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Optimal Excess-of-Loss Reinsurance Contract in a Dynamic Risk Model

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Abstract This paper studies the optimal excess-of-loss reinsurance contract between an insurer and a reinsurer in a dynamic risk model. The risk process is assumed to be a diffusion approximation process of the classical Cramer-Lundberg model which is perturbed by a Brownian motion. In addition to reinsurance, we assume that the insurer is allowed to invest his/her surplus into a financial market containing one risk-free rate of return and determines the reinsurance strategy by a self-reinsurance function. Our aim is to obtain the simultaneous equilibrium strategy in this reinsurance dynamic risk setting using the objective functions of insurer and reinsurance. By employing the dynamic programming approach, we derive the minimization of insurer's ruin probability and maximization of reinsurance's expected aggregate discounted net profits to have the optimal portfolio for the two parties treaties in a fixed term insurance contract. In order to provide a more explicit reinsurance contract and to facilitate our quantitative analysis, we study the cases when the reinsurance premium function is based on the standard-deviation principle and expected value principle from the integro-differential equations. Numerical examples are given to investigate the effects of model parameters on the equilibrium strategy.

Keywords Brownian motion, Compound Poisson risk process, Diffusion approximation process, Dynamic programming, Excess-of-loss reinsurance

AMS 2010 subject classifications Primary 91B30; Secondary 62P05

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1. Introduction

The most recent step in the evolution of an insurance portfolio has been a shift from an asset allocation-centered process to a more comprehensive risk allocation-based process. Development of risk models and increasing the reserve process is a current and important challenge for researchers. Nowadays, reinsurance is an important tool for financial risks management. Reinsurance contracts usually run for long time periods (at least for longer than the typical maturity of financial contracts) and are exposed to high frictional costs. This subject is an effective way to spread risk in the insurance business and as a result, reinsurance negotiations are costly, lengthy, and can be thought of as irreversible. Any insurance company, not withstanding its size, has to use reinsurance (risk transfer to other insurer) for stable performance.

Reinsurance is usually used to transfer and control risk because it allows insurance companies to provide more secure coverage with higher limits. Over the past two decades, the studies devoted to the search of an optimal reinsurance strategy have registered considerable advancements and relevance in the actuarial literature. The optimal problem under the Cramér-Lundberg model was first solved by [1] who showed that a band-type dividend strategy is optimal. [2] suggested a reinsurance contract under which whenever the surplus is negative, the reinsurer makes the required payment to bring the surplus back to zero. A great attention has been given to the classical proportional reinsurance and excess-of-loss reinsurance which are two popular types of reinsurance strategies (see [3]; [4]; [5] and references therein), which have been addressed under different optimization criteria. [6] considered an insurance company whose insurance business follows a diffusion perturbated classical risk process

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to find the controls that maximize expected utility of assets at a terminal time. [7] studied the reserve process of the insurance company is described by a stochastic differential equation driven by a Brownian motion and a Poisson random measure, representing the randomness from the financial market and the insurance claims, respectively. As a representative form of non-proportional reinsurance, many literatures proved that excess-of-loss reinsurance is the optimal reinsurance form under the expected value premium principle and various objective functions (see, [10] and [13], then this form of reinsurance has been received a great deal of attention of researchers.

The reinsurance contract with an excess-of-loss provision indicates that the reinsurer is responsible for losses over a certain limit point. The majority of previous studies on optimal excess-of-loss reinsurance problem assumed that the company takes reinsurance at initial time and aims to find the optimal limit point under different objectives and assumptions. The popular objective for the optimization problem of reinsurance is minimizing the ruin probability of insurance company. [8] studied a dynamic choice of excess-of-loss reinsurance retention level and the dividend distribution policy which maximizes the expected present value of the dividends in a diffusion model. [9] considered the problem of minimizing the probability of ruin by controlling the combinational quota-share and excess-of-loss reinsurance strategy. [10] investigated optimal risk control for the excess-of-loss reinsurance policy which minimizes the probability of ruin. [11] considered that the insurer purchases excess-of-loss reinsurance and invests its wealth in the constant elasticity of variance (CEV) stock market and studied the optimization problem of maximizing the exponential utility of terminal wealth under the controls of excess-of-loss reinsurance and investment. [12] considered the optimal combining quota-share and excess-of-loss reinsurance to maximize the expected exponential utility from terminal wealth and derived the closed form expressions of the optimal strategies and value function not only for the diffusion approximation risk model but also for the jump-diffusion risk model. [13] investigated the optimal excess-of-loss reinsurance with the surplus followed a Markov jump process with state-dependent income.

[14] and [15] employed a principal-agent model to study the optimal reinsurance premium from the viewpoint of the reinsurer, where proportional reinsurance and excess-of-loss reinsurance are discussed. [16] employed a new continuous time framework to analyze optimal reinsurance, in which an insurer and a reinsurer are two players in a stochastic Stackelberg differential game. [17] investigated a jump-diffusion risk process which the insurer is allowed to purchase excess-of-loss reinsurance and derived the closed-form expressions for the optimal strategy and the optimal value function. [18] studied the optimal excess-of-loss reinsurance problem when both the intensity of the claims arrival process and the claim size distribution are influenced by an exogenous stochastic factor. [19] discussed an optimal excess-of-loss reinsurance contract in a continuous time principal-agent framework where the surplus of the insurer (agent/he) is described by a classical Cramér-Lundberg (C-L) model.

[20] considered the optimal excess-of-loss reinsurance for an insurance company facing a constant fixed cost when reinsurance contract is signed. A reinsurance strategy which combines a proportional and an excess-of-loss reinsurance in a risk model is studied by [21]. Other equilibrium concepts are possible; for example, [22] investigated the Bowley equilibrium with risk sharing and optimal reinsurance formulations and focused on the common traits of Bowley optimality and Pareto efficiency under fairly general preferences. [23] provided estimators for the optimal excess of loss and stop-loss contracts and investigated their statistical properties under many premium principle assumptions and simulated data and real-life data are used to illustrate the main theoretical findings.

Although research on the optimal risk management strategy problem has been rapidly increasing in recent years, none of these contributions deals with finding the optimization problem with consideration of the insurer's strategy together with reinsurer's strategy simultaneously. We tend to believe that the insurer is primarily concerned about risk mitigation in entering a reinsurance contract, whereas the reinsurer is mainly concerned about profitability. In our formulation, the major contribution is considering optimization problem based on the objective functions of insurer and reinsurer simultaneously. We obtain the optimal risk management strategy in a dynamic risk model under the excess-of-loss reinsurance contract to minimize the insurer's ruin probability and maximize the reinsurer's expected aggregate discounted net profits in the financial market to have the optimal portfolio for the two parties treaties in a fixed term insurance contract.

The rest of this paper is organized as follows. In Section 2, we present the well-known compound Poisson risk process and give reformulation of the surplus process in terms of risk-free rate of return and reinsurance arrangement. Furthermore, we present the simultaneous optimization problem based on the objective functions of insurer and reinsurer. Section 3 derives the optimal reinsurance strategy for the controlled operator functions based on the reinsurance premium function using the Hamilton-Jacobi-Bellman equations and performs some delicate analyses on the equilibrium strategies. Sections 4 deals with finding the explicit solutions of the optimization problem from the integro-differential equations when the reinsurance premium function is based on the standard-deviation principle. In Section 5, we give the explicit reinsurance contract for expected value principle. In Section 6, we present a numerical example and offer detailed interpretations of model parameters effects on reinsurance. Conclusions are provided in Section 7.

2. Risk model and preliminaries

In the sequel, we will always work on a probability space $(\Omega, \mathcal{F}, P)$, endowed with the information filtration $\{\mathcal{F}_t\}_{t\geq 0}$ which carries all stochastic quantities to be defined in the following. An insurance company, having an initial capital, cashes premiums continuously and pays claims of random sizes at random times. According to the compound Poisson model, the surplus process R_t of a homogeneous insurance portfolio can be described by

$$R_t = u + ct - \sum_{i=1}^{N_t} X_i, \tag{1}$$

with an initial deterministic surplus $R(0) = u$ is the initial surplus, the positive amount c corresponds to the premium income rate, $\{X_i\,; i = 1, 2, \dots\}$ are a class of successive claim amounts which are represented by non-negative independent identically distributed random variables, denoting claim amounts, with continuous density function $f_X(x)$, $S_t = \sum_{i=1}^{N_t} X_i$ is the total successive claim amount up to time t which is a compound Poisson risk process. $\{N(t), t \geq 0\}$ is a Poisson process, with parameter $\lambda > 0$, which counts the claim occurrences until time t. We assume that the successive claim amounts $\{X_i\,; i = 1, 2, \dots\}$ have finite first moment μ and second moment σ^2. Moreover, we assume that c fulfills the net-profit condition $c > \lambda E(X)$. The net-profit condition indicates that the insurer is in profit. It is widely adopted in the literature of optimal reinsurance, see for example, [24], [25] and [26].

Motivated by the references mentioned in Section 1, we will present the dynamic forms that describe the surplus risk process (1) and give the objective functions of insurer and reinsurer. Moreover, we derive the optimal risk management strategy in the dynamic risk model under the excess-of-loss reinsurance contract.

The structure of the insurance portfolio is planned as follows. We assume throughout that the insurer has the option to purchase excess-of-loss reinsurance and is in a position to determine the price of reinsurance. In fact, we assume that the insurer determines the reinsurance strategy by the self-reinsurance function $a_\alpha(t) : [0, \infty) \to [0, \infty)$, at time t, where $\alpha \geq 0$ is the self-retention level parameter that uniquely identifies the function a such that $0 \leq a_\alpha(X) \leq X$ and $a_0(X) \equiv 0$. Thus, for the ith claim with the random value X_i the insurer pays the amount $a_\alpha(X_i)$ and the rest, which is the random value $X_i - a_\alpha(X_i)$, is left to the reinsurer for payment. But in this contract, the reinsurer must also make a profit. Therefore, the insurer pays him a part of the premium until the end of contract. In this article, we show this part of premium as the reinsurance premium function and denoted by $\mathbb{W}_\theta(.) : [0, \infty) \to [0, \infty)$, where $\theta \in [\theta_1, \theta_2]$ is the safety loading of the reinsurer that specifies the reinsurance premium rate, θ_1 and θ_2 satisfying $0 < \theta_1 < \theta_2$. In addition, we assume that the reinsurance premium function $\mathbb{W}_\theta(X)$ is strictly increasing in θ and X with $\mathbb{W}_\theta(0) = 0$, and that the reinsurance contract is non-cheap, i.e. $\lambda\mathbb{W}_\theta(X) > c$ for $\theta \in [\theta_1, \theta_2]$.

Now, we adapt the compound Poisson risk process in terms of reinsurance contract up to the time when the insurer goes bankrupt. To spread risk in the portfolio, the insurer purchases reinsurance. In our diffusion approximation process, both the reinsurer and the insurer can invest their idle assets in the financial market. Meanwhile, the insurer can purchase the reinsurance contract from the reinsurer to diversify its claim risk. Considering that both the insurer and reinsurer belong to the same large insurance company, they have a common interest goal, which is to achieve the maximum expectation of the weighted sum of their wealth processes and minimum corresponding variance. Therefore, the interests of both the insurer and the reinsurer should be considered when formulating reinsurance strategy. We assume that the insurer is allowed to invest his/her surplus into a financial market containing one risk-free rate of return $r \geq 0$. Therefore, the risk process can be written as the following dynamic risk process:

$$R_t = u + \int_0^t \big(rR_s + c - \lambda \mathbb{W}_\theta(X - a_\alpha(X))\big)ds - \sum_{i=1}^{N_t} a_\alpha(X_i), \qquad t \geq 0.$$

According to [27] this process can be approximated by the following diffusion process:

$$\begin{aligned} R_t &= u + \int_0^t \big(rR_s + c - \lambda E\big(a_\alpha(X)\big) - \lambda \mathbb{W}_\theta(X - a_\alpha(X))\big)ds \\ &\quad + \int_0^t \sqrt{\lambda E\big(a_\alpha^2(X)\big)} dB_s, \end{aligned} \tag{2}$$

where $\{B_t, t \geq 0\}$ is a standard Brownian motion.

2.1. *Simultaneous optimization problem*

To present the objective functions of insurer and reinsurance, we assume that the insurer and the reinsurer adjust their strategies dynamically, and rewrite their strategies as $\alpha = \{\alpha_t\}$ and $\theta = \{\theta_t\}$, where α_t and θ_t denote the insurer's strategy and reinsurance pricing strategy at time t, respectively.
Definition 1. The strategy (α, θ) is said to be admissible if
(i) $\{\alpha_t\}$ and $\{\theta_t\}$ are $\{\mathcal{F}_t\}$-progressively measurable.
(ii) The dynamic risk process (1.1) has a strong solution.
Let us denote the sets of admissible insurer's strategy and reinsurance pricing strategy by Δ and Θ, respectively. Given the insurer's strategy and reinsurer's pricing strategy $\alpha \in \Delta$ and $\theta \in \Theta$, the insurer's objective is to minimize his/her ruin probability:

$$O_I(u; \alpha, \theta) = P\big(\tau_0 < \infty | R_0 = u\big), \tag{3}$$

where $\tau_0 = \inf\{t > 0 : R_t \leq 0\}$ is the time to ruin of the insurer. In the meanwhile, the reinsurance's objective is to maximize his/her expected aggregate discounted net profits up to the time of the insurer's bankruptcy:

$$O_R(u; \alpha, \theta) = E\Big(\int_0^{\tau_0} e^{-\zeta t} \lambda g(\alpha_t, \theta_t) dt | R_0 = u\Big), \tag{4}$$

where $\lambda g(\alpha_t, \theta_t)$ denotes the reinsurer's net profit rate at time t with $g(\alpha, \theta) = \mathbb{W}_\theta\big(X - a_\alpha(X)\big) - E\big(X - a_\alpha(X)\big)$, and $\zeta > r$ is the discount rate of the reinsurer.
In our risk optimization problem, problems (3) and (4) lead to a stochastic differential problem between the insurer and the reinsurer, as formulated below.

$$\begin{aligned} O_I(u; \alpha^*, \theta^*) &\leq O_I(u; \alpha, \theta^*), \qquad for\ all \qquad \alpha \in \Delta, \\ O_R(u; \alpha^*, \theta^*) &\leq O_R(u; \alpha^*, \theta), \qquad for\ all \qquad \theta \in \Theta, \end{aligned} \tag{5}$$

where α^* and θ^* are the optimal strategies which lead to have an equilibrium portfolio. If two estimators α^* and θ^* exist to optimize the risk problem, we define $O_I(u) = O_I(u, \alpha^*, \theta^*)$, and $O_R(u) = O_R(u, \alpha^*, \theta^*)$ as the insurer and the reinsurer's equilibrium value functions, respectively.
In section 3, we will derive the optimal reinsurance strategies α^* and θ^* for the controlled operator functions based on the reinsurance premium function using the Hamilton-Jacobi-Bellman equations.

2.2. *Some conditions to have the equilibrium strategies*

For notational convenience, we denote

$$\begin{aligned} u_1 &= \frac{\lambda \mathbb{W}_{\theta_1}(X) - c}{r}, \qquad u_2 = \frac{\lambda \mathbb{W}_{\theta_2}(X) - c}{r}, \\ \nu_i &= \frac{\lambda g(0, \theta_i)}{\zeta}, \qquad i = 1, 2. \end{aligned} \tag{6}$$

Due to our assumptions on $\mathbb{W}$ and g, it is clear that $0 < u_1 < u_2$ and $0 < \nu_1 < \nu_2$. On the other hand, inspired by [28] we have the following Lemma.

Lemma 1
According to the notations defined for the insurer and the reinsurer's equilibrium value functions, we have $O_I(u) = 0$, $O_R(u) = u_2$, and $(\alpha^*, \theta^*) = (0, \theta_2)$ for $u \geq u_2$.

Proof
We prove the lemma in two steps.
(i) Let $(\alpha_t, \theta_t) = (0, \theta_2)$ for any $t \geq 0$. Then $a_\alpha(X) \equiv 0$ and the diffusion approximation process (2) becomes

$$\begin{aligned} dR_t &= \big(rR_t - \big(\lambda \mathbb{W}_{\theta_2}(X) - c\big)\big)dt \\ &= r\big(R_t - u_2\big)dt, \qquad\qquad R_0 = u. \end{aligned} \tag{7}$$

That is, $R_t = (u - u_2)e^{rt} + u_2$. When $u \geq u_2$, we have $R_t \geq 0$, and hence $\tau_0 = \infty$ and $O_I(u; 0, \theta_2) = 0 \leq O_I(u; \alpha, \theta_2)$ for $\alpha \in \Delta$.
(ii) On the other hand, since the function $g(\alpha, \theta)$ is increasing in θ, when $u \geq u_2$, for any $\theta \in \Theta$, we have

$$\begin{aligned} O_R(u; 0, \theta_2) &= E\big(\int_0^\infty e^{-\zeta t} \lambda g(0, \theta_2) dt | R_0 = u\big) = \frac{\lambda g(0, \theta_2)}{\zeta} \\ &\geq E\big(\int_0^{\tau_0} e^{-\zeta t} \lambda g(0, \theta_t) dt | R_0 = u\big) = O_R(u; 0, \theta). \end{aligned}$$

Therefore, for $u \geq u_2$, from the optimization problem (5) we can see that $(0, \theta)$ is the equilibrium strategy, $O_I(u) = O_I(u, 0, \theta_2) = 0$, and $O_R(u) = O_R(u, 0, \theta_2) = u_2$, and this completes the proof. □

Lemma 1 shows that, if the insurer has sufficient financial reserve in his/her insurance portfolio, at least u_2, he/she can transfer all the risks to the reinsurance to avoid bankruptcy. In this case, since reinsurance is in high demand, the reinsurer will charge the maximum price for the contract to increase his profit. Therefore, u_2 is a critical value of the financial reserve of the insurer to have the optimal reinsurance contract. Thus, u_2 is the insurer safety level. In the sequel, we focus on the more interesting case when $0 < u < u_2$.

3. Stochastic differentiable equations

In this section, we consider the optimization problem (5) to solve it by employing the dynamic programming approach. To do it, for $h \in \mathbb{C}^2(0, u_2)$, define the operator functions $\mathbb{L}^{\alpha,\theta}$ and $\mathbb{A}^{\alpha,\theta}$ as

$$\begin{cases} \mathbb{L}^{\alpha,\theta} h(u) = h'(u)\big(ru + c - \lambda \mathbb{W}_\theta\big(X - a_\alpha(X)\big) - \lambda E[a_\alpha(X)]\big) + \frac{1}{2} E[a_\alpha^2(X)] h''(u), \\ \mathbb{A}^{\alpha,\theta} h(u) = \mathbb{L}^{\alpha,\theta} h(u) - \zeta h(u) + \lambda g(\alpha, \theta), \end{cases}$$

where h' and h'' are the first and second order derivatives of function h with respect to u, respectively. Following standard dynamic programming techniques, if $O_I(u; \alpha, \theta), O_R(u; \alpha, \theta) \in \mathbb{C}^2(0, u_2)$ and the strategies (α^*, θ^*) exist, the objective functions satisfy the following HJB equations:

$$\inf_{\alpha \in \Delta} \mathbb{L}^{\alpha,\theta^*} O_I(u; \alpha^*, \theta^*) = 0, \qquad O_I(0; \alpha^*, \theta^*) = 1, \quad O_I(u_2; \alpha^*, \theta^*) = 0, \tag{8}$$

and

$$\sup_{\theta \in \Delta} \mathbb{A}^{\alpha^*,\theta^*} O_R(u; \alpha^*, \theta^*) = 0, \qquad O_R(0; \alpha^*, \theta^*) = 0, \quad O_R(u_2; \alpha^*, \theta^*) = \nu_2. \tag{9}$$

In the following theorem the solutions to HJB equations (8) and (9) coincide with the value functions (O_I, O_R).

Theorem 3.1
If $(\alpha^*, \theta^*) \in \Delta \times \Theta$ and $O_I(u;\alpha^*,\theta^*), O_R(u;\alpha^*,\theta^*) \in \mathbb{C}^2(0,u_2)$ satisfy the HJB equations (8) and (9), then (α^*, θ^*) is a pair of simultaneous equilibrium strategy and given by

$$\alpha^*(u) = \arg\inf_{\alpha} \mathbb{L}^{\alpha,\theta^*} O_I(u;\alpha^*,\theta^*), \tag{10}$$

and

$$\begin{aligned}\theta^*(u) &= \arg \sup_{\theta\in[\theta_1,\theta_2]} \mathbb{A}^{\alpha^*,\theta} O_R(u;\alpha^*,\theta^*) \\ &= \arg \sup_{\theta\in[\theta_1,\theta_2]} \left\{\lambda\big(1 - O'_R(u;\alpha^*,\theta^*)\big)\mathbb{W}_\theta\big(X - a_\alpha(X)\big)\right\} \end{aligned} \tag{11}$$

Proof
The proof of this theorem is standard and convenient, and is therefore omitted for simplicity. □

Theorem 3.1 states that to obtain an optimal strategy in the insurance portfolio, we need to obtain the values (α^*, θ^*) such that the equations (8) and (9) hold. But solving these equations without having the reinsurance premium function is impossible. In Sections 4, we provide the explicit solution of the optimization problem from the integro-differential equations for special case of reinsurance premium function.
In the following, we perform some delicate analyses on the equilibrium strategies.
Remark 3.1. When the insurer purchases reinsurance, i.e. $a^*_\alpha(X) < X$, then $\mathbb{W}_\theta\big(X - a^*_\alpha(X)\big) > 0$ and the equilibrium strategy for θ is given by

$$\theta^*(u) = \begin{cases} \theta_2, & if \quad O'_R(u;\alpha^*,\theta^*) \le 1, \\ \theta_1, & if \quad O'_R(u;\alpha^*,\theta^*) > 1. \end{cases} \tag{12}$$

It means that when the insurer purchases no reinsurance, i.e. $a^*_\alpha(X) = X$, then $\theta^*(u) = \theta_2$.
Remark 3.2. From Lemma 1, we have $\theta^*(u_2) = \theta_2$ and from equation (12) we have $O'_R(u_2;\alpha^*,\theta^*) \le 1$. Let $u_R = \sup\left\{u \in [0,u_2] : O'_R(u;\alpha^*,\theta^*) = 1\right\}$, with the condition $u_R = 0$ if $O'_R(u;\alpha^*,\theta^*) < 1$, $\forall u \in [0,u_2]$. If $O'_R(u;\alpha^*,\theta^*) > 1$ for $u \in [0,u_2]$, then from equation (12), we have

$$\theta^*(u) = \begin{cases} \theta_1, & if \quad u \in [0,u_R), \\ \theta_2, & if \quad u \in [u_R,u_2]. \end{cases} \tag{13}$$

In this case, u_2 is called the switching-over point at which the price of the reinsurance contract is adjusted from a minimal (maximal) level to a maximal (minimal) level. In addition, the equilibrium strategy for α is given by

$$\alpha^*(u) = \begin{cases} \alpha^*_1(u), & if \quad u \in [0,u_R), \\ \alpha^*_2(u), & if \quad u \in [u_R,u_2), \end{cases}$$

where $\alpha^*_i(u) = \arg\inf_\alpha \mathbb{L}^{\alpha,\theta_i} O_I(u;\alpha,\theta_i)$, $i = 1,2$, is the insurer's optimal strategy.
Equation (13) indicates that the reinsurer spends more (less) for the reinsurance contract when the insurer has a sufficient (an insufficient) financial reserve. It is obvious that when the insurer has a sufficient financial reserve, he/she can provide more reinsurance support. The large demand for reinsurance leads to a high reinsurance price $\theta = \theta_2$. Otherwise, when the insurer has an insufficient financial reserve and is faced with high bankruptcy risk, he/she cannot afford to overpay for reinsurance.
In the following, we present two Corollaries to have some details on the estimations.
Corollary 3.1. For $u \ge u_1$, $\alpha^*_1(u) = 0$ and for $u \ge u_2$, $\alpha^*_2(u) = 0$, where u_1 and u_2 are defined in (6).

Proof
The proof is similar to part (i) of Lemma 1, and is thus omitted. □

Corollary 3.2. Assume that for $u \in [0, u_R)$, $O'_R(u; \alpha^*, \theta^*) > 1$. Then $u_R \leq u_1$, where u_1 is defined in (6).

Proof
We prove this Corollary by contradiction. Assume that $u_R > u_1$, since for $u \in [0, u_R)$, $O'_R(u; \alpha^*, \theta) > 1$, according to Remark 3.2, for $u \in [u_1, u_R)$ we have $\theta^*(u) = \theta_1$. On the other hand, according to Remark 3.1 the insurer's optimal reinsurance strategy is given by $\alpha_1^*(u)$ with $\alpha_1^*(u) = 0$ for $u \in [u_1, u_R)$.
For $u \in [u_1, u_R)$, similar to the part (i) of Lemma 1 for $t < \tau_R$ we have $R_t = (u - u_1)e^{rt} + u_1$, where $\tau_R = \inf\{t \geq 0 : X_t = u_R\}$ is the first time when the process X_t arrives u_R, i.e. $\tau_R = \frac{1}{r} ln \frac{u_R - u_1}{u - u_1}$. Thus

$$
\begin{aligned}
O_R(u; \alpha^*, \theta^*) &= E\Big(\int_0^{\tau_0} e^{-\zeta t} \lambda g(\alpha^*, \theta^*) dt | R_0 = u\Big) \\
&= E\Big(\int_0^{\tau_R} e^{-\zeta t} \lambda g(0, \theta_1) dt | R_0 = u\Big) \\
&\quad + E\Big(\int_{\tau_R}^{\tau_0} e^{-\zeta t} \lambda g(\alpha_2^*, \theta_2) dt | R_0 = u\Big).
\end{aligned}
$$

On the other hand, let $\tilde{u} = \frac{u_1 + u}{2} \in (u_1, u_2)$. For $\theta \in (\theta_1, \theta_2)$, since

$$
u_1 = \frac{\lambda \mathbb{W}_{\theta_1}(X) - c}{r} < \frac{\lambda \mathbb{W}_{\theta}(X) - c}{r} < u_2 = \frac{\lambda \mathbb{W}_{\theta_2}(X) - c}{r}, \tag{14}
$$

there exists $\tilde{\theta} \in (\theta_1, \theta_2)$ such that $\frac{\lambda \mathbb{W}_{\tilde{\theta}}(X) - c}{r} = \tilde{u}$. Consider the following strategy:

$$
(\tilde{\alpha}(u), \tilde{\theta}(u)) = \begin{cases} (\alpha_1^*(u), \theta_1), & if \quad u \in [0, u_1), \\ (0, \tilde{\theta}), & if \quad u \in [u_1, u_R), \\ (\alpha_2^*(u), \theta_2), & if \quad u \in [u_R, u_2). \end{cases}
$$

Then similar to the part (i) of Lemma 1, we have $\tilde{X}_t = (u - \tilde{u})e^{rt} + \tilde{u}$ for $t < \tilde{\tau}_R$, where where $\tilde{X}$ is a risk process and $\tilde{\tau}_R$ is the first time when the process $\tilde{X}$ arrives u_R for the first time, i.e. $\tilde{\tau}_R = \frac{1}{r} ln \frac{u_R - \tilde{u}}{u - \tilde{u}}$. It is clear that $\tilde{\tau}_R > \tau_R$. Since the two risk processes X and $\tilde{X}$ have the same path after arriving u_R, under the two strategies the insurer has the same ruin probability, (i.e. $O_I(u; \tilde{\alpha}, \tilde{\theta}) = O_I(u; \alpha^*, \theta^*)$) and

$$
E\Big(\int_{\tilde{\tau}_R}^{\tilde{\tau}_0} e^{-\zeta t} \lambda g(\alpha_2^*, \theta_2) dt | R_0 = u\Big) = E\Big(\int_{\tau_R}^{\tau_0} e^{-\zeta t} \lambda g(\alpha_2^*, \theta_2) dt | R_0 = u\Big),
$$

where $\tilde{\tau}_0$ is the time to ruin with strategy $(\tilde{\alpha}, \tilde{\theta})$. On the other hand, since $f(\alpha, \theta)$ is strictly increasing in θ and $\tilde{\tau}_R > \tau_R$, then we have

$$
\begin{aligned}
O_R(u; \tilde{\alpha}, \tilde{\theta}) &= E\Big(\int_0^{\tilde{\tau}_R} e^{-\zeta t} \lambda g(0, \theta_1) dt | R_0 = u\Big) \\
&\quad + E\Big(\int_{\tilde{\tau}_R}^{\tilde{\tau}_0} e^{-\zeta t} \lambda g(\alpha_2^*, \theta_2) dt | R_0 = u\Big) \\
&> E\Big(\int_0^{\tau_R} e^{-\zeta t} \lambda g(0, \theta_1) dt | R_0 = u\Big) \\
&\quad + E\Big(\int_{\tau_R}^{\tau_0} e^{-\zeta t} \lambda g(\alpha_2^*, \theta_2) dt | R_0 = u\Big) = O_R(u; \alpha^*, \theta^*),
\end{aligned}
$$

which is a contradiction to the optimality of (α^*, θ^*), and this completes the proof. □

Thus, if $O_R'(u;\alpha^*,\theta^*) > 1$ for $u \in [0,u_R)$, we have $0 \leq u_R \leq u_1$, In this case, the equilibrium strategies are given by

$$(\alpha^*(u),\theta^*(u)) = \begin{cases} (\alpha_1^*(u),\theta_1), & if \quad u \in [0,u_R), \\ (\alpha_2^*(u),\theta_2), & if \quad u \in [u_R,u_2), \end{cases} \tag{15}$$

and $(\alpha^*(u),\theta^*(u)) = (\alpha_2^*(u),\theta_2)$ if $u_R = 0$.
From (15), if $u_R > 0$, the HJB equations (8) and (9) can be rewritten as
P_1:

$$\mathbb{A}^{\alpha_1^*,\theta_1} O_I(u;\alpha^*,\theta^*) = 0, \qquad u \in [0,u_R), \quad O_I(0;\alpha^*,\theta^*) = 1, \tag{16}$$

$$\mathbb{A}^{\alpha_2^*,\theta_2} O_I(u;\alpha^*,\theta^*) = 0, \qquad u \in [u_R,u_2), \quad O_I(u_2;\alpha^*,\theta^*) = 0, \tag{17}$$

$$O_I'(u_R^-;\alpha^*,\theta^*) = O_I'(u_R^+;\alpha^*,\theta^*), \quad O_I(u_R^-;\alpha^*,\theta^*) = O_I(u_R^+;\alpha^*,\theta^*), \tag{18}$$

$$\mathbb{L}^{\alpha_1^*,\theta_1} O_R(u;\alpha^*,\theta^*) = 0, \qquad u \in [0,u_R), \quad O_R(0;\alpha^*,\theta^*) = 0, \quad O_R'(u_R^;\alpha^*,\theta^*-) = 1, \tag{19}$$

$$\mathbb{L}^{\alpha_2^*,\theta_2} O_R(u;\alpha^*,\theta^*) = 0, \qquad u \in [u_R,u_2), \quad O_R'(u_R^+;\alpha^*,\theta^*) = 1, \quad O_R(u_2;\alpha^*,\theta^*) = \nu_2, \tag{20}$$

and

$$O_R(u_R^-;\alpha^*,\theta^*) = O_R(u_R^+;\alpha^*,\theta^*). \tag{21}$$

Otherwise, if $u_R = 0$ the HJB equations (8) and (9) becomes
P_2:

$$\mathbb{L}^{\alpha_2^*,\theta_2} O_I(u;\alpha^*,\theta^*) = 0, \qquad u \in [0,u_2), \quad O_I(0;\alpha^*,\theta^*) = 1, \quad O_I(u_2;\alpha^*,\theta^*) = 0, \tag{22}$$

$$\mathbb{A}^{\alpha_2^*,\theta_2} O_R(u;\alpha^*,\theta^*) = 0, \qquad u \in [0,u_2), \quad O_R(0;\alpha^*,\theta^*) = 0, \quad O_R(u_2;\alpha^*,\theta^*) = \nu_2. \tag{23}$$

where (22) is a ruin probability problem and is commonly observed in the literature of ruin probability optimization, see e.g. [29] and [24].

4. Explicit reinsurance contract for standard-deviation principle

In this section, we assume that the reinsurance premium function is based on the standard-deviation principle and show that under this specified function, the reinsurance strategy is a proportional reinsurance strategy. Consider the standard-deviation principle for the reinsurance premium function as $\mathbb{W}_\theta(X) = E(X) + \theta\sqrt{E(X^2)}$. Thus, the insurer and the reinsurer's premium rates are given by

$$\begin{cases} c = \lambda(\mu + \beta\sigma), \\ \mathbb{W}_\theta(X) = \mu + \theta\sigma, \end{cases}$$

where $\beta \in (0,\theta_1)$ and $\theta \in (\theta_1,\theta_2)$ are the safety loadings of the insurer and the reinsurer. From (6) we have $u_1 = \lambda\frac{\sigma_0(\theta_1-\beta)}{r}$, $u_2 = \lambda\frac{\sigma_0(\theta_2-\beta)}{r}$ and $\nu_i = \lambda\frac{\theta_i\sigma_0}{\zeta}$ for $i = 1,2$.

Lemma 2
Assume that for $u \in (0,u_2)$, the value function $O_I''(u)$ satisfies $O_I''(u) \geq 0$. Then the function $a_\alpha(X)$ is given by $a_\alpha(X) = \alpha X$, where $0 \leq \alpha \leq 1$ is the self-retention level parameter.

Proof
This lemma is a modified version of Proposition 1 given in [30] and thus the proof is omitted. □

This Lemma shows that under the standard-deviation principle, the reinsurance strategy is a proportional reinsurance strategy. Therefore, given a proportional reinsurance strategy $\{\alpha\}_{t\geq 0}$ the dynamics of the insurer's cash reserve (2) can be rewritten as

$$R_t = \big[rR_t + \big(\theta\alpha - (\theta_t - \beta)\big)\lambda\sigma_0\big]dt + \sqrt{\lambda}\alpha_t\sigma_0 dB_t,$$

with $R_0 = u$ and the operators $\mathbb{L}^{\alpha,\theta}$ and $\mathbb{A}^{\alpha,\theta}$ can be written as

$$\begin{cases} \mathbb{L}^{\alpha,\theta}h(u) = h'(u)\big(ru + \big(\theta\alpha - (\theta-\beta)\big)\lambda\sigma_0\big) + \frac{\lambda}{2}\alpha^2\sigma_0^2 h''(u), \\ \mathbb{A}^{\alpha,\theta}h(u) = \mathbb{L}^{\alpha,\theta}h(u) - \zeta h(u) + \lambda\sigma_0\theta(1-\alpha). \end{cases}$$

Lemma 3
Assume that $O_I'(u;\alpha^*,\theta) < 0$ and $O_I''(u;\alpha^*,\theta) > 0$. Then for $i = 1,2$, the insurer's optimal strategy, $\alpha_i^*(u)$, is given by

$$\alpha_i^*(u) = \min\big\{\frac{2r}{\lambda\theta_i\sigma_0}\big(\frac{(\theta_i-\beta)\lambda\sigma_0}{r} - u\big), 1\big\}. \tag{24}$$

Proof
If $O_I''(u;\alpha^*,\theta) > 0$ and $\mathbb{L}^{\alpha,\theta_i}O_I(u;\alpha,\theta_i)$, $i = 1,2$, is convex in α. Then using the first order condition, we have

$$\alpha_i^*(u) = \min\big\{\frac{-\theta_i}{\sigma_0}\frac{O_I'(u;\alpha^*,\theta)}{O_I''(u;\alpha^*,\theta)}, 1\big\}. \tag{25}$$

If $\alpha_i^*(u) < 1$, then the HJB equation (8) can be rewritten as

$$-\frac{\lambda}{2r}\theta_i^2\frac{\big(O_I'(u;\alpha^*,\theta)\big)^2}{O_I''(u;\alpha^*,\theta)} + \big(u - \frac{(\theta_i-\beta)\lambda\sigma_0}{r}\big)O_I'(u;\alpha^*,\theta) = 0. \tag{26}$$

Combining the equations (25) and (26) we have (24), and this completes the proof. □

Note that α^* is the decreasing function of β and u. Thus, with $\theta = \theta_i$, the insurer retains more claim risk and purchases less reinsurance when she has a smaller amount of financial reserve or the insurance business is less profitable with a smaller β. The reason is that with insufficient savings or financial reserve, the insurer can not afford to pay too much reinsurance and as a result needs to pay more damages by himself/herself. On the other hand, it is clear that $\alpha_2^*(u) \geq \alpha_1^*(u)$. That is, as the reinsurance contract becomes more expensive, the insurer tends to retain more risk of damage and the demand for reinsurance decreases.
To simplify our results, we make further the assumption $\theta_1 < 2\beta < \theta_2$. Therefore, from (24), it is clear that $0 \leq \alpha_1^*(u) < 1$ for $u \in [0,u_1]$, $0 \leq \alpha_2^*(u) < 1$ for $u \in (u_0,u_2]$, and $\alpha_2^*(u) = 1$ for $u \in [0,u_0]$, where $u_0 = \frac{\lambda\sigma_0}{r}\big(\frac{\theta_2}{2} - \beta\big)$.
Depending on the values of u_1 and u_0, we consider two different cases:
Case (1): $u_0 < u_1$.
Case (2): $u_0 \geq u_1$,
and in both cases we will obtain the explicit solutions to our optimization Problem.

4.1. Case (1)

In this case, $\theta_1 > \frac{\theta_2}{2}$. On the other hand, since $u_R \leq u_1$ (see Corollary 3.2), depending on the model parameters there are three situations for the value of u_R: (a) $u_0 \leq u_R \leq u_1$; (b) $0 < u_R < u_0$; and (c) $u_R = 0$ (i.e. $\theta^*(u) = \theta_2$). Figure 1 illustrates the insurer and the reinsurer's strategies for each situation.

Lemma 4
With $u_R \in (0,u_1]$, the equations (19) and (20) have the solutions $O_{R_1}(.;u_R) \in \mathbb{C}^2(0,u_2)$ and $O_{R2}(.;u_R) \in \mathbb{C}^2(u_R,u_2)$, respectively. Moreover,
(i) $O_{R_1}(u_R;u_R)$ and $O_{R_2}(u_R;u_R)$ are continuous in $u_{R_1} \in (0,u_1]$.
(ii) $O_{R_2}(u_1;u_1) < \nu_1$ and $O_{R_2}(0;0) = d_0$, where d_0 is defined in (36).

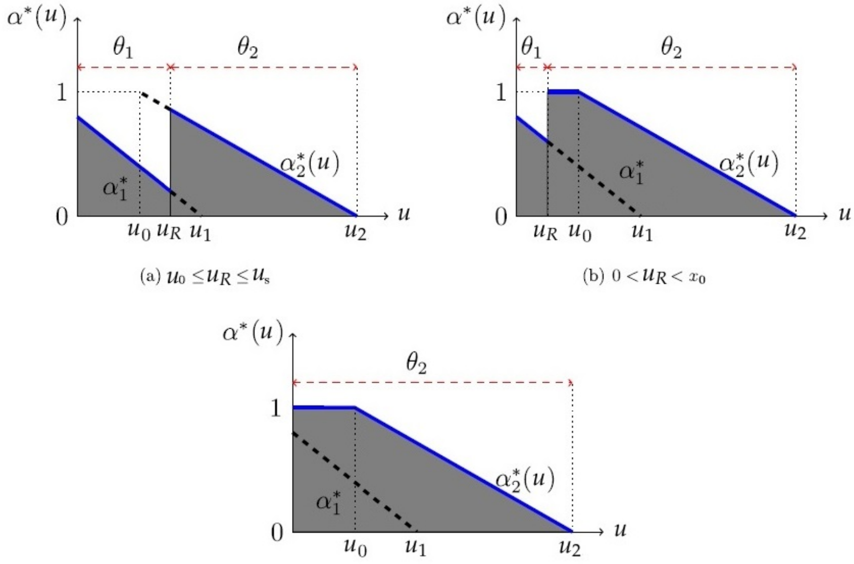

Figure 1. Two types of ruin.

Proof
(i) First, we determine the solution to (16). Substituting the equation (24) into (19) leads to

$$\begin{aligned} & r(u_1-u)O'_R(u;\alpha^*,\theta^*) + \frac{r}{\delta_1}(u_1-u)^2 O''_R(u;\alpha^*,\theta^*) \\ & +\lambda\theta_1\sigma_0 - 2r(u_1-u) = 0, \qquad u \in [0,u_R], \end{aligned} \tag{27}$$

where $\delta_1 = \frac{\lambda\theta_1^2}{2r}$. The equation (27) has the solution:

$$O_{R_1}(u,u_R) = Q_{11}(u_1-u)^{y_1} + Q_{12}(u_2-u)^{y_2} + k(u_1-u) + \nu_1, \qquad u \in [0,u_R], \tag{28}$$

where $k = -\frac{2r}{r+\zeta} \in (-1,0)$, $y_1 > 1$ and $y_2 < 0$ are the roots of equation $\frac{r}{\delta_1}y^2 - \big(\frac{1}{\delta_1}+1\big)ry - \zeta = 0$. Q_{11} and Q_{12} are determined by $O_{R_1}(0,u_R) = 0$ and $O'_{R_1}(u_R,u_R) = 1$, i.e.

$$\begin{cases} Q_{11}u_1^{y_1} + Q_{12}u_2^{y_2} + ku_1 + \nu_1 = 0, \\ -y_1Q_{11}(u_1 - _R)^{y_1-1} - y_2Q_{12}(u_1-u_R)^{y_2-1} - k = 1. \end{cases}$$

It is clear that $O_{R_1}(.;u_R) \in \mathbb{C}^2(0,u_R)$, Q_{11} and Q_{12} are continuous in u_R, Thus $O_{R_1}(u_R;u_R)$ is also continuous in u_R.
(ii) Now, we determine the solution to the equation (20) for u_R in two intervals $[u_0,u_1]$ and $[0,u_0)$.
(a) If $u_R \in [u_0,u_1]$, then using the boundary condition $O'_R(u_R;u_R) = 1$ and $O_R(u_2;u_R) = \nu_2$, inspired by (19), we see that the solution to the equation (20) is

$$O_{R_2}(u,u_R) = A(u_2-u)^m + k(u_2-u) + \nu_2, \qquad u \in [u_R,u_2], \tag{29}$$

where $m = \frac{\frac{r}{\delta_2}+1+\left(\left(\frac{r}{\delta_2}+1\right)^2+4\zeta\frac{r}{\delta_2}\right)^{\frac{1}{2}}}{\frac{2r}{\delta_2}} > \frac{\frac{r}{\delta_1}+1+\left(\left(\frac{r}{\delta_1}+1\right)^2+4\zeta\frac{r}{\delta_1}\right)^{\frac{1}{2}}}{\frac{2r}{\delta_1}} = y_1 > 1$, is the positive root of $\frac{r}{\delta_2}m^2 - \big(\frac{1}{\delta_2}+1\big)rm - \zeta = 0$ with $\delta_2 = \frac{\lambda\theta_2^2}{2r}$, and

$$A = -\frac{1+k}{m}(u_2-u_R)^{1-m} = -b(u_2-u_R)^{1-m} < 0, \tag{30}$$

where $b = \frac{1+k}{m}$. It is clear that $O_{R_2}(.;u_R) \in \mathbb{C}^2(u_R, u_2)$ and

$$O_{R_2}(u_R; u_R) = (k-b)(u_2 - u_R) + \nu_2, \tag{31}$$

is continuous in u_R. On the other hand, since $O''_{R_2}(u;u_R) = Am(m-1)(u_2-u)^{m-2} < 0$, from $\mathbb{A}^{\alpha_2^*,\theta_2} O_{R_2}(u_1, u_1) = 0$ and $O'_{R_2}(u_1+; u_1) = 1$, we have

$$\begin{aligned} O_{R_2}(u_1;u_1) &= \frac{\frac{r}{\delta_2}(u_2-u_1)^2 O''_{R_2}(u_1+;u_1) + \lambda\sigma_0\theta_2 - r(u_2-u_1)}{\zeta} \\ &< \frac{\lambda\sigma_0\theta_2 - r(u_2-u_1)}{\zeta} = \nu_1. \end{aligned}$$

(b) If $u_R \in [0, u_0]$, it is clear that $\alpha_2^*(u) = 1, \forall u \in [u_R, u_0]$, then the equation (20) can be written as

$$(ru + \lambda\sigma_0\beta)O'_R(u;\alpha^*,\theta^*) + \frac{\lambda}{2}\sigma_0^2 O''_R(u;\alpha^*,\theta^*) - \zeta O_R(u;\alpha^*,\theta^*) = 0, \tag{32}$$

for $u \in (u_R, u_0)$. Therefore, the solution to this equation is given by

$$O_R(u;u_R) = pg_1(u) + qg_2(u), \tag{33}$$

where g_1 and g_2 are the two classical solutions to the equation (32) with the conditions $g_1(0) = 0$, $g'_1(0) = 1$, $g_2(0) = 1$, $g'_2(0) = 0$, p and q are constants to be determined. On the other hand, we can see from (a) that, for $u \in [u_0, u_2]$, the equation (20) has the solution (28). Thus, from $O_{R_2}(u_0-;u_R) = O_{R_2}(u_0+;u_R)$ and $O'_{R_2}(u_0-;u_R) = O'_{R_2}(u_0+;u_R)$, we have

$$\begin{cases} pg_1(u_0) + qg_2(u_0) = B(u_2-u_0)^m + k(u_2-u_o) + \nu_2, \\ pg'_1(u_0) + qg'_2(u_0) = -mB(u_2-u_0)^{m-1} - k, \end{cases}$$

where

$$B = -\frac{pg'_1(u_0) + qg'_2(u_0) + k}{m}(u_2-u_0)^{1-m}. \tag{34}$$

The above equation follows that

$$\begin{aligned} p\big(g_1(u_0) + g'_1(u_0)\frac{u_2-u_0}{m}\big) + q\big(g_2(u_0) + g'_2(u_0)\frac{u_2-u_0}{m}\big) \\ = \big(1-\frac{1}{m}\big)k(u_2-u_0) + \nu_2. \end{aligned}$$

Combining this equation with $pg'_1(u_R) + qg'_2(u_R) = 1$, we have

$$(p,q)' = \begin{pmatrix} g'_1(u_R) & g'_2(u_R) \\ g_1(u_0) + \frac{u_2-u_0}{m}g'_1(u_0) & g_2(u_0) + \frac{u_2-u_0}{m}g'_2(u_0) \end{pmatrix}^{-1} \times \begin{pmatrix} 1 \\ \big(1-\frac{1}{m}\big)k(u_2-u_0) + \nu_2 \end{pmatrix}.$$

It is clear that p, q and B are continuous in u_R, thus $O_{R_2}(u_R;u_R)$ is continuous in $u_R \in [0, u_0]$.
(iii) Finally, when $u_R \to u_0-$, we have $pg'_1(u_0) + qg'_2(u_0) = 1$, $B = -b(u_2-u_0)^{1-m}$ and

$$O_{R_2}(u_0-;u_0-) = (k-b)(u_2-u_0) + \nu_2 = O_{R_2}(u_0+;u_0+), \tag{35}$$

where the second equality is due to (31). Thus $O_{R_2}(u_R;u_R)$ is continuous at $u_R = u_0$ and hence continuous in $[0, u_1]$. Specially, when $u_R = 0$, since $g'_1(0) = 1$ and $g'_2(0) = 0$, we have $p = 1$ and $O_{R_2}(0;0) = q = q_0$, where

$$q_0 = \frac{\big(1-\frac{1}{m}\big)k(u_2-u_0) + \nu_2 - g_1(u_0) - \big(\frac{u_2-u_0}{m}\big)g'_1(u_0)}{g_2(u_0) + \big(\frac{u_2-u_0}{m}\big)g'_2(u_0)}, \tag{36}$$

and this completes the proof. □

This Lemma shows that if there exists a point $u_R \in (0, u_1]$, such that $O_{R_1}(u_R; u_R) = O_{R_2}(u_R; u_R)$, then the solution to (19)-(21) is given by

$$O_R(u;\alpha^*,\theta^*) = \begin{cases} O_{R_1}(u;u_R), & if \quad u \in [0,u_R), \\ O_{R_2}(u;u_R), & if \quad u \in [u_R,u_2). \end{cases}$$

By substituting the insurer's optimal strategies (24) into (16)-(18), $O_I(u;\alpha^*,\theta^*)$ is also determined, and thus problem P_1 is solved. Otherwise, we consider problem P_2.
The main results are presented in the following theorem.

Theorem 4.1
Assume that $u_1 > u_0$ and for any $z \in [0, u_1 - u_0]$ define the function $H(z)$ as

$$\begin{aligned} H(z) &= (y_2\omega + m - y_2)bu_R\big(\frac{u_1}{u_R}\big)^{y_1} \\ &\quad -(y_1\omega + (m - y_1)bu_R\big(\frac{u_1}{u_R}\big)^{y_2} + (ku_1 + \nu_1)(y_2 - y_1). \end{aligned} \tag{37}$$

(i) If $q_0 > 0$, the problem P_1 has a solution. Moreover,
(a) if $H(u_1 - u_0) < 0$, the equilibrium strategies are given by

$$(\alpha^*(u),\theta^*(u)) = \begin{cases} \big(\frac{2r}{\lambda\theta_1\sigma_0}\big(\frac{(\theta_1-\beta)\lambda\sigma_0}{r} - u\big),\theta_1\big), & if \quad u \in [0,u_R), \\ \big(\frac{2r}{\lambda\theta_2\sigma_0}\big(\frac{(\theta_2-\beta)\lambda\sigma_0}{r} - u\big),\theta_2\big), & if \quad u \in [u_R,u_2), \end{cases} \tag{38}$$

(b) otherwise, if $H(u_1 - u_0) \geq 0$, the equilibrium strategies are given by

$$(\alpha^*(u),\theta^*(u)) = \begin{cases} \big(\frac{2r}{\lambda\theta_1\sigma_0}\big(\frac{(\theta_1-\beta)\lambda\sigma_0}{r} - u\big),\theta_1\big), & if \quad u \in [0,u_R), \\ (1,\theta_2) & if \quad u \in [u_R,u_0), \\ \big(\frac{2r}{\lambda\theta_2\sigma_0}\big(\frac{(\theta_2-\beta)\lambda\sigma_0}{r} - u\big),\theta_2\big), & if \quad u \in [u_0,u_2), \end{cases} \tag{39}$$

(ii) If $q_0 \leq 0$, the problem P_2 has a solution and the equilibrium strategies are given by

$$(\alpha^*(u),\theta^*(u)) = \begin{cases} (1,\theta_2) & if \quad u \in [0,u_0), \\ \big(\frac{2r}{\lambda\theta_2\sigma_0}\big(\frac{(\theta_2-\beta)\lambda\sigma_0}{r} - u\big),\theta_2\big), & if \quad u \in [u_0,u_2). \end{cases} \tag{40}$$

Proof
First we consider that $q_0 > 0$ and the case $u_R \in [u_0, u_1)$ to show that the problem P_1 admits the solutions (O_I, O_R), which are also the solutions to HJB equations (8) and (9).
(a) We construct an explicit solution to (19)-(21). In this case, from (28) and (29) a solution to (19) and (20) is given by

$$O_R(u;\alpha^*,\theta^*) = \begin{cases} Q_{11}(u_1-u)^{y_1} + Q_{12}(u_2-u)^{y_2} + k(u_1-u) + \nu_1, & u \in [0,u_R], \\ A(u_2-u)^m + k(u_2-u) + \nu_2, & u \in [u_R,u_2]. \end{cases} \tag{41}$$

Applying the conditions $O'_R(u_R-;\alpha^*,\theta^*) = 1$ and $O_R(u_R-;\alpha^*,\theta^*) = O_R(u_R+;\alpha^*,\theta^*)$ yields

$$\begin{cases} -y_1Q_{11}(u_1-u_R)^{y_1-1} - y_2Q_{12}(u_2-u_R)^{y_2-1} - k = 1, & u \in [0,u_R], \\ Q_{11}(u_1-u_R)^{y_1} + Q_{12}(u_2-u_R)^{y_2} + k(u_1-u_R) + \nu_1 = (k-b)(u_2-u_R) + \nu_2. & u \in [u_R,u_2]. \end{cases} \tag{42}$$

Let $u_R = u_1 - u_R \in (0, u_1 - u_0)$. Then (42) can be rewritten as

$$\begin{cases} -y_1 Q_{11} u_R^{y_1} + y_2 Q_{12} u_R^{y_2} = -mby_R, & u \in [0, u_R], \\ Q_{11} u_R^{y_1} + Q_{12} u_R^{y_2} = -my_R + n, & u \in [u_R, u_2], \end{cases} \tag{43}$$

where

$$\begin{aligned} n &= (k-b)(u_2 - u_1) + \nu_2 - \nu_1 \\ &= \big(k - \frac{1+k}{m}\big)\lambda\sigma_0 \frac{1}{r}(\theta_2 - \theta_1) + \frac{1}{\zeta}\lambda\sigma_0(\theta_2 - \theta_1) \\ &= \lambda\sigma_0(\theta_2 - \theta_1)\big[\big(k\big(1 - \frac{1}{m}\big) - \frac{1}{m}\big)\frac{1}{r} + \frac{1}{\zeta}\big] \\ &= \lambda\sigma_0(\theta_2 - \theta_1)\big[\frac{r-\zeta}{\zeta(r+\zeta)} + \frac{1}{m}\frac{r-\zeta}{r(r+\zeta)}\big] < 0. \end{aligned} \tag{44}$$

Solving (43) leads to

$$\begin{cases} Q_{11} = -\frac{1}{y_1 - y_2}\big[y_2(n - by_R) + mby_R\big] y_R^{-y_1}, \\ Q_{12} = \frac{1}{y_1 - y_2}\big[y_1(n - by_R) + mby_R\big] y_R^{-y_2}. \end{cases}$$

Since $m > y_1 > 1$, it is clear that $Q_{11} < 0$. With boundary condition $O_R(0; \alpha^*, \theta^*) = 0$, we see that y_R satisfies $H(y_R) = 0$. On the other hand, the direct calculation shows that $\lim_{z \to 0^+} H(z) = \infty$ and

$$\begin{aligned} H'(z) &= -y_1 y_2 n u_R^{y_1} z^{-y_1 - 1} + (1 - y_1)(m - y_2) b u_R^{y_1} z^{-y_1} \\ &\quad + y_1 y_2 n u_R^{y_2} z^{-y_2 - 1} - (m - y_1)(1 - y_2) b u_R^{y_2} z^{-y_2} \\ &= y_1 y_2 n \frac{1}{u_1}\big(\big(\frac{u_1}{u_R}\big)^{y_1+1} - \big(\frac{u_1}{u_R}\big)^{y_1+1}\big) \\ &\quad + (1 - y_1)(m - y_2) b u_R^{y_1} z^{-y_1} - (m - y_1)(1 - y_2) b u_R^{y_2} z^{-y_2} < 0, \end{aligned}$$

i.e. the function H is strictly decreasing. Therefore, $H(u_1 - u_0) \leq 0$ if and only if the function given in (37) has a unique solution $u_R \in (0, u_1 - u_0]$, with which $u_R \in [u_0, u_1)$ and Q_{11} and Q_{12} are also uniquely determined.
(b) Once u_R is determined, we construct a solution to the equations (16)-(18). For $u \in (0, u_R)$, with $\theta^*(u) = \theta$, (16) has the solution

$$O_I(u; \alpha^*, \theta^*) = e_1 - e_2(u_1 - u)^{\delta_1 + 1}, \tag{45}$$

where e_1 and e_2 are constants. Note that, $O_I(0; \alpha^*, \theta^*) = 1$ indicates that $e_1 = 1 + e_2 u_1^{\delta_1 + 1}$.
For $u \in (u_R, u_2)$, with boundary condition $O_I(u_2; \alpha^*, \theta^*) = 0$, the equation (17) has the solution

$$O_I(u; \alpha^*, \theta^*) = -a_0(u_2 - u)^{\delta_2 + 1}, \tag{46}$$

where a_0 is a constant to be determined. Combining (45) and (46), we have

$$O_I(u; \alpha^*, \theta^*) = \begin{cases} 1 + e_2 u_1^{\delta_1 + 1} - e_2(u_1 - u)^{\delta_1 + 1}, & u \in [0, u_R], \\ -a_0(u_2 - u)^{\delta_2 + 1}, & u \in [u_R, u_2]. \end{cases}$$

The value-matching at point u_R implies that

$$\begin{cases} 1 + e_2 u_1^{\delta_1 + 1} - e_2(u_1 - u_R)^{\delta_1 + 1} = -a_0(u_2 - u_R)^{\delta_2 + 1}, \\ (\delta_1 + 1)e_1(u_1 - u_R)^{\delta_1} = a_0(\delta_2 + 1)(u_2 - u_R)^{\delta_2}. \end{cases} \tag{47}$$

By solving the equations, e_2 is uniquely characterized as

$$e_2 = -\big[u_1^{\delta_1+1} - (u_1-u_R)^{\delta_1+1} + (u_1-u_R)^{\delta_1}(u_2-u_R)\frac{\delta_1+1}{\delta_2+1}\big]^{-1} < 0. \tag{48}$$

After computing e_2, from the first equation (47), a_0 will be uniquely characterized. Moreover, from the second equation (47), e_1 will be uniquely characterized.
(c) Finally, we prove the optimality of (α^*, θ^*). It is clear that (α^*, θ^*) are admissible, by verification Theorem 3.1. we just need to show that (O_I, O_R) are the solutions to (8) and (9). On the other hand, since $O_I(u;\alpha^*,\theta^*)$ is convex, and hence for $\alpha \in [0,1]$, $0 = \mathbb{L}^{\alpha^*,\theta^*} O_I(u;\alpha^*,\theta^*) \leq \mathbb{L}^{\alpha,\theta^*} O_I(u;\alpha^*,\theta^*)$. Since $O_R(u;\alpha^*,\theta^*)$ is concave on (u_R, u_2), $O'_R(u;\alpha^*,\theta^*) < O'_R(u_R;\alpha^*,\theta^*) < 1$ for all $u \in (u_R, u_2)$. Once we prove that $O'_R(u;\alpha^*,\theta^*) > 1$ for $u \in (0, u_R)$, then $0 = \mathbb{A}^{\alpha^*,\theta^*} O_R(u;\alpha^*,\theta^*) \leq \mathbb{A}^{\alpha,\theta^*} O_R(u;\alpha^*,\theta^*)$, for $\theta \in [\theta_1, \theta_2]$, and thus prove our argument.
(ii) If $H(u_1 - u_0) > 0$, we are left with $u_R \in (0, u_R)$.
(a) We construct an explicit solution to (19)-(21). In this case,

$$O_R(u;\alpha^*,\theta^*) = \begin{cases} Q_{11}(u_1-u)^{y_1} + Q_{12}(u_2-u)^{y_2} + k(u_1-u) + \nu_1, & u \in [0, u_R), \\ H(u;u_R) = pg_1(u) + qg_2(u), & u \in [u_R, u_0), \\ B(u_2-u)^m + k(u_2-u) + \nu_2, & u \in [u_0, u_2], \end{cases} \tag{49}$$

where B is given in (34) and Q_{11}, Q_{12} and u_R are determined by

$$Q_{11}u_1^{y_1} + Q_{12}u_1^{y_2} + ku_1 + \nu_1 = 0, \tag{50}$$

$$Q_{11}y_1(u_1-u_R)^{y_1-1} + Q_{12}y_2(u_1-u_R)^{y_2-1} + k = -1, \tag{51}$$

$$Q_{11}(u_1-u_R)^{y_1} + Q_{12}(u_1-u_R)^{y_2} + k(u_1-u_R) + \nu_1 = H(u_R;u_R). \tag{52}$$

Let $y_R = u_1 - u_R \in (u_1 - u_0, u_1)$. Then (51) and (52) become

$$\begin{cases} Q_{11}y_1y_R^{y_1} + Q_{12}y_2y_R^{y_2} = -mby_R, \\ Q_{11}y_R^{y_1} + Q_{12}y_R^{y_2} = H(u_R;u_R) - ky_R - \nu_1. \end{cases} \tag{53}$$

Solving the equations, we have

$$\begin{cases} Q_{11} = -\frac{1}{y_1-y_2}\big[mby_R + y_2\big(H(u_R;u_R) - ky_R - \nu_1\big)\big]y_R^{-y_1}, \\ Q_{12} = \frac{1}{y_1-y_2}\big[mby_R + y_1\big(H(u_R;u_R) - ky_R - \nu_1\big)\big]y_R^{-y_2}. \end{cases} \tag{54}$$

Let us denote $H_1(u) = H(u;u)$ for $u \in [0, u_0)$. Thus, (50) becomes $H_2(y_R) = 0$ with $u_R = u_1 - y_R$, where

$$\begin{aligned} H_2(z) \quad = \quad & \big[mbz + y_2\big(H_1(u_1-z) - kz - \nu_1\big)\big]\big(\frac{u_1}{z}\big)^{y_1} \\ & -\big[mbz + y_1\big(H_1(u_1-z) - kz - \nu_1\big)\big]\big(\frac{u_1}{z}\big)^{y_2} \\ & +(ku_1+\nu_1)(y_2-y_1). \end{aligned} \tag{55}$$

We proceed to show that (55) admits a solution $u_R \in (u_1 - u_0, u_1)$. From (35), $H_1(u_0) = (k-s)(u_2-u_0) + \nu_2$ and

$$H_1(u_0) - k(u_1-u_0) - \nu_1 = n - s(u_1-u_0).$$

Thus, we have

$$\begin{aligned} H_2(u_1-u_0) &= \big[ny_2+(m-y_2)b(u_1-u_0)\big]\big(\frac{u_1}{u_1-u_0}\big)^{y_1} \\ &\quad -\big[ny_1+(m-y_1)b(u_1-u_0)\big]\big(\frac{u_1}{u_1-u_0}\big)^{y_2} \\ &\quad +(ku_1+\nu_1)(y_2-y_1)=H(u_1-u_0)>0. \end{aligned} \tag{56}$$

On the other hand

$$\begin{aligned} H_2(u_1) &= \big[mbu_1+y_2\big(H_1(0)-ku_1-\nu_1\big)\big] \\ &\quad -\big[mbu_1+y_1\big(H_1(0)-ku_1-\nu_1\big)\big] \\ &\quad +(ku_1+\nu_1)(y_2-y_1) \\ &\quad =(y_2-y_1)H_1(0)=(y_2-y_1)q_0<0. \end{aligned} \tag{57}$$

Therefore, we may determine z_R and hence u_R by solving the equation $H_2(z)=0$. Substituting the value of u_R into (34) and (54), we obtain p, q, A, Q_{11} and Q_{12}.
(b) We construct a solution to (16)- (18). For $u\ in (u_R,u_0)$, with $\theta^*(u)=\theta_2$ and $\alpha^*(u)=1$, the equation (8) becomes

$$(ru+\lambda\sigma_0\beta)O_I'(u,\alpha^*,\theta^*)+\frac{1}{2}\lambda\sigma_0^2O_I''(u,\alpha^*,\theta^*)=0,$$

which admits solution

$$O_I(u,\alpha^*,\theta^*)=O_I(u_R,\alpha^*,\theta^*)+a_1\int_{u_R}^{u}e^{-\frac{r}{\lambda\sigma_0^2}\left(z+\frac{\lambda\sigma_0\beta}{r}\right)^2}dz. \tag{58}$$

Combining (45), (46) and (58), a solution to (16)-(18) is given by

$$O_I(u;\alpha^*,\theta^*)=\begin{cases} 1+e_3u_1^{\delta_1+1}-e_3(u_1-u)^{\delta_1+1}, & u\in[0,u_R), \\ O_I(u_R;\alpha^*,\theta^*)+a_1\int_{u_R}^{u}e^{-\frac{r}{\lambda\sigma_0^2}\left(z+\frac{\lambda\sigma_0\beta}{r}\right)^2}dz, & u\in[u_R,u_0), \\ -a_2(u_2-u)^{\delta_2+1}, & u\in[u_0,u_2], \end{cases} \tag{59}$$

where e_3, a_1 and a_2 are determined by the conditions $O_I'(u_R-;\alpha^*,\theta^*)=O_I'(u_R+;\alpha^*,\theta^*)$, $O_I'(u_0+;\alpha^*,\theta^*)=O_I'(u_0-;\alpha^*,\theta^*)$ and $O_I(u_0+;\alpha^*,\theta^*)=O_I(u_0-;\alpha^*,\theta^*)$. That is

$$\begin{cases} e_3=a_1(\delta_1+1)^{-1}(u_1-u_R)^{-\delta_1}e^{-\frac{r}{\lambda\sigma_0^2}\left(z+\frac{\lambda\sigma_0\beta}{r}\right)^2}=a_1\Omega_1, \\ a_2=a_0(\delta_2+1)^{-1}(u_2-u_0)^{-\delta_2}e^{-\frac{r}{\lambda\sigma_0^2}\left(z+\frac{\lambda\sigma_0\beta}{r}\right)^2}=a_1\Omega_2, \\ a_1=-\big[\Omega_1\big(u_1^{\delta_1+1}-(u_1-u_R)^{\delta_1+1}\big)+\int_{u_R}^{u_0}e^{-\frac{r}{\lambda\sigma_0^2}\left(z+\frac{\lambda\sigma_0\beta}{r}\right)^2}dz+\Omega_2(u_2-u_{u_0})^{\delta_2+1}\big]^{-1}. \end{cases} \tag{60}$$

(c) We show that O_R and O_I given in (49) and (59) are the solutions to (8) and (9). It is clear that $a_1<0$ and hence $a_2<0$. On the other hand, since O_I is convex, therefore, $0=\mathbb{L}^{\alpha^*,\theta^*}O_I(u;\alpha^*,\theta^*)\le\mathbb{L}^{\alpha,\theta^*}O_I(u;\alpha^*,\theta^*)$, $\forall\alpha\in[0,1]$. We just need to show that $\mathbb{A}^{\alpha^*,\theta^*}O_R(u;\alpha^*,\theta^*)\ge\mathbb{A}^{\alpha^*,\theta}O_R(u;\alpha^*,\theta^*)$, which is proved in the following steps:
Step 1: We claim that $O_R'(u_0;\alpha^*,\theta^*)<1$. Otherwise, if $O_R'(u_0;\alpha^*,\theta^*)ge1$, from (34) it is clear that $B<0$

and $O''_R(u_0;\alpha^*,\theta^*)<0,\ u\in(u_0,u_2)$. Since the function O_R is twice continuously differentiable at u_0 (see Remark 4.1 (ii)), we also have $O''_R(u_0-;\alpha^*,\theta^*)=O''_R(u_0+;\alpha^*,\theta^*)<0$ and $O'_R(u;\alpha^*,\theta^*)>1,\ u\in(u_0-\epsilon,u_0)$ with $\epsilon>0$ begin small. Moreover, $O'_R(u_R;\alpha^*,\theta^*)=1$, there exists $\bar{u}\in(u_R,u_0)$ such that $O'_R(\bar{u};\alpha^*,\theta^*)>1$, $O''_R(\bar{u};\alpha^*,\theta^*)=0$, and $O'''_R(\bar{u};\alpha^*,\theta^*)\leq 0$. On the other hand, by taking derivative on (32) and letting $u=\bar{u}$, we have

$$(r-\zeta)O'_R(\bar{u};\alpha^*,\theta^*)=-\frac{\lambda}{2}\sigma_0^2O'''_R(\bar{u};\alpha^*,\theta^*)<0, \tag{61}$$

which is a contradiction.
Step 2: We show that $O'_R(u;\alpha^*,\theta^*)<1$, for $u\in(u_R,u_2)$. If $B<0$, then O_R is strictly concave in (u_0,u_2) and, for $u\in(u_0,u_2)$, $O'_R(u;\alpha^*,\theta^*)\leq O'_R(u_0;\alpha^*,\theta^*)\leq 1$; if $B\geq 0$, O_R is convex and hence $O'_R(u;\alpha^*,\theta^*)\leq O'_R(u_2;\alpha^*,\theta^*)-k<1$ for $u\in(u_0,u_2)$. On the other hand, by performing exactly the same procedure as above, we are able to show that $O'_R(u;\alpha^*,\theta^*)<1$, for $u\in(u_R,u_0)$.
Step 3: We show that $O'_R(u;\alpha^*,\theta^*)>1$, for $u\in(0,u_R)$. Since $O'_R(u;\alpha^*,\theta^*)<1$, for $u\in(u_R,u_2)$ and $O'_R(u_R;\alpha^*,\theta^*)=1$, we have $O''_R(u_R+;\alpha^*,\theta^*)\leq 0$. If $O''_R(u_R+;\alpha^*,\theta^*)=0$, similar to (61) we have $O'''_R(u_R+;\alpha^*,\theta^*)>0$, and hence $O''_R(u;\alpha^*,\theta^*)>0$, and $O'_R(u;\alpha^*,\theta^*)>1$ for $u\in(u_R,u_R+\epsilon)$ with $\epsilon>0$ begin small, which contradicts argument (ii). Thus, we have $O''_R(u_R+;\alpha^*,\theta^*)<0$. According to Remark 4.1 (i), we have $O''_R(u_R-;\alpha^*,\theta^*)<0$. By a similar proof as in (c) in part (i), we see that $O''_R(u;\alpha^*,\theta^*)<0$ for $u\in(0,u_R)$ and hence $O'_R(u;\alpha^*,\theta^*)>1$.
With the above observations, we have

$$\mathbb{A}^{\alpha^*,\theta^*}O_R(u;\alpha^*,\theta^*)\geq\mathbb{A}^{\alpha^*,\theta}O_R(u;\alpha^*,\theta^*),$$

for $u\in[0,u_R)\cup[u_0,u_2]$.
Moreover, when $u\in(u_R,u_0)$, with $\alpha^*(u)=1$ we have

$$\begin{aligned}\mathbb{A}^{\alpha^*,\theta}O_R(u;\alpha^*,\theta^*) &= (ru+\beta\lambda\sigma_0)O'_R(u;\alpha^*,\theta^*)+\frac{\lambda}{2}\sigma_0^2O''_R(u;\alpha^*,\theta^*)-\zeta O_R(u;\alpha^*,\theta^*)\\ &= \mathbb{A}^{\alpha^*,\theta_2}O_R(u;\alpha^*,\theta^*),\end{aligned}$$

for $\theta\in[\theta_1,\theta_2]$, and this completes the proof of part (i).
Now we prove the part (ii) with $q_0\leq 0$. From (46), (58) and and boundary condition $O_I(0;\alpha^*,\theta^*)=1$, a solution to the (22) is given by

$$O_I(u;\alpha^*,\theta^*)=\begin{cases}1+a_1\int_0^u e^{-\frac{r}{\lambda\sigma_0^2}\left(u_R+\frac{\lambda\sigma_0\beta}{r}\right)^2}du_R, & u\in[0,u_0),\\ -a_2(u_2-u)^{\delta_2+1}, & u\in[u_0,u_2],\end{cases} \tag{62}$$

where a_1 and a_2 are determined by the conditions $O_I(u_0-;\alpha^*,\theta^*)=O_I(u_0+;\alpha^*,\theta^*)$ and $O'_I(u_0-;\alpha^*,\theta^*)=O'_I(u_0+;\alpha^*,\theta^*)$, i.e.

$$a_2=-\frac{1}{e^{-\frac{r}{\lambda\sigma_0^2}}\left(u_0+\frac{\lambda\sigma_0\beta}{r}\right)^2(\delta_2+1)(u_2-u_0)^{\delta_2}\int_0^{u_0}e^{-\frac{r}{\lambda\sigma_0^2}\left(u_R+\frac{\lambda\sigma_0\beta}{r}\right)^2}du_R+(u_2-u_0)^{\delta_2+1}}<0,$$

and

$$a_1=a_2e^{\frac{r}{\lambda\sigma_0^2}\left(u_0+\frac{\lambda\sigma_0\beta}{r}\right)^2}(\delta_2+1)(u_2-u_0)^{\delta_2}<0.$$

From the equation (23), similar to our analyses in the previous proof, we have

$$O_R(u;\alpha^*,\theta^*)=\begin{cases}pg_1(u)+qg_2(u), & u\in[0,u_0),\\ B(u_2-u)^m+k(u_1-u)+\nu_2, & u\in[u_0,u_2].\end{cases}$$

□

4.2. Case (2):

When $u_1 \leq u_0$, similar to the analyses in Section 4.1, we are able to construct the solutions (O_I, O_R) to the problems P_1 and P_2 along with the equilibrium strategies (α^*, θ^*), and and then verify their optimality. An illustration of the equilibrium strategies is provided in Figure 2. The results are presented in the following theorem.

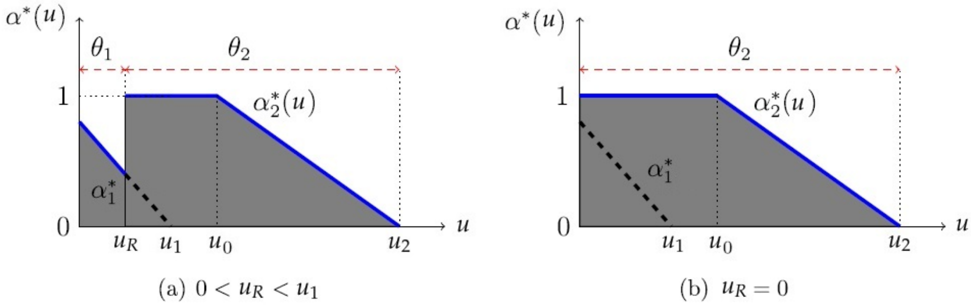

Figure 2. Optimal equilibrium strategies (α^*, θ^*) for $(a) \quad 0 < u_R < u_1$ and $(b) \quad u_0 > u_1$.

Theorem 4.2
Assume that $u_1 \leq u_0$ and $q_0 > 0$. The equilibrium strategy is given by (39); otherwise, if $q_0 \leq 0$, the equilibrium strategy is given by (40).

Proof
Since the proof of this theorem is very similar to that of Theorem 4.1 (ii), therefore, we omit it. □

As compared to the case $u_1 > u_0$, in this case the reinsurance contract has a larger upper bound θ_2 for the safety loading, indicating that the reinsurance contract becomes more costly. Thus, with other model parameters being fixed, there is always a noreinsurance zone $[u_R, u_0]$ in which the insurer will not purchase reinsurance.
Consider the model parameters as given in Table 1. Let $\beta = 0.28, 0.2$ and 0.225, we $u_R = 0.325, 0.24$ and 0, respectively. Figure 3 plots the equilibrium value functions $\big(O_I(u), O_R(u)\big)$. As expected, the function $O_I(u)$ is strictly decreasing and convex, and the function $O_R(u)$ is strictly increasing. When the insurance safety loading increases (i.e., the insurance business is more profitable), the insurer has a smaller ruin probability and the reinsurance contract becomes more valuable. These results are consistent with our common sense.

Table 1. Model parameters for the standard-deviation principle

Parameters	β	θ_1	θ_2	μ	σ_0	r	ζ	λ
Values	0.2	0.3	0.6	1.0	0.5	0.08	0.2	0.1

5. Explicit reinsurance contract for expected value principle

In this section, we assume that the reinsurance premium function is based on the expected value principle $\mathbb{W}_\theta(X) = (1+\theta)E(X)$. Thus, the insurer and the reinsurer's premium rates are given by

$$\begin{cases} c = \lambda\mu(1+\beta), \\ \mathbb{W}_\theta(X) = \lambda\mu(1+\theta), \end{cases}$$

where $\beta \in (0, \theta_1)$ and $\theta \in (\theta_1, \theta_2)$ are the safety loadings of the insurer and the reinsurer, respectively. From (6) we have $u_1 = \lambda \frac{\mu_0(\theta_1 - \beta)}{r}$, $u_2 = \lambda \frac{\mu_0(\theta_2 - \beta)}{r}$ and $\nu_i = \lambda \frac{\theta_i \mu_0}{\zeta}$ for $i = 1, 2$.
Similar to Lemma 4, we present the following result without proof.

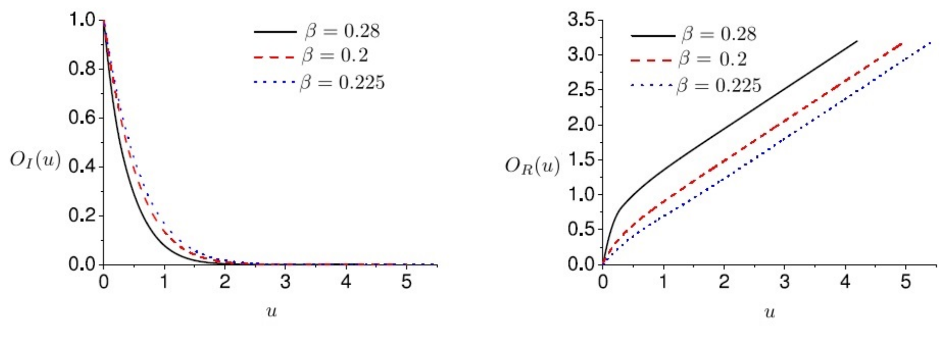

Figure 3. The equilibrium value functions $O_I(u)$ and $O_R(u)$ for $\beta = 0.28, 0.2$ and 0.225.

Lemma 5
Assume that for $u \in (0, u_2)$, the value function $O''_I(u)$ satisfies $O''_I(u) \geq 0$. Then the function $a_\alpha(X)$ is given by $a_\alpha(X) = \min\{X, \alpha_0 X + \alpha\}$, where $\alpha_0 \in [0, 1]$ and $\alpha > 0$ are constants.

In particular, when $\alpha_0 = 0$, the insurer's strategy is an excess-of-loss reinsurance strategy that is uniquely specified by the retention level α. In this case, for each claim X_i, the insurer pays the amount $\min\{X_i, \alpha\}$ and the reinsurer covers the rest $X_i - \min\{X_i, \alpha\}$. With a larger α, the insurer pays a larger proportion for each claim and takes more risk by herself. The excess-of-loss reinsurance is well adopted in practice and theoretical research. Thus, in this section we consider the excess-of-loss reinsurance and call α as the insurer's reinsurance strategy. Given a dynamic strategy $\{\alpha_t\}_{t\geq 0}$, the equation (2) becomes

$$dX_t = \big(rX_t + \lambda\big(\theta\mu_{\alpha_t} + (\beta - \theta)\mu\big)\big)dt + \sqrt{\lambda}\sigma_{\alpha_t} dB_t, \qquad X_0 = u, \tag{63}$$

where

$$\begin{cases} \mu_\alpha = E\big(X \wedge \alpha\big) = \int_0^\alpha \bar{F}_Z(y)dy, \\ \sigma_\alpha = \sqrt{E\big(X \wedge \alpha\big)^2} = \sqrt{\int_0^\alpha 2y\bar{F}_Z(y)dy}, \end{cases}$$

and $\bar{F}_Z(x) = 1 - F_Z(y)$. For any test function, we follow the standard dynamic programming techniques. If $h \in \mathbb{C}^2(0, u_2)$ and the strategies (α^*, θ^*) exist, the objective functions satisfy the following HJB equations:

$$\mathbb{L}^{\alpha,\theta} g(u) = \big(ru + \lambda\big(\theta\mu(\alpha) - (\theta - \beta)\mu\big)\big)g'(u) + \frac{\lambda}{2}\sigma^2(\alpha)g''(u), \tag{64}$$

and

$$\mathbb{A}^{\alpha,\theta} g(u) = \mathbb{L}^{\alpha,\theta} g(u) - \zeta g(u) + \lambda\theta(\mu - \mu_\alpha), \tag{65}$$

To construct solutions $O_I(u; \alpha^*, \theta^*)$ and $O_R(u; \alpha^*, \theta^*)$, to $P1$-$P2$, we first characterize the insurer's reinsurance strategy $\alpha^*_i(u)$, $i = 1, 2$.

Lemma 6
Assume that for $u \in (0, u_2)$, the value function $O''_I(u)$ satisfies $O''_I(u) < 0$. Then $\alpha^*_i(u) \in [0, \alpha^{0*}_i]$ is uniquely determined by the following non-linear equation:

$$k_i(\alpha) = -\frac{\alpha}{\theta_i}\big(ru + \lambda\big(\theta_i\mu_\alpha + (\eta - \theta_i)\mu\big)\big) + \frac{\lambda}{2}\sigma^2_\alpha = 0, \tag{66}$$

where α^{0*}_i is the solution of equation (66) when $r = 0$. It is clear that $\alpha^*_1(u) = 0$ if and only if $u \geq u_1$ and $\alpha^*_2(u) = 0$ if and only if $u \geq u_2$. Moreover, when $\alpha^*_i(u) > 0$, $\alpha^*_i(u)$ is strictly decreasing in u, η and r, and $\alpha^*_1(u) < \alpha^*_2(u)$.

Proof
By the first order condition, $\frac{d}{d\alpha}\mathbb{L}^{\alpha,\theta_i}O_I(u)|_{\alpha=\alpha_i^*}$, we have

$$\lambda\theta_i\bar{F}(\alpha_i^*)O_I'(u)+\lambda\alpha_i^*\bar{F}(\alpha_i^*)O_I''(u)=0,$$

which gives that $\frac{O_I'(u)}{O_I''(u)}=-\frac{\alpha_i^*}{\theta_i}$. Combining the above equation with $\mathbb{L}^{\alpha_i^*,\theta_i}O_I(u)=0$, we have $k_i(\alpha_i*)=0$. We now proceed to show that (66) has unique solution. Direct calculation shows $k_i(0)=0$,

$$\lim_{\alpha\to\infty}k_i(\alpha)=\lim_{\alpha\to\infty}\big(-\frac{\alpha}{\theta_i}\big(ru+\lambda\beta\mu\big)+\frac{1}{2}\lambda\sigma_0^2\big)=-\infty,$$

and

$$k_i'(0)=-\frac{r}{\theta_i}\big(u-\frac{\lambda\mu}{r}(\theta_i-\eta)\big),\qquad k_i''(\alpha)=-\lambda\bar{F}_X(\alpha)<0.$$

Thus, $k_1(\alpha)=0$ is a unique solution for $\alpha_1^*(u)\in(0,\infty)$ if $u<u_1$ and the solution $\alpha_1^*(u)=0$ if $u\geq u_1$; $k_2(\alpha)=0$ is a unique solution for $\alpha_2^*(u)\in(0,\infty)$ if $u<u_1$ and the solution $\alpha_2^*(u)=0$ if $u\geq u_2$. In addition, when $r=0$, we see that $k_i(\alpha)=0$ admits a unique solution $\alpha_i^0>0$. Finally, we observe that $k_i(\alpha)$ is strictly decreasing in r, u and β, thus if $\alpha_i^*(u)>0$, it is strictly decreasing in r, u and β with $\alpha_i^*(u)\leq\alpha_i^0$. Similarly, we see that $\alpha_i^*(u)$ is strictly increasing in θ and hence $\alpha_1^*(u)<\alpha_2^*(u)$. □

Lemma 7
For $u_R\in(0,u_2)$, the equation (19) admits a solution $O_{R_1}(.;u_R)\in\mathbb{C}^2(0,u_R)$ and the equation (20) admits a solution $O_{R_2}(.;u_R)\in\mathbb{C}^2(0,u_2)$. Moreover, the two functions $O_{R_1}(u_R;u_R)$ and $O_{R_2}(u_R;u_R)$ are continuous in $u_R\in[0,u_1]$ with $O_{R_2}''(u_R+;u_R)<0$ and $\lim_{u_R\to u_1}O_{R_2}(u_R;u_R)<\nu_1$.

Proof
We prove this lemma in three steps:
(i) First, we show that equations (19) and (20) admit solutions. To this end, given $u_R\in(0,u_2)$, consider two stochastic processes $\{\hat{R}_t\}_{t\geq0}$ and $\{R_t\}_{t\geq0}$. The former has initial state $\hat{R}_0=\hat{r}\in(0,u_R)$, and is absorbed at 0 and reflected at u_R; whereas the latter has initial state $\bar{R}_0=\bar{r}\in(0,u_1)$, and is reflected at u_R and absorbed at u_2. The dynamics of $\{\hat{R}_t\}$ and $\{R_t\}$ are defined by

$$\begin{cases} d\hat{R}_t=\big(r\hat{R}_t+\lambda\big(\theta_1\mu\big(\alpha_1^*\hat{R}_t\big)+(\beta-\theta_1)\mu\big)\big)dt+\sqrt{\lambda}\sigma\big(\alpha_1^*\hat{R}_t\big)dB_t-d\hat{B}_t,\\ d\bar{R}_t=\big(r\bar{R}_t+\lambda\big(\theta_1\mu\big(\alpha_1^*\bar{R}_t\big)+(\beta-\theta_1)\mu\big)\big)dt+\sqrt{\lambda}\sigma\big(\alpha_1^*\hat{R}_t\big)dB_t-d\bar{B}_t,\\ \hat{R}_0=\hat{r}\in[0,u_R],\qquad \bar{R}_0=\bar{r}\in[0,u_R],\end{cases}$$

where $\hat{B}_t$ and $\bar{B}_t$ are F-adapted, nondecreasing, left continuous processes with $\hat{B}_0=\bar{B}_0=0$. Define

$$\begin{cases} O_{R_1}(\hat{r};u_R)=E\big(\int_0^{\hat{\tau}_0}e^{-\zeta t}\lambda\theta_1\big(\mu-\mu\big(\alpha_1^*\hat{R}_t\big)dt+\int_0^{\hat{\tau}_0}e^{-\zeta t}d\hat{B}_t\big),\\ O_{R_2}(\bar{r};u_R)=E\big(e^{-\zeta t}\nu_2+\int_0^{\bar{\tau}_0}e^{-\zeta t}\lambda\theta_2\big(\mu-\mu\big(\alpha_2^*\bar{R}_t\big)dt+\int_0^{\bar{\tau}_0}e^{-\zeta t}d\bar{B}_t\big),\end{cases}$$

where $\hat{\tau}_0=\inf\{t\geq0:\hat{R}_t=0\}$ and $\bar{\tau}_0=\inf\{t\geq0:\bar{R}_t=u_2\}$ are F-stopping times. According to Lemma 6, the volatilities $\sqrt{\lambda}\sigma\big(\alpha_1^*(u)\big)$ and $\sqrt{\lambda}\sigma\big(\alpha_2^*(u)\big)$ are bounded away from 0 in $(0,u_R)$ and (u_R,u_2), respectively. Also, since $O_{R_1}(.;u_R)$ and $O_{R_2}(.;u_R)$ are continuously dependent on u_R, we see that $O_{R_1}(.;u_R)$ and $O_{R_2}(.;u_R)$ are continuous in $[0,u_1]$.
(ii) We prove that $O_{R_2}''(u_R+;u_R)<0$, $u_R\in(0,u_1)$, by contradiction. Assume that $O_{R_2}''(u_R+;u_R)>0$. Since $O_{R_2}'(u_R+;u_R)=1$, either of the following two cases hold:
(a) $O_{R_2}'(u;u_R)$ has a local maximum at $u_h\in(u_1,u_2)$ such that $O_{R_2}'(u_h;u_R)>1$, $O_{R_2}''(u_h;u_R)=0$ and

$O'''_{R_2}(u_h;u_R) \leq 0$.
(b) $O'_{R_2}(u;u_R)$ has a global maximum at u_2 such that $O'_{R_2}(u_2-;u_R) > 1$ and $O'''_{R_2}(u_2-;u_R) \geq 0$.
For case (a), by evaluating $\frac{d}{du}\mathbb{L}^{\alpha_2^*,\theta_2}O_{R_2}(u,u_R)|_{u=u_h} = 0$, and the fact that $\frac{\alpha_2^*(u_h)}{du} < 0$ (see Lemma 6), we have

$$\big(r + \lambda\theta_2\mu\big(\alpha_2^*(u_h)\big)\frac{\alpha_2^*(u_h)}{du} - \zeta\big)O'_{R_2}(u_h;u_R) - \lambda\theta_2\mu\big(\alpha_2^*(u_h)\big)\frac{\alpha_2^*(u_h)}{du} \geq 0,$$

on the other hand

$$(r-\zeta)O'_{R_2}(u_h;u_R) + \big(O'_{R_2}(u_h;u_R) - 1\big)\lambda\theta_2\mu\big(\alpha_2^*(u_h)\big)\frac{\alpha_2^*(u_h)}{du} < 0,$$

which is a contradiction. For case (b), we have $O'_{R_2}(u,u_R) > 1$, $u \in (u_R, u_2]$. However,

$$\begin{aligned} 0 &= \mathbb{L}^{\alpha_2^*,\theta_2}O_{R_2}(u_R,u_R) > \big(ru_R + \lambda\big(\theta_2\mu\big(\alpha_2^*(u_R)\big) + (\beta-\theta_2)\mu\big)\big) \\ &\quad -\zeta O_{R_2}(u_R,u_R) + \lambda\theta_2\big(\mu - \mu\big(\alpha_2^*(u_R)\big)\big). \end{aligned}$$

Thus, $O_{R_2}(u_2,u_R) - O_{R_2}(u_R,u_R) > \frac{\lambda\theta_2\mu}{\zeta} - \frac{ru_R+\lambda\beta\mu}{\zeta} = \frac{r}{\zeta}(u_2-u_R) < u_2 - u_R$, which is a contradiction.
On the other hand, assume that $O''_{R_2}(u_R+,u_R) = 0$. With $O'_{R_2}(u_R+,u_R) = 1$, direct calculation shows

$$\begin{aligned} 0 &= \lambda\frac{\sigma^2\big(\alpha_2^*(u_R)\big)}{2}O'''_{R_2}(u_R+,u_R) \\ &\quad +\big(r + \lambda\theta_2\mu\big(\alpha_2^*(u_R)\big)(\alpha_2^*)'(u_R) - \zeta\big)O'_{R_2}(u_R+,u_R) \\ &\quad -\lambda\theta_2\mu\big(\alpha_2^*(u_R)\big)(\alpha_2^*)'(u_R) \\ &= \lambda\frac{\sigma^2\big(\alpha_2^*(u_R)\big)}{2}O'''_{R_2}(u_R+,u_R) + r - \zeta, \end{aligned}$$

which gives that $O'''_{R_2}(u_R+,u_R) > 0$. Thus, we have $O''_{R_2}(u_R+,u_R) > 0$, $u \in (u_R, u_R+\epsilon)$ with $\epsilon > 0$ being small. By performing exactly the same procedure as above, we see that contradiction exists, and thus the assumption $O''_{R_2}(u_R+,u_R) = 0$ does not hold. Argument (ii) is proved.
(iii) From $O''_{R_2}(u_R+,u_R) = 1$ and $\mathbb{L}^{\alpha_2^*,\theta_2}O_{R_2}(u_R,u_R) = 0$, we have

$$\begin{aligned} O_{R_2}(u_R,u_R) &= \frac{ru_R + \lambda\beta\mu + \frac{\lambda}{2}\sigma^2\big(\alpha_2^*(u_R)\big)O''_{R_2}(u_R+,u_R)}{\zeta} \\ &< \frac{ru+\lambda\beta\mu}{\zeta} = \nu_1. \end{aligned}$$

Specially, by letting $u_R = u_1$, we have $O_{R_2}(u_1,u_1) < \frac{ru+\lambda\beta\mu}{\zeta} = \nu_1$. These arguments complete the proof. □

Theorem 5.1
(i) If $O_{R_1}(0,0) > 0$, the equilibrium strategy is given by (15) where u_R solves $O_{R_1}(u_R,u_R) = O_{R_2}(u_R,u_R)$. Moreover, the equilibrium value

$$O_R(u) = \begin{cases} O_{R_1}(u;u_R), & u \in [0,u_R], \\ O_{R_2}(u;u_R), & u \in [u_R,u_2], \end{cases} \tag{67}$$

and

$$O_I(u) = \begin{cases} O_I(u_R) - O'_I(u_R)\int_u^{u_R}\exp\big(-\int_y^{u_R}\frac{\theta_1}{\alpha_1^*(z)}dz\big)dy, & u \in [0,u_R], \\ O_I(u_R) + O'_I(u_R)\int_u^{u_R}\exp\big(-\int_y^{u_R}\frac{\theta_1}{\alpha_2^*(z)}dz\big)dy, & u \in [u_R,u_2], \end{cases} \tag{68}$$

where

$$\begin{cases} O'_I(u_R) = -\frac{1}{\int_0^{u_R}\exp\left(-\int_y^{u_R}\frac{\theta_1}{\alpha_1^*(z)}dz\right)dy+\int_u^{u_R}\exp\left(-\int_y^{u_R}\frac{\theta_1}{\alpha_2^*(z)}dz\right)dy}, \\ O_I(u_R) = 1 + O'_I(u_R)\int_0^{u_R}\exp\big(-\int_y^{u_R}\frac{\theta_1}{\alpha_1^*(z)}dz\big)dy. \end{cases} \tag{69}$$

(ii) If $O_{R_1}(0,0) \le 0$, the equilibrium strategy is given by (15) with $u_R = 0$. Moreover,

$$\begin{cases} O_R(u) = E\big(\nu_2 e^{-\zeta\tau_2} I(\tau_2 < \tau_0) + \int_0^{\tau_2\wedge\tau_0} e^{-\zeta\tau_2}\lambda\theta_2\big(\mu - \mu\big(\alpha_2^*(R_t)\big)\big)dt\big), \\ O_I(u) = \frac{1}{\int_0^{u_2}\exp\left(-\int_y^{u_2}\frac{\theta_2}{\alpha_2^*(z)}dz\right)dy}\int_u^{u_2}\exp\big(-\int_y^{u_2}\frac{\theta_2}{\alpha_2^*(z)}dz\big)dy, \end{cases} \tag{70}$$

where $\tau_2 = \inf\{t \ge 0 : R_t \ge u_2\}$.

Proof
(i) For $O_{R_1}(0,0) > 0$, we prove this theorem in three steps:
(a) $\big(O_R(u), O_I(u)\big)$ are the solutions to $P1$.
(b) Proving that $O'_R(u) \le 1$, $u \in (u_R, u_2)$.
(c) Proving that $O'_R(u) > 1$, $u \in (0, u_R)$.
Then, since $O_I(u)$ given in (68) is strictly convex and that (α^*, θ^*) are admissible, similar to the proof of Theorem 4.1, we see that $O_I(u)$, $O_R(u)$ and (α^*, θ^*) are the equilibrium strategies.
Since $O_{R_1}(0,0) = 0 < O_{R_2}(0,0)$ and $O_{R_1}(u_1,u_1) = \nu_1 > O_{R_2}(u_1,u_1)$, by continuity there exists $u_R \in (0,u_1)$ such that $O_{R_1}(u_R,u_R) = O_{R_2}(u_R,u_R)$. From Lemma 7, we see that $O_R(u)$ is the solution to equations (19)-(21). Also, it is clear that $O_I(u)$ satisfies the equations (16)-(18).
(b) With some calculations, we get

$$\begin{aligned} &\lambda\frac{\sigma^2\big(\alpha_i^*(u)\big)}{2}O'''_R(u) + \big(\lambda\alpha_i^*(u)\mu\big(\alpha_i^*(u)\big)\big(\alpha_i^*(u)\big)' \\ &+ru + \lambda\big(\theta_i\mu\big(\alpha_i^*(u)\big) + (\beta-\theta_i)\mu\big)\big)O''_R(u) \\ &+\big(r + \lambda\theta_i\mu\big(\alpha_i^*(u)\big)\big(\alpha_i^*\big)'(u) - \zeta\big)O'_R(u) \\ &-\lambda\theta_i\mu\big(\alpha_i^*(u)\big)\big(\alpha_i^*\big)'(u) = 0. \end{aligned} \tag{71}$$

We prove by contradiction and assume that $\sup_{u\in(u_R,u_2]} O'_R(u) > 1$. Assume there exists local maximum point $u_m \in (u_R, u_2)$ such that $O'_R(u_m) > 1$, $O''_R(u_m) = 0$ and $O'''_R(u_m) \le 0$. Combining (71) and the fact that $\frac{d}{du}\alpha_2^*(u)|_{u=u_m} < 0$ (see Lemma 6), we have

$$\begin{aligned} 0 &\ge \lambda\frac{\sigma^2\big(\alpha_2^*(u)\big)}{2}O'''_R(u_m) \\ &= \lambda\theta_2\mu\big(\alpha_2^*(u_m)\frac{d}{du}\alpha_2^*(u_m)(1-O'_R(u_m)) + (\zeta - r)O'_R(u_m) \\ &\ge (\zeta - r)O'_R(u_m) > 0, \end{aligned}$$

which is a contradiction. Otherwise, assume $O'_R(u)$ has global maximum point at u_2 such that $O'_R(u_m) > 1$. Let $\bar{u} = \sup\{u \in (u_R, u_2) : O'_R(u) = 1\}$. Then $O'_R(\bar{u}) = 1$, $O''_R(\bar{u}) \ge 0$ and

$$\begin{aligned} 0 &= \mathbb{A}^{\alpha_2^*,\theta_2}O_R(\bar{u}) \ge \big(ru + \lambda\big(\theta_2\mu\big(\alpha_2^*(\bar{u})\big) + (\beta-\theta_2)\mu\big)\big) \\ &\quad -\zeta O_R(\bar{u}) + \lambda\big(\mu - \big(\alpha_2^*(\bar{u})\big)\big)\theta_2 \\ &= r(\bar{u} - u_2) + \zeta O_R(u_2) - \zeta O_R(\bar{u}), \end{aligned}$$

which gives that $\frac{O_R(u_2)-O_R(\bar{u})}{u_2-\bar{u}} \le \frac{r}{\zeta} < 1$, and this contradicts the fact that $O'_R(u) > 1$, for $u \in (\bar{u}, u_2)$.
(c) Since $O'_R(u_R) = 1$ and $O''_R(u_R+) < 0$ (see Lemma 7), we $O''_R(u_R-) < 0$ and hence $O'_R(u) > 1$, for $u \in$

$(u_R-\epsilon, u_R)$ with $\epsilon>0$ begin small. We claim that u_R is strictly concave on $(0,u_R)$ and thus our argument holds. Otherwise, let $\bar{\bar{u}}=\sup\{u\in(0,u_2): O''_R(u)=0\}$. Then $O'''_R(\bar{\bar{u}})\le 0$, $O''_R(\bar{\bar{u}})=0$ and $O'_R(\bar{\bar{u}})>1$. Therefore,

$$\begin{aligned}0 &\geq \lambda\frac{\sigma^2\big(\alpha_2^*(\bar{\bar{u}})\big)}{2}O'''_R(\bar{\bar{u}})\\ &= \lambda\theta_1\mu\big(\alpha_1^*(\bar{\bar{u}})\frac{d}{du}\alpha_2^*(\bar{\bar{u}})(1-O'_R(\bar{\bar{u}}))+(\zeta-r)O'_R(\bar{\bar{u}})>0,\end{aligned}$$

which is a contradiction.
(ii) For the case $O'_{R_2}(0,0)\le 0$, since $O_I(u)$ given in (70) is strictly convex, we just need to show that $O'_R(u)<1$, for $u\in(0,u_2)$. First, we show that $O'_R(0)\le 1$. Otherwise, if $O'_R(0)>1$, then the inequality

$$\begin{aligned}O_R(u) &= E\big(\nu_2 e^{-\zeta\tau_2}I(\tau_2<\tau_0)+\int_0^{\tau_2}e^{-\zeta t}\lambda\theta_2\big(\mu-\mu\big(\alpha_2^*(\bar{R}_t)\big)\big)dt\big)\\ &\quad -O'_R(0)\int_0^{\tau_2}e^{-\zeta t}d\bar{B}_t\big)\\ &< E\big(\nu_2 e^{-\zeta\tau_2}I(\tau_2<\tau_0)+\int_0^{\tau_2}e^{-\zeta t}\lambda\theta_2\big(\mu-\mu\big(\alpha_2^*(\bar{R}_t)\big)\big)dt\big)\\ &\quad -\int_0^{\tau_2}e^{-\zeta t}d\bar{B}_t\big)=O_{R_2}(u,0).\end{aligned}$$

Thus, $0=O_R(0)<O_{R_2}(u,0)\le 0$, which is a contradiction, and this completes our proof. □

With Theorem 5.1 we may determine the solutions to our optimization problem numerically by using the Monte Carlo simulation or finite difference method. As an example, we suppose that the claims $\{X_i\}_{i\in N}$ has the Pareto distribution with density $\frac{F_X(y)}{dy}=3(1+y)^{-4}$, $y\ge 0$. In this density, $\mu=0.9706$ and $\sigma=0.9853$. The default parameters are given in Table 2.

Table 2. Model parameters for the expected value principle

Parameters	β	θ_1	θ_2	r	ζ	λ
Values	0.17	0.2	0.3	0.05	0.13	0.3

When $\beta=0.17$, we have $O_{R_1}(0,0)=0.0406$ and $u_R=0.070$. According to Theorem 5.1, at equilibrium the reinsurance is priced with $\theta^*=0.3$ when insurer's cash reserve is more than $u_R=0.070$, and is adjusted down to $\theta^*=0.2$ when the insurer's cash reserve is less than $u_R=0.070$. When $\beta=0.10$, we have $O_{R_1}(0,0)=-0.096$ and $u_R=0$. Therefore, the reinsurance contract is provided at the peak price with $\theta^*=0.3$. Figure 4 plots the equilibrium strategy and the equilibrium value functions. In line with our findings in Section 4, we see that, with a larger β the insurance business becomes more profitable, the insurer is able to afford for more insurance protection to decrease her ruin probability. Correspondingly, the contract becomes more valuable.

6. Numerical example

In this section, we conduct a numerical study to investigate the effects of model parameters on the equilibrium strategy. To simplify calculation, we assume that both the insurer and the reinsurer adopt the standard-deviation principle, and set the default model parameters as given in Table 1. Since the insurer's optimal reinsurance strategy provided in Theorem 4.1 has a simple structure and is easy to understand, in this section we mainly focus on the impact of model parameters on the reinsurance contract, which is uniquely characterized by the point u_R. Figure 5 shows the impacts of parameters β, r, λ and ζ on u_R. When the insurer's capital reserve is more (or less) than u_R, as stated in Section 4, the reinsurance contract is in high (or low) price region. Figure 5(a) plots u_R as the function of the insurer's safety loading $\beta\in(\frac{\theta_1}{2},\frac{\theta_2}{2})=(0.16,0.3)$. This figure clearly shows that the safety loading 0.2 where

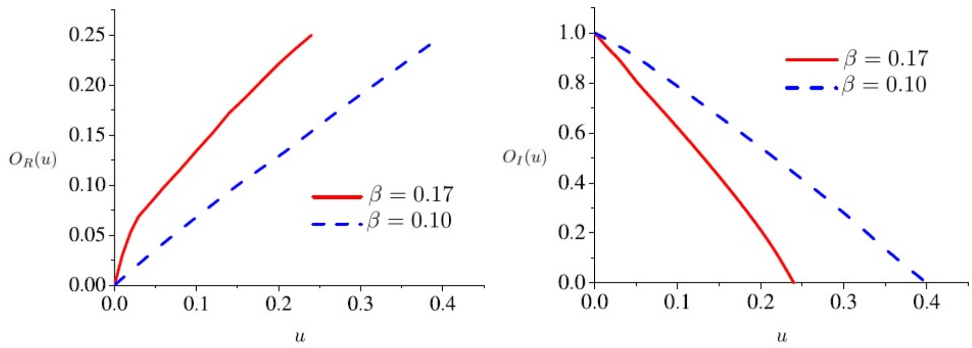

Figure 4. Equilibrium strategies and value functions when $\beta = 0.17$ and $\beta = 0.10$.

$q_0 = 0$, is critical for the reinsurance contract. For $\beta > 0.2$, we have $q_0 > 0$. Moreover, u_R is an increasing function of β, indicating that the reinsurance contract becomes cheaper as the insurance business becomes more profitable. In fact, as β increases, the insurer becomes more profitable and is at a lower risk level, thus less risk premium is imposed on the reinsurance and the reinsurance contract becomes cheaper. We can also see that, when $\beta > 0.282$, since the price of reinsurance is low, therefore the insurer chooses to buy reinsurance to control the risk. When the insurer is less profitable with $\beta \in (0.2, 0.282)$, as the reinsurance becomes more expensive, the insurer buys some reinsurance if the price is low $(u < u_R)$ or he/she has a sufficient initial reserve $(u > u_0)$. Finally, when $\beta < 0$, $q_0 < 0$, the insurance business is in a bad state such that reinsurance contract is offered at peak prices θ_2. Paying the reinsurance puts a heavy financial burden on the insurer, so he/she will buy reinsurance only if he/she has a sufficient initial reserve $(u > u_0)$. Figure 5(b) that r has similar impacts on the point u_R. That is, when the insurer gets a higher return on its investment, the reinsurance contract becomes cheaper and the insurer has more demand for it.

Figure 5(c) shows that, when $\lambda < 1.01$, $q_0 < 0$ and hence $u_R = 0$. when $\lambda > 1.01$, $q_0 > 0$ and u_R is positive and slightly increases as λ increases. Since λ can be used to measure the insurer's business, this result shows that the insurer's business scale has scale slight effect on the price of reinsurance. In addition, we see that, as λ increases, the no reinsurance purchase region enlarges. That is with a larger business scale, the insurer is less inclined to search reinsurance protection One possible reason is that with a larger business scale, the insurer becomes more stable and has fewer assets willingness to buy reinsurance Figure 5(d) shows that, when $\zeta < 0.1$, $q_0 > 0$ and u_R is positive and a decreasing function of ζ. Moreover, as λ increases, the purchase area without reinsurance becomes large. Since ζ represents capital cost of reinsurer, this result shows that with higher capital cost, reinsurer focuses more on short term interest and tends to increase the price of reinsurance, which leads to less demand by insurers for reinsurance. Figure 6 illustrates the impact of θ_1 and θ_2 on the reinsurance price. We can see that u_R is increasing in θ_1 and decreasing in θ_2. This can be explained as follows. When θ_1 increases and gets closer to θ_2, a reinsurance contract with minimal price becomes more acceptable to the reinsurer and thus the low price region enlarges. Moreover, as θ_2 increases, the reinsurer naturally prefers to increase the high price area for more profit. We also note that the slope of the line connecting u_R is much steeper in panel (a) than it is in panel (b). This result shows that the change in the peak price has less effect on the insurer when it has a small insurer amount of cash reserve (note that the price in low price region remains unchanged). Otherwise, if u_R decreases sharply in panel (b), insurer with an insufficient cash reserve will probably give up the buy of reinsurance, which is not in the reinsurer's interest.

7. Conclusions

In this work, we investigated an optimal excess-of-loss reinsurance contract between an insurer and a reinsurer in a dynamic risk model which is assumed to be a diffusion approximation process of the classical Cramer-Lundberg model. We assumed that the insurer is allowed to invest his/her surplus into a financial market containing one

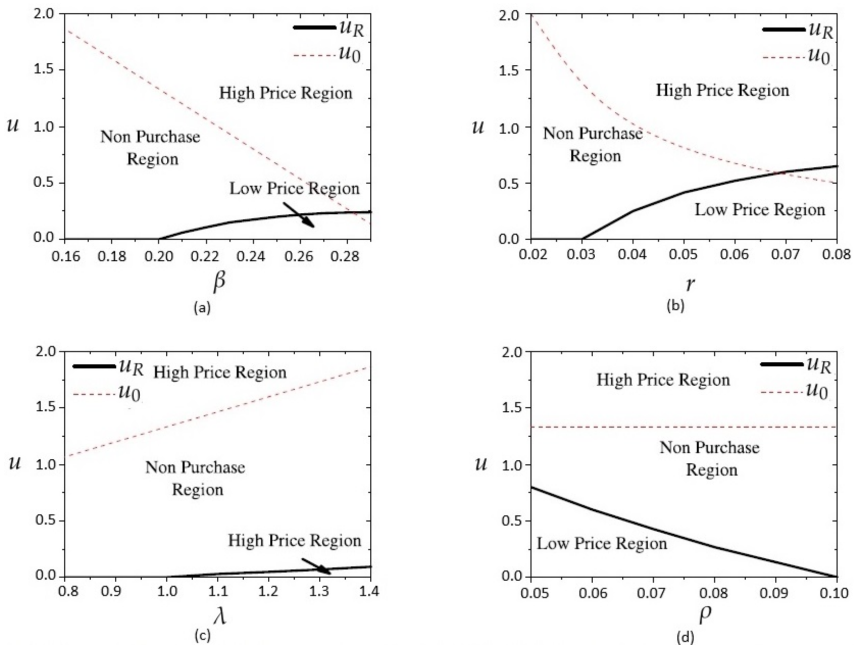

Figure 5. The impact of β, r, λ and ζ on u_R and u_0.

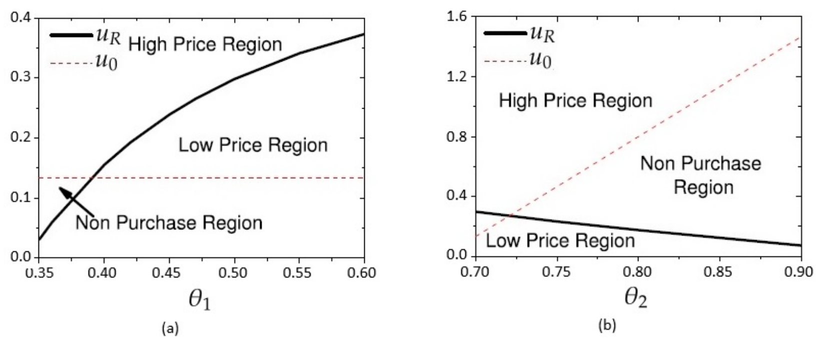

Figure 6. The impact of θ_1 and θ_2 on u_R and u_0.

risk-free rate of return and determines the reinsurance strategy by a self-reinsurance function. The simultaneous equilibrium strategy is obtained using the objective functions of insurer and reinsurance. Lemma 1 showed that, if the insurer has sufficient financial reserve in his/her insurance portfolio, at least u_2, he/she can transfer all the risks to the reinsurance to avoid bankruptcy. In this case, since reinsurance is in high demand, the reinsurer will charge the maximum price for the contract to increase his profit. In Theorem 3.1, we derived an optimal strategy in the insurance portfolio, which needs to obtain the values (α^*, θ^*) such that the equations (8) and (9) hold. In Sections 4 and 5, we provided the explicit solution of the optimization problem when the reinsurance premium function is based on the standard-deviation principle and standard-deviation principle, respectively. In our numerical example, the effects of model parameters on the equilibrium strategy for standard-deviation principle are studied.

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