiv 0 (\text{mod } 3) \\ f_e(\alpha\beta) + \left\lceil \frac{i-4}{3} \right\rceil + 2, & \text{if } i \equiv 1 (\text{mod } 3) \end{cases}$$

The edge weight from the above function will give edge weight sets in the following.

$$w(\alpha\gamma_i) = i : 1 \leq i \leq n$$

$$w(\alpha\beta) = n + i$$

$$w(\beta\gamma_i) = n + 1 + i : 1 \leq i \leq n$$

We can see that the all edge weights on triangular book graph are distinct. From the labeling on TB_n, it gives $res(TB_n) \leq k$. Since we have $res(TB_n) \geq k$ and $res(TB_n) \leq k$. It concludes that $res(TB_n) = k$. □

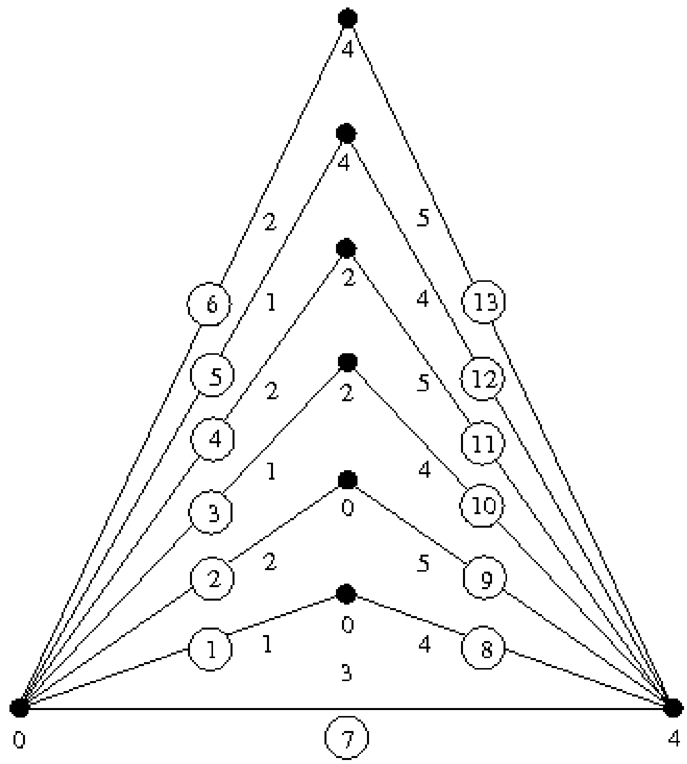

Figure 2. The example of edge irregular reflexive 5 labeling on TB_6.

Theorem 3
Let J_n be a jahangir graph. For every natural number $n \geq 3$,

$$res(J_n) = \begin{cases} n+1, & \text{if } n \text{ odd} \\ n, & \text{if } n \text{ even} \end{cases}$$

Proof
Let J_n, for $n \geq 3$, be a jahangir graph with the vertex set $V(J_n) = \{\alpha, x_i, y_i : 1 \leq i \leq n\}$ and the edge set $E(J_n) = \{\alpha x_i, x_i y_i : 1 \leq i \leq n\} \cup \{y_i x_{i+1}, y_n x_1 : 1 \leq i \leq n-1\}$. The cardinality of vertex and edge set of J_n are $2n+1$ and $3n$, respectively. Based on lower bound Lemma of $res(G)$, we have $res(J_n) \geq \lceil \frac{|E(G)|}{3} \rceil = \lceil \frac{3n}{3} \rceil = n$. For odd n, it means $n = 2a+1$, such that $3n$ will be $3(2a+1) = 6a+3 \equiv 3(mod\, 6)$, where a is a natural number. So we have $res(J_n) \geq \lceil \frac{|E(G)|}{3} \rceil + 1 = \lceil \frac{3n}{3} \rceil + 1 = n+1$.
Furthermore, we show an upper bound of reflexive edge strength of J_n, by defining the following functions. We divide two cases for odd n and even n as follows.
Case 1. For odd n

$$f_v(\alpha) = n+1$$

$$f_v(x_i) = 2\lceil \frac{i}{2} \rceil - 2 : 1 \leq i \leq n$$

$$f_v(y_i) = \begin{cases} 0, & \text{if } 1 \leq i \leq n-1 \\ n-1, & \text{if } i = n \end{cases}$$

$$f_e(\alpha x_i) = \begin{cases} n, & \text{if } i \text{ odd} \\ n+1, & \text{if } i \text{ even} \end{cases}$$

$$f_e(x_iy_i) = \begin{cases} i, & \text{if } 1 \leq i \leq n-2,\ i \text{ odd} \\ i+1, & \text{if } 2 \leq i \leq n-1,\ i \text{ even} \end{cases}$$

$$f_e(x_ny_n) = 1$$

$$f_e(y_ix_{i+1}) = 2\lceil\frac{i}{2}\rceil : 1 \leq i \leq n-1$$

$$f_e(y_nx_1) = n+1$$

The edge weight sets in the following can be obtained using the edge weight from the function above.

$$w(x_iy_i) = 2i-1 : 1 \leq i \leq n$$

$$w(y_ix_{i+1}) = 2i : 1 \leq i \leq n-1$$

$$w(y_nx_1) = 2n$$

$$w(\alpha x_i) = 2n+i : 1 \leq i \leq n$$

Case 2. For even n

$$f_v(\alpha) = n$$

$$f_v(x_i) = 2\lceil\frac{i}{2}\rceil - 2 : 1 \leq i \leq n-1$$

$$f_v(x_n) = n$$

$$f_v(y_i) = \begin{cases} 0, & \text{if } 1 \leq i \leq 2 \\ 2, & \text{if } 3 \leq i \leq n-2 \end{cases}$$

$$f_v(y_{n-1}) = n-2$$

$$f_v(y_n) = n$$

$$f_e(\alpha x_i) = \begin{cases} n-1, & \text{if } i \text{ odd} \\ n, & \text{if } i \text{ even} \end{cases}$$

$$f_e(x_iy_i) = \begin{cases} 1, & \text{if } i = 1 \\ i-2, & \text{if } 3 \leq i \leq n-2,\ i \text{ odd} \\ 3, & \text{if } i = 2 \\ i-1, & \text{if } 4 \leq i \leq n-2,\ i \text{ even} \end{cases}$$

$$f_e(x_{n-1}y_{n-1}) = 1$$

$$f_e(x_ny_n) = n-1$$

$$f_e(y_i x_{i+1}) = \begin{cases} 2, & \text{if } 1 \leq i \leq 2 \\ 2\lceil \frac{i-2}{2} \rceil, & \text{if } 3 \leq i \leq n-2 \end{cases}$$

$$f_e(y_{n-1}x_n) = n$$
$$f_e(y_n x_1) = n-2$$

The edge weight sets in the following can be obtained using the edge weight from the function above.

$$w(x_i y_i) = 2i - 1 : 1 \leq i \leq n-1$$
$$w(x_n y_n) = 3n - 1$$
$$w(y_i x_{i+1}) = 2i : 1 \leq i \leq n-2$$
$$w(y_{n-1} x_n) = 3n - 2$$
$$w(y_n x_1) = 2n - 2$$
$$w(\alpha x_i) = 2n - 2 + i : 1 \leq i \leq n-1$$
$$w(\alpha x_n) = 3n$$

We can see that the all edge weights on jahangir graph are distinct. From the labeling on J_n, it gives $res(J_n) \leq n+1$ for odd n and $res(J_n) \leq n$ for even n . Since we have $res(J_n) \geq n+1$ and $res(J_n) \leq n+1$ for odd n, and also $res(J_n) \geq n$ and $res(J_n) \leq n$ for even n. It concludes that $res(J_n) = n+1$ for odd n and $res(J_n) = n$ for even n. □

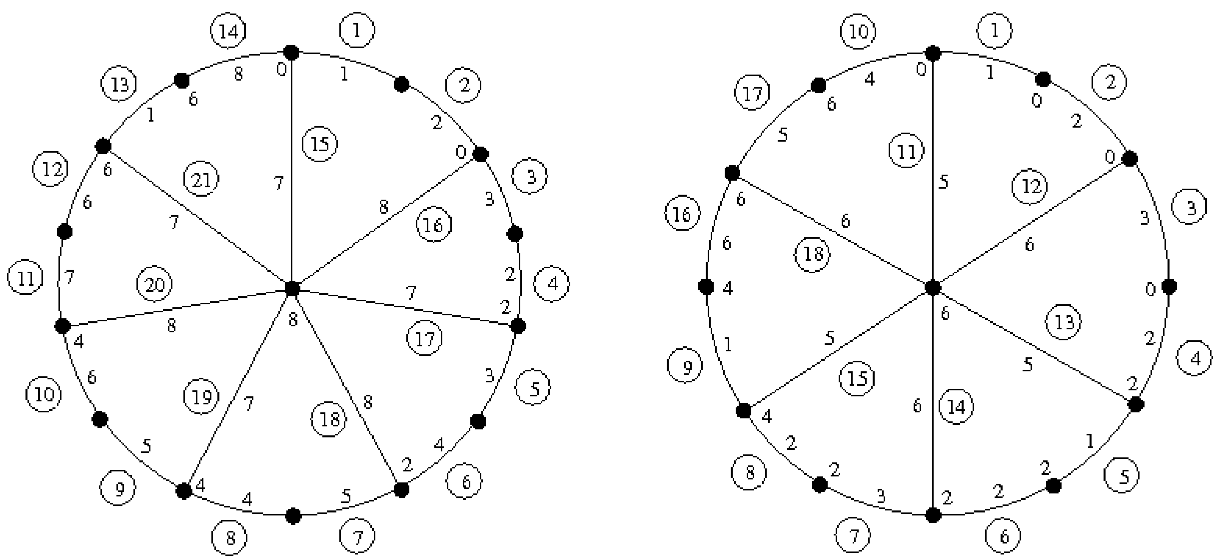

Figure 3. The example of edge irregular reflexive labeling on J_7 and J_6.

Theorem 4
Let H_n be a helm graph. For every natural number $n \geq 3$,

$$res(H_n) = \begin{cases} n+1, & \text{if } n \text{ odd} \\ n, & \text{if } n \text{ even} \end{cases}$$

Proof
Let H_n, for $n \geq 3$, be a helm graph with the vertex set $V(H_n) = \{a, x_i, y_i : 1 \leq i \leq n\}$ and the edge set $E(H_n) = \{ax_i, x_iy_i : 1 \leq i \leq n\} \cup \{x_ix_{i+1}, x_nx_1 : 1 \leq i \leq n-1\}$. The order and size of H_n are $2n+1$ and $3n$, respectively. Based on lower bound Lemma of $res(G)$, we have $res(H_n) \geq \lceil \frac{|E(G)|}{3} \rceil = \lceil \frac{3n}{3} \rceil = n$. For odd n, it means $n = 2b+1$, such that $3n$ will be $3(2b+1) = 6b+3 \equiv 3(mod\, 6)$, where b is a natural number. So we have $res(H_n) \geq \lceil \frac{|E(G)|}{3} \rceil + 1 = \lceil \frac{3n}{3} \rceil + 1 = n+1$.
Furthermore, we show an upper bound of reflexive edge strength of H_n, by defining the following functions. We divide two cases for odd n and even n as follows.
Case 1. For odd n

$$f_v(a) = n+1$$

$$f_v(x_i) = 2\lceil \frac{i}{2} \rceil - 2 : 1 \leq i \leq n$$

$$f_v(y_i) = \begin{cases} 0, & \text{if } i = 1 \\ 2, & \text{if } 2 \leq i \leq n \end{cases}$$

$$f_e(ax_i) = \begin{cases} n, & \text{if } 1 \leq i \leq n,\ i \text{ odd} \\ n+1, & \text{if } 2 \leq i \leq n-1,\ i \text{ even} \end{cases}$$

$$f_e(x_ix_{i+1}) = \begin{cases} 2, & \text{if } 1 \leq i \leq n-2,\ i \text{ odd} \\ 1, & \text{if } 2 \leq i \leq n-1,\ i \text{ even} \end{cases}$$

$$f_e(x_nx_1) = n+1$$

$$f_e(x_iy_i) = \begin{cases} 1, & \text{if } i = 1 \\ i, & \text{if } 2 \leq i \leq n-1,\ i \text{ even} \\ i-2, & \text{if } 3 \leq i \leq n,\ i \text{ odd} \end{cases}$$

The edge weight sets in the following can be obtained using the edge weight from the function above.

$$w(x_ix_{i+1}) = \begin{cases} 2i, & \text{if } 1 \leq i \leq n-2,\ i \text{ odd} \\ 2i-1, & \text{if } 2 \leq i \leq n-1,\ i \text{ even} \end{cases}$$

$$w(x_nx_1) = 2n$$

$$w(ax_i) = 2n + i : 1 \leq i \leq n$$

$$w(x_iy_i) = \begin{cases} 2i-1, & \text{if } 1 \leq i \leq n,\ i \text{ odd} \\ 2i, & \text{if } 2 \leq i \leq n-1,\ i \text{ even} \end{cases}$$

Case 2. For even n

$$f_v(a) = n$$

$$f_v(x_i) = 2\lceil \frac{i}{2} \rceil - 2 : 1 \leq i \leq n-1$$

$$f_v(x_n) = n$$

$$f_v(y_i) = \begin{cases} 0, & \text{if } i = 1 \\ 2, & \text{if } 2 \leq i \leq n-1 \\ n, & \text{if } i = n \end{cases}$$

$$f_e(ax_i) = \begin{cases} n-1, & \text{if } 1 \leq i \leq n-1,\ i \text{ odd} \\ n, & \text{if } 2 \leq i \leq n,\ i \text{ even} \end{cases}$$

$$f_e(x_i x_{i+1}) = \begin{cases} 2, & \text{if } 1 \leq i \leq n-3,\ i \text{ odd} \\ 1, & \text{if } 2 \leq i \leq n-2,\ i \text{ even} \end{cases}$$

$$f_e(x_{n-1}x_n) = n$$

$$f_e(x_n x_1) = n-2$$

$$f_e(x_i y_i) = \begin{cases} 1, & \text{if } i = 1 \\ i, & \text{if } 2 \leq i \leq n-2,\ i \text{ even} \\ i-2, & \text{if } 3 \leq i \leq n-1,\ i \text{ odd} \end{cases}$$

$$f_e(x_n y_n) = n-1$$

The edge weight sets in the following can be obtained using the edge weight from the function above.

$$w(x_i x_{i+1}) = \begin{cases} 2i, & \text{if } 1 \leq i \leq n-3,\ i \text{ odd} \\ 2i-1, & \text{if } 2 \leq i \leq n-2,\ i \text{ even} \end{cases}$$

$$w(x_{n-1}x_n) = 3n-2$$

$$w(x_n x_1) = 2n-2$$

$$w(ax_i) = 2n+i : 1 \leq i \leq n-1$$

$$w(ax_n) = 3n$$

$$w(x_i y_i) = \begin{cases} 2i-1, & \text{if } 1 \leq i \leq n-1,\ i \text{ odd} \\ 2i, & \text{if } 2 \leq i \leq n-2,\ i \text{ even} \end{cases}$$

$$w(x_n y_n) = 3n-1$$

We can see that the all edge weights on helm graph are distinct. From the labeling on H_n, it gives $res(H_n) \leq n+1$ for odd n and $res(H_n) \leq n$ for even n . Since we have $res(H_n) \geq n+1$ and $res(H_n) \leq n+1$ for odd n, and also $res(H_n) \geq n$ and $res(H_n) \leq n$ for even n. It concludes that $res(H_n) = n+1$ for odd n and $res(H_n) = n$ for even n. □

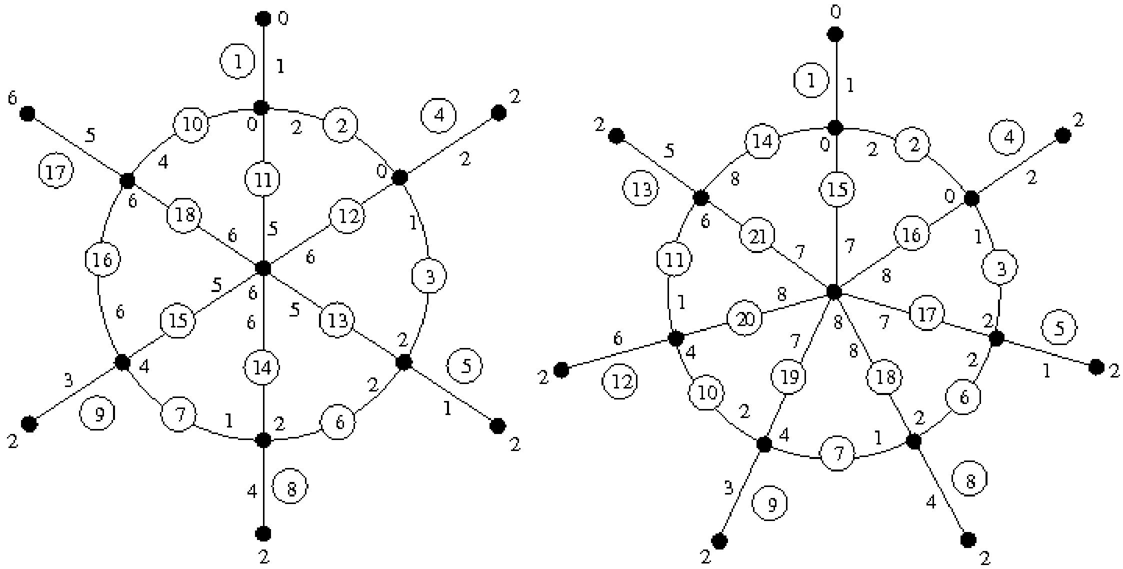

Figure 4. The example of edge irregular reflexive labeling on H_6 and H_7.

Illustrations of irregular edge reflexive k labeling shown to support the proof of Theorem 1 to Theorem 4 are presented in Figure 1 to Figure 4. Figure 1 shows the irregular edge reflexive k labeling on graphs B_4 and B_5. Graph B_4 obtain $k = 5$ and graph B_5 obtain $k = 6$. Figure 2 shows the irregular edge reflexive k labeling on graph TB_6. Graph TB_6 obtain $k = 5$. Figure 3 shows the irregular edge reflexive k labeling on graphs J_7 and J_6. Graph J_7 obtain $k = 8$ and graph J_6 obtain $k = 6$. Figure 4 shows the irregular edge reflexive k labeling on graph H_6 and H_7. Graph H_6 obtain of $k = 6$ and graph H_7 obtain of $k = 8$. The edge weights of the graphs in Figure 1 to Figure 4 are all different, which is in accordance with the definition of edge irregular reflexive k labeling. Therefore, Theorem 1 to Theorem 4 are true and valid.

3. Discussion

Reflexive Edge Strength (RES) labeling has several potentially relevant practical applications, both in theoretical and applied contexts. One of the most important applications is in cryptography, where RES values can be used to generate secure and efficient encryption keys. The unique combination of vertex labels and edge labels in a graph can be used to build graph-based cryptographic algorithms that provide high levels of security with controlled computational complexity. In addition, RES has relevance in network design, such as route optimization in computer or communication networks, where this labeling can help distribute the workload efficiently. In coding theory, RES can be used to design data coding schemes that are more robust to interference.

This research specifically discusses the RES value on graphs with dominant nodes, where the RES value is only affected by the combination of a node label and an edge label. This happens because the dominant node label is constant and contributes directly to all edge weights associated with it. This uniform contribution results in fewer combinations available to generate different edge weights compared to graphs without dominant nodes. This finding not only highlights the unique complexity that arises in graphs with dominant nodes, but also provides deep insights into how the dominant structure affects the labeling process and weight distribution. These findings open opportunities for the development of more efficient labeling methods, while expanding the potential applications of RES in various fields, including cryptography, network design, and coding theory.

4. Concluding Remarks

This research shows that the graph structure strongly affects the Reflexive Edge Strength (RES) value, such as in book, triangular book, and Jahangir graphs, with the role of dominant nodes varying depending on the graph type. RES values have been determined using analytical and deductive approaches, with potential applications in network optimization, molecular structure analysis, code theory, and cryptography. Further research can explore the RES value on graphs of other operations or structures, and develop a general framework for its calculation.

The conclusion of this study is that the structure of each graph greatly influences reflexive edge strength in graphs such as book graphs, triangular book graphs, and Jahangir graphs. The role of dominant vertices in determining reflexive edge strength in a graph varies depending on the type of graph. This study investigates these dynamics within the context of vertex-dominant graphs. As a result, the following open problem is still open for other researchers to determine reflexive edge strength of graphs apart from the previous results.

Open Problem 1
Determine exact value of res (reflexive edge strength) on other graphs, which includes those resulting from operations, such as Cartesian products, joints, corona products, and etc.

Open Problem 2
Characterization of RES values on different graph families includes analysis of the relationship between graph structure and RES values, as well as identification of patterns that simplify computations on graphs in certain families, opening opportunities for further development of graph theory.

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