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The New Topp-Leone Exponentiated Half Logistic-Gompertz-G Family of Distributions with Applications

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Abstract This research introduces a new family of distributions (FoD) titled the Topp-Leone Exponentiated-Half-Logistic-Gompertz-G (TL-EHL-Gom-G) distribution. The study explores a variety of statistical properties of the developed family, such as the quantile function, series expansion, order statistics, entropy, stochastic orders and moments. Through Monte Carlo simulations, various estimation techniques were compared, including the least squares (LS), Anderson Darling (AD), maximum likelihood (ML) and Cramér-von-Mises (CVM) methods via root mean square error (RMSE) and average bias (Abias). The results indicated that the ML estimation method performed better than other methods, hence, the selection for estimating the model parameters. To showcase the usefulness, robustness and applicability of the model, we applied it to three real-life data, including dataset with censored observations. The TL-EHL-Gom-W distribution, a special case of the TL-EHL-Gom-G FoD showed superiority over nested and non-nested models.

Keywords Maximum likelihood, Moments, Censoring, Exponentiated-Half-Logistic Distribution, Stochastic Ordering, Gompertz Distribution, Entropy, Topp-Leone Distribution.

AMS 2010 subject classifications 62E30; 60E05; 62E15

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1. Introduction

Lifetime distributions have garnered great attention from statisticians because of their wide applicability across diverse fields, including insurance, risk management, reliability analysis, medicine, engineering and public health. Among these distributions, the Topp-Leone distribution (Topp and Leone [31]) stands out as a prominent model in the study of lifetimes and failure times. It is a bounded distribution with domain between 0 and 1 and possesses a J-shaped probability density function (pdf). It is not flexible because its hazard rate function (hrf) is always bathtub-shaped. However, the distribution is very useful in situations where the risk of an event is initially high, decreases over time and then increases again towards the end. This characteristic makes the Topp-Leone distribution a popular choice in many fields such as reliability analysis, survival analysis and risk modeling. For a comprehensive proof of the bathtub hrf of the Topp-Leone distribution, see Glaser [12]. The following are some recent Topp-Leone generalizations with wide applications: Topp-Leone exponential-G family of distributions (FoD) by Sanusi et al. [27], odd Weibull-Topp-Leone-G power series FoD by Vasileva et al. [32], Topp-Leone odd exponential half logistic-G FoD by Chipepa and Oluyede [6], Topp-Leone Gompertz-G FoD by Oluyede et al. [22], Topp-Leone-Harris-G FoD by Oluyede et al. [25], Cosine Topp-Leone FoD by Nanga et al. [19], gamma Topp-Leone type II exponentiated half logistic-G FoD by Oluyede and Moakofi [24], and

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Topp-Leone type I heavy-tailed-G power series class of distributions by Nkomo et al. [20], just to mention but a few.

Al-Shomrani et al. [4] introduced the Topp-Leone-G (TL-G) family of distributions (FoD) with cumulative distribution (cdf)

$$F_{TL-G}(z;b,\underline{\psi}) = [1-\overline{G}^2(z;\underline{\psi})]^b \tag{1}$$

and probability density function (pdf)

$$f_{TL-G}(z;b,\underline{\psi}) = 2bg(z;\underline{\psi})\overline{G}(z;\underline{\psi})[1-\overline{G}^2(z;\underline{\psi})]^{b-1}, \tag{2}$$

for $b>0$, and parent parameter vector $\underline{\psi}$, where $\overline{G}(z;\underline{\psi}) = 1-G(z;\underline{\psi})$. Cordeiro [9] introduced the exponentiated-half-logistic-G (EHL-G) FoD with cdf

$$F_{EHL-G}(z;\alpha,\lambda,\underline{\psi}) = \left(\frac{1-\overline{G}^{\lambda}(z;\underline{\psi})}{1+\overline{G}^{\lambda}(z;\underline{\psi})}\right)^{\alpha}, \tag{3}$$

for $\alpha,\lambda>0$, and parent parameter vector $\underline{\psi}$. This family provides enhanced modeling capabilities to handle skewed data. In this research we set $\lambda=1$.

The Gompertz distribution (Gompertz [13]) offers a versatile framework for modelling various phenomena and has applications in computer science, reliability and demography. Some recent generalizations of the Gompertz distribution include the Marshall-Olkin exponential Gompertz distribution by Khaleel et al. [14], Marshall-Olkin Gompertz-G FoD by Chipepa and Oluyede, [7], and the Topp-Leone Gompertz exponentiated-half-logistic-G FoD by Dingalo et al. [11]. Alizadeh et al. [3] introduced the Gompertz-G (Gom-G) FoD with cdf

$$F_{Gom-G}(z;\gamma,\underline{\psi}) \quad = \quad 1-e^{\frac{1}{\gamma}[1-\overline{G}^{-\gamma}(z;\underline{\psi})]}, \tag{4}$$

for $\gamma>0$, and parent parameter vector $\underline{\psi}$. The pdf of the Gompertz distribution is capable of exhibiting either right or left skewness, making it well-suited for modeling extreme data.

If we let Equation (4) to be the parent cdf of Equation (3), we get the EHL-Gom-G FoD with cdf

$$F_{EHL-Gom-G}(z;\gamma,\alpha,\underline{\psi}) \quad = \quad \left(\frac{1-\exp\left\{\frac{1}{\gamma}[1-\overline{G}^{-\gamma}(z;\underline{\psi})]\right\}}{1+\exp\left\{\frac{1}{\gamma}[1-\overline{G}^{-\gamma}(z;\underline{\psi})]\right\}}\right)^{\alpha}$$

for $\gamma,\alpha>0$, and parent parameter vector $\underline{\psi}$. In this paper, we will set $K_G(z;\gamma,\alpha,\underline{\psi}) = \left(\frac{1-\exp\left\{\frac{1}{\gamma}[1-\overline{G}^{-\gamma}(z;\underline{\psi})]\right\}}{1+\exp\left\{\frac{1}{\gamma}[1-\overline{G}^{-\gamma}(z;\underline{\psi})]\right\}}\right)^{\alpha}$.

Different estimation techniques are important when developing new generalized distributions because each technique has distinct advantages for parameter estimation than others. For example, maximum likelihood (ML) estimation provides efficient and consistent estimates, which are especially useful for large samples. The least squares approach on the other hand reduces the sum of squared differences between observed and predicted values, ensuring a close fit to the data. Finding the best technique to estimate the parameters in a new FoD is critical, particularly for model applications. For further information on different methods of estimation see Dey et al. [10], Ali et al. [2], and Warahena-Liyanage et al. [33], among others.

The motivations for developing the TL-EHL-Gom-G FoD:

- The TL-EHL-Gom-G FoD offers better fitting capabilities compared to earlier families such as TL-G, Gom-G and EHL-G. This enhancement is primarily attributed to the inclusion of an additional shape parameter, which allows for greater flexibility in modeling diverse data patterns.

- The new family aims to establish a modeling framework that can adapt to various hazard rate behaviors.
- The TL-EHL-Gom-G FoD is capable of modeling a wide array of data, including highly skewed datasets.

The layout of the paper is as follows: The new TL-EHL-Gom-G FoD, sub-models, quantile function, hrf, as well as the series expansion and special cases are presented in Section 2. Section 3 consists of additional statistical properties of the new FoD, including the moments, distribution of order statistics, Rényi entropy and stochastic orders. Section 4 discusses the various parameter estimation techniques of the new FoD. Section 6 presents the Monte Carlo simulation (MCS) results using various estimation techniques. Section 7 showcases applications of the TL-EHL-Gom-W distribution a special case of the TL-EHL-Gom-G FoD to real-world scenarios. Section 7 provides a summary of the key findings followed by conclusions.

2. The New Model, Statistical Properties and Special Cases

We present the new model, several statistical properties and special cases of the TL-EHL-Gom-G FoD in this section.

2.1. The New Model

Taking Equation (5) to be the parent distribution of Equation (1), we get the TL-EHL-Gom-G FoD with cdf

$$F_{TL-EHL-Gom-G}(z;\Phi,\underline{\psi}) = \left(1-\left[1-K_G(z;\gamma,\alpha,\underline{\psi})\right]^2\right)^b \tag{5}$$

and pdf

$$\begin{aligned} f_{TL-EHL-Gom-G}(z;\Phi,\underline{\psi}) &= \frac{4\alpha b g(z;\underline{\psi})\exp\{\frac{1}{\gamma}[1-\overline{G}^{-\gamma}(z;\underline{\psi})]\}}{[\overline{G}(z;\underline{\psi})]^{\gamma+1}} \\ &\times \left(\frac{\left[1-\exp\{\frac{1}{\gamma}[1-\overline{G}^{-\gamma}(z;\underline{\psi})]\}\right]^{\alpha-1}}{\left[1+\exp\{\frac{1}{\gamma}[1-\overline{G}^{-\gamma}(z;\underline{\psi})]\}\right]^{\alpha+1}}\right) \\ &\times \left[1-K_G(z;\gamma,\alpha,\underline{\psi})\right] \\ &\times \left(1-\left[1-K_G(z;\gamma,\alpha,\underline{\psi})\right]^2\right)^{b-1}, \end{aligned} \tag{6}$$

for $\Phi = b,\gamma,\alpha > 0$, $\underline{\psi}$ is a parent parameter vector, where $K_G(z;\gamma,\alpha,\underline{\psi}) = \left(\frac{1-\exp\left\{\frac{1}{\gamma}[1-\overline{G}^{-\gamma}(z;\underline{\psi})]\right\}}{1+\exp\left\{\frac{1}{\gamma}[1-\overline{G}^{-\gamma}(z;\underline{\psi})]\right\}}\right)^\alpha$.

2.2. Quantile Function

The quantile function (QF) is a very important statistical tool, serving multiple purposes. It is utilized not only for generating random numbers but also for deriving various statistical metrics, including extreme quantiles, the median, kurtosis and skewness. The QF of the TL-EHL-Gom-G FoD is

$$Q_Z(p) = G^{-1}\left(1-\left[1-\gamma\log\left(\frac{1-(1-[1-p^{\frac{1}{b}}]^{\frac{1}{2}})^{\frac{1}{\alpha}}}{1+(1-[1-p^{\frac{1}{b}}]^{\frac{1}{2}})^{\frac{1}{\alpha}}}\right)\right]^{\frac{-1}{\gamma}}\right),$$

for $p \in [0,1]$, where G is the parent cdf.

2.3. Series Expansion and Representation

In this subsection, we express the pdf of the TL-EHL-Gom-G FoD as an infinite linear combination of exponentiated-G (Exp-G) FoD. Therefore,

$$f_{TL-EHL-Gom-G}(z;\Phi,\underline{\psi}) = \sum_{k=0}^{\infty}\omega_{k+1}g_{k+1}(z;\underline{\psi}),$$

where $g_{k+1}(z;\underline{\psi}) = (k+1)g(z;\underline{\psi})G^{k}(z;\underline{\psi})$ is the Exp-G distribution with parameter $(k+1)$ and linear component

$$\begin{aligned}\omega_{k+1} &= \sum_{i,j,l,m,p,q=0}^{\infty}\frac{4\alpha b(-1)^{i+j+l+q+r}}{p!}\left(\frac{l+m+1}{\gamma}\right)^{p}\binom{b-1}{i}\binom{2i+1}{j}\\ &\times \binom{\alpha j+\alpha-1}{l}\binom{-(\alpha j+\alpha+1)}{m}\binom{p}{q}\binom{q-\gamma-1}{k}\frac{1}{(r+1)}. \qquad (7)\end{aligned}$$

It follows that the TL-EHL-Gom-G FoD can be expressed as a linear combination of the Exp-G distribution with parameter $(k+1)$. *For all derivations, please refer to the Webb Appendix.*

2.4. Special Cases

In this section, we provide two special cases of the TL-EHL-Gom-G FoD, including Topp-Leone Exponentiated-Half-Logistic-Gompertz-Weibull (TL-EHL-Gom-W) and Topp-Leone Exponentiated-Half-Logistic-Gompertz-Burr XII (TL-EHL-Gom-BXII) distributions. We also present the pdfs, hrfs as well as skewness and kurtosis for these special cases

Table 1. Baseline Distributions

Baseline	cdf	pdf
Weibull(W)	$G(z;\beta) = 1-e^{-z^{\beta}}$	$g(z;\beta) = \beta z^{\beta-1}e^{-z^{\beta}};\ z,\beta>0$
Burr XII(BXII)	$G(z;c,k) = 1-(1+z^{c})^{-k}$	$g(z;k) = ckz^{c-1}(1+z^{c})^{-k-1};\ z,c,k>0$

Table 1 presents baseline distributions for the TL-EHL-Gom-G FoD.

2.4.1. TL-EHL-Gom-W Distribution Using the Weibull baseline in Table 1, the TL-EHL-Gom-W distribution has cdf

$$F(z;\Phi,\beta) = \left(1-[1-K_W(z;\gamma,\alpha,\beta)]^{2}\right)^{b}$$

and pdf

$$\begin{aligned}f(z;\Phi,\beta) &= \frac{4\alpha b\beta z^{\beta-1}e^{-z^{\beta}}\exp\left\{\frac{1}{\gamma}[1-e^{\gamma z^{\beta}}]\right\}}{[e^{-z^{\beta}}]^{\gamma+1}}\\ &\times \left(\frac{\left[1-\exp\left\{\frac{1}{\gamma}[1-e^{\gamma z^{\beta}}]\right\}\right]^{\alpha-1}}{\left[1+\exp\left\{\frac{1}{\gamma}[1-e^{\gamma z^{\beta}}]\right\}\right]^{\alpha+1}}\right)\\ &\times [1-K_W(z;\gamma,\alpha,\beta)]\left(1-[1-K_W(z;\gamma,\alpha,\beta)]^{2}\right)^{b-1},\end{aligned}$$

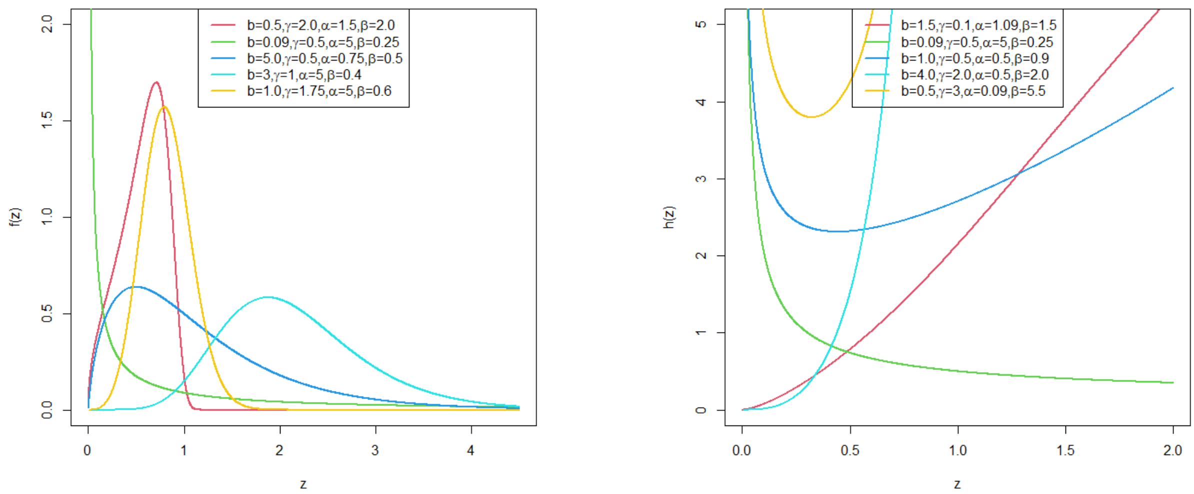

Figure 1. TL-EHL-Gom-W pdf and hrf plots

for $\Phi, \beta > 0$, where $K_W(z;\gamma,\alpha,\beta) = \left(\frac{1-\exp\left\{\frac{1}{\gamma}[1-e^{\gamma z^{\beta}}]\right\}}{1+\exp\left\{\frac{1}{\gamma}[1-e^{\gamma z^{\beta}}]\right\}}\right)^{\alpha}$.
Figure 1 displays density plots of the TL-EHL-Gom-W with various shapes, such as reverse-J, almost symmetric and right-skewed. Plots of the hrf show monotonic and non-monotonic shapes. The 3D plots of skewness and kurtosis in Figure 2 show that the TL-EHL-Gom-W has the capacity to take different shapes for different parameter values demonstrating that our proposed model is flexible.

2.4.2. TL-EHL-Gom-BXII Distribution Using the Burr XII baseline in Table 1, the TL-EHL-Gom-BXII distribution has cdf

$$F(z;\Phi,c,k) = \left(1 - [1 - K_{BXII}(z;\gamma,\alpha,c,k)]^2\right)^b,$$

and pdf

$$\begin{aligned} f(z;\Phi,c,k) &= \frac{4\alpha bckz^{c-1}(1+z^c)^{-k-1}\exp\left\{\frac{1}{\gamma}[1-(1+z^c)^{\gamma k}]\right\}}{[(1+z^c)^{-k}]^{\gamma+1}} \\ &\times \left(\frac{\left[1-\exp\left\{\frac{1}{\gamma}[1-(1+z^c)^{\gamma k}]\right\}\right]^{\alpha-1}}{\left[1+\exp\left\{\frac{1}{\gamma}[1-(1+z^c)^{\gamma k}]\right\}\right]^{\alpha+1}}\right) \\ &\times [1-K_{BXII}(z;\gamma,\alpha,c,k)]\left(1-[1-K_{BXII}(z;\gamma,\alpha,c,k)]^2\right)^{b-1}, \end{aligned}$$

for $\Phi, c, k > 0$, where $K_{BXII}(z;\gamma,\alpha,c,k) = \left(\frac{1-\exp\left\{\frac{1}{\gamma}[1-(1+z^c)^{\gamma k}]\right\}}{1+\exp\left\{\frac{1}{\gamma}[1-(1+z^c)^{\gamma k}]\right\}}\right)^{\alpha}$.
If we set $k = 1$, we obtain a Topp-Leone exponentiated Half Logistic Gompertz-log-logistic distribution. If we also set $c = 1$, obtain a Topp-Leone exponentiated half logistic Gompertz-Lomax distribution. The TL-EHL-Gom-BXII density plots shown in Figure 3 exhibit a variety of shapes, such as reverse-J, almost symmetric and right-skewed shapes. Plots of the hrf show monotonic and non-monotonic shapes. The 3D plots presented in Figure 4 illustrate

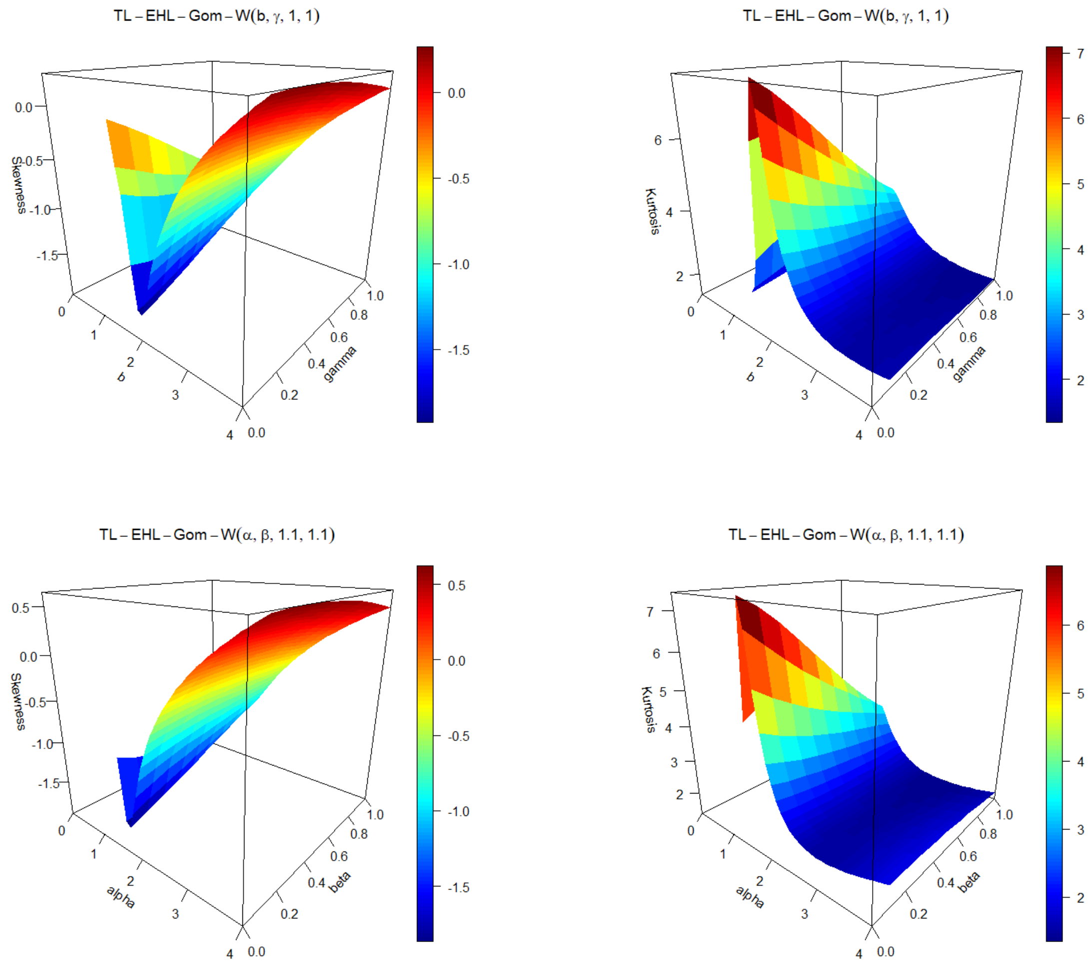

Figure 2. TL-EHL-Gom-W SK and KS Plots

that the TL-EHL-Gom-BXII distribution is capable of assuming a variety of shapes for different parameter values, highlighting the flexibility of our proposed model.

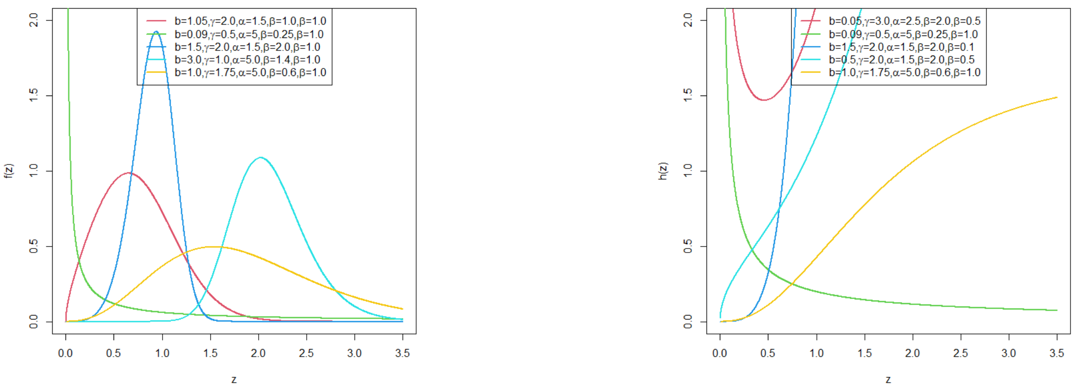

Figure 3. TL-EHL-Gom-BXII pdf and hrf Plots

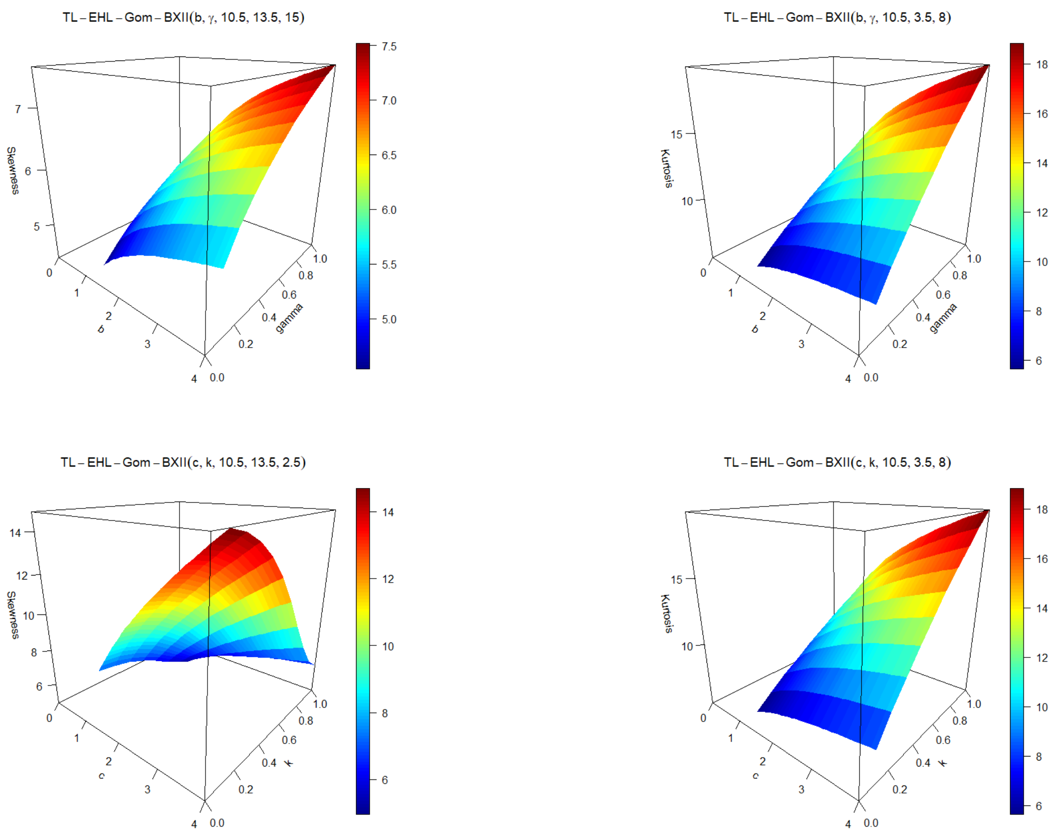

Figure 4. TL-EHL-Gom-BXII SK and KS Plots

3. Additional Statistical Properties of the TL-EHL-Gom-G FoD

We derive additional statistical properties of the TL-EHL-Gom-G FoD, including moments, order statistics, Rényi entropy and stochastic orders in this section.

3.1. Moments, Generating Functions and Conditional Moments

If Z ~TL-EHL-Gom-G distribution and Y ~Exp-G$(k+1)$, then the r^{th} moment $\mu_r{}'$ of the TL-EHL-Gom-G FoD is obtained as

$$\mu_r{}' = E(Z^r) = \int_0^\infty z^r f(z)dz = \sum_{k=0}^{\infty} \omega_{k+1} E(Y^r),$$

where $f(z) = f_{TL-EHL-Gom-G}(z;\Phi,\underline{\psi})$, $E(Y^r)$ is the r^{th} moment of the Exp-G distribution with parameter $(k+1)$ and ω_{k+1} is given by Equation (7). The mgf for the TL-EHL-Gom-G is

$$M_Z(t) = E(e^{tZ}) = \sum_{r=0}^{\infty} \frac{t^r}{r!} E(Z^r) = \sum_{k=0}^{\infty} \omega_{k+1} M_Z(t),$$

where $M_Z(t)$ is the moment generating function (mgf) of Y and ω_{k+1} is as given in Equation (7). The r^{th} conditional moment of the TL-EHL-Gom-G distribution is obtained as

$$E(Z^r|Z>t) = \frac{1}{\overline{F}(t)} \int_t^\infty z^r f(z)dz = \sum_{k=0}^{\infty} \omega_{k+1} E(Y^r I_{\{Y^r>t\}}),$$

where $E(Y^r I_{\{Y^r>t\}}) = \int_t^\infty y^r g_{k+1}(z;\underline{\psi})dy$.

3.2. Distribution of Order Statistics

Order statistics has many applications in statistics, including modeling auctions, analyzing insurance policies and optimizing production processes, among others.

Let $Z_1, ..., Z_n$ be independent and identically distributed TL-EHL-Gom-G random variables. The pdf of the l^{th} OS $Z_{\{l:n\}}$ of the TL-EHL-Gom-G FoD is

$$f_{l:n}(z) = \frac{\Gamma(n+1)}{\Gamma(l)\Gamma(n-l+1)} \sum_{m=0}^{\infty} \sum_{k=0}^{n-l} (-1)^k \binom{n-l}{k} \omega_{m+1} g_{m+1}(z;\underline{\psi}), \tag{8}$$

where $\Gamma(a) = (a)!$, $f(z)$ is the pdf of TL-EHL-Gom-G, $g_{m+1} = (m+1)g(z;\underline{\psi})G^m(z;\underline{\psi})$ and

$$\begin{aligned} \omega_{m+1} &= \sum_{i,j,p,q,r,s=0}^{\infty} \frac{4\alpha b(-1)^{i+j+p+s+m}}{r!} \left(\frac{p+q+1}{\gamma}\right)^r \binom{bl+bk-1}{i} \\ &\times \binom{2i+1}{j} \binom{\alpha j+\alpha-1}{p} \binom{-(\alpha j+\alpha+1)}{q} \binom{r}{s} \\ &\times \binom{-\gamma s-\gamma-1}{m} \frac{1}{(m+1)}. \end{aligned} \tag{9}$$

For all derivations, please refer to the Webb Appendix.

3.3. Rényi Entropy

In this subsection, we derive the Rényi Entropy (Rényi [26]) of the TL-EHL-Gom-G FoD. Entropy measures the amount of uncertainty associated with a random variable. Rényi Entropy is a generalization of Shannon entropy (Shannon [29]). The Rényi Entropy of order v of the TL-EHL-Gom-G FoD is

$$I_R(v) = (1-v)^{-1} \log\left(\sum_{r=0}^{\infty} \omega_{r+1}^{*} e^{(1-v)I_{REG}}\right),$$

where

$$\begin{aligned} \omega_{r+1}^{*} &= \sum_{i,j,l,m,p,q=0}^{\infty} \frac{4\alpha b(-1)^{i+j+l+q+r}}{p!}\left(\frac{l+m+1}{d}\right)^{p}\binom{b-1}{i}\binom{2i+1}{j} \\ &\times \binom{\alpha j+v(\alpha-1)}{l}\binom{-[\alpha j+v(\alpha+1)]}{m}\binom{p}{q}\binom{\gamma q-v(\gamma+1)}{r} \\ &\times \frac{1}{(\frac{r}{v}+1)^{v}} \end{aligned} \tag{10}$$

and

$$I_{REG} = \frac{1}{1-v}\log\left(\int_{0}^{\infty}\left(\frac{r}{v}+1\right)g(z;\underline{\psi})G^{\frac{r}{v}}(z;\underline{\psi})dz\right)^{v} \tag{11}$$

is the Rényi Entropy of the Exp-G distribution with parameter $(\frac{r}{v}+1)$. This means that we can derive the Rényi Entropy of the TL-EHL-Gom-G FoD from the Rényi Entropy of the Exp-G distribution. *For all derivations, please refer to the Webb Appendix.*

3.4. Stochastic Orders

Stochastic ordering has numerous applications in statistics. It serves as a valuable tool for comparing stochastic models, deriving probability inequalities, and establishing bounds and inequalities in reliability analysis.

In statistics, three commonly used orders are the hazard rate (hr) order, the stochastic (st) order and the likelihood ratio (lr) order. Let Z_1 and Z_2 be two random variables with cdf's $F_{Z_1}(u)$ and $F_{Z_2}(u)$. We say that Z_1 is stochastically smaller than Z_2 if $\overline{F}_{Z_1}(u) \leq \overline{F}_{Z_2}(u)$ or equivalently $F_{Z_1}(u) \geq F_{Z_2}(u)$, $\forall u$, where $\overline{F}_{Z_1}(u)$ and $\overline{F}_{Z_2}(t)$ are survival functions of Z_1 and Z_2. The hr order is $Z_1 <_{hr} Z_2$ if $h_{Z_1}(u) \geq h_{Z_2}(u)$ and the lr order is $Z_1 <_{lr} Z_2$ if $\frac{f_{Z_1}(u)}{f_{Z_2}(u)}$ is decreasing $\forall u$, where $h_Z(u)$ is the hrf. It is well established that $Z_1 <_{lr} Z_2 \Rightarrow Z_1 <_{hr} Z_2 \Rightarrow Z_1 <_{st} Z_2$, (Shaked and Shanthikumar [28]).

Theorem 1
Let $Z_1 \sim$ TL-EHL-Gom-G $(b_1, \gamma, \alpha; \underline{\psi})$ and $Z_2 \sim$ TL-EHL-Gom-G $(b_2, \gamma, \alpha; \underline{\psi})$, if $b_1 \leq b_2$, then $\frac{f(z;b_1,\gamma,\alpha,\underline{\psi})}{f(z;b_2,\gamma,\alpha,\underline{\psi})}$ is decreasing in z.

Proof: Let Z_1 and Z_2 be random variables following the TL-EHL-Gom-G FoD with with pdfs

$$
\begin{aligned}
f(z; b_1, \gamma, \alpha, \underline{\psi}) &= \frac{4\alpha b_1 g(z;\underline{\psi}) \exp\{\frac{1}{\gamma}[1-\overline{G}^{-\gamma}(z;\underline{\psi})]\}}{[\overline{G}(z;\underline{\psi})]^{\gamma+1}} \\
&\times \left(\frac{\left[1-\exp\{\frac{1}{\gamma}[1-\overline{G}^{-\gamma}(z;\underline{\psi})]\}\right]^{\alpha-1}}{\left[1+\exp\{\frac{1}{\gamma}[1-\overline{G}^{-\gamma}(z;\underline{\psi})]\}\right]^{\alpha+1}} \right) \\
&\times \left[1-K_G(z;\gamma,\alpha,\underline{\psi})\right] \\
&\times \left(1-\left[1-K_G(z;\gamma,\alpha,\underline{\psi})\right]^2\right)^{b_1-1}
\end{aligned}
$$

and

$$
\begin{aligned}
f(z; b_2, \gamma, \alpha, \underline{\psi}) &= \frac{4\alpha b_2 g(z;\underline{\psi}) \exp\{\frac{1}{\gamma}[1-\overline{G}^{-\gamma}(z;\underline{\psi})]\}}{[\overline{G}(z;\underline{\psi})]^{\gamma+1}} \\
&\times \left(\frac{\left[1-\exp\{\frac{1}{\gamma}[1-\overline{G}^{-\gamma}(z;\underline{\psi})]\}\right]^{\alpha-1}}{\left[1+\exp\{\frac{1}{\gamma}[1-\overline{G}^{-\gamma}(z;\underline{\psi})]\}\right]^{\alpha+1}} \right) \\
&\times \left[1-K_G(z;\gamma,\alpha,\underline{\psi})\right] \\
&\times \left(1-\left[1-K_G(z;\gamma,\alpha,\underline{\psi})\right]^2\right)^{b_2-1}.
\end{aligned}
$$

Then,

$$
\begin{aligned}
\frac{f(z; b_1, \gamma, \alpha, \underline{\psi})}{f(z; b_2, \gamma, \alpha, \underline{\psi})} &= \left(\frac{b_1}{b_2}\right) \frac{\left(1-\left[1-K_G(z;\gamma,\alpha,\underline{\psi})\right]^2\right)^{b_1-1}}{\left(1-\left[1-K_G(z;\gamma,\alpha,\underline{\psi})\right]^2\right)^{b_2-1}} \\
&= \left(\frac{b_1}{b_2}\right) \left(1-\left[1-K_G(z;\gamma,\alpha,\underline{\psi})\right]^2\right)^{b_1-b_2}.
\end{aligned}
$$

$$
\begin{aligned}
\frac{d}{dz}\left(\frac{f(z; b_1, \gamma, \alpha, \underline{\psi})}{f(z; b_2, \gamma, \alpha, \underline{\psi})}\right) &= \left(\frac{b_1}{b_2}\right)(b_1-b_2) \left(1-\left[1-K_G(z;\gamma,\alpha,\underline{\psi})\right]^2\right)^{b_1-b_2-1} \\
&\times \frac{4\alpha g(z;\underline{\psi}) \exp\{\frac{1}{\gamma}[1-\overline{G}^{-\gamma}(z;\underline{\psi})]\}}{[\overline{G}(z;\underline{\psi})]^{\gamma+1}} \\
&\times \left(\frac{\left[1-\exp\{\frac{1}{\gamma}[1-\overline{G}^{-\gamma}(z;\underline{\psi})]\}\right]^{\alpha-1}}{\left[1+\exp\{\frac{1}{\gamma}[1-\overline{G}^{-\gamma}(z;\underline{\psi})]\}\right]^{\alpha+1}} \right) \\
&\times \left[1-K_G(z;\gamma,\alpha,\underline{\psi})\right],
\end{aligned}
$$

which is less than zero if $b_1 < b_2$. Therefore, $Z_1 <_{lr} Z_2$, $Z_1 <_{hr} Z_2$ and $Z_1 <_{st} Z_2$. We conclude that the random variables Z_1 and Z_2 are stochastically ordered.

3.5. Identifiability

Identifiability ensures that model parameters can be uniquely estimated from observed data. This characteristic is fundamental for the reliability and validity of any statistical model. To establish identifiability of the TL-EHL-GOM-G FoD, we let $\Delta_1 = (b_1, \gamma_1, \alpha_1, \underline{\psi}_1)$ and $\Delta_2 = (b_2, \gamma_2, \alpha_2, \underline{\psi}_2)$. The likelihood ratio is given by

$$\frac{f_1(\Delta_1)}{f_2(\Delta_2)} = \frac{\frac{4\alpha_1 b_1 g(z;\underline{\psi}_1)\exp\{\frac{1}{\gamma_1}[1-\overline{G}^{-\gamma_1}(z;\underline{\psi}_1)]\}}{[\overline{G}(z;\underline{\psi}_1)]^{\gamma_1+1}}\left(\frac{\left[1-\exp\{\frac{1}{\gamma_1}[1-\overline{G}^{-\gamma_1}(z;\underline{\psi}_1)]\}\right]^{\alpha_1-1}}{\left[1+\exp\{\frac{1}{\gamma_1}[1-\overline{G}^{-\gamma_1}(z;\underline{\psi}_1)]\}\right]^{\alpha_1+1}}\right)}{\frac{4\alpha_2 b_2 g(z;\underline{\psi}_2)\exp\{\frac{1}{\gamma_2}[1-\overline{G}^{-\gamma_2}(z;\underline{\psi}_2)]\}}{[\overline{G}(z;\underline{\psi}_2)]^{\gamma_2+1}}\left(\frac{\left[1-\exp\{\frac{1}{\gamma_2}[1-\overline{G}^{-\gamma_2}(z;\underline{\psi}_2)]\}\right]^{\alpha_2-1}}{\left[1+\exp\{\frac{1}{\gamma_2}[1-\overline{G}^{-\gamma_2}(z;\underline{\psi}_2)]\}\right]^{\alpha_2+1}}\right)}$$

$$\times \frac{\left[1-\left(\frac{1-\exp\left\{\frac{1}{\gamma_1}[1-\overline{G}^{-\gamma_1}(z;\underline{\psi}_1)]\right\}}{1+\exp\left\{\frac{1}{\gamma_1}[1-\overline{G}^{-\gamma_1}(z;\underline{\psi}_1)]\right\}}\right)^{\alpha_1}\right]\left(1-\left[1-\left(\frac{1-\exp\left\{\frac{1}{\gamma_1}[1-\overline{G}^{-\gamma_1}(z;\underline{\psi}_1)]\right\}}{1+\exp\left\{\frac{1}{\gamma_1}[1-\overline{G}^{-\gamma_1}(z;\underline{\psi}_1)]\right\}}\right)^{\alpha_1}\right]^2\right)^{b_1-1}}{\left[1-\left(\frac{1-\exp\left\{\frac{1}{\gamma_2}[1-\overline{G}^{-\gamma_2}(z;\underline{\psi}_2)]\right\}}{1+\exp\left\{\frac{1}{\gamma_2}[1-\overline{G}^{-\gamma_2}(z;\underline{\psi}_2)]\right\}}\right)^{\alpha_2}\right]\left(1-\left[1-\left(\frac{1-\exp\left\{\frac{1}{\gamma_2}[1-\overline{G}^{-\gamma_2}(z;\underline{\psi}_2)]\right\}}{1+\exp\left\{\frac{1}{\gamma_2}[1-\overline{G}^{-\gamma_2}(z;\underline{\psi}_2)]\right\}}\right)^{\alpha_2}\right]^2\right)^{b_2-1}}.$$

The log-likelihood ratio is

$$\ln\left(\frac{f_1(\Delta_1)}{f_2(\Delta_2)}\right) = \ln\left(\frac{\alpha_1 b_1}{\alpha_2 b_2}\right) + \ln\left(\frac{g(z;\underline{\psi}_1)}{g(z;\underline{\psi}_2)}\right) + \ln\left(\frac{[\overline{G}(z;\underline{\psi}_2)]^{\gamma_2+1}}{[\overline{G}(z;\underline{\psi}_1)]^{\gamma_1+1}}\right)$$

$$+ \ln\left(\frac{\exp\{\frac{1}{\gamma_1}[1-\overline{G}^{-\gamma_1}(z;\underline{\psi}_1)]\}}{\exp\{\frac{1}{\gamma_2}[1-\overline{G}^{-\gamma_2}(z;\underline{\psi}_2)]\}}\right) + \ln\left(\frac{\left(\frac{\left[1-\exp\{\frac{1}{\gamma_1}[1-\overline{G}^{-\gamma_1}(z;\underline{\psi}_1)]\}\right]^{\alpha_1-1}}{\left[1+\exp\{\frac{1}{\gamma_1}[1-\overline{G}^{-\gamma_1}(z;\underline{\psi}_1)]\}\right]^{\alpha_1+1}}\right)}{\left(\frac{\left[1-\exp\{\frac{1}{\gamma_2}[1-\overline{G}^{-\gamma_2}(z;\underline{\psi}_2)]\}\right]^{\alpha_2-1}}{\left[1+\exp\{\frac{1}{\gamma_2}[1-\overline{G}^{-\gamma_2}(z;\underline{\psi}_2)]\}\right]^{\alpha_2+1}}\right)}\right)$$

$$+ \ln\left(\frac{\left[1-K_G(z;\gamma_1,\alpha_1,\underline{\psi}_1)\right]}{\left[1-K_G(z;\gamma_2,\alpha_2,\underline{\psi}_2)\right]}\right) + \ln\left(\frac{\left(1-\left[1-K_G(z;\gamma_1,\alpha_1,\underline{\psi}_1)\right]^2\right)^{b_1-1}}{\left(1-\left[1-K_G(z;\gamma_2,\alpha_2,\underline{\psi}_2)\right]^2\right)^{b_2-1}}\right)$$

(12)

$$
\begin{aligned}
= &\quad \ln\left(\frac{\alpha_1 b_1}{\alpha_2 b_2}\right) + \ln[g(z;\underline{\psi}_1)] - \ln[g(z;\underline{\psi}_2)] + (\gamma_2+1)\ln[\overline{G}(z;\underline{\psi}_2)] \\
- &\quad (\gamma_1+1)\ln[\overline{G}(z;\underline{\psi}_1)] + \frac{1}{\gamma_1}[1-\overline{G}^{-\gamma_1}(z;\underline{\psi}_1)] - \frac{1}{\gamma_2}[1-\overline{G}^{-\gamma_2}(z;\underline{\psi}_2)] \\
+ &\quad (\alpha_1-1)\ln\left[1-\exp\{\frac{1}{\gamma_1}[1-\overline{G}^{-\gamma_1}(z;\underline{\psi}_1)]\}\right] \\
- &\quad (\alpha_1+1)\ln\left[1+\exp\{\frac{1}{\gamma_1}[1-\overline{G}^{-\gamma_1}(z;\underline{\psi}_1)]\}\right] \\
- &\quad (\alpha_2-1)\ln\left[1-\exp\{\frac{1}{\gamma_2}[1-\overline{G}^{-\gamma_2}(z;\underline{\psi}_2)]\}\right] \\
+ &\quad (\alpha_2-1)\ln\left[1+\exp\{\frac{1}{\gamma_2}[1-\overline{G}^{-\gamma_2}(z;\underline{\psi}_2)]\}\right] \\
+ &\quad \ln\left[1-\left(\frac{1-\exp\left\{\frac{1}{\gamma_1}[1-\overline{G}^{-\gamma_1}(z;\underline{\psi}_1)]\right\}}{1+\exp\left\{\frac{1}{\gamma_1}[1-\overline{G}^{-\gamma_1}(z;\underline{\psi}_1)]\right\}}\right)^{\alpha_1}\right] \\
- &\quad \ln\left[1-\left(\frac{1-\exp\left\{\frac{1}{\gamma_2}[1-\overline{G}^{-\gamma_2}(z;\underline{\psi}_2)]\right\}}{1+\exp\left\{\frac{1}{\gamma_2}[1-\overline{G}^{-\gamma_2}(z;\underline{\psi}_2)]\right\}}\right)^{\alpha_2}\right] \\
+ &\quad (b_1-1)\ln\left(1-\left[1-\left(\frac{1-\exp\left\{\frac{1}{\gamma_1}[1-\overline{G}^{-\gamma_1}(z;\underline{\psi}_1)]\right\}}{1+\exp\left\{\frac{1}{\gamma_1}[1-\overline{G}^{-\gamma_1}(z;\underline{\psi}_1)]\right\}}\right)^{\alpha_1}\right]^2\right) \\
- &\quad (b_2-1)\ln\left(1-\left[1-\left(\frac{1-\exp\left\{\frac{1}{\gamma_2}[1-\overline{G}^{-\gamma_2}(z;\underline{\psi}_2)]\right\}}{1+\exp\left\{\frac{1}{\gamma_2}[1-\overline{G}^{-\gamma_2}(z;\underline{\psi}_2)]\right\}}\right)^{\alpha_2}\right]^2\right). \qquad (13)
\end{aligned}
$$

Differentiating the log-likelihood with respect to z, we get

$$
\begin{aligned}
\frac{d}{dz}\left(\ln\left[\frac{f_1(\Delta_1)}{f_2(\Delta_2)}\right]\right) &= \ln\left(\frac{\alpha_1 b_1}{\alpha_2 b_2}\right) + \frac{g'(z;\underline{\psi}_1)}{g(z;\underline{\psi}_1)} - \frac{g'(z;\underline{\psi}_2)}{g(z;\underline{\psi}_2)} - (\gamma_2+1)\frac{g(z;\underline{\psi}_2)}{\overline{G}(z;\underline{\psi}_2)} \\
&+ (\gamma_1+1)\frac{g(z;\underline{\psi}_1)}{\overline{G}(z;\underline{\psi}_1)} - [1-\overline{G}^{-\gamma_1-1}(z;\underline{\psi}_1)]g(z;\underline{\psi}_1) \\
&- [1-\overline{G}^{-\gamma_2-1}(z;\underline{\psi}_2)]g(z;\underline{\psi}_2) \\
&+ (\alpha_1-1)\frac{\exp\{\frac{1}{\gamma_1}[1-\overline{G}^{-\gamma_1}(z;\underline{\psi}_1)]\}[1-G(z;\underline{\psi}_1)]^{-\gamma_1-1}g(z;\underline{\psi}_1)}{\left[1-\exp\{\frac{1}{\gamma_1}[1-\overline{G}^{-\gamma_1}(z;\underline{\psi}_1)]\}\right]} \\
&+ (\alpha_1+1)\frac{\exp\{\frac{1}{\gamma_1}[1-\overline{G}^{-\gamma_1}(z;\underline{\psi}_1)]\}[1-G(z;\underline{\psi}_1)]^{-\gamma_1-1}g(z;\underline{\psi}_1)}{\left[1+\exp\{\frac{1}{\gamma_1}[1-\overline{G}^{-\gamma_1}(z;\underline{\psi}_1)]\}\right]} \\
&- (\alpha_2-1)\frac{\exp\{\frac{1}{\gamma_2}[1-\overline{G}^{-\gamma_2}(z;\underline{\psi}_2)]\}[1-G(z;\underline{\psi}_2)]^{-\gamma_2-1}g(z;\underline{\psi}_2)}{\left[1-\exp\{\frac{1}{\gamma_2}[1-\overline{G}^{-\gamma_2}(z;\underline{\psi}_2)]\}\right]} \\
&- (\alpha_2-1)\frac{\exp\{\frac{1}{\gamma_2}[1-\overline{G}^{-\gamma_2}(z;\underline{\psi}_2)]\}[1-G(z;\underline{\psi}_2)]^{-\gamma_2-1}g(z;\underline{\psi}_2)}{\left[1+\exp\{\frac{1}{\gamma_2}[1-\overline{G}^{-\gamma_2}(z;\underline{\psi}_2)]\}\right]} \\
&+ \frac{D_1}{\left[1-\left(\frac{1-\exp\left\{\frac{1}{\gamma_1}[1-\overline{G}^{-\gamma_1}(z;\underline{\psi}_1)]\right\}}{1+\exp\left\{\frac{1}{\gamma_1}[1-\overline{G}^{-\gamma_1}(z;\underline{\psi}_1)]\right\}}\right)^{\alpha_1}\right]} - \frac{D_2}{\left[1-\left(\frac{1-\exp\left\{\frac{1}{\gamma_2}[1-\overline{G}^{-\gamma_2}(z;\underline{\psi}_2)]\right\}}{1+\exp\left\{\frac{1}{\gamma_2}[1-\overline{G}^{-\gamma_2}(z;\underline{\psi}_2)]\right\}}\right)^{\alpha_2}\right]} \\
&+ (b_1-1)\frac{W_1}{\left(1-\left[1-\left(\frac{1-\exp\left\{\frac{1}{\gamma_1}[1-\overline{G}^{-\gamma_1}(z;\underline{\psi}_1)]\right\}}{1+\exp\left\{\frac{1}{\gamma_1}[1-\overline{G}^{-\gamma_1}(z;\underline{\psi}_1)]\right\}}\right)^{\alpha_1}\right]^2\right)} \\
&- (b_2-1)\frac{W_2}{\left(1-\left[1-\left(\frac{1-\exp\left\{\frac{1}{\gamma_2}[1-\overline{G}^{-\gamma_2}(z;\underline{\psi}_2)]\right\}}{1+\exp\left\{\frac{1}{\gamma_2}[1-\overline{G}^{-\gamma_2}(z;\underline{\psi}_2)]\right\}}\right)^{\alpha_2}\right]^2\right)}, \qquad (14)
\end{aligned}
$$

where $D_1 = \frac{2\alpha_1 g(z;\underline{\psi}_1)\exp\{\frac{1}{\gamma_1}[1-\overline{G}^{-\gamma_1}(z;\underline{\psi}_1)]\}}{[\overline{G}(z;\underline{\psi}_1)]^{\gamma_1+1}[1-\exp\{\frac{1}{\gamma_1}[1-\overline{G}^{-\gamma_1}(z;\underline{\psi}_1)]\}]}$, $D_2 = \frac{2\alpha_2 g(z;\underline{\psi}_2)\exp\{\frac{1}{\gamma_2}[1-\overline{G}^{-\gamma_2}(z;\underline{\psi}_2)]\}}{[\overline{G}(z;\underline{\psi}_2)]^{\gamma_2+1}[1-\exp\{\frac{1}{\gamma_2}[1-\overline{G}^{-\gamma_2}(z;\underline{\psi}_2)]\}]}$,

$$
\begin{aligned}
W_1 &= \frac{4\alpha_1 g(z;\underline{\psi}_1)\exp\{\frac{1}{\gamma_1}[1-\overline{G}^{-\gamma_1}(z;\underline{\psi}_1)]\}}{[\overline{G}(z;\underline{\psi}_1)]^{\gamma_1+1}}\left(\frac{\left[1-\exp\{\frac{1}{\gamma_1}[1-\overline{G}^{-\gamma_1}(z;\underline{\psi}_1)]\}\right]^{\alpha_1-1}}{\left[1+\exp\{\frac{1}{\gamma_1}[1-\overline{G}^{-\gamma_1}(z;\underline{\psi}_1)]\}\right]^{\alpha_1+1}}\right) \\
&\times \left[1-\left(\frac{1-\exp\left\{\frac{1}{\gamma}[1-\overline{G}^{-\gamma_1}(z;\underline{\psi}_1)]\right\}}{1+\exp\left\{\frac{1}{\gamma_1}[1-\overline{G}^{-\gamma_1}(z;\underline{\psi}_1)]\right\}}\right)^{\alpha_1}\right],
\end{aligned}
$$

and

$$W_2 = \frac{4\alpha_2 g(z;\underline{\psi}_2)\exp\{\frac{1}{\gamma_2}[1-\overline{G}^{-\gamma_2}(z;\underline{\psi}_2)]\}}{[\overline{G}(z;\underline{\psi}_2)]^{\gamma_2+1}}\left(\frac{\left[1-\exp\{\frac{1}{\gamma_2}[1-\overline{G}^{-\gamma_2}(z;\underline{\psi}_2)]\}\right]^{\alpha_2-1}}{\left[1+\exp\{\frac{1}{\gamma_2}[1-\overline{G}^{-\gamma_2}(z;\underline{\psi}_2)]\}\right]^{\alpha_2+1}}\right)$$
$$\times \left[1-\left(\frac{1-\exp\left\{\frac{1}{\gamma}[1-\overline{G}^{-\gamma_2}(z;\underline{\psi}_2)]\right\}}{1+\exp\left\{\frac{1}{\gamma_2}[1-\overline{G}^{-\gamma_2}(z;\underline{\psi}_2)]\right\}}\right)^{\alpha_2}\right].$$

If $\frac{d}{dz}\left(\ln\left[\frac{f_1(\Delta_1)}{f_2(\Delta_2)}\right]\right)=0$, then the TL-EHL-GOM-G FoD is identifiable and can be uniquely determined for any dataset if and only if $\Delta_1=\Delta_2$.

4. Parameter Estimation

We examine various approaches used to estimate the TL-EHL-Gom-G FoD parameters. These include the Anderson-Darling (AD), least squares (LS), Cramér-von-Mises (CVM) and the maximum likelihood (ML).

4.1. ML Estimation

4.1.1. Estimation in the Absence of Censoring Let $Z\sim$TL-EHL-Gom-G$(\Phi,\underline{\psi})$ and let $\mathbf{\Theta}=(\Phi,\underline{\psi})^T$ be the vector of model parameters, then the log-likelihood function $\ell(\mathbf{\Theta})$ is

$$\begin{aligned}\ell(\mathbf{\Theta}) &= n\ln(4\alpha b)+\sum_{i=1}^{n}\ln[g(z_i;\underline{\psi})]+\sum_{i=1}^{n}\{\frac{1}{\gamma}[1-\overline{G}^{-\gamma}(z_i;\underline{\psi})]\}\\ &- (\gamma+1)\sum_{i=1}^{n}\ln[\overline{G}(z_i;\underline{\psi})]+\sum_{i=1}^{n}\ln\left(\frac{\left[1-\exp\{\frac{1}{\gamma}[1-\overline{G}^{-\gamma}(z_i;\underline{\psi})]\}\right]^{\alpha-1}}{\left[1+\exp\{\frac{1}{\gamma}[1-\overline{G}^{-\gamma}(z_i;\underline{\psi})]\}\right]^{\alpha+1}}\right)\\ &+ \sum_{i=1}^{n}\ln\left[1-K_G(z_i;\gamma,\alpha,\underline{\psi})\right]+(b-1)\sum_{i=1}^{n}\ln\left(1-\left[1-K_G(z_i;\gamma,\alpha,\underline{\psi})\right]^2\right),\end{aligned}$$

where $K_G(z_i;\gamma,\alpha,\underline{\psi})=\left(\frac{1-\exp\left\{\frac{1}{\gamma}[1-\overline{G}^{-\gamma}(z_i;\underline{\psi})]\right\}}{1+\exp\left\{\frac{1}{\gamma}[1-\overline{G}^{-\gamma}(z_i;\underline{\psi})]\right\}}\right)^{\alpha}$. To obtain the ML parameter estimates of our proposed model, we solve the nonlinear system of equations $\left(\frac{\partial\ell}{\partial b},\frac{\partial\ell}{\partial\gamma},\frac{\partial\ell}{\partial\alpha},\frac{\partial\ell}{\partial\underline{\psi}_k}\right)^T=\mathbf{0}$, using numerical methods like Newton-Raphson procedure. *see the Webb Appendix for the score vector.*

4.1.2. Estimation in the Presence of Censoring In survival studies, censored observations are common, meaning only partial information is available during the study period. Interval censoring occurs when follow-up or inspections are required, with right censoring being a specific type often used in medical research. This happens when the study ends before all subjects experience the event of interest. Consider a study with n patients, each having an independent censoring time Y_t ($t=1,2,\ldots,n$), representing the time from entry to study completion, and a failure time X_t ($i=1,2,\ldots,n$) for the i-th patient. Assuming X_t and Y_t are independent and follow the TL-EHL-Gom-G FoD, let $T_t=\min(X_t,Y_t)$. The observed data is (T_t,η_t), where $\eta_t=1$ indicates failure and $\eta_t=0$ indicates censoring. The log-likelihood function is:

$$\ell = \sum_{k=1}^{n}\eta_t\log\big(f(t_t)\big)+\sum_{k=1}^{n}(1-\eta_t)\log\big(S(t_t)\big), \tag{15}$$

where $f(\cdot)$ is the pdf and $S(\cdot) = 1 - F(\cdot)$ is the survival function. Equation (15) can be optimised using numerical methods like Newton-Raphson procedure to obtain the ML estimates of the TL-EHL-Gom-G FoD.

4.2. LS Estimation

The LS (Swain et al. [30]) method can be categorized into two methods: ordinary least squares (OLS) and weighted least squares (WLS). OLS parameter estimates are obtained by minimizing

$$OLS(\mathbf{\Theta}) = \sum_{i=1}^{n}\left[\left(1-\left[1-K_G(z_{(i)};\gamma,\alpha,\underline{\psi})\right]^2\right)^b - p_i\right]^2,$$

with respect to b, λ, α, and $\underline{\psi}$, where $p_i = \frac{i}{n+1}$. OLS parameter estimates are found by solving the nonlinear system of equations

$$\left[\frac{\partial OLS(\mathbf{\Theta})}{\partial b}, \frac{\partial OLS(\mathbf{\Theta})}{\partial \gamma}, \frac{\partial OLS(\mathbf{\Theta})}{\partial \alpha}, \frac{\partial OLS(\mathbf{\Theta})}{\partial \underline{\psi}_k}\right]^T = \mathbf{0},$$

using numerical methods like Newton-Raphson procedure.

WLS parameter estimates of the TL-EHL-Gom-G are obtained by minimizing

$$WLS(\mathbf{\Theta}) = \sum_{i=1}^{n} W^*\left[\left(1-\left[1-K_G(z_{(i)};\gamma,\alpha,\underline{\psi})\right]^2\right)^b - p_i\right]^2,$$

with respect to b, λ, α, and $\underline{\psi}$, where $W^* = \frac{(n+1)^2(n+2)}{i(n-i+1)}$. WLS parameter estimates are found by solving the nonlinear system of equations

$$\left[\frac{\partial WLS(\mathbf{\Theta})}{\partial b}, \frac{\partial WLS(\mathbf{\Theta})}{\partial \gamma}, \frac{\partial WLS(\mathbf{\Theta})}{\partial \alpha}, \frac{\partial WLS(\mathbf{\Theta})}{\partial \underline{\psi}_k}\right]^T = \mathbf{0},$$

using numerical methods like Newton-Raphson procedure.

4.3. Cramér-von-Mises Estimation (CVM)

The CVM Technique was proposed by MacDonald [16]. The CVM parameter estimates are obtained by minimizing

$$CVM(\mathbf{\Theta}) = \frac{1}{12n} + \sum_{i=1}^{n}\left[\left(1-\left[1-K_G(z_{(i)};\gamma,\alpha,\underline{\psi})\right]^2\right)^b - q_i\right]^2,$$

with respect to b, λ, α, and $\underline{\psi}$, where $q_i = \frac{2i-1}{2n}$. CVM parameter estimates are found by solving the nonlinear system of equations

$$\left[\frac{\partial CVM(\mathbf{\Theta})}{\partial b}, \frac{\partial CVM(\mathbf{\Theta})}{\partial \gamma}, \frac{\partial CVM(\mathbf{\Theta})}{\partial \alpha}, \frac{\partial CVM(\mathbf{\Theta})}{\partial \underline{\psi}_k}\right]^T = \mathbf{0},$$

using numerical methods like Newton-Raphson procedure.

4.4. Anderson-Darling Estimation (AD)

The AD method of estimation was developed in 1952 by Anderson and Darling [5]. The parameters of the AD are obtained by minimizing

$$\begin{aligned} AD(\mathbf{\Theta}) &= -n - \frac{1}{n}\sum_{i=1}^{n} r_i [\log \left(1 - \left[1 - K_G(z_{(i)};\gamma,\alpha,\underline{\psi})\right]^2\right)^b \\ &- \frac{1}{n}\sum_{i=1}^{n} r_i \log\left[1 - \left(1 - \left[1 - K_G(z_{(i)};\gamma,\alpha,\underline{\psi})\right]^2\right)^b\right], \end{aligned}$$

with respect to b, λ, α, and $\underline{\psi}$, where $r_i = 2i - 1$. AD parameter estimates are found by solving the nonlinear system of equations

$$\left[\frac{\partial AD(\mathbf{\Theta})}{\partial b}, \frac{\partial AD(\mathbf{\Theta})}{\partial \gamma}, \frac{\partial AD(\mathbf{\Theta})}{\partial \alpha}, \frac{\partial AD(\mathbf{\Theta})}{\partial \underline{\psi}_k}\right]^T = \mathbf{0},$$

5. Simulation Study

We present simulation results for various estimation techniques discussed in Section 4. Several simulations were run for varying sample sizes in order to assess the performance of the TL-EHL-Gom-W, a special case of the TL-EHL-Gom-G FoD. The number of repeated samples was set at $N = 3000$. The RMSE and ABias were used as metrics to assess the estimators' performance. The RMSE and ABias expressions are

ABias$(\hat{\mathbf{\Omega}}) = \frac{\sum_{i=1}^{N}\hat{\mathbf{\Omega}}_i}{N} - \mathbf{\Omega}$, and RMSE$(\hat{\mathbf{\Omega}}) = \sqrt{\frac{\sum_{i=1}^{N}(\hat{\mathbf{\Omega}}_i - \mathbf{\Omega})^2}{N}}$.

Tables 2 and 3 show Abias and RMSE simulation results for all estimation methods for some selected parameter values. The superscript in both tables indicates the rank of each estimator. For instance, the RMSE of b computed using the ML estimation method for $n = 100$ ranks fifth among all the other estimators in Table 2. The cumulative sum of the ranks is indicated by the row $\sum$ranks.

Table 4 presents the combined rankings for RMSE and Abias. The superscript denotes the combined rank for each estimation method. The partial sum of the ranks is shown by the row with the label $\sum$ranks. From Tables 2 and 3, we can conclude that, generally, the RMSE decreases as sample size (n) increases. However, the Abias occasionally decreases with increasing n. From the results in Table 4, the ML estimation method ranks the highest, while the WLS method ranks the lowest.

Figures 5 and 6 exhibit the RMSE plots, which illustrate how the RMSEs decrease with increasing n for different estimation techniques. For all the five estimation techniques examined, the graphs show that as n increases, the RMSE consistently decrease, indicating improved accuracy in parameter estimation.

Table 2. **Simulation results for** $b = 2100.8$,$\gamma = 1200.8$,$\alpha = 150.0$,$\beta = 200.7$

				RMSE					Abias		
n		ML	AD	CVM	LS	WLS	ML	AD	CVM	OLS	WLS
100	b	1.3496 $^{(5)}$	0.3032 $^{(1)}$	0.5797 $^{(2)}$	0.5863 $^{(3)}$	1.6572 $^{(4)}$	0.2270 $^{(1)}$	0.5894 $^{(2)}$	0.9759 $^{(4)}$	0.9902 $^{(5)}$	0.8668 $^{(3)}$
100	γ	2.1852 $^{(5)}$	0.6356 $^{(3)}$	0.5223 $^{(2)}$	0.5017 $^{(1)}$	1.5258 $^{(4)}$	0.5384 $^{(1)}$	1.0827 $^{(5)}$	0.8288 $^{(3)}$	0.8617 $^{(4)}$	0.7599 $^{(2)}$
100	α	4.9437 $^{(5)}$	0.3912 $^{(1)}$	0.5229 $^{(2)}$	0.5293 $^{(3)}$	1.5058 $^{(4)}$	2.1596 $^{(5)}$	0.6950 $^{(1)}$	0.8949 $^{(3)}$	0.9078 $^{(4)}$	0.8071 $^{(2)}$
100	β	0.2590 $^{(1)}$	0.6351 $^{(5)}$	0.5328 $^{(3)}$	0.5278 $^{(2)}$	1.5293 $^{(5)}$	0.0308 $^{(1)}$	1.1000 $^{(5)}$	0.9911 $^{(4)}$	0.8117 $^{(3)}$	0.6897 $^{(2)}$
	$\sum$ranks	16	9	9	9	17	8	13	14	16	9
200	b	0.2020 $^{(1)}$	0.2031 $^{(2)}$	0.2979 $^{(3)}$	0.3411 $^{(4)}$	0.9987 $^{(5)}$	0.0523 $^{(1)}$	0.5582 $^{(2)}$	0.8084 $^{(3)}$	0.9205 $^{(5)}$	0.8470 $^{(4)}$
200	γ	0.7517 $^{(4)}$	0.3941 $^{(3)}$	0.3374 $^{(2)}$	0.3258 $^{(1)}$	1.0147 $^{(5)}$	0.1839 $^{(1)}$	-1.1021 $^{(5)}$	0.7908 $^{(2)}$	0.8624 $^{(4)}$	0.8304 $^{(3)}$
200	α	4.0684 $^{(5)}$	0.2446 $^{(1)}$	0.2933 $^{(2)}$	0.3147 $^{(3)}$	0.9299 $^{(4)}$	1.2189 $^{(5)}$	0.6869 $^{(1)}$	0.8211 $^{(3)}$	0.8756 $^{(4)}$	0.8161 $^{(2)}$
200	β	0.0237 $^{(1)}$	0.3889 $^{(4)}$	0.3616 $^{(3)}$	0.3268 $^{(2)}$	1.0073 $^{(5)}$	0.0043 $^{(1)}$	1.0001 $^{(5)}$	0.9980 $^{(4)}$	0.8158 $^{(3)}$	0.7784 $^{(2)}$
	$\sum$ranks	11	10	10	10	19	8	13	12	16	11
400	b	0.1412 $^{(1)}$	0.1903 $^{(2)}$	0.2153 $^{(3)}$	0.2249 $^{(4)}$	0.7712 $^{(5)}$	0.0210 $^{(1)}$	0.5557 $^{(2)}$	0.7871 $^{(3)}$	0.8323 $^{(4)}$	0.8459 $^{(5)}$
400	γ	0.2100 $^{(1)}$	0.2470 $^{(4)}$	0.2105 $^{(2)}$	0.2432 $^{(3)}$	0.6953 $^{(5)}$	0.0656 $^{(1)}$	0.9103 $^{(5)}$	0.7767 $^{(3)}$	0.8672 $^{(4)}$	0.7724 $^{(2)}$
400	α	3.2436 $^{(5)}$	0.1905 $^{(1)}$	0.2189 $^{(3)}$	0.2183 $^{(2)}$	0.7052 $^{(4)}$	0.4941 $^{(1)}$	0.7366 $^{(2)}$	0.8381 $^{(4)}$	0.8332 $^{(5)}$	0.7498 $^{(3)}$
400	β	0.0178 $^{(1)}$	0.2840 $^{(4)}$	0.2635 $^{(3)}$	0.2543 $^{(2)}$	0.6852 $^{(5)}$	0.0030 $^{(1)}$	0.9991 $^{(5)}$	0.9750 $^{(4)}$	0.8000 $^{(3)}$	0.6929 $^{(2)}$
	$\sum$ranks	8	11	11	11	19	4	14	14	16	12
800	b	0.1412 $^{(1)}$	0.1901 $^{(4)}$	0.1369 $^{(2)}$	0.1371 $^{(3)}$	0.4683 $^{(5)}$	0.0210 $^{(1)}$	0.5677 $^{(2)}$	0.7828 $^{(3)}$	0.8453 $^{(5)}$	0.7955 $^{(4)}$
800	γ	0.1166 $^{(1)}$	0.1635 $^{(4)}$	0.1298 $^{(2)}$	0.1571 $^{(3)}$	0.4413 $^{(5)}$	0.0011 $^{(1)}$	-1.0068 $^{(5)}$	0.7554 $^{(3)}$	0.8540 $^{(4)}$	0.7121 $^{(2)}$
800	α	1.0044 $^{(5)}$	0.1166 $^{(1)}$	0.1362 $^{(3)}$	0.1333 $^{(2)}$	0.4350 $^{(4)}$	0.0899 $^{(1)}$	0.7351 $^{(2)}$	0.8504 $^{(5)}$	0.8369 $^{(4)}$	0.7594 $^{(3)}$
800	β	0.0074 $^{(1)}$	0.1739 $^{(4)}$	0.1624 $^{(2)}$	0.1651 $^{(3)}$	0.4539 $^{(5)}$	0.0013 $^{(1)}$	0.8991 $^{(4)}$	-1.0001 $^{(5)}$	0.7999 $^{(3)}$	0.7980 $^{(2)}$
	$\sum$ranks	8	13	9	11	19	4	13	16	16	11
1000	b	0.0378 $^{(1)}$	0.0687 $^{(2)}$	0.0878 $^{(4)}$	0.0813 $^{(3)}$	0.3217 $^{(5)}$	0.0035 $^{(1)}$	0.5665 $^{(2)}$	0.7568 $^{(4)}$	0.7227 $^{(3)}$	0.7642 $^{(5)}$
1000	γ	0.1041 $^{(1)}$	0.1183 $^{(4)}$	0.1109 $^{(3)}$	0.1098 $^{(2)}$	0.3284 $^{(5)}$	0.0005 $^{(1)}$	0.0886 $^{(2)}$	0.7938 $^{(4)}$	0.8599 $^{(5)}$	0.6530 $^{(3)}$
1000	α	0.0806 $^{(1)}$	0.0807 $^{(2)}$	0.0920 $^{(4)}$	0.0897 $^{(3)}$	0.3005 $^{(5)}$	0.0740 $^{(1)}$	0.7188 $^{(2)}$	0.8168 $^{(5)}$	0.8015 $^{(4)}$	0.7369 $^{(3)}$
1000	β	0.0064 $^{(1)}$	0.1203 $^{(2)}$	0.1233 $^{(4)}$	0.1230 $^{(3)}$	0.3255 $^{(5)}$	0.0004 $^{(1)}$	-1.1000 $^{(5)}$	0.9623 $^{(4)}$	0.8000 $^{(2)}$	0.8201 $^{(3)}$
	$\sum$ranks	4	10	15	11	20	4	11	17	14	14
1200	b	0.0242 $^{(1)}$	0.0687 $^{(2)}$	0.0878 $^{(4)}$	0.0813 $^{(3)}$	0.3217 $^{(5)}$	0.0029 $^{(1)}$	0.5967 $^{(3)}$	0.5965 $^{(2)}$	0.7227 $^{(4)}$	0.8642 $^{(5)}$
1200	γ	0.0249 $^{(1)}$	0.1183 $^{(4)}$	0.1109 $^{(3)}$	0.0998 $^{(2)}$	0.3284 $^{(5)}$	0.0002 $^{(1)}$	0.0809 $^{(3)}$	0.0781 $^{(2)}$	0.8116 $^{(5)}$	0.7853 $^{(4)}$
1200	α	0.0220 $^{(1)}$	0.0807 $^{(2)}$	0.0902 $^{(4)}$	0.0897 $^{(3)}$	0.3005 $^{(5)}$	0.0646 $^{(1)}$	0.7019 $^{(3)}$	0.6119 $^{(2)}$	0.8015 $^{(4)}$	0.8369 $^{(5)}$
1200	β	0.0063 $^{(1)}$	0.1122 $^{(4)}$	0.1219 $^{(3)}$	0.1203 $^{(2)}$	0.3255 $^{(5)}$	0.0028 $^{(1)}$	-1.0003 $^{(5)}$	0.8200 $^{(3)}$	0.6234 $^{(2)}$	0.8201 $^{(4)}$
	$\sum$ranks	4	12	14	10	20	4	14	9	15	18

6. Applications

We provide three applications to showcase the flexibility, applicability and versatility of the TL-EHL-Gom-W distribution. From the Monte Carlo simulation results, the ML estimation technique showed superior performance compared to other estimation techniques, hence, we employ it to estimate the model parameters. The process of estimating the parameters begins with the careful selection of initial values to ensure convergence of the optimization algorithm. For β, a shape parameter of the baseline Weibull distribution, we used the method of moments and fitted a one parameter Weibull distribution to the data, leveraging the relationship between sample and theoretical moments. Since γ, is a scale parameter, we chose the sample mean as a starting point, ensuring consistency with the data's scale. Since α and b are shape parameters influencing skewness and tail behavior, we initialized them close to 1, corresponding to a symmetric and moderate-tailed distribution, and iteratively adjusted them basing on the model's fit. To refine the estimates, we employed an iterative strategy, using fitted values as new starting points for subsequent optimizations, ensuring convergence to the global maximum of the likelihood function. Finally, robustness checks through sensitivity analyses validated the stability of the estimates, confirming their reliability and insensitivity to initial value choices. We achieved this by systematically varying initial values and assessing convergence. We compared estimates across different starting points, and log-likelihood values for consistency.

To evaluate the model performance, several goodness-of-fit (GoF) measures were used. These include the BIC, AIC, $-2\log(\text{Ł})$, W^* and A^* to evaluate model performance. A high p-value of the K-S statistic and small GoF values are an indication of a good fit. In addition, the empirical cumulative distribution function (ECDF) plots, total time on test (TTT) plots, profile plots, probability-probability (PP) plots, Kaplan-Meier (K-M) survival

Table 3. **Simulation results for** $b = 10.1$,$\gamma = 21.5$,$\alpha = 10.1$,$\beta = 0.6$

		RMSE					Abias				
n		MLE	AD	CVM	LS	WLS	MLE	AD	CVM	OLS	WLS
100	b	0.9737 $^{(5)}$	1.0877 $^{(4)}$	0.5345 $^{(3)}$	0.4971 $^{(1)}$	0.5075 $^{(2)}$	0.1089 $^{(1)}$	0.5714 $^{(2)}$	0.8967 $^{(4)}$	0.8121 $^{(3)}$	0.9493 $^{(5)}$
100	γ	3.0204 $^{(5)}$	1.9752 $^{(4)}$	0.4418 $^{(1)}$	0.5264 $^{(3)}$	0.4975 $^{(2)}$	0.3683 $^{(1)}$	1.0557 $^{(5)}$	0.8006 $^{(2)}$	0.8729 $^{(3)}$	0.9070 $^{(4)}$
100	α	0.6049 $^{(1)}$	1.3017 $^{(5)}$	0.4968 $^{(4)}$	0.4858 $^{(3)}$	0.4691 $^{(2)}$	0.1221 $^{(1)}$	0.8087 $^{(3)}$	0.8531 $^{(4)}$	0.8594 $^{(5)}$	0.7707 $^{(2)}$
100	β	0.0393 $^{(4)}$	1.8257 $^{(5)}$	0.4596 $^{(2)}$	0.5171 $^{(3)}$	0.4499 $^{(1)}$	0.0018 $^{(1)}$	0.9000 $^{(4)}$	0.9581 $^{(5)}$	0.8542 $^{(3)}$	0.6382 $^{(2)}$
	$\sum$ranks	15	18	10	10	7	4	14	15	14	13
200	b	0.6333 $^{(5)}$	0.7314 $^{(4)}$	0.2827 $^{(1)}$	0.3014 $^{(2)}$	0.3315 $^{(3)}$	0.0971 $^{(1)}$	0.5640 $^{(2)}$	0.8845 $^{(4)}$	0.8047 $^{(3)}$	0.9405 $^{(5)}$
200	γ	2.7364 $^{(4)}$	1.1686 $^{(5)}$	0.2939 $^{(2)}$	0.2883 $^{(1)}$	0.3520 $^{(3)}$	0.0899 $^{(1)}$	1.0287 $^{(5)}$	0.8087 $^{(3)}$	0.7904 $^{(2)}$	0.9331 $^{(4)}$
200	α	0.5974 $^{(1)}$	0.8287 $^{(5)}$	0.2940 $^{(2)}$	0.2988 $^{(3)}$	0.3086 $^{(4)}$	0.1129 $^{(1)}$	0.7386 $^{(2)}$	0.8290 $^{(4)}$	0.8382 $^{(5)}$	0.7606 $^{(3)}$
200	β	0.0241 $^{(1)}$	1.1180 $^{(5)}$	0.3399 $^{(4)}$	0.3185 $^{(3)}$	0.3164 $^{(2)}$	0.0010 $^{(1)}$	0.8900 $^{(4)}$	0.9491 $^{(5)}$	0.8544 $^{(3)}$	0.6002 $^{(2)}$
	$\sum$ranks	11	19	9	9	12	4	13	16	13	14
400	b	0.5902 $^{(4)}$	0.4788 $^{(5)}$	0.2287 $^{(2)}$	0.2522 $^{(3)}$	0.2149 $^{(1)}$	0.0772 $^{(1)}$	0.5700 $^{(2)}$	0.8376 $^{(5)}$	0.8133 $^{(4)}$	0.8095 $^{(3)}$
400	γ	2.5343 $^{(4)}$	0.8541 $^{(5)}$	0.2191 $^{(2)}$	0.2143 $^{(1)}$	0.2797 $^{(3)}$	0.0719 $^{(1)}$	1.0197 $^{(5)}$	0.8016 $^{(3)}$	0.7817 $^{(2)}$	0.8052 $^{(4)}$
400	α	0.4932 $^{(1)}$	0.5837 $^{(5)}$	0.2222 $^{(3)}$	0.2332 $^{(4)}$	0.2092 $^{(2)}$	0.0266 $^{(1)}$	0.7120 $^{(3)}$	0.8489 $^{(5)}$	0.8311 $^{(4)}$	0.7032 $^{(2)}$
400	β	0.0206 $^{(1)}$	0.8165 $^{(5)}$	0.2309 $^{(3)}$	0.2160 $^{(2)}$	0.2464 $^{(4)}$	0.0008 $^{(1)}$	0.9120 $^{(5)}$	0.8000 $^{(4)}$	0.7000 $^{(3)}$	0.6326 $^{(2)}$
	$\sum$ranks	10	20	10	10	10	4	15	17	13	11
800	b	0.4784 $^{(4)}$	0.3245 $^{(5)}$	0.1244 $^{(2)}$	0.1188 $^{(1)}$	0.1416 $^{(3)}$	0.0681 $^{(1)}$	0.5390 $^{(2)}$	0.7819 $^{(4)}$	0.7431 $^{(3)}$	0.8681 $^{(5)}$
800	γ	2.0025 $^{(1)}$	0.5569 $^{(5)}$	0.1371 $^{(2)}$	0.1576 $^{(4)}$	0.1476 $^{(3)}$	0.0399 $^{(1)}$	1.0054 $^{(5)}$	0.8003 $^{(3)}$	0.7766 $^{(2)}$	0.8005 $^{(4)}$
800	α	0.1204 $^{(1)}$	0.3627 $^{(5)}$	0.1311 $^{(3)}$	0.1251 $^{(2)}$	0.1351 $^{(4)}$	0.0147 $^{(1)}$	0.7022 $^{(2)}$	0.8182 $^{(5)}$	0.7884 $^{(4)}$	0.7448 $^{(3)}$
800	β	0.0196 $^{(1)}$	0.5000 $^{(5)}$	0.1582 $^{(4)}$	0.1581 $^{(3)}$	0.1429 $^{(2)}$	0.0004 $^{(1)}$	0.9087 $^{(5)}$	0.7112 $^{(4)}$	0.7000 $^{(3)}$	0.6329 $^{(2)}$
	$\sum$ranks	7	20	11	10	12	4	14	16	12	14
1000	b	0.0290 $^{(1)}$	0.2423 $^{(5)}$	0.0843 $^{(3)}$	0.0834 $^{(2)}$	0.0904 $^{(4)}$	0.0033 $^{(1)}$	0.5076 $^{(2)}$	0.7487 $^{(4)}$	0.7435 $^{(3)}$	0.7654 $^{(5)}$
1000	γ	1.0919 $^{(5)}$	0.3835 $^{(4)}$	0.1104 $^{(3)}$	0.1023 $^{(1)}$	0.1090 $^{(2)}$	0.0500 $^{(1)}$	0.9000 $^{(5)}$	0.7751 $^{(3)}$	0.7021 $^{(2)}$	0.8119 $^{(4)}$
1000	α	0.0596 $^{(1)}$	0.2641 $^{(5)}$	0.0889 $^{(2)}$	0.0897 $^{(3)}$	0.0901 $^{(4)}$	0.0088 $^{(1)}$	0.7004 $^{(2)}$	0.7941 $^{(5)}$	0.7801 $^{(3)}$	0.7923 $^{(4)}$
1000	β	0.0163 $^{(1)}$	0.3536 $^{(5)}$	0.1118 $^{(3)}$	0.1120 $^{(4)}$	0.1014 $^{(2)}$	0.0002 $^{(1)}$	0.9009 $^{(5)}$	0.4120 $^{(2)}$	0.7000 $^{(4)}$	0.5509 $^{(3)}$
	$\sum$ranks	8	19	11	10	12	4	14	14	12	16
1200	b	0.0232 $^{(1)}$	0.1212 $^{(5)}$	0.0345 $^{(2)}$	0.0432 $^{(3)}$	0.0510 $^{(4)}$	0.0004 $^{(1)}$	0.4076 $^{(3)}$	0.4349 $^{(5)}$	0.4244 $^{(4)}$	0.2217 $^{(2)}$
1200	γ	1.0662 $^{(5)}$	0.1008 $^{(3)}$	0.0941 $^{(2)}$	0.0510 $^{(1)}$	0.1010 $^{(4)}$	0.0064 $^{(1)}$	0.8900 $^{(5)}$	0.3428 $^{(2)}$	0.5123 $^{(4)}$	0.4231 $^{(3)}$
1200	α	0.0289 $^{(1)}$	0.1533 $^{(5)}$	0.0709 $^{(4)}$	0.0429 $^{(2)}$	0.0433 $^{(3)}$	0.0074 $^{(1)}$	0.6004 $^{(5)}$	0.1239 $^{(2)}$	0.3828 $^{(3)}$	0.5229 $^{(4)}$
1200	β	0.0149 $^{(1)}$	0.0814 $^{(4)}$	0.0781 $^{(3)}$	0.0611 $^{(2)}$	0.0931 $^{(5)}$	0.0001 $^{(1)}$	0.8301 $^{(5)}$	0.5230 $^{(4)}$	0.3420 $^{(3)}$	0.2155 $^{(2)}$
	$\sum$ranks	8	17	11	8	16	4	18	13	14	11

Table 4. **Partial and Overall Ranks of all Estimation Techniques**

Parameters	n	ML	AD	CVM	OLS	WLS
$b = 2100.8$,$\gamma = 1200.8$,$\alpha = 150.0$,$\beta = 200.7$	100	19 $^{(1)}$	32 $^{(5)}$	25 $^{(4)}$	24 $^{(3)}$	20 $^{(2)}$
	200	15 $^{(1)}$	32 $^{(5)}$	25 $^{(3)}$	22 $^{(2)}$	26 $^{(4)}$
	400	14 $^{(1)}$	35 $^{(5)}$	27 $^{(4)}$	23 $^{(3)}$	21 $^{(2)}$
	800	11 $^{(1)}$	34 $^{(5)}$	27 $^{(4)}$	22 $^{(2)}$	26 $^{(3)}$
	1000	12 $^{(1)}$	33 $^{(5)}$	25 $^{(3)}$	22 $^{(2)}$	28 $^{(4)}$
	1200	12 $^{(1)}$	35 $^{(5)}$	245 $^{(4)}$	22 $^{(2)}$	27 $^{(4)}$
$b = 10.1$,$\gamma = 21.5$,$\alpha = 10.1$,$\beta = 0.6$	100	24 $^{(3)}$	22 $^{(1)}$	23 $^{(2)}$	25 $^{(4)}$	26 $^{(5)}$
	200	19 $^{(1)}$	23 $^{(3)}$	22 $^{(2)}$	26 $^{(4)}$	30 $^{(5)}$
	400	12 $^{(1)}$	25 $^{(2.5)}$	25 $^{(2.5)}$	27 $^{(4)}$	31 $^{(5)}$
	800	12 $^{(1)}$	26 $^{(3)}$	25 $^{(2)}$	27 $^{(4)}$	30 $^{(5)}$
	1000	8 $^{(1)}$	21 $^{(2)}$	32 $^{(4)}$	25 $^{(3)}$	34 $^{(5)}$
	1200	8 $^{(1)}$	36 $^{(4)}$	23 $^{(3)}$	25 $^{(2)}$	38 $^{(5)}$
$\sum$ranks		14	45.5	36.5	36	49
Overall rank		1	4	3	2	5

plots, density plots and hrf plots are among the graphs that are used to evaluate model performance.

We compare our proposed model with nested and the following non-nested models: Topp-Leone-Weibull (TLW)

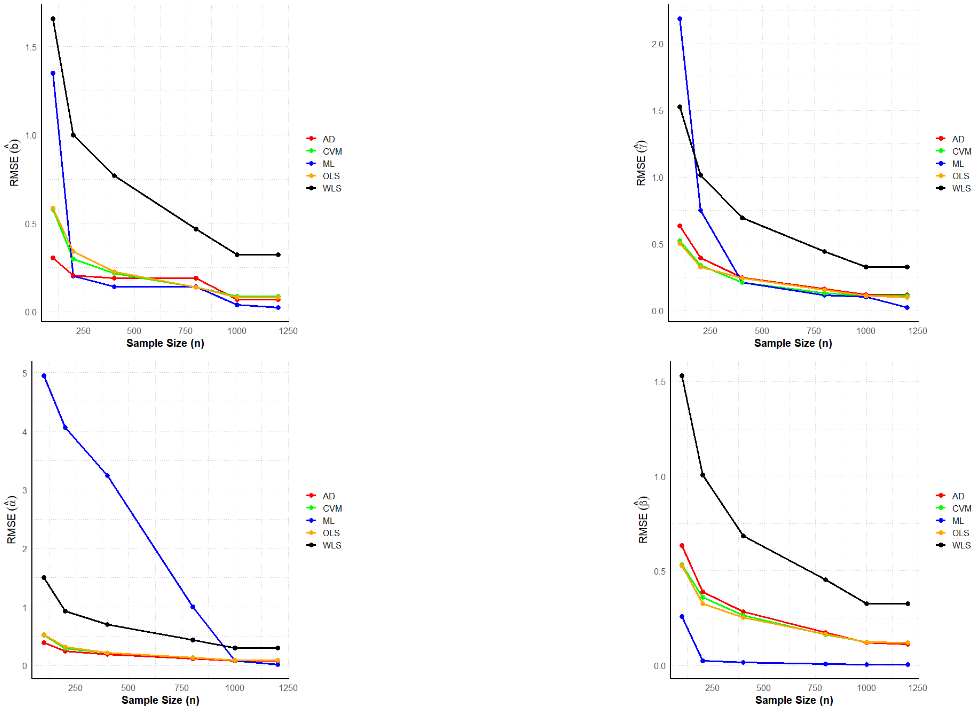

Figure 5. Plots of RMSEs for b, γ, α, β from Table 2

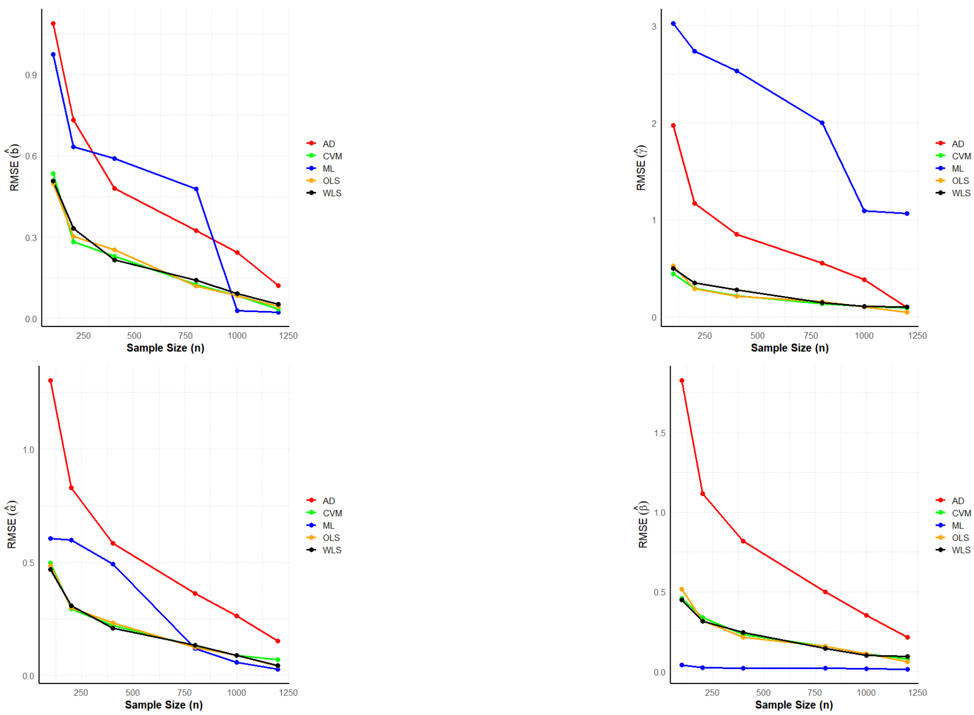

Figure 6. Plots of RMSEs for b, γ, α, β from Table 3

distribution by Al-Shomrani et al. [4], Gompertz-Weibull (GomW) distribution by Alizadeh et al. [3], Topp-Leone-Gompertz exponentiated half logistic-Weibull (TLGomEHLW) distribution by Dingalo et al. [11], Topp-Leone odd exponential half logistic-Weibull (TLOEHLW) distribution by Chipepa and Oluyede [6], exponentiated Lindley odd log-logistic-Weibull (ELOLLW) distribution by Korkmarz [15], exponentiated half logistic generalized-Weibull-Poisson (EHLWP) distribution by Chipepa et al. [8], Marshall-Olkin Gompertz-Weibull (MOGomW) distribution by Chipepa and Oluyede, [7], and Topp-Leone type I heavy-tailed log-logistic-Poisson (TLHTLLoGP) distribution by Nkomo et al. [20] *See the Webb Appendix for the pdfs of the non nested-models.*

6.1. *Active Repair Times Data*

This dataset presents the number of hours spent actively repairing an aerial communication transceiver, consisting of a total of forty data points. Chipepa et al. [8] analyzed the dataset. *See the dataset in the Webb Appendix.*

Table 5. **Active Repair Times Data: Parameter Estimates and GoF Statistics**

			Estimates						Statistics			
Distribution	b	γ	α	β	-2log(L)	AIC	CAIC	BIC	W^*	A^*	K−S	p−value
TL-EHL-Gom-W	3.5953×10^{4}	3.3738	14.2230	1.9485×10^{-2}	178.7795	186.7795	187.9223	193.535	0.0603	0.3852	0.0977	0.8400
	(2.9148×10^{-8})	(1.2879×10^{-2})	(1.4685×10^{-4})	(2.3458×10^{-3})								
TL-EHL-Gom-W	2.5741×10^{3}	4.7839×10^{-9}	3.1708×10^{-2}	1	465.6312	471.6312	472.2978	476.6978	0.0969	0.8168	0.40767	3.3640×10^{-06}
	(1.5822×10^{-8})	(2.3727×10^{-3})	(2.6015×10^{-3})	(-)								
TL-EHL-Gom-W	1	1	6.1530	0.1863	186.3838	190.3838	190.7081	193.7616	0.1058	0.7617	0.1413	0.4019
	(-)	(-)	(0.7476)	(0.0155)								
TL-EHL-Gom-W	1	1	1	0.1226	361.8551	363.8551	363.9763	365.4104	0.0739	0.4950	0.9431	2.2000×10^{-16}
	(-)	(-)	(-)	0.0153								
	b	α										
TLW	11.3196	0.3976			181.7470	185.7470	186.0713	189.1247	0.0698	0.496	0.1285	0.5238
	(1.8062)	(0.0378)										
	γ	β										
GomW	2.2532×10^{-09}	0.5184			228.5625	232.5625	232.8868	235.9403	0.1045	0.7609	0.5108	1.7270×10^{-09}
	(1.3834×10^{-02})	(5.2612×10^{-02})										
	b	γ	α	λ								
TLGomEHLW	1.2238	9.0780×10^{-9}	2.4665	45.555	186.4203	194.4203	195.5632	201.1758	0.1064	0.7607	0.1374	0.4366
	(86.339)	$(2.8452\times10^{-})$	(1.2866)	(4.2514×10^{-2})								
	b	λ	γ	ω								
TLOEHLW	698.0000	199.3900	1.8397×10^{-2}	15.609	179.4677	187.4677	188.6105	194.2232	0.0583	0.3817	0.11117	0.7060
	(4.6420×10^{-7})	(1.3586×10^{-7})	(4.0085×10^{-4})	(1.7104×10^{-2})								
	β	λ	θ	γ								
ELOLLW	1.5284×10^{-5}	3.3741×10^{-2}	86.9660	3.9604	191.0227	199.0227	200.1656	1205.7782	0.1455	1.048	0.1290	0.5181
	(1.6163)	(1.1408×10^{-2})	(5.6672×10^{-5})	(0.1089)								
	α	β	δ	θ								
EHLWP	5.2951×10^{-2}	8.6893	1.0849×10^{4}	8.4405	181.4291	189.4291	190.5719	196.1846	0.0704	0.4709	0.1141	0.6755
	(1.2810×10^{-2})	(0.1649)	(1.2024×10^{-5})	(4.5549)								
	θ	γ	δ	λ								
MOGomW	1.5541	3.3092	8.6841×10^{5}	3.3572×10^{-2}	186.0072	194.0072	195.1501	200.7628	0.1020	0.6921	0.1176	0.6380
	(0.9241)	(0.8086)	(1.1673×10^{-7})	(8.2157×10^{-3})								
	b	θ	δ	β								
TLHTLLoGP	61.7020	42.7310×10^{02}	1.3015	0.1018	186.3095	194.3095	195.4523	201.065	0.1051	0.7635	0.1303	0.5056
	(7.2010×10^{-05})	(1.0643×10^{-06})	(1.2460×10^{-02})	(1.1676×10^{-02})								

Table 5 results show that TL-EHL-Gom-W outperforms the nested and selected non-nested models. This is evident from its largest p-value and lowest values of all GoF statistics. The profile log-likelihood plots for active repair times data are presented in Figure 7. The plots show the parameters are identifiable.

The density plot displayed in Figure 8 shows that our proposed model is the best fit because its density closely aligns with the empirical histogram. Additionally, the PP plot of our proposed model closely follows the 45-degree line, further confirming its fit. Figure 9 shows the K-M survival and ECDF plots for the active repair times data. Both graphs indicate that our model is a good fit. This is evidenced by the close proximity between the observed data and the fitted distribution. Figure 10 shows concave TTT scaled plot concurs with the hrf plot which depicts an upside-down bathtub shape for active repair times data.

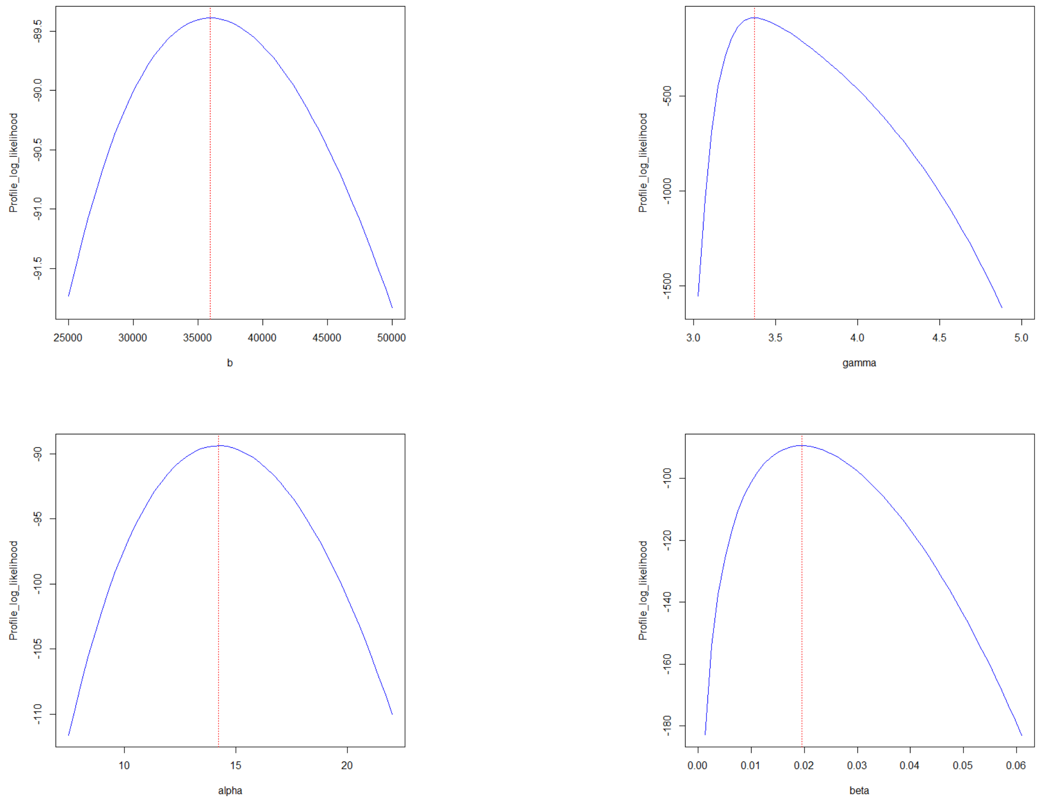

Figure 7. TL-EHL-Gom-W Profile log-likelihood Plots for Active Repair Times Data

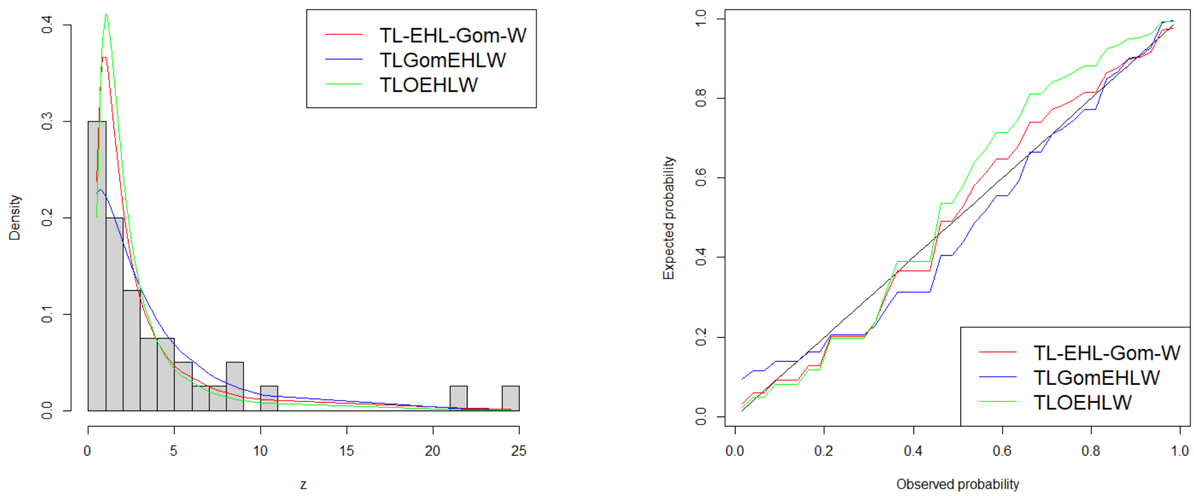

Figure 8. Fitted Density and PP Plots for Active Repair Times Data

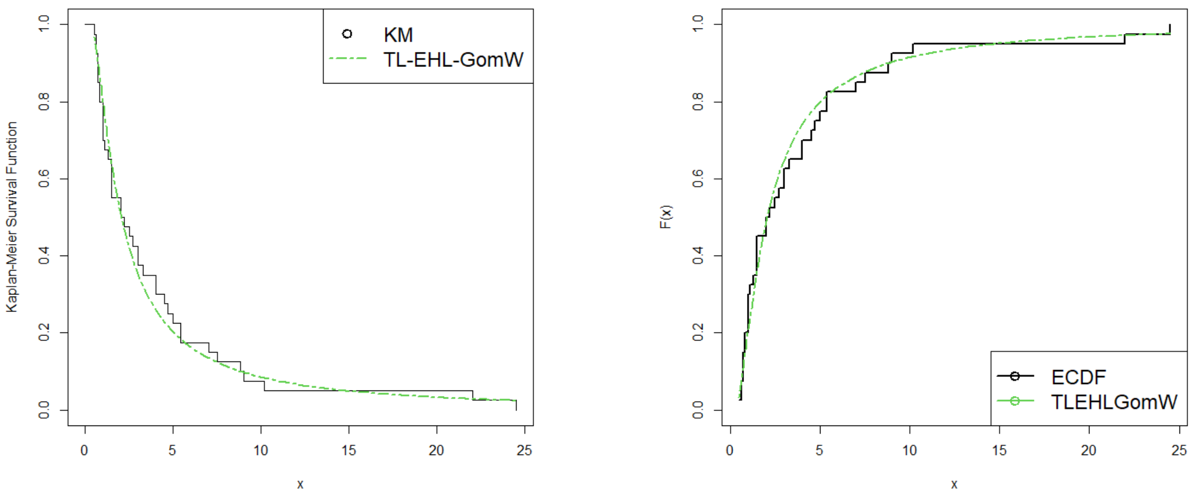

Figure 9. Active Repair Times Data: Fitted K-M Survival and ECDF Plots

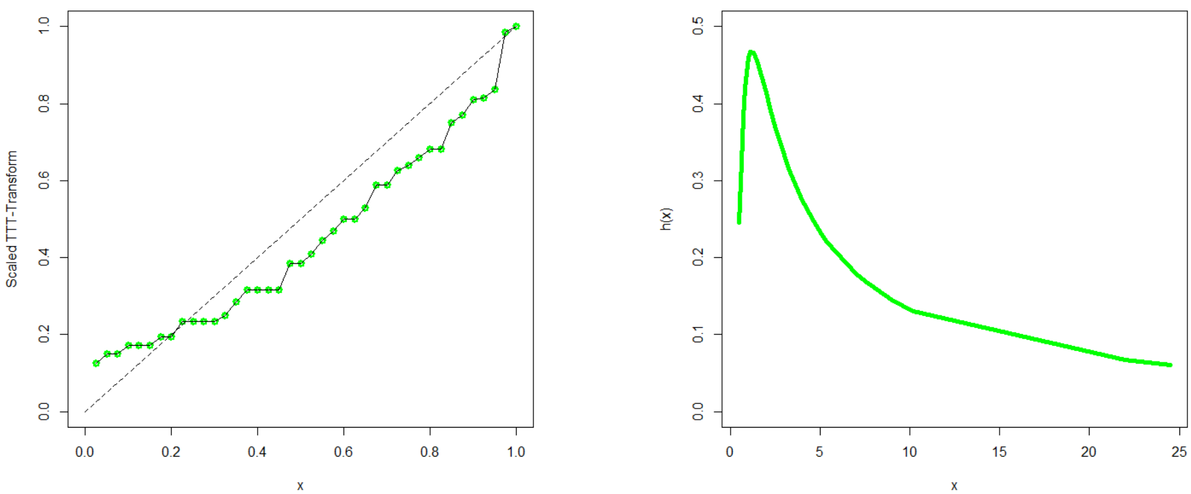

Figure 10. Active Repair Times Data: Fitted TTT scaled Plot and Hrf Plots

6.2. Prices of Vehicles Data

The dataset comprises the prices of 428 ($\times 10^4$ dollars) new vehicles from 2004 (Kiplinger's Personal Finance, December 2003). The data are presented in the table. The data was analysed by Oluyede et al. [21]. *See the dataset in the Webb Appendix.*

Results presented in Table 6 shows that the TL-EHL-Gom-W model outperforms both the nested models and selected non-nested models, since it has the highest p value of the K-S statistic and the lowest values across all the GoF statistics for prices of vehicles data. Figure 11 presents the profile log-likelihood plots for the parameters of prices of vehicles data, confirming that the parameters are identifiable.

Table 6. **Prices of Vehicles Data**

	Estimates				Statistics							
Distribution	b	γ	α	β	-2log(L)	AIC	CAIC	BIC	W^*	A^*	K−S	p−value
TL-EHL-Gom-W	15.3520	2.6391	53.5210	7.2518×10^{-2}	1488.9250	1496.9250	1497.0190	1513.1610	0.0319	0.2545	0.0212	0.9904
	(1.2580×10^{-4})	(1.0973×10^{-2})	(5.9099×10^{-5})	(2.6167×10^{-3})								
TL-EHL-Gom-W	2.5741×10^{3}	16.9980	10.0790	1	2280.1280	2286.1280	2286.186	2298.307	0.0760	0.5232	0.2454	2.2000×10^{-16}
	(5.4353×10^{-8})	(1.5132×10^{-3})	(2.2749×10^{-3})	(-)								
TL-EHL-Gom-W	45.5301	1	1	0.2594	1499.9360	1503.9360	1503.964	1512.0540	0.0840	0.6846	0.0356	0.6498
	(2.6625)	(-)	(-)	()	0.0059							
TL-EHL-Gom-W	1	1	1	0.1695	3718.6520	3720.6520	3720.6610	3724.7110	0.4136	2.8472	0.9095	2.2000×10^{-16}
	(-)	(-)	(-)	(0.0060)								
	b	α										
TLW	31.8518	0.6228			1495.0020	1499.0020	1499.0300	1507.1200	0.0477	0.4281	0.0276	0.9000
	(1.8965)	(0.0148)										
	γ	β										
GomW	4.2629×10^{-09}	6.3671×10^{-01}			2482.7810	2486.7810	2486.8090	2494.8990	0.4476	3.0618	0.6566	2.2000×10^{-16}
	(1.3577×10^{-02})	(2.1643×10^{-02})										
	b	γ	α	λ								
TLGomEHLW	182.4500	2.4097×10^{-8}	21.9960	63.0930	1497.0600	1505.0600	1505.1550	1521.2970	0.0630	0.5377	0.0317	0.7842
	(6.0744×10^{-6})	(2.1685×10^{-2})	(1.1014×10^{-2})	(2.7989×10^{-2})								
	b	λ	γ	ω								
TLOEHLW	9.9957	0.0065	5.0381	0.1118	1526.7310	1534.7310	1534.825	1550.967	0.3005	2.1270	0.0527	0.1854
	(0.0040)	(0.0017)	(0.2581)	(0.0060)								
	β	λ	θ	γ								
ELOLLW	1.6488×10^{-6}	8.5205×10^{-2}	8.3055	1.8390	1638.4200	1646.4200	1646.5140	1662.656	1.4473	9.1097	0.0989	0.0005
	(3.5866×10^{-1})	(3.8293×10^{-3})	(2.1282×10^{-5})	(5.9652×10^{-2})								
	α	β	δ	θ								
EHLWP	1.3884×10^{-1}	6.2539	1.7143×10^{3}	8.4404	1497.2010	1505.2010	1505.2950	1521.4370	0.0403	0.3714	0.0288	0.8691
	(9.1686×10^{-3})	(3.7461×10^{-2})	(1.1649×10^{-4})	(1.2481)								
	θ	γ	δ	λ								
MOGomW	1.3633	2.9431	7.6515×10^{4}	8.7882×10^{-2}	1541.1610	1549.1610	1549.255	1565.397	0.4454	3.0701	0.0563	0.1321
	(3.7784×10^{-1})	(3.8483×10^{-1})	(2.7883×10^{-7})	(9.5467×10^{-3})								
	b	θ	δ	β								
TLHTLLoGP	18.9300	1.5577×10^{-04}	0.9880	1.4830	1493.2470	1501.2470	1501.3410	1517.4830	0.1041	0.6761	0.0327	0.7499
	(20.414)	(6.2475×10^{-02})	(0.3792)	(0.5172)								

Figure 12 shows that the fitted density is closer to the sample histogram, and the fitted PP plot is closer to the empirical line. This means that our proposed model is an adequate fit for prices of vehicles data. The K-M survival plot and ECDF plot are presented in Figure 13. We conclude that our model fits the data well because the fitted and observed distributions for both graphs are fairly close to each other. Figure 14 shows that the TTT scaled plot concurs with the hrf plot which depicts an upside-down bathtub shape for prices of vehicles data.

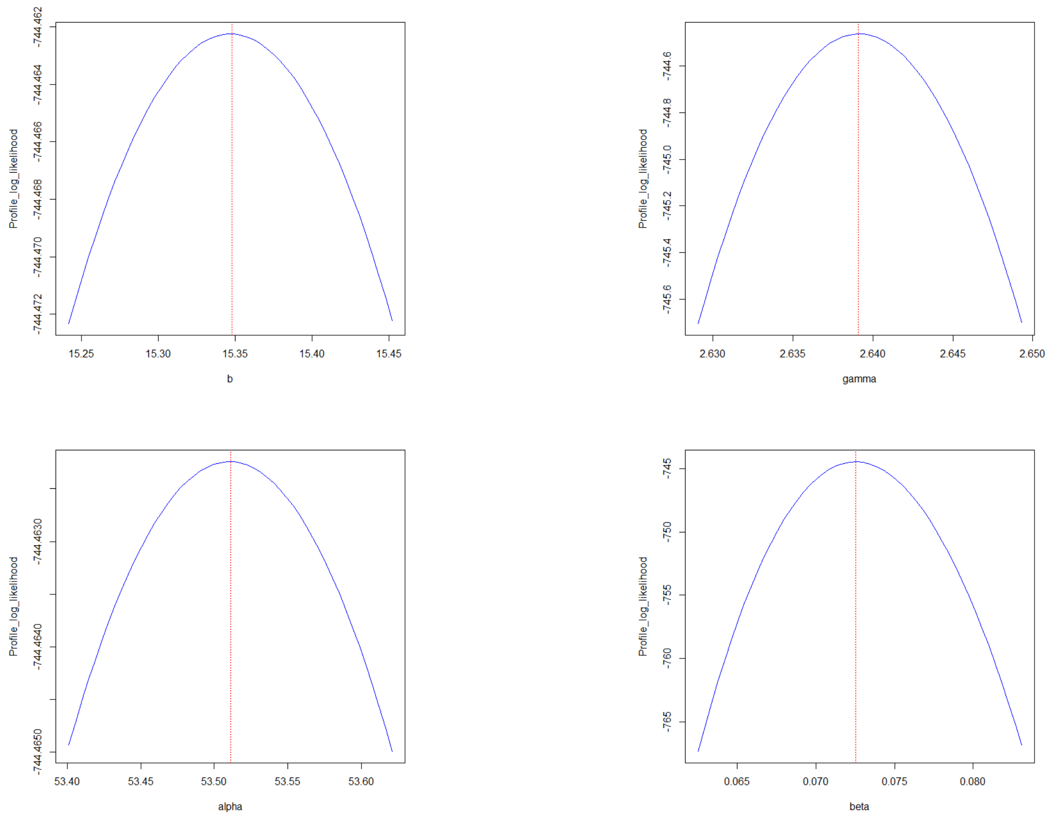

Figure 11. TL-EHL-Gom-W Profile log-likelihood Function Plots for Prices of Vehicles Data

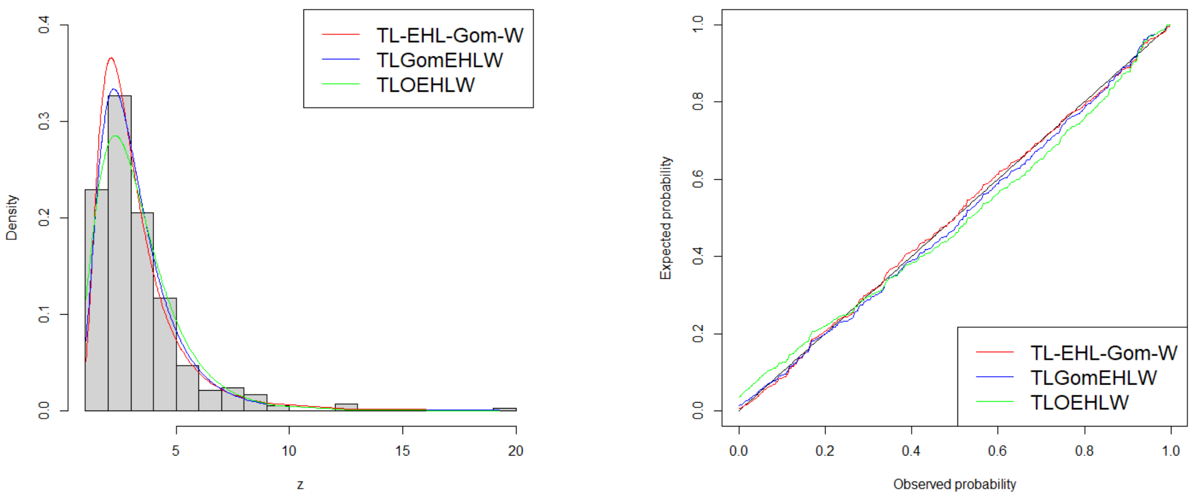

Figure 12. Fitted Density Plot and PP Plots for Prices of Vehicles Data

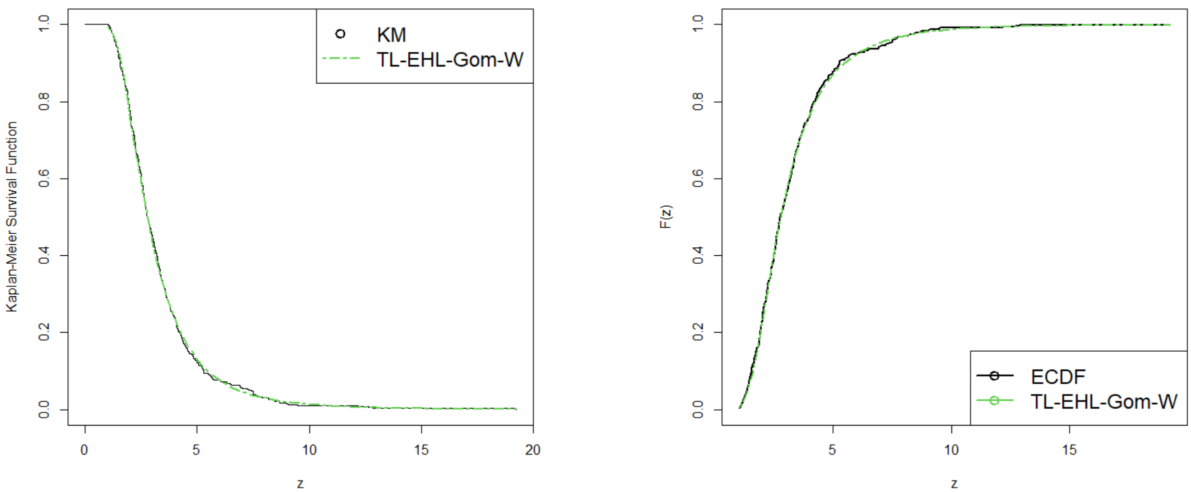

Figure 13. Prices of Vehicles Data: Fitted K-M Survival and ECDF Plots

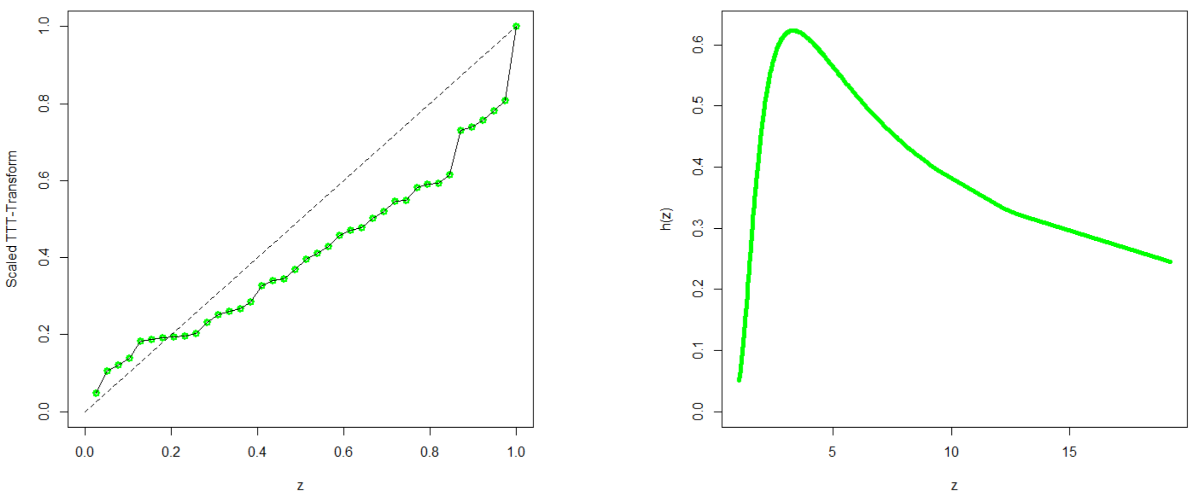

Figure 14. Prices of Vehicles Data: Fitted TTT Scaled and Hrf Plots

6.3. Adjuvant Chemoradiotherapy (CRT) Survival Data (Complete Case)

The dataset records the time (in months) until death following adjuvant CRT surgery for 32 patients. The data was analyzed by Martinez [18]. *See the dataset in the Webb Appendix.*

Results presented in Table 7 shows that the TL-EHL-Gom-W model is the best model compared to the selected nested and non-nested models since it has the highest p value of the K-S statistic and the lowest values of all the GoF statistics for adjuvant CRT survival data. Figure 15 presents the profile log-likelihood plots, confirming that the parameters are identifiable for adjuvant CRT survival data.

Table 7. **Adjuvant CRT Survival Data**

	Estimates				Statistics							
Distribution	b	γ	α	β	-2log(L)	AIC	CAIC	BIC	W^*	A^*	K−S	p−value
TL-EHL-Gom-W	0.1837	0.2541	1.6319×10^{02}	0.3885	204.0441	212.4411	222.304	218.304	0.0296	0.2388	0.09721	0.9930
	(6.1446×10^{-02})	(1.3687×10^{-03})	(1.0011×10^{-04})	(2.1612×10^{-01})								
TL-EHL-Gom-W	2.5742×10^{3}	1.1789×10^{-10}	1.9282×10^{-02}	1	1999.1710	2005.1710	2006.0290	2009.5690	0.5254	3.1206	0.9999	2.2000×10^{-16}
	()	()	()	(-)								
TL-EHL-Gom-W	4.6994×10^{02}	1	1	1.7293×10^{-01}	210.1241	214.1241	214.5379	217.0556	0.0496	3808	0.084	0.8603
	(2.5510×10^{-07})	(-)	(-)	(4.9111×10^{-03})								
TL-EHL-Gom-W	1	1	1	0.0721	441.6746	443.6746	443.8079	445.1403	0.0305	0.2507	0.9532	2.2000×10^{-16}
	(-)	(-)	(-)	(0.0096)								
	b	α										
TLW	3.6249e+02	4.2255×10^{-01}			210.5345	214.5345	214.9483	217.466	0.0537	0.4062	0.0886	0.9442
	(2.0477×10^{-06})	(1.2138×10^{-02})										
	γ	β										
GomW	4.2130×10^{-09}	0.2779			345.8834	349.8834	350.2972	352.8149	0.0304	0.2496	0.8034	2.2000×10^{-16}
	(2.4690×10^{-02})	(3.2862×10^{-02})										
	b	γ	α	λ								
TLGomEHLW	7.2110×10^{3}	0.1626	0.1756	0.3224	210.2899	218.2899	219.7714	224.1529	0.0516	0.3934	0.0867	0.9052
	(1.1046×10^{-05})	(0.5623)	(2.1570)	(0.2508)								
	b	λ	γ	ω								
TLOEHLW	0.1837	0.2541	1.6319×10^{2}	0.3885	207.4411	215.6552	217.1366	221.5181	0.0343	0.2606	0.094	0.9144
	(6.1446×10^{-02})	(0.5369)	(1.0011×10^{-04})	(0.2161)								
	β	λ	θ	γ								
ELOLLW	0.0392	0.0299	5.1850	2.8202	208.1613	216.1613	217.6428	222.0243	0.0630	0.393	0.0994	0.8789
	(12.9304)	(0.0259)	(0.0982)	(0.3903)								
	α	β	δ	θ								
EHLWP	0.8396	0.0488	3.5541	95.2975	208.0771	216.0771	217.5586	221.9401	0.0580	0.3701	0.0984	0.8863
	(0.4125)	(0.0969)	(2.0770)	(0.0040)								
	θ	γ	δ	λ								
MOGomW	6.0782×10^{-02}	4.8134	1.1211×10^{3}	9.8100×10^{-02}	210.1389	218.1389	219.6204	224.0018	0.0593	0.4130	0.1060	0.8277
	(9.4975×10^{-02})	(1.9500)	(1.5039×10^{-03})	(3.6171×10^{-02})								
	b	θ	δ	β								
TLHTLLoGP	1.0093×10^{2}	7.3397×10^{-08}	0.14941	6.7964	216.4627	224.4624	225.9439	230.3254	0.0865	0.6036	0.1607	0.3428
	(1.3888×10^{-02})	(3.6997×10^{-02})	(3.6757×10^{-02})	(1.5848)								

Figure 16 demonstrates that the fitted density aligns closely with the sample histogram, and the fitted PP plot is close to the empirical line. This indicates that our proposed model is a good fit for the adjuvant CRT survival data. The K-M survival plot and ECDF plot are presented in Figure 17. We conclude that our model fits the data well because the fitted and observed distributions for both graphs are fairly close to each other. Figure 18 shows concave TTT scaled plot concurs with the hrf plot which depicts an increasing shape for adjuvant CRT survival data.

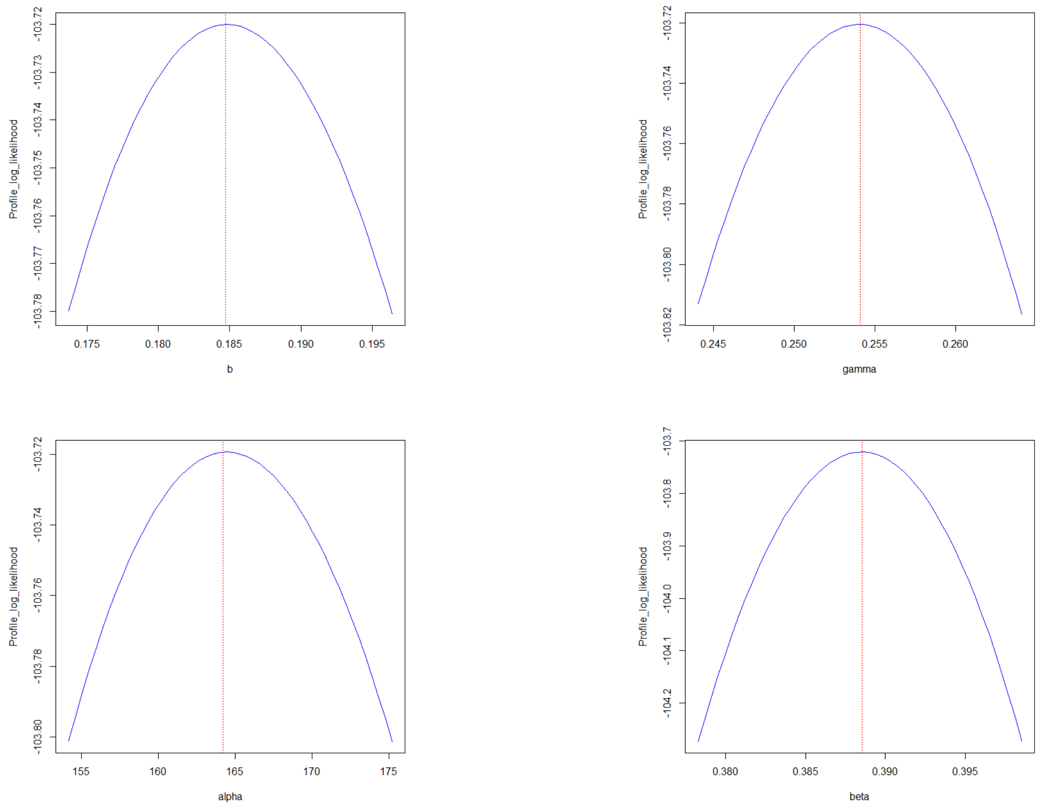

Figure 15. TL-EHL-Gom-W Profile log-likelihood Function Plots for Adjuvant CRT Data

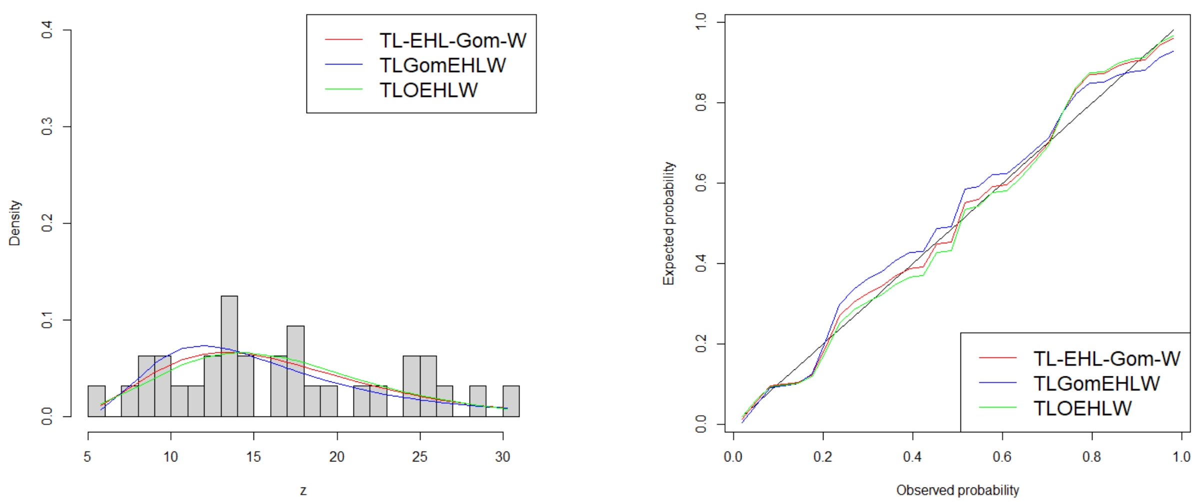

Figure 16. Fitted Density Plot and PP Plots for Adjuvant CRT Data

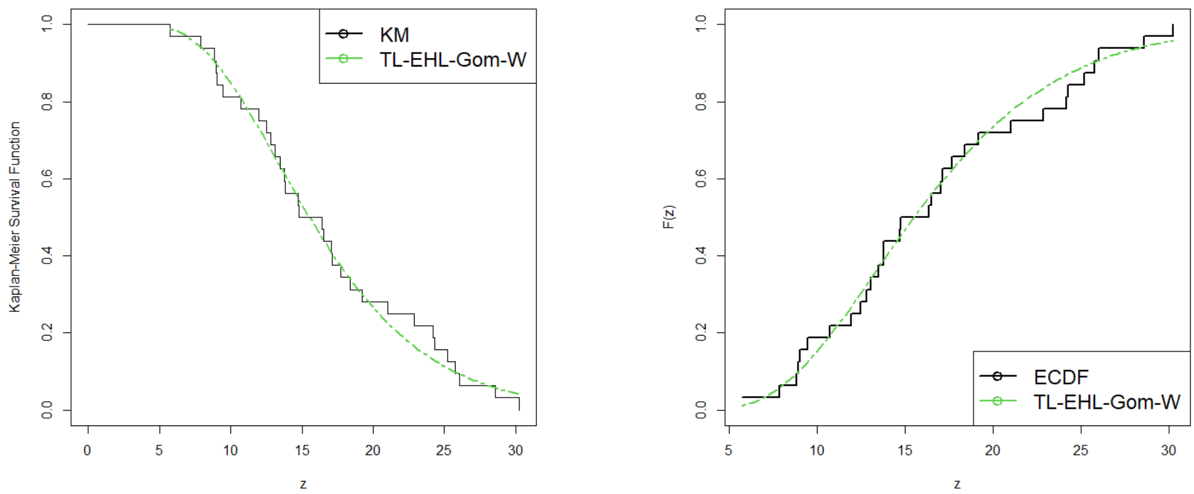

Figure 17. Adjuvant CRT Data: Fitted K-M Survival and ECDF Plots

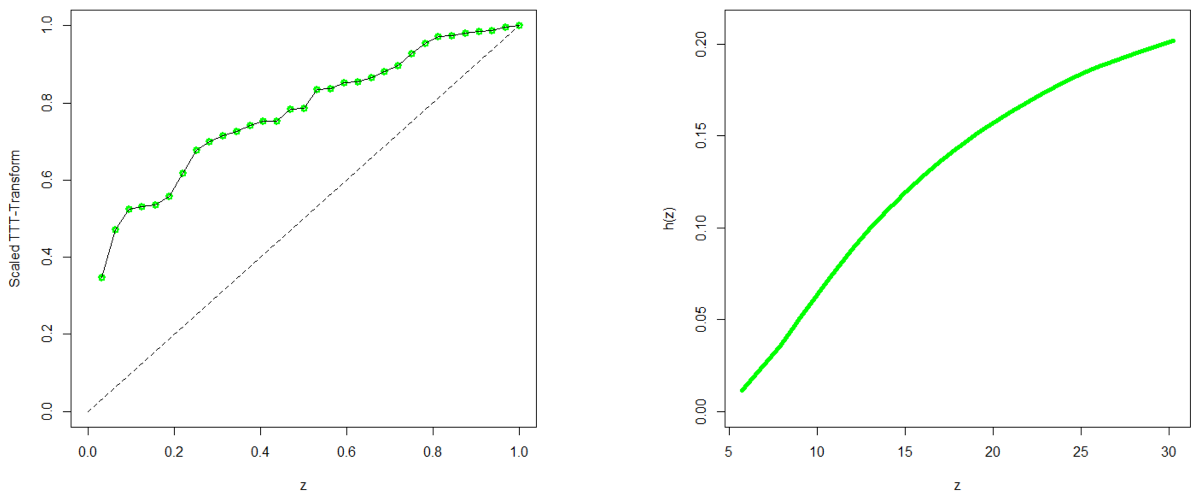

Figure 18. Adjuvant CRT Data: Fitted TTT Scaled and Hrf Plots

6.4. Adjuvant Chemoradiotherapy (CRT) Survival Censored Data

The dataset records the time (in months) until death following adjuvant CRT surgery for 76 patients including censored observations. The data is provided in the Webb Appendix.

The results presented in Table 8 shows that the TL-EHL-Gom-W model is the best model compared to the selected nested and non-nested models, since it has the lowest values of all the GoF statistics for adjuvant CRT survival censored data.

Table 8. **Adjuvant CRT Survival Data : Parameter Estimates and GoF Statistics (Censored Case)**

	Estimates				Statistics			
Distribution	b	γ	α	β	-2log(L)	AIC	CAIC	BIC
TL-EHL-Gom-W	46.6045	1.9667	8.2901	0.0509	306.9361	314.9361	328.259	320.259
	(0.0816)	(0.3177)	(5.5511)	(0.0146)				
TL-EHL-Gom-W	2.7742	0.2297	2.3321	1	316.4319	322.4319	332.4241	329.4241
	(0.8532)	(2.5643)	(2.5430)	(-)				
TL-EHL-Gom-W	1	1	27.5705	0.1273	311.6572	315.6572	322.3187	320.3187
	(-)	(-)	(5.2666)	(0.0095)				
TL-EHL-Gom-W	1	1	1	0.0280	724.0273	726.0273	729.358	728.358
	(-)	(-)	(-)	(0.0044)				
	b	α						
TLW	95.0718	0.2538			309.2842	313.2842	319.9457	317.9457
	(24.9907)	(0.0210)						
	γ	β						
GomW	8.9954×10^{-10}	5.0421×10^{-02}			535.1701	543.1701	556.493	552.493
	(7.1199×10^{-03})	(8.6110×10^{-03})						
	a	b	α	β				
TLGomEHLW	2.7314	5.5428×10^{-9}	4.7490	2.7748×10^{-1}	311.8048	319.8048	333.1277	329.1277
	(1.0151)	(1.4456×10^{-2})	(1.3845)	(2.2264×10^{-2})				
	β	λ	θ	γ				
ELOLLW	1.0372	0.10673	0.1389	0.7371	323.6976	331.6976	345.0205	341.0205
	(0.3505)	(0.1216)	(0.1664)	(0.2515)				
	b	λ	γ	ω				
TLOEHLW	10.0000	6.9169×10^{-3}	7.1819	1.6401	316.2847	324.2847	337.6076	333.6076
	(4.0941)	(3.0898×10^{-3})	(2.8636×10^{-5})	(2.6046×10^{-1})				
	α	β	δ	θ				
EHLWP	1.9889×10^{-1}	4.7380×10^{-1}	3.1308×10^{-1}	7.7059×10^{-15}	312.7382	320.7382	334.0611	330.0611
	(0.0435)	(0.8963)	(0.3455)	(0.9014)				
	θ	γ	δ	λ				
MOGomW	4.8354×10^{-3}	7.6115	2.4722×10^{2}	4.9748×10^{-2}	320.6328	328.6328	341.9557	337.9557
	(3.6400×10^{-3})	(9.2961)	(2.9080×10^{-3})	(1.0841×10^{-2})				
	b	θ	δ	β				
TLHTLLoGP	4.6994×10^{2}	11.1000	0.1000	0.1729	210.1241	218.1241	231.4470	227.4470
	(2.5510×10^{-07})	(0.0001)	(5.0214×10^{-19})	(4.9111×10^{-03})				

6.5. *Likelihood Ratio (LR) Test*

Table 9 presents the LR test results for active repair times, prices of vehicles data and adjuvant CRT survival data. The results demonstrates that the TL-EHL-Gom-W model is superior to the nested models at a 5% level of significance because all the p-values are less than 0.05 for all three datasets.

Table 9. **Results of the LR Test for Active Repair Time and Prices of Vehicles Data**

Model	df	Active Repair Times Data χ^2(p-value)	Prices of Vehicles Data χ^2(p-value)	Adjuvant CRT Survival Data χ^2(p-value)
TL-EHL-Gom-W$(b,\gamma,\alpha,1)$	1	286.8517(<0.05)	791.203(<0.05)	1795.126(<0.05)
TTL-EHL-Gom-W$(b,1,1,\beta)$	2	7.6043(<0.05)	11.011(<0.05)	6.08(<0.05)
TL-EHL-Gom-W$(1,1,1,\beta)$	3	183.0756(<0.05)	2229.727(<0.05)	237.6305(<0.05)

7. Conclusions

This research introduces a new FoD called the TL-EHL-Gom-G distribution. Sub-families and some special cases of the new family are presented, offering different parameterizations and flexibility. The study derives important structural properties of the TL-EHL-Gom-G FoD, including moments, distribution of order statistics, Rényi entropy and stochastic orders. These derived properties provide a comprehensive understanding of the behaviour and characteristics of the new distribution. To estimate the parameters of the TL-EHL-Gom-G FoD, different estimation techniques, namely, ML, AD, CVM, OLS and WLS are evaluated using MCS via RMSE and Abias. Simulation results indicate that the ML estimation technique demonstrates superior performance compared to other methods. Consequently, it was selected for parameter estimation. Furthermore, the TL-EHL-Gom-W distribution, a special case of the TL-EHL-Gom-G FoD was applied to three real-world datasets, including censored data. Selected competing models are considered for comparison. The TL-EHL-Gom-W distribution demonstrates superior performance compared to the nested and non nested models. The authors are hopeful that the newly developed TL-EHL-Gom-G FoD will be applicable in various domains, such as economics, insurance, biology, and engineering.

Appendix

Please click on the link below to access the Webb Appendix. https://drive.google.com/file/d/1LMr4ah7i7FdFidkPnALLkBRYuk8vkiHk/view?usp=sharing

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Declaration of competing interest

The authors assert that there are no conflicts of interest among them.

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