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Enhancing Overlap Measures in Censored Data Analysis: A Focus on Pareto Distributions and Adaptive Type-II Censoring

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Abstract This article explores the application of the adaptive type-II progressive hybrid censoring scheme to calculate three widely recognized statistical measures of overlap for two Pareto distributions with distinct parameters. By utilizing this innovative censoring technique, we derive estimators for these measures and provide their asymptotic bias and variance. When small sample sizes pose challenges in assessing estimator precision or bias due to the lack of closed-form expressions for variances and exact sampling distributions, Monte Carlo simulations are employed to enhance reliability. Additionally, we construct confidence intervals for these measures using both the bootstrap method and Taylor approximation. Our approach offers improved accuracy and efficiency in estimating overlap measures, addressing key challenges in censored data analysis. To demonstrate the practical relevance of our proposed estimators, we include an illustrative application involving real data from an Iranian insurance company.

Keywords Key Words: Bootstrap method; Overlap measures; adaptive type-II progressive hybrid censoring

AMS 2010 subject classifications 62E10, 62N01, 62N02, 62G30

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1. Introduction

Life-testing experiments pose challenges in controlling test duration and conserving experimental units while ensuring efficient estimation. Censoring techniques offer a solution by removing active units and stopping the experiment before all units fail. Progressive censoring is crucial, as it involves removing units at predetermined or random time points during the experiment, accounting for potential losses or removals.

Over the years, progressive censoring has been extensively studied, with models falling into two categories: progressive Type-I censoring, concluding the experiment at predefined times, and progressive Type-II censoring, ending after a predetermined number of failures. Both approaches provide flexibility by allowing unit removal at non-terminal times.

Progressive Type-I censoring involves fixed durations at specific time points, potentially resulting in few or no observed failures for units with long lifetimes. In contrast, progressive Type-II censoring, although flexible, may lead to extended test durations when units have extended lifetimes, which is considered a drawback.

Kundu and Joarder (2006) introduced two progressive hybrid censoring schemes, offering alternatives to traditional progressive Type-II censoring by ending experiments at a certain time T. These schemes adapt to the data, allowing fewer than m observations in Type-I hybrid censoring or extended testing in Type-II hybrid censoring.

During real-life experiments, it is imperative to acknowledge that a fixed censoring scheme may not always be a practical approach. Any intentional or unintentional alternation during the experiment can significantly impact the

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Figure 1. Adaptive type-II progressive hybrid censoring model as proposed by Ng et al. (2009). (a) Experiment ends before time T. (b) Experiment ends after time T.

results. However, Ng et al. (2009) have introduced a new model depicted in Figure (1) that allows the censoring scheme to be changed as required during the experiment. This model is called adaptive type-II progressive hybrid censoring (Adaptive-IIPH), in which a threshold time T switches between the original and modified schemes.

Assume there are n units in a life-testing experiment, and the effective sample size $m(< n)$ is predetermined, along with the censoring scheme $(K_1, K_2, \ldots, K_m)$; however, the values of some of the K_i may change as the experiment progresses. Assuming the experimenter has provided an ideal total test time T. If the m-th failure occurs before time T (see Figure 1(a)), the experiment proceeds similarly to type-II progressive censoring. It halts at time X_m with a pre-fixed censoring scheme $(K_1, K_2, \ldots, K_m)$. Otherwise, if the experimental time has passed T, but the number of observed failures has not yet reached m, we do not remove any items from the experiment by setting $K_{j+1} = K_{j+2} = \cdots = K_{m-1} = 0$ and $K^* = n - m - \sum_{i=1}^{j} K_i$. This setting can be seen as a design that guarantees m observed failure times while keeping the total test time not too far away from the ideal test time T (depicted in Figure 1(b)). Note that if we set $T = 0$, we will have a traditional type-II censoring method. However, if $T \to \infty$, the Adaptive-IIPH process becomes a progressive type-II censoring technique.

Adaptive-IIPH significantly impacts real-life applications, as evidenced by its widespread use in literature. Most recently, Alslman and Helu (2023) developed new methods for estimating the stress strength of the Inverse Weibull distribution using the Adaptive-IIPH censoring scheme. Asadi et al. (2022) employed Adaptive-IIPH censoring to conduct accelerated life tests on virus-containing microdroplets, monitoring Virus-MD persistence during coughs at different time points. Alotaibi et al. (2022) utilized Adaptive-IIPH censoring for testing sodium sulfur battery lifetimes in a chemical application employing the XLindley distribution. Furthermore, Helu and Samawi (2021) applied Adaptive-IIPH censoring to radar-evaluated rainfall data from 52 cumulus clouds in South Florida, highlighting its versatile utility in various fields.

Estimating the proportion of machines or electronic devices with similar failure time ranges is crucial in reliability analysis, especially when dealing with different sources or stress levels. Various overlap coefficients (OVL), such as Matusia's measure ρ, Morisita's measure λ, and Weitzman's measure Δ, are utilized to assess the degree of similarity between these distributions. These coefficients represent the common area between two probability density functions. The depiction of OVL for two distributions in Figure 2 displays the natural interpretations of OVL as a fraction of probability mass under either density, represented by the shaded area in Figure 2.

OVL has found widespread use in various practical applications as well. It has been utilized in quantitative ecology, as demonstrated by Gastwirth (1975). Furthermore, OVL has been applied to electromyographic assessment of muscular asymmetry by Ferrario et al. (2000) and in treatment assessment during clinical trials, as discussed by Mizuno et al. (2005).

For a deeper exploration of the various applications of overlap coefficients, interested readers can refer to the works of Wang and Tian (2017) and Martinez-Camblor (2022).

The mathematical form of the OVL measures are as follows: Suppose two samples of observations are drawn from two continuous distributions $f_1(x)$ and $f_2(x)$. Then the overlap measures are defined as follows:

$$\text{Matusita's Measure [12]:} \quad \rho = \int \sqrt{f_1(x) f_2(x)}\, dx,$$

$$\text{Morisita's Measure [13]:} \quad \lambda = \frac{2\int f_1(x) f_2(x)\, dx}{\int [f_1(x)]^2 dx + \int [f_2(x)]^2 dx},$$

$$\text{Weitzman's Measure [14]:} \quad \Delta = \int \min(f_1(x), f_2(x))\, dx.$$

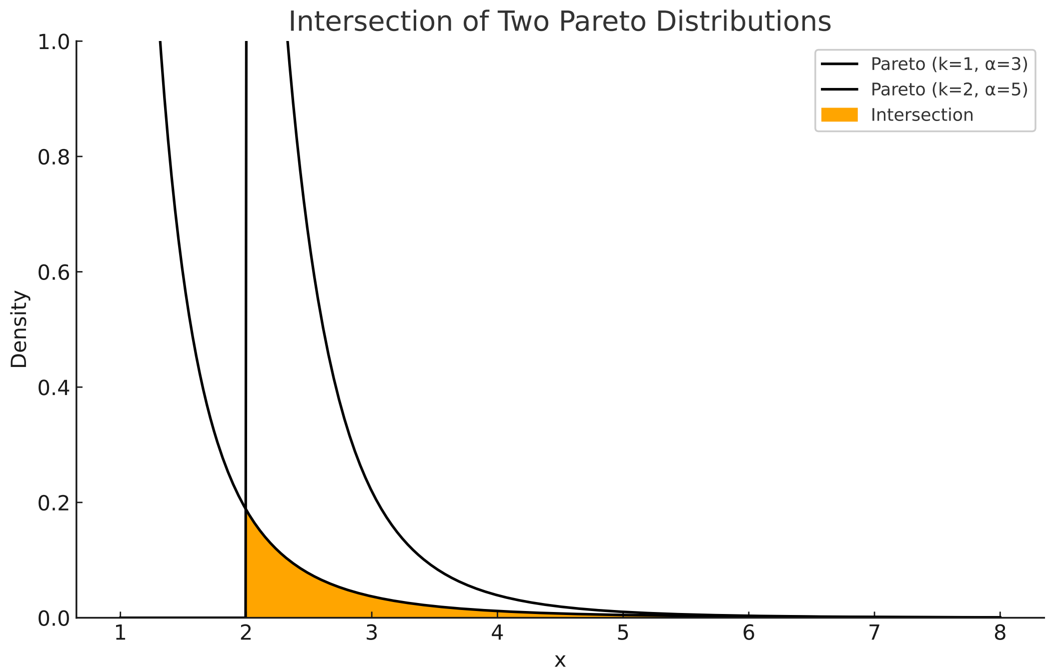

Figure 2. Overlap of two densities.

It is possible to adapt these measures for discrete distributions by using summations. They can also be extended to multivariate distributions. They are quantified on a scale from 0 to 1, with values near 0 indicating significant inequality (or disagreement) and 1 suggesting exact equality (perfect agreement) between density functions.

The mathematical structure of these measures is intricate, and there are no results available on the exact sampling distributions of their estimators. Prior work includes Smith (1982) on discrete Weitzman's measure, Mishra et al. (1986) on sampling properties under homogeneity assumptions, Mulekar and Mishra's (1994) simulations on normal densities, and Lu et al.'s (1989) study of sampling variability. Additionally, Dixon (1993) applied bootstrapping and jackknife techniques, while Mulekar and Mishra (2000) addressed inference problems.

The sampling behavior of a nonparametric estimator of OVL was analyzed by Helu and Samawi (2011). Samawi et al. (2017) conducted a study investigating the similarities and distinctions between the maximum of the Youden index (J) and overlap coefficient (OVL), highlighting the advantages of OVL over J.

In this article, our primary focus lies in making inferences regarding the measure of overlap (OVL) while utilizing Adaptive-IIPH censoring data from two independent Pareto distributions with different parameters.

In Section 2, we introduce the Pareto distribution and derive the measures, as well as introduce the estimators. Moving on to Section 3, we discuss the approximate biases to establish confidence intervals via the delta method and bootstrap techniques for the OVL measures. In Section 4, we present the outcomes of our simulations and engage in a comprehensive discussion. Finally, in Section 5, we showcase a practical example using real-life data.

2. The model

Suppose that the lifetime U of products have a Pareto distribution with shape parameter θ. The probability density function (pdf) is given by

$$f\left(u\right)=\frac{1}{\theta}u^{-\left(\frac{1}{\theta}+1\right)},u\geq 1,\theta>0. \tag{1}$$

The cumulative distribution function (cdf) corresponding to (1) for $u>0$, is

$$F(u)=1-u^{-\frac{1}{\theta}}.$$

Making use of the transformation $X=\log\left(U\right)$, we have a new lifetime variable X following an exponential distribution with scale parameter θ $\left(Exp\left(\theta\right)\right)$, with pdf and cdf as follows:

$$g(x)=\frac{1}{\theta}e^{-\frac{x}{\theta}},\quad \text{for } x>0,\theta>0 \tag{2}$$

and

$$G(x)=1-e^{-\frac{x}{\theta}},\quad \text{for } x>0,\theta>0 \tag{3}$$

Similar to Helu and Samawi (2011), let $f_1(x)$ and $f_2(x)$ be two Pareto densities with scale parameters θ_1 and θ_2 respectively. Define $R=\frac{\theta_1}{\theta_2}$, the continuous version of the three proposed overlap measures can be expressed as a function of R as follows:

$$\rho=\frac{2\sqrt{R}}{1+R}, \tag{4}$$

$$\lambda=\frac{4R}{\left(1+R\right)^2}, \tag{5}$$

and

$$\Delta=1-R^{\frac{1}{1-R}}\left|1-\frac{1}{R}\right|,R\neq 1. \tag{6}$$

According to Mulekar and Mishra (2000), ρ, λ, and Δ are not monotone for all $R>0$. However, they exhibit certain properties, such as symmetry in R, meaning that $OVL(R)=OVL(\frac{1}{R})$. They also remain invariance under linear transformations, $Y=aX+b$, $a\neq 0$ and attain the maximum value of 1 at $R=1$.

2.1. Maximum likelihood estimates

The OVL measures ρ,λ and Δ are functions of θ_1 and θ_2. In order to draw any inference about the OVL measures, we need to estimate the unknown parameters, θ_1 and θ_2. In this section we obtain the maximum likelihood estimates $(MLEs)$ of the parameters θ_1 and θ_2 based on Adaptive-IIPH censored samples. Let $\mathbf{U}=U_{1:m_1:n_1}<U_{2:m_1:n_1}<...<U_{m_1:m_1:n_1}$ be an Adaptive-IIPH censoring sample from Pareto(θ_1) under the censoring scheme $\{n_1,m_1,K_1,...,K_{J_1},0,...,0,K^*=n_1-m_1-\sum_{i=1}^{J_1}K_i\}$ such that $U_{J_1:m_1:n_1}<T_1<U_{J_1+1:m_1:n_1}$. And, $\mathbf{V}=\{V_{1:m_2:n_2}<V_{2:m_2:n_2}<...<V_{m_2:m_2:n_2}\}$ be an Adaptive-IIPH censoring sample from Pareto(θ_2) under the scheme
$\{n_2,m_2,L_1,...,L_{J_2},0,...,0,L^*=n_2-m_2-\sum_{i=1}^{J_2}L_i\}$ such that $V_{J_{2:m_2:n_2}}<T_2<V_{J_2+1:m_2:n_2}$. For simplicity, let $U_i=U_{i:m_1:n_1}$ and $V_i=V_{i:m_2:n_2}$. Then the joint likelihood function of the Adaptive-IIPH censored sample (see Balakrishnan and Cramer, 2014) can be written as

$$\begin{aligned}L(\theta_1,\theta_2|\mathbf{X},\mathbf{Y})=C_1C_2[1-F_1(u_{m_1})]^{K^*}\prod_{i=1}^{m_1}f_1(u_i)\prod_{i=1}^{J_1}[1-F_1(u_i)]^{K_i}\\ [1-F_2(v_{m_2})]^{L^*}\prod_{i=1}^{m_2}f_2(v_i)\prod_{i=1}^{J_2}[1-F_2(v_i)]^{L_i},\end{aligned} \tag{7}$$

where,

$$C_1 = n_1(n_1 - K_1 - 1)(n_1 - K_1 - K_2 - 2)...(n_1 - K_1 - K_2 - ... - K_{m_1-1} - m_1 + 1)$$

$$C_2 = n_2(n_2 -_1 -1)(n_2 -_1 -L_2 - 2)...(n_2 - L_1 - L_2 - ... - L_{m_2-1} - m_2 + 1)$$

$$f_1(u) = \frac{1}{\theta_1} u^{-\left(\frac{1}{\theta_1}+1\right)}, \qquad F_1(u) = 1 - u^{-\frac{1}{\theta_1}}, \quad \text{for } u > 0, \tag{8}$$

$$f_2(v) = \frac{1}{\theta_2} v^{-\left(\frac{1}{\theta_2}+1\right)}, \qquad F_2(v) = 1 - v^{-\frac{1}{\theta_2}}, \quad \text{for } v > 0. \tag{9}$$

After substituting Eq.8 and Eq. 9 into Eq.7, then taking the log-likelihood function, we get the following:

$$\begin{aligned} l \quad \propto \quad & -m_1 \log(\theta_1) - \frac{\left(K^* \log(u_{m_1}) + \sum_{i=1}^{m_1} \log(u_i) + \sum_{i=1}^{J_1} K_i \log(u_i)\right)}{\theta_1} \\ & -m_2 \log(\theta_2) - \frac{\left(L^* \log(v_{m_2}) + \sum_{i=1}^{m_2} \log(v_i) + \sum_{i=1}^{J_2} L_i \log(v_i)\right)}{\theta_2}. \end{aligned} \tag{10}$$

Using the transformation $X = \log(U)$ and $Y = \log(V)$, Eq. 10 becomes

$$\begin{aligned} l \quad \propto \quad & -m_1 \log(\theta_1) - \frac{\left(K^* x_{m_1} + \sum_{i=1}^{m_1} x_i + \sum_{i=1}^{J_1} K_i x_i\right)}{\theta_1} \\ & -m_2 \log(\theta_2) - \frac{\left(L^* y_{m_2} + \sum_{i=1}^{m_2} y_i + \sum_{i=1}^{J_2} L_i y_i\right)}{\theta_2}. \end{aligned} \tag{11}$$

The MLEs of the parameters θ_1 and θ_2 can be obtained by taking the first derivative of Eq.11 with respect to θ_1 and θ_2 and equating the normal equations to 0 to get

$$\hat{\theta}_1 = \frac{K^* x_{m_1} + \sum_{i=1}^{m_1} x_i + \sum_{i=1}^{J_1} K_i x_i}{m_1}, \tag{12}$$

$$\hat{\theta}_2 = \frac{L^* y_{m_2} + \sum_{i=1}^{m_2} y_i + \sum_{i=1}^{J_2} L_i y_i}{m_2}. \tag{13}$$

Viveros and Balakrishnan (1994; page 88) showed that when the underlying distribution is an exponential with unknown mean θ, and when data $W_{1:m:n} < W_{2:m:n} < \cdots < W_{m:m:n}$ are based on progressively type-II censored sample with censoring scheme $\mathbf{K} = (K_1, K_2, \ldots, K_m)$, $\hat{\theta} = \frac{\sum_{i=1}^{m}(K_i+1)w_i}{m}$ is the MLE of θ, and $\hat{\theta} \sim Gamma(m, \frac{\theta}{m})$ in which $Gamma(.,.)$ denote the Gamma distribution. Cramer and Iliopolous (2010; Theorems 5 and 7) showed that the MLE when data are based on Adaptive-IIPH coincide with the MLE in deterministic progressive type-II censoring schemes. Thus, the distribution of this particular random variable is invariant with respect to random (fixed) progressive type-II censoring procedure. Thus, we obtain $\hat{\theta}_i \sim G(m_i, \frac{\theta_i}{m_i}); i = 1, 2.$ Consequently, the means and variances of the MLEs in (12) and (13) are

$$E(\hat{\theta}_1) = \theta_1, \qquad E(\hat{\theta}_2) = \theta_2, \tag{14}$$

$$Var(\hat{\theta}_1) = \frac{\theta_1^2}{m_1}, \qquad Var(\hat{\theta}_2) = \frac{\theta_2^2}{m_2}, \tag{15}$$

therefore, the MLE of R is $\hat{R} = \frac{\hat{\theta}_1}{\hat{\theta}_2}$. Hence, $\frac{\theta_2}{\theta_1}\hat{R}$ has F-distribution with $2m_1$ and $2m_2$ degrees of freedom ($F_{2m_1,2m_2}$). Thus, by using the delta method of approximation, the variance of $\hat{R}$ can be approximated by:

$$Var(\hat{R}) = \frac{m_2^2(m_1+m_2-1)}{m_1(m_2-1)^2(m_2-2)}R^2. \tag{16}$$

Clearly, an unbiased estimator of R is given by $\hat{R}^* = \frac{(m_2-1)}{m_2}\hat{R}$ with variance $Var(\hat{R}^*) = \frac{(m_1+m_2-1)}{m_1(m_2-2)}R^2$ and hence $Var(\hat{R}^*) < Var(\hat{R})$. Since the OVL measures are functions of R, therefore, based on the MLE estimate of R, the OVL measures can be estimated by

$$\hat{\rho} = \frac{2\sqrt{\hat{R}^*}}{1+\hat{R}^*}, \tag{17}$$

$$\hat{\lambda} = \frac{4\hat{R}^*}{\left(1+\hat{R}^*\right)^2}, \tag{18}$$

and,

$$\hat{\Delta} = 1 - \hat{R}^{*\frac{1}{1-\hat{R}^*}}\left|1-\frac{1}{\hat{R}^*}\right|, \hat{R}^* \neq 1. \tag{19}$$

3. Asymptotic properties of OVL

Using the delta method, the asymptotic variance and bias for OVL measures are as follows: Let $OVL = g(\hat{R}^*)$, then the asymptotic variance are given by

$$Var(\hat{\rho}) = \sigma_{\hat{\rho}}^2 \cong \frac{(m_1+m_2-1)}{m_1(m_2-2)}\frac{R(1-R)^2}{(1+R)^4}, \tag{20}$$

$$Var(\hat{\lambda}) = \sigma_{\hat{\lambda}}^2 \cong \frac{16(m_1+m_2-1)}{m_1(m_2-2)}\frac{R^2(1-R)^2}{(1+R)^6}, \tag{21}$$

$$Var(\hat{\Delta}) = \sigma_{\hat{\Delta}}^2 \cong \frac{(m_1+m_2-1)}{m_1(m_2-2)}\frac{R^{\frac{2}{1-R}}(\log R)^2}{(1-R)^2}. \tag{22}$$

with the asymptotic bias

$$Bias(\hat{\rho}) \cong \frac{(m_1+m_2-1)}{m_1(m_2-2)} \times \frac{\sqrt{R}(3R^2-6R-1)}{2(1+R)^3}, \tag{23}$$

$$Bias(\hat{\lambda}) \cong \frac{(m_1+m_2-1)}{m_1(m_2-2)} \times \frac{4R^2(R-2)}{(1+R)^4}, \tag{24}$$

and,

$$Bias(\hat{\Delta}) \cong \left\{\begin{array}{lr} H(R)\frac{(m_1+m_2-1)}{m_1(m_2-2)}, & R>1 \\ -H(R)\frac{(m_1+m_2-1)}{m_1(m_2-2)}, & R<1 \end{array}\right\}, \tag{25}$$

where, $H(R) = R^2\left[\frac{R^{\frac{2R-1}{1-R}}R\{2R-\log R-2\}\log R-(R-1)^2}{(R-1)^3}\right]$.

Consistent estimators for the above variances and biases can be obtained by substituting R by $\hat{R}^*$ in the above formulas.

3.1. Interval estimation

Two types of interval estimation for the OVL measure are considered, namely the asymptotic confidence interval and the bootstrap confidence interval that were introduced by Efron (1992). For a large sample, normal approximation to the sampling distribution using the delta-method, works fairly well. Therefore, the asymptotic $100(1-\alpha)\%$ confidence interval for the OVL measures is given by:
$\left\{\widehat{OVL} \mp \hat{\sigma}_{\widehat{OVL}} Z_{\alpha/2}\right\}$, where $Z_{\alpha/2}$ is the $\frac{\alpha}{2}$ upper quantile of the standard normal distribution.

There is an obvious bias involved in all OVL measure estimates, however, for large samples, they work fairly well. Thus, the bias corrected interval can be computed as follows:

$$\left(\widehat{OVL} - Bias(\widehat{OVL})\right) \pm \hat{\sigma}_{\widehat{OVL}} Z_{\alpha/2}. \tag{26}$$

However, uniform bootstrap resampling approach for estimating bootstrap confidence intervals as described by Efron (1992), is designed for one sample case. For a two-sample case, the uniform resampling rules will apply to each sample separately and independently (see Helu and Samawi, 2011).
Let $\mathbf{X} = (X_1, X_2, \cdots, X_{m_1})$ and $\mathbf{Y} = (Y_1, Y_2, \cdots, Y_{m_2})$ be two independent
Adaptive-IIPH samples drawn from $f_1(x)$ and $f_2(y)$ respectively. Assume that the parameter of interest is the OVL coefficient. Let S be an estimate of OVL based on the mentioned two random samples. For B uniform re-samples, say $(X^*_{i1}, X^*_{i2}, ..., X^*_{im_1})$ and $(Y^*_{i1}, Y^*_{i2}, ..., Y^*_{im_2}), i = 1, 2, ..., B$, let $S^*_1, S^*_2, ..., S^*_B$ be the re-sampling realization of S. Then, the uniform re-sampling approximation to the $100(1-\alpha)\%$ bootstrap confidence limits can be obtained as follows: Let $S^*_{(1)}, S^*_{(2)}, ..., S^*_{(B)}$ be the order statistics of $S^*_1, S^*_2, ..., S^*_B$. Define $\omega_1 = integer(B(\alpha))$ and $\omega_2 = integer(B(1-\alpha))$. Then the uniform re-sampling approximation of the $100(1-\alpha)\%$ confidence interval is $\left(\frac{S^*_{(\omega_1)} + S^*_{(\omega_1+1)}}{2}, \frac{S^*_{(\omega_2)} + S^*_{(\omega_2+1)}}{2}\right)$.

4. Simulation Study

This simulation study aims to rigorously compare the performance of maximum likelihood estimators for the measures of overlap. These estimators are derived from diverse sets of Adaptive-IIHP censoring samples, as described by Ng et al. (2009), generated from two independent Pareto distributions. The algorithm proceeds as follows:

1. Generate two independent progressive type-II censored samples, denoted as $U_1, U_2, ..., U_{m_1}$ and $V_1, V_2, ..., V_{m_2}$ from Pareto(θ_1) and Pareto(θ_2), respectively. Use censoring schemes $\mathbf{K} = (K_1, K_2, \ldots, K_{m_1})$ and $\mathbf{L} = (L_1, L_2, \ldots, L_{m_2})$ as proposed by Balakrishnan and Cramer(2014).
2. Determine the values of J_1 and J_2, such that $U_{J_1} < T_1 < U_{J_1+1}$ and $V_{J_2} < T_2 < V_{J_2+1}$. Then, remove $U_{J_1+2}, \ldots, U_{m_1}$ and $V_{J_2+2}, \ldots, V_{m_2}$.
3. Generate the first $m_1 - j_1 - 1$ order statistics from the truncated distribution $\frac{f_1(u)}{1-F_1(u_{J_1+1})}$ as $U_{J_1+2}, \ldots, U_{m_1}$, and adjust the censoring scheme to $\mathbf{K} = (K_1, \ldots, K_{J_1}, 0, \ldots, 0, K^* = n_1 - m_1 - \sum_{i=1}^{J_1} K_i)$. Similarly, generate the first $m_2 - j_2 - 1$ order statistics from the truncated distribution $\frac{f_2(v)}{1-F_2(v_{J_2+1})}$ as $V_{J_2+2}, \ldots, V_{m_2}$, and update the censoring scheme to $\mathbf{L} = (L_1, \ldots, L_{J_2}, 0, \ldots, 0, L^* = n_2 - m_2 - \sum_{i=1}^{J_2} L_i)$. Use the transformation $X = \log(U)$ and $Y = \log(V)$.
4. Calculate $\hat{\theta}_1$ and $\hat{\theta}_2$, and subsequently obtain the estimates of the measures of overlap $\hat{\rho}$, $\hat{\lambda}$, and $\hat{\Delta}$.

In this study, we executed a total of 10000 simulations, each corresponding to one of four distinct values of R. Specifically:

1. When $R = 0.005$, the resulting parameter values are as follows: $\rho = 0.14, \lambda = 0.02$, and $\Delta = 0.03$.
2. For $R = 0.05$, we observed $\rho = 0.42, \lambda = 0.18,$, and $\Delta = 0.19$.

3. When $R = 0.2$, the associated parameter values are $\rho = 0.70, \lambda = 0.50$, and $\Delta = 0.42$.
4. Lastly, $R = 0.8$ yielded parameter values of $\rho = 0.98, \lambda = 0.96$, and $\Delta = 0.85$.

These simulations are conducted based on four distinct sets of population parameters: $(\theta_1, \theta_2) = (0.005, 1), (0.1, 2), (0.1, 0.5)$, and $(4, 5)$. This comprehensive range of parameter combinations allowed us to explore varying degrees of similarity between the two Pareto distributions in our analysis. Additionally, three primary stopping times are considered: $T_1 = X_{\lfloor \frac{m}{4} \rfloor}$,$T_2 = X_{\lfloor \frac{4m}{5} \rfloor}$, and $T_3 = X_m + 2$.

We then computed the associated approximate 95% confidence intervals, bias ($|Bias|$), mean squared error (MSE), length of the confidence intervals (L) and coverage probability (Cov) using Taylor and bootstrap approximation techniques. The bootstrap approximation is based on $B = 1000$ resamples. For illustrative purposes we generated the censoring samples using $n = n_1 = n_2 = 20, 30, m = m_1 = m_2 = 5, 10, 20$, and set $\mathbf{K} = \mathbf{L}$, employing three censoring schemes:

- Scheme-I: $(n - m, 0^{*(m-1)})$, known as scheme-I, where $n - m$ units are removed just after the first failure.
- Scheme-II: $\left(0^{*(m-1)}, n - m\right)$, known as scheme-II, where $n - m$ units are removed after the last failure.
- Scheme-III: $\left(\frac{n-m}{2}, 0^{*(m-2)}, \frac{n-m}{2}\right)$, known as scheme-III, where $\frac{n-m}{2}$ units are removed after the first and last failures. For brevity, we use the notation 0^{*p} to denote p successive zeros. Thus, the scheme $(9, 0, 0, 0, 0, 0)$ is denoted by $(9, 0^{*5})$.

4.1. Data analysis and comparison study

In this study, we investigate the behavior of overlap estimators when applied to samples drawn from two Pareto distributions with varying degrees of similarity. Our research sheds light on the crucial relationship between the similarity of the two distributions and the accuracy of the estimators, utilizing Adaptive-IIPH censored data.

Most favorable estimators aim for minimal bias ($|Bias|$), smallest mean square error (MSE), and shortest confidence intervals (L). These desirable properties are most evident when there is a substantial disparity between the two Pareto density distributions, when $\rho = 0.14, \lambda = 0.019$ and $\Delta = 0.03$ as shown in Tables 6 and 10. Conversely, when the source distributions become more congruent, we consistently observe an increase in $|Bias|$, MSE, and L across all OVL estimators (see Tables 6-9). Notably, this trend inversely affects the coverage probability (Cov) for $\hat{\rho}$ and $\hat{\lambda}$, decreasing as source distributions become more alike (see Table 8 and 9). However, for $\hat{\Delta}$, coverage improves with the increasing similarity between the two densities.

Furthermore, as $\hat{\rho}$, $\hat{\lambda}$, and $\hat{\Delta}$ approach 1, indicating strong agreement between source distributions, we notice a consistent pattern: an increase in $|Bias|$, MSE, and L, coupled with a decrease in coverage for $\hat{\rho}$ and $\hat{\lambda}$.

It's important to highlight the consistent behavior of $\hat{\Delta}$ regarding coverage. As $\hat{\Delta}$ approaches 0 or 1, its coverage is the closest to the nominal level. This highlights the remarkable stability of the $\hat{\Delta}$ estimator in scenarios where the source distributions either fully align or diverge.

It is noteworthy that when there exists a substantial disagreement between the two Pareto densities, there are minimal differences between the three stopping times. Additionally, when the ratio of m/n is large ($\geq 2/3$), $|Bias|$, MSE, L, and coverage probability show noticeable improvement.

Shifting our focus to the bootstrap method, results presented in Tables 10-13 align with the observations made in Tables 6-9), except for instances when the two densities have perfect agreement. In such cases, all three OVL estimates exhibit similar behavior: $|Bias|$ and Cov increase while MSE and L values decrease. Furthermore, the bootstrap results indicate no significant impact from varying censoring schemes or OVL values, except for the consistent coverage values, which remain stable regardless of the source distributions aligning or diverging.

5. Real life data

The dataset utilized in this section originates from the research conducted by [27]. Their study aimed to conduct a comparative analysis of claim and indemnity amounts within the Iran insurance company. We have introduced modifications to the dataset by applying the natural logarithm to each value. Table 1 provides an overview of the claim and indemnity amounts.

To assess the validity of the Pareto model for both claim and indemnity amounts, we have employed the Kolmogorov-Smirnov (K-S), Anderson-Darling (A-D), and chi-square tests with respective parameters $\theta_1 = 0.6343$ and $\theta_2 = 0.6059$. Table 2 provides a detailed results of these tests with a significance level of 0.05.

Table 1. Claim and indemnity amount groups

Claim amount	1.18	1.21	1.00	2.02	1.83	2.56	3.50	2.90	1.35	1.41
	2.05	1.95	3.66	3.02	2.64	1.59	1.74	1.71	1.19	
Indemnity amount	1.01	1.15	1.25	1.25	1.33	1.39	1.55	1.62	1.65	1.70
	1.84	1.86	1.88	1.97	2.43	2.54	2.76	2.94	3.45	3.54

Table 2. Test statistic and p-value associated with each test for claim and indemnity amounts

	K-S(p-value)	A-D(p-value)	chi-squared(p-value)
Claim amount	0.2028 (0.3654)	1.2301 (0.3102)	3.0753 (0.2149)
Indemnity amount	0.2173 (0.2611)	1.350 (0.2133)	4.8387 (0.0890)

Table 2 indicates that the Pareto model fits both data sets well. In addition, the fitted pdfs and Q-Q plots are plotted for both data sets and reported in Figures 3 - 6, confirming that the Pareto distribution is a good fit for the claim and indemnity data of Iran insurance company.

Three different artificial Adaptive-IIHP censored data are created for both sets using the same censoring schemes as those in Section 4. The associated stopping time for each scheme and the generated censored samples are given in Table 3 and 4.

The estimates of the OVLs are calculated based on $m_1 = 14$, $m_2 = 15$. The corresponding MLEs, $|Bias|$, asymptotic variance (Var) and 95% confidence intervals for OVLs, using Taylor approximation and bootstrap methods, are reported in Table 5. This table reveals a remarkable proximity of the OVL estimates to 1, signifying a substantial agreement between the density distributions of the two sample sets.

In terms of $|Bias|$, Var, and L, the performance of $\hat{\Delta}$ is notably inferior compared to $\hat{\rho}$ and $\hat{\lambda}$. $\hat{\rho}$ exhibits superior performance across these metrics. This observation aligns with the findings of the simulation study in the previous section, indicating that $\hat{\rho}$ and $\hat{\lambda}$ excel when there is a high degree of similarity between the two density distributions. Furthermore, the estimates based on scheme-I closely approximate those obtained from the complete data set.

6. Concluding Remarks

In this study, we thoroughly examined the behavior of overlap estimators in the context of Adaptive Type-II Progressive Hybrid Censoring, focusing on samples drawn from two Pareto distributions with varying degrees of similarity. Our findings demonstrate that the overlap estimators exhibit desirable properties, such as minimal bias, low mean squared error (MSE), and narrow confidence intervals, particularly when there is significant disparity between the two Pareto distributions. As the similarity between the distributions increases, the performance of

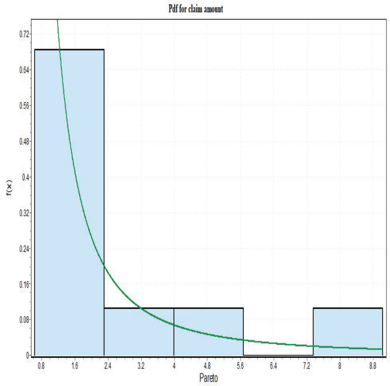

Figure 3. Estimated pdf of the claim amount data

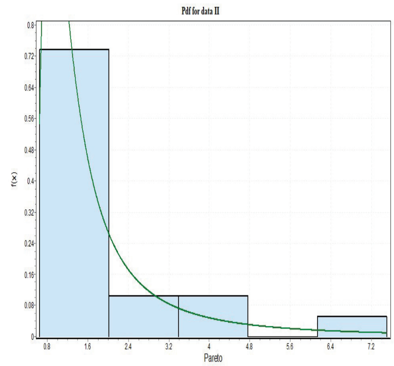

Figure 4. Estimated pdf of the indemnity amount data

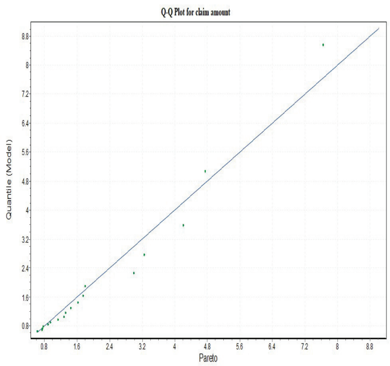

Figure 5. Q-Q plot for the claim amount data

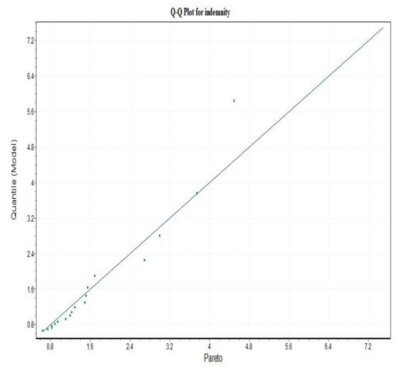

Figure 6. Q-Q plot for the indemnity amount data

these estimators becomes less robust. Notably, the estimator $\hat{\Delta}$ showed remarkable stability in scenarios where the source distributions were either perfectly aligned or highly divergent, maintaining coverage levels close to the nominal value. This highlights the robustness of $\hat{\Delta}$ in extreme cases. Moreover, our analysis revealed that the ratio of m/n plays a critical role in the performance of these estimators, with higher ratios leading to improvements in bias, MSE, and coverage probability. The implications of our study are significant for applications in reliability engineering, risk analysis, and other fields where understanding the overlap between distributions is crucial. By providing a detailed examination of the overlap measures under a novel censoring scheme, our work offers

Table 3. Artificial Adaptive-IIHP censored samples for claim amount group

scheme	T	Adaptive-IIHP censored data I						
I	T_1	1.00	1.18	1.21	1.35	1.41	1.59	1.71
		2.02	2.05	2.56	2.64	2.90	3.02	3.66
II	T_2	1.00	1.18	1.19	1.21	1.346	1.41	1.59
		1.71	1.83	2.02	2.56	2.64	2.90	3.02
III	T_3	1.00	1.18	1.19	1.21	1.35	1.41	1.59
		1.71	1.95	2.02	2.05	2.56	2.64	2.90

Table 4. Artificial Adaptive-IIHP censored samples for the indemnity amount

scheme	T	Adaptive-IIHP censored data II							
I	T_1	1.01	1.15	1.25	1.55	1.62	1.97	1.65	1.70
		1.84	1.86	1.88	2.43	2.54	2.76	3.54	
II	T_2	1.01	1.15	1.25	1.33	1.39	1.55	1.62	1.65
		1.70	1.84	1.87	1.97	2.94	3.45	3.54	
III	T_3	1.15	1.25	1.33	1.55	1.62	1.64	1.70	1.84
		1.86	1.88	1.97	2.43	2.54	2.76	3.54	

Table 5. Results based on the real data of Efron (1988)

				Asymptotic Inference 95% confidence	Bootstrap Inference 95% confidence
Scheme	Coeff	MLEs ($\|Bias\|$)	Asymptotic variance	L	L
Complete	ρ	0.9999(0.0140)	0.00019	0.0015	0.0275
	λ	0.9999(0.0279)	0.00008	0.0030	0.0542
	Δ	0.9983(0.0204)	0.01545	0.4807	0.1711
1	ρ	0.999(4.22E-6)	0.0004	0.0081	0.0668
	λ	0.999(0.00002)	0.0016	0.0161	0.1292
	Δ	0.992(0.0208)	0.0216	0.5656	0.2672
2	ρ	0.998(0.0214)	0.0001	0.0460	0.0709
	λ	0.996(0.0425)	0.0005	0.0917	0.1368
	Δ	0.956(0.0274)	0.0207	0.5646	0.2726
3	ρ	0.9952(0.0225)	0.0004	0.0750	0.0811
	λ	0.9904(0.0445)	0.0015	0.1494	0.1557
	Δ	0.9277(0.0268)	0.0206	0.5629	0.2937

valuable insights into the practical application of these estimators. Future research could expand upon our findings by exploring different types of distributions beyond the Pareto model. Additionally, investigating the behavior of overlap measures in the presence of outliers or under different censoring schemes could further enhance the versatility and applicability of these methods in various statistical contexts.

Table 6. Taylor approximation: absolute value of bias ($|Bias|$), length L, mean squared error (MSE) & coverage probability (Cov), when $R = 0.005, \rho = 0.14, \lambda = 0.019$ and $\Delta = 0.030$

3* (n,m)	Scheme		$T_1 = X_{[\frac{m}{4}]}$				$T_2 = X_{[\frac{4\times m}{5}]}$				$T_3 = X_{[m]}+2$			
		Estimate	Bias	MSE	L	Cov	Bias	MSE	L	Cov	Bias	MSE	L	Cov
(20,6)	I	ρ	0.0078542	0.0022360	0.1758549	0.9707	0.0078052	0.0022176	0.1747471	0.9647	0.0078746	0.0022641	0.1760899	0.9618
		λ	0.0001180	0.0002286	0.0509102	0.9399	0.0001187	0.0002295	0.0504984	0.9290	0.0001275	0.0002458	0.0515873	0.9267
		Δ	0.0006033	0.0003425	0.0657005	0.9435	0.0005986	0.0003410	0.0651585	0.9372	0.0006128	0.0003567	0.0661448	0.9364
	II	ρ	0.0080241	0.0023867	0.1787519	0.9412	0.0082482	0.0026189	0.1819413	0.8741	0.0078733	0.0022634	0.1760616	0.9621
		λ	0.0001563	0.0002986	0.0544833	0.8936	0.0002437	0.0004514	0.0601193	0.8366	0.0001275	0.0002457	0.0515730	0.9265
		Δ	0.0006515	0.0004045	0.0685246	0.9093	0.0007308	0.0005202	0.0725072	0.8318	0.0006126	0.0003566	0.0661289	0.9367
	III	ρ	0.0080106	0.0023719	0.1785624	0.9437	0.0081101	0.0024947	0.1797937	0.9014	0.0078735	0.0022643	0.1760595	0.9615
		λ	0.0001511	0.0002894	0.0541273	0.8911	0.0001967	0.0003704	0.0571060	0.8564	0.0001279	0.0002464	0.0515933	0.9261
		Δ	0.0006465	0.0003973	0.0682749	0.9141	0.0006886	0.0004602	0.0702694	0.8728	0.0006129	0.0003571	0.0661397	0.9350
(20,12)	I	ρ	0.0033405	0.0009200	0.1159606	0.9637	0.0033676	0.0009354	0.1168271	0.9609	0.0033644	0.0009347	0.1167056	0.9584
		λ	0.0000421	0.0000822	0.0329093	0.9456	0.0000438	0.0000855	0.0334671	0.9496	0.0000440	0.0000859	0.0334443	0.9370
		Δ	0.0002507	0.0001336	0.0430840	0.9499	0.0002553	0.0001376	0.0436474	0.9476	0.0002551	0.0001378	0.0435999	0.9432
	II	ρ	0.003370	0.0009479	0.1167397	0.9223	0.0034459	0.0010032	0.1189505	0.8753	0.0033655	0.0009355	0.1167393	0.9557
		λ	0.0000479	0.0000932	0.0339441	0.8918	0.0000585	0.0001131	0.0359894	0.8663	0.0000442	0.0000862	0.0334763	0.9358
		Δ	0.0002597	0.0001446	0.0439014	0.9103	0.0002770	0.0001644	0.0456821	0.8623	0.0002554	0.0001382	0.0436276	0.9424
	III	ρ	0.0033566	0.0009356	0.1163822	0.9405	0.0034052	0.0009675	0.1178605	0.9221	0.0033662	0.0009360	0.1167591	0.9556
		λ	0.0000453	0.0000883	0.0334863	0.9182	0.0000503	0.0000978	0.0346566	0.9018	0.0000443	0.0000864	0.0334947	0.9353
		Δ	0.0002557	0.0001397	0.0435374	0.9256	0.0002654	0.0001499	0.0446191	0.9100	0.0002556	0.0001383	0.0436438	0.9421
(30,10)	I	ρ	0.0041341	0.0011492	0.1288188	0.9630	0.0041476	0.0011576	0.1291974	0.9600	0.0041582	0.0011641	0.1294887	0.9574
		λ	0.0000548	0.0001068	0.0368360	0.9395	0.0000560	0.0001091	0.0371085	0.9415	0.0000571	0.0001111	0.0373226	0.9351
		Δ	0.0003129	0.0001698	0.0480068	0.9437	0.0003154	0.0001723	0.0482685	0.9431	0.0003174	0.0001744	0.0484701	0.9432
	II	ρ	0.0042162	0.0012195	0.1308356	0.9030	0.0043120	0.0013160	0.1328536	0.8133	0.0041587	0.0011645	0.1295033	0.9562
		λ	0.0000697	0.0001348	0.0391727	0.8798	0.0000999	0.0001895	0.0424486	0.8193	0.0000572	0.0001112	0.0373354	0.9351
		Δ	0.0003353	0.0001971	0.0499097	0.8903	0.0003678	0.0002426	0.0522910	0.8200	0.0003175	0.0001745	0.0484816	0.9421
	III	ρ	0.0042003	0.0012062	0.1304463	0.9162	0.0042699	0.0012744	0.1319757	0.8475	0.0041596	0.0011652	0.1295246	0.9554
		λ	0.0000667	0.0001293	0.0387267	0.8900	0.0000859	0.0001644	0.0410303	0.8389	0.0000573	0.0001115	0.0373585	0.9355
		Δ	0.0003310	0.0001919	0.0495466	0.9008	0.0003536	0.0002225	0.0512662	0.8241	0.0003178	0.0001748	0.0485009	0.9419
(30,20)	I	ρ	0.0019180	0.0005261	0.0886037	0.9626	0.0019127	0.0005238	0.0883645	0.9549	0.0019163	0.0005260	0.0885168	0.9501
		λ	0.0000230	0.0000450	0.0251553	0.9483	0.0000229	0.0000449	0.0250449	0.9430	0.0000232	0.0000454	0.0251548	0.9408
		Δ	0.0001439	0.0000753	0.0330218	0.9539	0.0001432	0.0000749	0.0328913	0.9474	0.0001439	0.0000756	0.0329975	0.9439
	II	ρ	0.0019321	0.0005394	0.0891016	0.9011	0.0019428	0.0005503	0.0894549	0.8449	0.0019166	0.0005263	0.0885256	0.9494
		λ	0.0000257	0.0000502	0.0258171	0.8952	0.0000282	0.0000550	0.0263598	0.8492	0.0000232	0.0000455	0.0251675	0.9389
		Δ	0.0001482	0.0000806	0.0335448	0.8919	0.0001517	0.0000852	0.0339554	0.8379	0.0001440	0.0000757	0.0330073	0.9428
	III	ρ	0.0019254	0.0005332	0.0888633	0.9304	0.0019269	0.0005363	0.0888798	0.9030	0.0019167	0.0005264	0.0885313	0.9494
		λ	0.0000244	0.0000478	0.0255071	0.9192	0.0000253	0.0000495	0.0256661	0.8929	0.0000233	0.0000456	0.0251735	0.9386
		Δ	0.0001462	0.0000781	0.0332990	0.9259	0.0001472	0.0000797	0.0333960	0.8911	0.0001440	0.0000757	0.0330124	0.9421

Table 7. Taylor approximation: absolute value of bias ($|Bias|$), length L, mean squared error (MSE) & coverage probability (Cov), when $R = 0.05, \rho = 0.42, \lambda = 0.17$ and $\Delta = 0.18$

3* (n,m)	Scheme		$T_1 = X_{[\frac{m}{4}]}$				$T_2 = X_{[\frac{4\times m}{5}]}$				$T_3 = X_{[m]}+2$			
		Estimate	Bias	MSE	L	Cov	Bias	MSE	L	Cov	Bias	MSE	L	Cov
(20,6)	I	ρ	0.0273201	0.0163161	0.4789260	0.9619	0.0271328	0.0161493	0.4759521	0.9560	0.0273937	0.0162840	0.4773923	0.9507
		λ	0.0084135	0.0129086	0.4053003	0.9365	0.0083651	0.0127724	0.4012606	0.9288	0.0087626	0.0131856	0.4059027	0.9259
		Δ	0.0058290	0.0081485	0.3339108	0.9504	0.0057760	0.0080665	0.3311823	0.9439	0.0058787	0.0082620	0.3341260	0.9430
	II	ρ	0.0279587	0.0164845	0.4775929	0.9197	0.0286451	0.0163809	0.4699172	0.8367	0.0273888	0.0162804	0.4773311	0.9508
		λ	0.0099660	0.0143434	0.4151969	0.8923	0.0124432	0.0160490	0.4216586	0.8172	0.0087589	0.0131808	0.4058061	0.9258
		Δ	0.0061334	0.0087775	0.3395973	0.9175	0.0065319	0.0096422	0.3445424	0.8471	0.0058771	0.0082594	0.3340595	0.9429
	III	ρ	0.0279142	0.0164916	0.4781303	0.9251	0.0282322	0.0163689	0.4722302	0.8712	0.0273895	0.0162778	0.4772490	0.9503
		λ	0.0098012	0.0142197	0.4147523	0.8805	0.0112280	0.0152451	0.4171807	0.8477	0.0087702	0.0131886	0.4057888	0.9241
		Δ	0.0061068	0.0087205	0.3393136	0.9227	0.0063219	0.0091918	0.3407829	0.8748	0.0058784	0.0082626	0.3340419	0.9442
(20,12)	I	ρ	0.0116199	0.0068820	0.3187838	0.9582	0.0117264	0.0069527	0.3202740	0.9540	0.0117142	0.0069386	0.3198609	0.9512
		λ	0.0032471	0.0052171	0.2694835	0.9444	0.0033571	0.0053646	0.2728329	0.9424	0.0033634	0.0053619	0.2723700	0.9366
		Δ	0.0024657	0.0033972	0.2218541	0.9540	0.0025038	0.0034673	0.2238907	0.9520	0.0025010	0.0034638	0.2235717	0.9466
	II	ρ	0.0117456	0.0069133	0.3181839	0.9106	0.0120437	0.0070365	0.3196569	0.8539	0.0117188	0.0069406	0.3198806	0.9504
		λ	0.0035561	0.0055478	0.2728770	0.8914	0.0040841	0.0061022	0.2811110	0.8528	0.0033719	0.0053712	0.2725007	0.9352
		Δ	0.0025255	0.0035261	0.2234054	0.9112	0.0026490	0.0037706	0.2281668	0.8642	0.0025029	0.0034677	0.2236442	0.9456
	III	ρ	0.0116895	0.0068992	0.3184015	0.9324	0.0118811	0.0069995	0.3201207	0.9095	0.0117216	0.0069418	0.3198935	0.9509
		λ	0.0034208	0.0054063	0.2713832	0.9169	0.0036973	0.0057253	0.2770016	0.8989	0.0033767	0.0053766	0.2725793	0.9350
		Δ	0.0024991	0.0034698	0.2226910	0.9210	0.0025735	0.0036129	0.2260068	0.8126	0.002504	0.0034700	0.2236873	0.9451
(30,10)	I	ρ	0.0143865	0.0085236	0.3530566	0.9563	0.0144394	0.0085549	0.3535985	0.9524	0.0144792	0.0085762	0.3539921	0.9493
		λ	0.0041484	0.0065787	0.2992959	0.9390	0.0042147	0.0066583	0.3007278	0.9362	0.0042677	0.0067144	0.3017433	0.9343
		Δ	0.0030625	0.0042403	0.2461547	0.9502	0.0030824	0.0042775	0.2470228	0.9477	0.0030974	0.0043053	0.2476755	0.9453
	II	ρ	0.0147184	0.0086244	0.3527090	0.8832	0.0150592	0.0085933	0.3488100	0.7834	0.0144814	0.0085774	0.3540051	0.9489
		λ	0.0048862	0.0073364	0.3072310	0.8737	0.0059910	0.0081839	0.3128539	0.7872	0.0042712	0.0067185	0.3018035	0.9345
		Δ	0.0032135	0.0045543	0.2502874	0.8790	0.0033980	0.0049437	0.2536116	0.8228	0.0030983	0.0043071	0.2477085	0.9462
	III	ρ	0.0146558	0.0086061	0.3527600	0.8971	0.0149177	0.0086128	0.3504874	0.8239	0.0144849	0.0085787	0.3540076	0.9477
		λ	0.0047470	0.0071992	0.3057655	0.8861	0.0055228	0.0078528	0.3107367	0.8178	0.0042783	0.0067261	0.3018908	0.9342
		Δ	0.0031849	0.0044956	0.2494919	0.8880	0.0033200	0.0047815	0.2521915	0.8323	0.0030998	0.0043102	0.2477537	0.9449
(30,20)	I	ρ	0.0066768	0.0039628	0.2439366	0.9577	0.0066565	0.0039460	0.2433648	0.9536	0.0066712	0.0039548	0.2435842	0.9470
		λ	0.0018216	0.0029779	0.2076300	0.9483	0.0018121	0.0029597	0.2067274	0.9423	0.0018301	0.0029832	0.2073364	0.9392
		Δ	0.0014223	0.0019569	0.1703983	0.9567	0.0014159	0.0019463	0.1698096	0.9491	0.0014214	0.0019568	0.1701624	0.9449
	II	ρ	0.0067381	0.0039771	0.2435491	0.8911	0.0067836	0.0039813	0.2429978	0.8299	0.0066723	0.0039550	0.2435737	0.9455
		λ	0.0019715	0.0031429	0.2098567	0.8897	0.0020998	0.0032727	0.2113613	0.8356	0.0018331	0.0029865	0.2073773	0.9377
		Δ	0.0014513	0.0020197	0.1713835	0.8881	0.0014740	0.0020695	0.1720283	0.8291	0.0014220	0.0019580	0.1701791	0.9451
	III	ρ	0.0067093	0.0039704	0.2437122	0.9212	0.0067173	0.0039642	0.2431984	0.8929	0.0066730	0.0039553	0.2435749	0.9451
		λ	0.0019017	0.0030673	0.2088217	0.9182	0.0019479	0.0031125	0.2089981	0.8878	0.0018343	0.0029880	0.2074016	0.9374
		Δ	0.0014377	0.0019905	0.1709112	0.8950	0.0014436	0.0020055	0.1708676	0.8905	0.0014222	0.0019586	0.1701912	0.9441

Table 8. Taylor approximation: absolute value of bias ($|Bias|$), length L, mean squared error (MSE) & coverage probability (Cov), when $R = 0.2, \rho = 0.7, \lambda = 0.5$ and $\Delta = 0.42$

3* (n,m)	Scheme		$T_1 = X_{[\frac{m}{4}]}$				$T_2 = X_{[\frac{4\times m}{5}]}$				$T_3 = X_{[m]+2}$			
			Bias	MSE	L	Cov	Bias	MSE	L	Cov	Bias	MSE	L	Cov
(20,5)	I	ρ	0.0716065	0.0370737	0.6896547	0.900	0.0717385	0.0370382	0.6889196	0.899	0.0716633	0.0368222	0.6858989	0.889
		λ	0.0721967	0.0683973	0.9429780	0.887	0.0726218	0.0684928	0.9429267	0.887	0.0728847	0.0681256	0.9382438	0.881
		Δ	0.0222726	0.0384894	0.7412371	0.954	0.0224152	0.0386204	0.7424052	0.951	0.0223548	0.0387315	0.7427990	0.947
	II	ρ	0.0708315	0.0352696	0.6648605	0.843	0.0695202	0.0331312	0.6349343	0.765	0.0716628	0.036812	0.6857445	0.890
		λ	0.0741018	0.0656036	0.9042171	0.823	0.0755394	0.0620459	0.8573204	0.740	0.0729056	0.0681142	0.9380299	0.880
		Δ	0.0220182	0.0390480	0.7394233	0.923	0.0213825	0.0397410	0.7376245	0.881	0.0223262	0.038738	0.7428223	0.947
	III	ρ	0.0558320	0.0269591	0.5945166	0.858	0.0545661	0.0256274	0.0304478	0.798	0.0559050	0.0277328	0.6069438	0.895
		λ	0.0588837	0.0516343	0.8243695	0.839	0.0585119	0.0486076	0.7858314	0.771	0.0574315	0.0527647	0.8420961	0.884
		Δ	0.0177059	0.0306268	0.6616996	0.924	0.0171406	0.0304478	0.6536668	0.880	0.0175740	0.0302032	0.6602110	0.946
(20,10)	I	ρ	0.0301493	0.0144200	0.4535761	0.921	0.0301890	0.0143789	0.4526856	0.918	0.0302199	0.0143381	0.4517276	0.912
		λ	0.0309158	0.0289533	0.6471111	0.915	0.0310988	0.0289323	0.6464284	0.912	0.0312608	0.0288975	0.6455462	0.907
		Δ	0.0094653	0.0161709	0.4902054	0.955	0.0094945	0.0162263	0.4908642	0.956	0.0095151	0.0162800	0.4915098	0.956
	II	ρ	0.0300324	0.0138784	0.4426637	0.857	0.0296106	0.0132986	0.4304742	0.794	0.0302209	0.0143310	0.4515825	0.912
		λ	0.0317188	0.0278827	0.6295322	0.848	0.0318566	0.0264000	0.6060906	0.773	0.0312777	0.0288885	0.6453528	0.907
		Δ	0.0095651	0.0163261	0.4897359	0.923	0.0095148	0.016301	0.4862114	0.871	0.0095177	0.0162838	0.4915289	0.956
	III	ρ	0.0300806	0.0140614	0.4464017	0.879	0.0298814	0.0137517	0.4400462	0.845	0.0302199	0.0143242	0.4514475	0.913
		λ	0.0314674	0.0282501	0.6356161	0.873	0.0315991	0.0274776	0.6235039	0.832	0.0312889	0.0288765	0.6451425	0.905
		Δ	0.0095485	0.0162767	0.4899529	0.926	0.0095441	0.0162695	0.4881879	0.898	0.0095194	0.0162859	0.4915235	0.947
(30,10)	I	ρ	0.0301362	0.0143943	0.4530857	0.919	0.0301874	0.0143580	0.4522663	0.916	0.0302199	0.0143381	0.4517276	0.912
		λ	0.0309343	0.0288916	0.6461867	0.912	0.0311370	0.0288954	0.6457965	0.910	0.0312608	0.0288975	0.6455462	0.907
		Δ	0.0094673	0.0161706	0.4900649	0.953	0.0095007	0.0162344	0.4908796	0.950	0.0095151	0.0162800	0.4915098	0.945
	II	ρ	0.0299532	0.0136703	0.4383383	0.834	0.0292229	0.0127344	0.4179451	0.732	0.0302211	0.0143349	0.4516607	0.912
		λ	0.0319459	0.0274364	0.6222027	0.821	0.0320912	0.0251123	0.5839731	0.709	0.0312701	0.0288946	0.6454683	0.907
		Δ	0.0095902	0.0163688	0.4892762	0.882	0.0093565	0.0163694	0.4840696	0.885	0.0095166	0.0162823	0.4915269	0.947
	III	ρ	0.0299872	0.0137968	0.4409804	0.850	0.0295406	0.0130943	0.4259188	0.768	0.0302203	0.0143277	0.4515170	0.912
		λ	0.0317714	0.0276851	0.6264288	0.837	0.0321122	0.0260237	0.5989469	0.754	0.0312827	0.0288826	0.6452509	0.906
		Δ	0.0095710	0.0163303	0.4893699	0.891	0.0094781	0.0163720	0.4861507	0.824	0.0095185	0.0162848	0.4915263	0.950
(30,20)	I	ρ	0.0141938	0.0044456	0.3133374	0.938	0.0141488	0.0065994	0.3131467	0.935	0.0141637	0.0065852	0.3127598	0.931
		λ	0.0146660	0.0139679	0.4575892	0.938	0.0145704	0.0138924	0.4562539	0.930	0.0146346	0.0138839	0.4560091	0.925
		Δ	0.0044456	0.0139679	0.3396541	0.957	0.0044277	0.0075940	0.3387812	0.956	0.0044382	0.0076116	0.3390946	0.947
	II	ρ	0.0141373	0.0064204	0.3082446	0.863	0.0140679	0.0062692	0.3039185	0.801	0.0141623	0.0065813	0.3126556	0.929
		λ	0.0148938	0.0135253	0.4485924	0.855	0.0150231	0.0131398	0.4405622	0.793	0.0146389	0.0138742	0.4558187	0.923
		Δ	0.0044732	0.0076520	0.3388453	0.891	0.0044842	0.0076584	0.3378817	0.831	0.0044386	0.0076119	0.3390736	0.947
	III	ρ	0.0141373	0.0064204	0.3082445	0.863	0.0141133	0.0064347	0.3086631	0.868	0.0141622	0.0065798	0.3126168	0.929
		λ	0.0148938	0.0135253	0.4485924	0.855	0.0148066	0.0135222	0.4486665	0.859	0.0146414	0.0138711	0.4557565	0.923
		Δ	0.0044732	0.0076520	0.3388454	0.926	0.0044578	0.0076257	0.3383269	0.892	0.0044390	0.0076123	0.3390736	0.946

Table 9. Taylor approximation: absolute value of bias ($|Bias|$), length L, mean squared error (MSE) & coverage probability (Cov), when $R = 0.8, \rho = 0.98, \lambda = 0.96$ and $\Delta = 0.85$

3* (n,m)	Scheme		$T_1 = X_{[\frac{m}{4}]}$ $\|Bias\|$	MSE	L	Cov	$T_2 = X_{[\frac{4 \times m}{5}]}$ $\|Bias\|$	MSE	L	Cov	$T_3 = X_{[m]+2}$ $\|Bias\|$	MSE	L	Cov
(20,6)	I	ρ	0.0812619	0.0205577	0.3894151	0.750	0.0810067	0.0205280	0.3888261	0.749	0.0803111	0.0207049	0.3926703	0.750
		λ	0.1387748	0.0626163	0.6977093	0.746	0.1382774	0.0625305	0.6966740	0.745	0.1366032	0.0627903	0.7020468	0.746
		Δ	0.0185057	0.0733602	1.0410969	0.976	0.0187023	0.0733798	1.0412381	0.977	0.0179505	0.0731603	1.0395517	0.974
	II	ρ	0.0757181	0.0216555	0.4142439	0.7629	0.0537394	0.0158385	0.3770439	0.790	0.0802754	0.0207099	0.3928302	0.750
		λ	0.1252449	0.0631945	0.7249431	0.7494	0.086879	0.0450793	0.6505005	0.762	0.1365192	0.0627890	0.7022496	0.745
		Δ	0.0136132	0.0712540	1.0234630	0.9515	0.0071651	0.0529574	0.8838795	0.898	0.0179349	0.0731500	1.0394659	0.975
	III	ρ	0.0590967	0.0141739	0.3397946	0.774	0.0560893	0.0153106	0.3648882	0.788	0.0620064	0.013542	0.3237355	0.759
		λ	0.1004796	0.0430642	0.6075121	0.765	0.0923564	0.0446436	0.6383941	0.767	0.1075356	0.042248	0.5868227	0.756
		Δ	0.0107952	0.0556002	0.9091876	0.956	0.0083894	0.0539651	0.893871	0.924	0.0137865	0.0565591	0.9181924	0.973
(20,12)	I	ρ	0.0336988	0.0045628	0.1922379	0.773	0.0335295	0.0045755	0.1925301	0.771	0.0333632	0.0045893	0.1931137	0.773
		λ	0.0614920	0.0154610	0.3614935	0.772	0.0611270	0.0154636	0.3616523	0.770	0.0607800	0.0154833	0.3624353	0.772
		Δ	0.0093371	0.0303215	0.6775147	0.979	0.0091161	0.0302996	0.6772292	0.978	0.0089251	0.0302767	0.3769169	0.976
	II	ρ	0.0321574	0.0051141	0.2093392	0.783	0.0307018	0.0058211	0.2312049	0.806	0.0333401	0.0045949	0.1931994	0.773
		λ	0.0573738	0.0166939	0.3871886	0.778	0.0532569	0.0182991	0.4200836	0.793	0.0607211	0.0154918	0.3624882	0.771
		Δ	0.0070480	0.0297689	0.6707346	0.950	0.0053693	0.0290633	0.6620333	0.911	0.0088786	0.0302700	0.6768339	0.976
	III	ρ	0.0326782	0.0049298	0.2035297	0.778	0.0319035	0.0053122	0.2154539	0.793	0.0333202	0.0046017	0.1933555	0.772
		λ	0.0587563	0.0162939	0.3785423	0.775	0.0565313	0.0171801	0.3968142	0.786	0.0606675	0.0155055	0.3626866	0.771
		Δ	0.0078522	0.0299597	0.6730938	0.960	0.0067652	0.0295933	0.6686354	0.940	0.0088357	0.0302630	0.6767484	0.974
(30,10)	I	ρ	0.0336360	0.0045955	0.1932728	0.773	0.0334671	0.0045954	0.1930931	0.772	0.0333632	0.0045893	0.1931137	0.773
		λ	0.0613047	0.0155428	0.3631435	0.772	0.0609635	0.0155067	0.3624600	0.770	0.0607800	0.0154833	0.3624353	0.772
		Δ	0.0092964	0.0302925	0.6771687	0.977	0.0090529	0.0302793	0.6769795	0.976	0.0089251	0.0302767	0.6769169	0.974
	II	ρ	0.0315784	0.0053300	0.216039	0.787	0.0290956	0.0063966	0.2484373	0.801	0.0333520	0.0045913	0.1931229	0.772
		λ	0.0558305	0.0171604	0.3970567	0.780	0.0491309	0.0193890	0.4433215	0.787	0.0607525	0.0154848	0.3623991	0.771
		Δ	0.0064249	0.0295431	0.6679282	0.939	0.0040293	0.0283724	0.6531454	0.868	0.0089111	0.0302739	0.6768811	0.974
	III	ρ	0.0319478	0.0052097	0.2123686	0.787	0.0300684	0.0059971	0.2362642	0.808	0.0333304	0.0045981	0.1932623	0.772
		λ	0.0567872	0.0169152	0.3918320	0.781	0.0516861	0.0185810	0.4263932	0.790	0.0606956	0.0154975	0.3625554	0.771
		Δ	0.0068682	0.0296759	0.6695963	0.946	0.0046989	0.0288378	0.6591284	0.896	0.0088116	0.0302666	0.6767912	0.974
(30,20)	I	ρ	0.0157053	0.0012227	0.1023877	0.773	0.0157279	0.0012593	0.1043478	0.776	0.0156734	0.0012672	0.1044796	0.774
		λ	0.0297394	0.0044292	0.1976157	0.773	0.0297213	0.0045462	0.2011134	0.776	0.0295968	0.0045672	0.2012317	0.774
		Δ	0.0042661	0.0141598	0.4649246	0.980	0.0053669	0.0141353	0.4645130	0.978	0.0041457	0.0141286	0.4643954	0.975
	II	ρ	0.0152284	0.0014716	0.1147454	0.7888	0.0148511	0.0016746	0.1247492	0.810	0.0156638	0.0012727	0.1047359	0.775
		λ	0.0283352	0.0051574	0.2186248	0.7842	0.0272243	0.0057232	0.2352612	0.791	0.0295677	0.0045838	0.2016696	0.775
		Δ	0.0040723	0.0139703	0.4616445	0.9459	0.0034117	0.0138066	0.4587685	0.909	0.0040910	0.0141246	0.4643273	0.974
	III	ρ	0.0152284	0.0013625	0.1093403	0.789	0.0152934	0.0014704	0.1148985	0.791	0.0156599	0.0012745	0.1048255	0.776
		λ	0.0283351	0.0048435	0.2095048	0.784	0.0284677	0.0051597	0.2190266	0.790	0.0295566	0.0045893	0.2018241	0.776
		Δ	0.0040723	0.0140551	0.4631190	0.962	0.0041435	0.0139732	0.4617028	0.946	0.0040641	0.0141233	0.4643049	0.974

Table 10. Bootstrap results: absolute value of bias ($|Bias|$), length L, mean squared error (MSE) & coverage probability (Cov), when $R = 0.005, \rho = 0.14, \lambda = 0.019$ and $\Delta = 0.030$

3*	(n,m)	Scheme		$T_1 = X_{[\frac{m}{4}]}$				$T_2 = X_{[\frac{4\times m}{5}]}$				$T_3 = X_{[m]+2}$			
				$\|Bias\|$	MSE	L	Cov	$\|Bias\|$	MSE	L	Cov	$\|Bias\|$	MSE	L	Cov
	(20,5)	I	ρ	0.0057719	0.0021693	0.1718689	0.937	0.0060681	0.0023285	0.1767447	0.938	0.0063525	0.0025073	0.1797093	0.950
			λ	0.0039358	0.0003375	0.0633661	0.937	0.0041810	0.0003834	0.0661163	0.938	0.0043293	0.0005461	0.0673430	0.950
			Δ	0.0037974	0.0004283	0.0731760	0.937	0.0040114	0.0004724	0.0758094	0.938	0.0041714	0.0005809	0.0770663	0.950
		II	ρ	0.0082319	0.0031671	0.2095153	0.945	0.0131046	0.0060223	0.2746659	0.947	0.0063667	0.0025124	0.1799279	0.949
			λ	0.0059441	0.0006063	0.0843087	0.945	0.0105502	0.0017741	0.1310053	0.947	0.0043435	0.0005450	0.0674629	0.949
			Δ	0.0055539	0.0006954	0.0931233	0.945	0.0093248	0.0016828	0.1334303	0.947	0.0041827	0.0005814	0.0771835	0.949
		III	ρ	0.0079313	0.0030251	0.2053284	0.944	0.0111045	0.0048143	0.2485872	0.939	0.0064118	0.0025226	0.1808699	0.948
			λ	0.0056887	0.0005595	0.0818932	0.944	0.0085835	0.0012233	0.1112053	0.939	0.0043854	0.0005398	0.0679887	0.948
			Δ	0.0053357	0.0006527	0.0908633	0.944	0.0077320	0.0012374	0.1166626	0.939	0.0042177	0.0005804	0.0776895	0.948
	(20,10)	I	ρ	0.0027247	0.000859	0.1149772	0.951	0.0029247	0.0009988	0.1166813	0.945	0.0030117	0.0010065	0.1200596	0.942
			λ	0.0017041	0.0000969	0.0369412	0.951	0.0018071	0.0001251	0.0378388	0.945	0.0018786	0.0001246	0.0392068	0.942
			Δ	0.0017315	0.0001432	0.0457371	0.951	0.0018366	0.0001757	0.0466124	0.945	0.0019038	0.0001766	0.0481555	0.942
		II	ρ	0.0039793	0.0013400	0.1397905	0.948	0.0054716	0.0020278	0.1650620	0.948	0.0030345	0.0010107	0.1205079	0.941
			λ	0.0025686	0.0001772	0.0474050	0.948	0.0036613	0.0003309	0.0595455	0.948	0.0018938	0.0001248	0.0393854	0.941
			Δ	0.0025604	0.0002427	0.0570842	0.948	0.0035662	0.0004092	0.0694439	0.948	0.0019192	0.0001771	0.0483513	0.941
		III	ρ	0.0034443	0.0011387	0.1296862	0.947	0.0041116	0.0015030	0.1412575	0.940	0.0030488	0.0010151	0.1207990	0.943
			λ	0.0021937	0.0001415	0.0430086	0.947	0.0026542	0.0002145	0.0483154	0.940	0.0019035	0.0001255	0.0395058	0.943
			Δ	0.0022046	0.0001997	0.0523862	0.947	0.0026382	0.0002833	0.0578976	0.940	0.0019287	0.0001779	0.0484832	0.943
	(30,10)	I	ρ	0.0034625	0.0011171	0.1290878	0.952	0.0035330	0.0011183	0.1302099	0.960	0.0036491	0.0011874	0.1328062	0.955
			λ	0.0021874	0.0001373	0.0427940	0.952	0.0022231	0.0001455	0.0431090	0.960	0.0023068	0.0001494	0.0443288	0.955
			Δ	0.0022027	0.0001949	0.0521607	0.952	0.0022414	0.0002013	0.0525420	0.960	0.0023200	0.0002098	0.0538257	0.955
		II	ρ	0.0057561	0.0020389	0.1698244	0.946	0.0091697	0.0035282	0.2225386	0.956	0.0036538	0.0011857	0.1328795	0.955
			λ	0.0038776	0.0003315	0.0619040	0.946	0.0067314	0.0007644	0.0920474	0.956	0.0023099	0.0001491	0.0443387	0.955
			Δ	0.0037657	0.0004109	0.0718598	0.946	0.0062372	0.0008297	0.1001930	0.956	0.0023234	0.0002094	0.0538427	0.955
		III	ρ	0.0053110	0.0018380	0.1623589	0.948	0.0076540	0.0028450	0.2005434	0.954	0.0036705	0.0011861	0.1331724	0.955
			λ	0.0035336	0.0002845	0.0580703	0.948	0.0054279	0.0005503	0.0785065	0.954	0.0023216	0.0001491	0.0444427	0.955
			Δ	0.0034563	0.0003610	0.0680675	0.948	0.0051285	0.0006295	0.0878331	0.954	0.0023349	0.0002094	0.0539614	0.955
	(30,20)	I	ρ	0.0017891	0.0005424	0.0877089	0.945	0.0017255	0.0004811	0.0887829	0.955	0.0017623	0.0005162	0.0898892	0.949
			λ	0.0010244	0.0000549	0.0267887	0.945	0.0010205	0.0000478	0.0272002	0.955	0.0010433	0.0000508	0.0275441	0.949
			Δ	0.0010708	0.0000854	0.0340056	0.945	0.0010590	0.0000751	0.0345083	0.955	0.0010829	0.0000801	0.0349269	0.949
		II	ρ	0.0027930	0.0008605	0.1127807	0.951	0.0035817	0.0012324	0.1318379	0.945	0.0017852	0.0005201	0.0904284	0.952
			λ	0.0016880	0.0000970	0.0360656	0.951	0.0022949	0.0001614	0.0442643	0.945	0.0010574	0.0000515	0.0277290	0.952
			Δ	0.0017317	0.0001432	0.0447414	0.951	0.0022980	0.0002225	0.0536219	0.945	0.0010977	0.0000809	0.0351495	0.952
		III	ρ	0.0023316	0.0007117	0.1019545	0.948	0.0025873	0.0008291	0.1109394	0.951	0.0017928	0.0005225	0.0906322	0.951
			λ	0.0013793	0.0000762	0.0319410	0.948	0.0016046	0.0000945	0.0355510	0.951	0.0010623	0.0000518	0.0278004	0.951
			Δ	0.0014268	0.0001153	0.0400356	0.948	0.0016357	0.0001391	0.0440948	0.951	0.0011027	0.0000813	0.0352344	0.951

Table 11. Bootstrap results: absolute value of bias ($|Bias|$), length L, mean squared error (MSE) & coverage probability (Cov), when $R = 0.05, \rho = 0.42, \lambda = 0.17$ and $\Delta = 0.18$

3*	(n,m)	Scheme		$T_1 = X_{[\frac{m}{4}]}$				$T_2 = X_{\left[\frac{4\times m}{5}\right]}$				$T_3 = X_{[m]}+2$			
				$|Bias|$	MSE	L	Cov	$|Bias|$	MSE	L	Cov	$|Bias|$	MSE	L	Cov
	(20,6)	I	ρ	0.0094978	0.0129907	0.4235773	0.937	0.0097378	0.0136084	0.4323874	0.938	0.0104861	0.0132736	0.4396057	0.950
			λ	0.0194719	0.0138432	0.4140023	0.937	0.0202544	0.0146998	0.4247872	0.938	0.0213622	0.0147013	0.4287571	0.950
			Δ	0.0129532	0.0083320	0.3305778	0.937	0.0135303	0.0088960	0.3395280	0.938	0.0142336	0.0090488	0.3437481	0.950
		II	ρ	0.0126486	0.0171257	0.4917633	0.945	0.0191223	0.0247619	0.5792681	0.947	0.0104968	0.0132940	0.4400079	0.949
			λ	0.0259875	0.0194883	0.4928500	0.945	0.0378904	0.0321148	0.6001988	0.947	0.0214023	0.0147710	0.4293260	0.949
			Δ	0.0178690	0.0119328	0.3985085	0.945	0.0270900	0.0203964	0.5010937	0.947	0.0142657	0.0090909	0.3441934	0.949
		III	ρ	0.0122549	0.0166390	0.4843928	0.944	0.0165280	0.0218074	0.5471864	0.939	0.0105486	0.0133754	0.4417017	0.948
			λ	0.0252023	0.0187734	0.4843507	0.944	0.0333555	0.0275673	0.5609553	0.939	0.0215318	0.0149129	0.4315642	0.948
			Δ	0.0172891	0.0114536	0.3910387	0.944	0.0230782	0.0172850	0.4621257	0.939	0.0143736	0.0091768	0.3459848	0.948
	(20,12)	I	ρ	0.0050470	0.0058923	0.3024659	0.957	0.0054515	0.0065995	0.3056943	0.945	0.0056654	0.0066636	0.3136868	0.942
			λ	0.0101427	0.0053418	0.2787546	0.957	0.0106353	0.0062976	0.2824768	0.945	0.0110002	0.0063983	0.2914379	0.942
			Δ	0.0063694	0.0032874	0.2224686	0.957	0.0067738	0.0038402	0.2256195	0.945	0.0069784	0.0038800	0.2323326	0.942
		II	ρ	0.0069741	0.0086450	0.3589377	0.948	0.0091878	0.0119409	0.4112578	0.948	0.0057017	0.0066987	0.3147307	0.941
			λ	0.0142612	0.0085010	0.3387726	0.948	0.0187339	0.0128143	0.3966464	0.948	0.0110855	0.0064175	0.2924645	0.941
			Δ	0.0091621	0.0051477	0.2698208	0.948	0.0123369	0.0077661	0.3171392	0.948	0.0070336	0.0038944	0.2331630	0.941
		III	ρ	0.0061811	0.0075306	0.3364148	0.947	0.0072120	0.0093945	0.3612012	0.940	0.0057249	0.0067243	0.3154023	0.943
			λ	0.0125412	0.0071878	0.3144940	0.947	0.0145194	0.0095797	0.3412721	0.940	0.0111350	0.0064462	0.2931757	0.943
			Δ	0.0079828	0.0043711	0.2505796	0.947	0.0094188	0.0057851	0.2723580	0.940	0.0070667	0.0039113	0.2337210	0.943
	(30,10)	I	ρ	0.0062764	0.0074177	0.3350111	0.952	0.0064567	0.0072630	0.3380036	0.960	0.0066154	0.0078107	0.3435340	0.955
			λ	0.0125689	0.0070565	0.3136631	0.952	0.0127941	0.0071344	0.3153762	0.960	0.0131511	0.0075272	0.3222788	0.955
			Δ	0.0080038	0.0042902	0.2496368	0.952	0.0081689	0.0043136	0.2515303	0.960	0.0084151	0.0045634	0.2566152	0.955
		II	ρ	0.0095358	0.0120137	0.4209876	0.946	0.0142513	0.0177257	0.5140453	0.956	0.0066236	0.0078020	0.3437599	0.955
			λ	0.0196853	0.0128972	0.4086553	0.946	0.0288333	0.0212418	0.5156304	0.956	0.0131709	0.0075149	0.3223657	0.955
			Δ	0.0129611	0.0078073	0.3263921	0.946	0.0197366	0.0131431	0.4201960	0.956	0.0084293	0.0045565	0.2567261	0.955
		III	ρ	0.0089342	0.0110818	0.4063776	0.948	0.0121448	0.0152767	0.4782279	0.954	0.0066512	0.0078042	0.3444821	0.955
			λ	0.0183643	0.0116627	0.3919223	0.948	0.0249387	0.0176339	0.4730264	0.954	0.0132333	0.0075152	0.3229916	0.955
			Δ	0.0120308	0.0070491	0.3125371	0.948	0.0168537	0.0107832	0.3819409	0.954	0.0084704	0.0045572	0.2572593	0.955
	(30,20)	I	ρ	0.0036759	0.0038644	0.2354817	0.945	0.0035030	0.0034464	0.2382119	0.955	0.0035674	0.0037094	0.2411036	0.949
			λ	0.0065141	0.0032788	0.2108528	0.945	0.0064497	0.0029083	0.2142501	0.955	0.0065897	0.0030976	0.2166533	0.949
			Δ	0.0041307	0.0020634	0.1696722	0.945	0.0040288	0.0018320	0.1720391	0.955	0.0041215	0.0019586	0.1740503	0.949
		II	ρ	0.0053210	0.0059082	0.2972158	0.951	0.0063791	0.0079782	0.3402727	0.945	0.0036095	0.0037318	0.2424851	0.952
			λ	0.0101865	0.0053265	0.2729093	0.951	0.0129548	0.0078616	0.3205403	0.945	0.0066767	0.0031270	0.2179933	0.952
			Δ	0.0064683	0.0032858	0.2180943	0.951	0.0082746	0.0047494	0.2550044	0.945	0.0041767	0.0019747	0.1751001	0.952
		III	ρ	0.0045583	0.0049761	0.2709806	0.948	0.0048714	0.0056654	0.2922562	0.951	0.0036235	0.0037476	0.2430010	0.951
			λ	0.0085267	0.0043581	0.2461344	0.948	0.0096118	0.0051763	0.2692787	0.951	0.0067057	0.0031430	0.2184961	0.951
			Δ	0.0054068	0.0027121	0.1972199	0.948	0.0060411	0.0031786	0.2148612	0.951	0.0041951	0.0019842	0.1754943	0.951

Table 12. Bootstrap results: absolute value of bias ($|Bias|$), length L, mean squared error (MSE) & coverage probability (Cov), when $R = 0.2, \rho = 0.7, \lambda = 0.5$ and $\Delta = 0.42$

3*	(n,m)	Scheme		$T_1 = X_{[\frac{m}{4}]}$				$T_2 = X_{[\frac{4\times m}{5}]}$				$T_3 = X_{[m]}+2$			
				$\|Bias\|$	MSE	L	Cov	$\|Bias\|$	MSE	L	Cov	$\|Bias\|$	MSE	L	Cov
	(20,6)	I	ρ	0.0117477	0.0153060	0.4520037	0.938	0.0124744	0.0159014	0.4582487	0.938	0.0123077	0.0151538	0.4711041	0.953
			λ	0.0205008	0.0305288	0.6375286	0.938	0.0214902	0.0309625	0.6459255	0.938	0.0216617	0.0292849	0.6600009	0.953
			Δ	0.0164230	0.0221759	0.5580971	0.938	0.0170966	0.0225292	0.5678643	0.938	0.0174720	0.0210157	0.5780715	0.953
		II	ρ	0.0173613	0.0189994	0.5056227	0.957	0.02832960	0.0226172	0.5720952	0.958	0.0123457	0.0151013	0.4713209	0.952
			λ	0.0304653	0.0351571	0.6961822	0.957	0.0492816	0.0381734	0.7637851	0.958	0.0216993	0.0292323	0.6604776	0.952
			Δ	0.0238102	0.0257647	0.6240167	0.957	0.0393319	0.0282494	0.6979926	0.958	0.0174671	0.0210003	0.5786043	0.952
		III	ρ	0.0167805	0.0186198	0.4996590	0.951	0.0236159	0.0205869	0.5471290	0.974	0.0124580	0.0151167	0.4721985	0.951
			λ	0.0294620	0.0347652	0.6898956	0.951	0.0416742	0.0364771	0.7410927	0.974	0.0218503	0.0292777	0.6617918	0.951
			Δ	0.0229306	0.0254761	0.6168704	0.951	0.0330614	0.0271488	0.6735641	0.974	0.0175051	0.0210673	0.5800995	0.951
	(20,12)	I	ρ	0.0061780	0.0093439	0.3784086	0.957	0.0064308	0.0091113	0.3839179	0.957	0.0067700	0.0102060	0.3896721	0.955
			λ	0.0109088	0.0204537	0.5486037	0.957	0.0115208	0.0195308	0.5550169	0.957	0.0121018	0.0214703	0.5625002	0.955
			Δ	0.0093439	0.0137712	0.4566329	0.957	0.0098415	0.0133135	0.4608642	0.957	0.0101267	0.0146796	0.4707832	0.955
		II	ρ	0.0077081	0.0111095	0.4034235	0.948	0.0110582	0.0140089	0.4477545	0.948	0.005830	0.0087364	0.3624388	0.941
			λ	0.0135373	0.0230542	0.5792616	0.948	0.0195532	0.0280009	0.6321406	0.948	0.010222	0.0188477	0.5275483	0.941
			Δ	0.0109682	0.0159236	0.4893388	0.948	0.0153288	0.0200989	0.5478277	0.948	0.008782	0.0126957	0.4342903	0.941
		III	ρ	0.0066184	0.0099493	0.3835531	0.947	0.0080926	0.0115844	0.4054907	0.940	0.0058606	0.0087640	0.3630609	0.943
			λ	0.0115890	0.0209350	0.5541380	0.947	0.0146447	0.0240177	0.5805475	0.940	0.0102687	0.0189043	0.5283669	0.943
			Δ	0.0094657	0.0141743	0.4620990	0.947	0.0116148	0.0168391	0.4907971	0.940	0.0088164	0.0127407	0.4351819	0.943
	(30,10)	I	ρ	0.0063220	0.0098149	0.3813168	0.952	0.0065258	0.0092576	0.3864415	0.960	0.0067700	0.0102060	0.3896721	0.955
			λ	0.0111349	0.0207490	0.5524642	0.952	0.0137229	0.0198060	0.5582560	0.960	0.0121018	0.0214703	0.5625002	0.955
			Δ	0.0095183	0.0140212	0.4607232	0.952	0.0099746	0.0135339	0.4643317	0.960	0.0101267	0.0146796	0.4707832	0.955
		II	ρ	0.0111383	0.0141172	0.4539873	0.946	0.0193504	0.0183464	0.5289440	0.975	0.0067787	0.0102024	0.3901201	0.955
			λ	0.0194449	0.0280094	0.6410094	0.946	0.0337041	0.0333715	0.7223986	0.975	0.0121252	0.0214542	0.5629440	0.955
			Δ	0.0157761	0.0201779	0.5586898	0.946	0.0271128	0.0245698	0.6482635	0.975	0.0101429	0.0146647	0.4710832	0.955
		III	ρ	0.0101599	0.0133057	0.4425067	0.948	0.0159464	0.0165252	0.5018451	0.962	0.0068184	0.0102117	0.3909943	0.955
			λ	0.0177450	0.0266800	0.6278937	0.948	0.0278609	0.0312857	0.6940647	0.962	0.0121964	0.0214546	0.5639524	0.955
			Δ	0.0145597	0.0190826	0.5437723	0.948	0.0225508	0.0229354	0.6162352	0.962	0.0101831	0.0146654	0.4719769	0.955
	(30,20)	I	ρ	0.0037043	0.0056780	0.2852449	0.945	0.0036912	0.0050746	0.2872892	0.955	0.0035733	0.0055372	0.2909235	0.949
			λ	0.0065500	0.0123430	0.4193755	0.945	0.0065839	0.0111278	0.4246590	0.955	0.0069573	0.0119694	0.4288250	0.949
			Δ	0.0054071	0.0075049	0.3276875	0.945	0.0051606	0.0067049	0.3318073	0.955	0.0052665	0.0072067	0.3357459	0.949
		II	ρ	0.0053870	0.0082616	0.3477700	0.951	0.0067238	0.0101398	0.3842908	0.945	0.0037242	0.0055508	0.2924221	0.952
			λ	0.0111466	0.0175064	0.5065918	0.951	0.0137173	0.0215014	0.5568929	0.945	0.0069851	0.0120183	0.4310227	0.952
			Δ	0.0079416	0.0113462	0.4116783	0.951	0.0100717	0.0148207	0.4668933	0.945	0.0053283	0.0072512	0.3376696	0.952
		III	ρ	0.0046563	0.0071328	0.3219558	0.948	0.0049234	0.0077694	0.3408670	0.951	0.0037335	0.0055702	0.2929728	0.951
			λ	0.0096027	0.0152661	0.4711108	0.948	0.0097987	0.0167407	0.4991944	0.951	0.0069020	0.0120601	0.4318050	0.951
			Δ	0.0067645	0.0096235	0.3764679	0.948	0.0072995	0.0108885	0.4052772	0.951	0.0053489	0.0072820	0.3383823	0.951

Table 13. Bootstrap results: absolute value of bias ($|Bias|$), length L, mean squared error (MSE) & coverage probability (Cov), when $R = 0.8, \rho = 0.98, \lambda = 0.96$ and $\Delta = 0.85$

3* (n,m)	Scheme		$T_1 = X_{[\frac{m}{4}]}$				$T_2 = X_{[\frac{4\times m}{5}]}$				$T_3 = X_{[m]}+2$			
			$\|Bias\|$	MSE	L	Cov	$\|Bias\|$	MSE	L	Cov	$\|Bias\|$	MSE	L	Cov
(20,6)	I	ρ	0.0362452	0.0102964	0.3063098	0.957	0.0374006	0.0106006	0.3086740	0.962	0.0396492	0.0106478	0.3185233	0.966
		λ	0.0617701	0.0303975	0.5086038	0.957	0.0639635	0.0308772	0.5117311	0.962	0.0674422	0.0210869	0.5255692	0.966
		Δ	0.0677677	0.0396635	0.5626429	0.957	0.0702554	0.0398479	0.5653044	0.962	0.0749248	0.0402544	0.5759449	0.966
	II	ρ	0.0467406	0.0160881	0.3833546	0.957	0.0602978	0.0251238	0.4682431	0.960	0.0396720	0.0106420	0.3188646	0.964
		λ	0.0768004	0.0252446	0.6083686	0.957	0.0967046	0.0666216	0.7044750	0.960	0.0674298	0.0210926	0.5260352	0.964
		Δ	0.0770783	0.0550396	0.6343539	0.957	0.0935555	0.0755426	0.6644194	0.960	0.0747381	0.0402872	0.5762856	0.964
	III	ρ	0.0465337	0.0160957	0.3821003	0.955	0.0428451	0.0170800	0.4347689	0.974	0.0336891	0.0084849	0.2806474	0.958
		λ	0.0766281	0.0253317	0.6066268	0.955	0.068911	0.0356669	0.6618134	0.974	0.0581917	0.0254326	0.4735348	0.958
		Δ	0.0763721	0.0551338	0.6331692	0.955	0.0663987	0.0281874	0.6648531	0.974	0.0650003	0.02543260	0.5382088	0.958
(20,12)	I	ρ	0.0203397	0.0029166	0.1751036	0.971	0.0208215	0.0027522	0.1790417	0.978	0.0215274	0.0032947	0.1864027	0.975
		λ	0.0367598	0.0096950	0.3149225	0.971	0.0375917	0.0092349	0.3218977	0.978	0.0386286	0.0108660	0.3330923	0.975
		Δ	0.0473882	0.0160446	0.4210892	0.971	0.0466914	0.0156428	0.4275851	0.978	0.0485550	0.0176450	0.4353935	0.975
	II	ρ	0.0283422	0.0056509	0.2363034	0.966	0.0365777	0.0088207	0.2973536	0.967	0.0216245	0.0033079	0.1873600	0.974
		λ	0.0498818	0.0177545	0.4095758	0.966	0.0626858	0.0266581	0.4975537	0.967	0.0387847	0.0109123	0.3346475	0.974
		Δ	0.0583501	0.0258102	0.4925148	0.966	0.0684537	0.0360490	0.5165736	0.967	0.0486358	0.0177315	0.4316039	0.974
	III	ρ	0.0256520	0.0046325	0.2153062	0.968	0.0245736	0.0076393	0.3120757	0.985	0.0217263	0.0033313	0.1881909	0.979
		λ	0.0455859	0.0147933	0.3778610	0.968	0.0397132	0.0199333	0.5125891	0.985	0.0389510	0.0109848	0.3359925	0.979
		Δ	0.0548202	0.0222757	0.4690797	0.968	0.0361879	0.0176676	0.4504559	0.985	0.0486859	0.0178301	0.4276171	0.979
(30,10)	I	ρ	0.0206886	0.0030118	0.1777876	0.971	0.0211355	0.0028474	0.1815704	0.978	0.0215274	0.0032947	0.1864027	0.975
		λ	0.0373451	0.0099889	0.3192320	0.971	0.0381079	0.0095286	0.3259365	0.978	0.0386286	0.0108660	0.3330923	0.975
		Δ	0.0479382	0.0164325	0.4243346	0.971	0.0471415	0.0160237	0.4306349	0.978	0.0485550	0.0176450	0.4353935	0.975
	II	ρ	0.0313163	0.0069072	0.2591363	0.965	0.0448860	0.0184974	0.3585237	0.975	0.0215706	0.0032950	0.1868645	0.977
		λ	0.0545513	0.0213105	0.4430167	0.965	0.0750240	0.0388637	0.5778535	0.975	0.0386985	0.0108710	0.3338478	0.977
		Δ	0.0622430	0.0298801	0.5166619	0.965	0.0782035	0.0487759	0.6136568	0.975	0.0485744	0.0176700	0.4359764	0.977
	III	ρ	0.0294368	0.0060963	0.2441454	0.965	0.0330044	0.0115788	0.3721305	0.975	0.0216841	0.0033105	0.1876902	0.977
		λ	0.0516392	0.0190040	0.4211409	0.965	0.0530268	0.0282407	0.589123	0.975	0.0388869	0.0109200	0.3351916	0.977
		Δ	0.0598637	0.0272054	0.5009776	0.965	0.0496806	0.0287254	0.609433	0.975	0.0486067	0.0177410	0.437122	0.977
(30,20)	I	ρ	0.0108596	0.0010398	0.1034992	0.973	0.0112099	0.0008851	0.1001572	0.981	0.0113659	0.0010517	0.1043244	0.976
		λ	0.0202895	0.0036895	0.1940241	0.973	0.0210902	0.0031436	0.1883089	0.981	0.0212949	0.0036870	0.1955236	0.976
		Δ	0.0311329	0.0076617	0.3178154	0.973	0.0343604	0.0075596	0.3145213	0.981	0.0332594	0.0073536	0.3206024	0.976
	II	ρ	0.0166170	0.0021866	0.1484602	0.974	0.0213490	0.0032993	0.1817661	0.969	0.0114955	0.0010519	0.1049895	0.977
		λ	0.0303940	0.0073928	0.2710037	0.974	0.0385094	0.0108234	0.3253150	0.969	0.0215325	0.0036904	0.1967327	0.977
		Δ	0.0413493	0.0128827	0.3854701	0.974	0.0484320	0.0173575	0.4287055	0.969	0.0334982	0.0071816	0.3218615	0.977
	III	ρ	0.0140299	0.0016392	0.1283727	0.973	0.0130339	0.0036929	0.2122456	0.985	0.0115421	0.0010583	0.1052925	0.978
		λ	0.0259133	0.0056444	0.2371304	0.973	0.0213336	0.0107525	0.3694064	0.985	0.0216161	0.0037126	0.1972620	0.978
		Δ	0.0371079	0.0104482	0.3568513	0.973	0.0166728	0.0109394	0.4040187	0.985	0.0336998	0.0071208	0.3223643	0.978

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