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# Bayesian Accelerated Life Testing Models for the Log-normal and Gamma Distributions under Dual-stresses

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**Abstract** In this paper, a Bayesian approach to accelerated life testing models with two stressors is presented. Lifetimes are assumed to follow either a log-normal distribution or a gamma distribution, which have been mostly overlooked in the Bayesian literature when considering multiple stressors. The generalized Eyring relationship is used as the time transformation function, which allows for the use of one thermal stressor and one non-thermal stressor. Due to the mathematically intractable posteriors of these models, Markov chain Monte Carlo methods are utilized to obtain posterior samples on which to base inference. The models are applied to a real dataset, where model comparison metrics are calculated and estimates are provided of the model parameters, predictive reliability, and mean time to failure. The robustness of the models is also investigated in terms of the prior specification. A study on simulated datasets provides further insights into the models in terms of fit and the accuracy of parameter estimates.

**Keywords** Accelerated Life Testing; Bayes; Generalized Eyring Relationship; Markov Chain Monte Carlo; Reliability

**AMS 2010 subject classifications** 62F15, 62N05, 62P30, 65C05

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## 1. Introduction

In the modern era, many products are designed to last for years. Classical reliability testing to quantify the life characteristics of products may not be viable in terms of financial and time constraints faced by manufacturers. This problem has lead to the development of accelerated life testing (ALT), where products are tested under conditions that are more severe than their normal operating conditions in order to induce early failures. The accelerated failure data can then be extrapolated, with the use of a time transformation function (TTF) specifying a relationship between the accelerated stressors and the parameters of the life distribution, to estimate the reliability of the products under their normal operating conditions. The accelerated testing conditions are obtained by applying or varying one or more stressors at higher than normal levels. Typically used stressors include temperature, pressure, voltage, wattage, humidity, cycles, and use rate [11, 5].

The most commonly used TTFs in constant stress ALT experiments are the Arrhenius relationship, inverse power law, and Eyring relationship. However, these TTFs allow for the use of only one accelerated stressor, where most products normally operate under various possible stressors. In practice, multiple-stressor models are necessary and often more relevant, since multiple-stressor ALT experiments can be used to investigate interactions between stressors, explore various failure modes, and allow for further acceleration compared to single-stressor experiments – particularly in cases of testing equipment limitations [16]. While multiple-stressor ALT models have been investigated to some extent in the frequentist setup, they are less common in the Bayesian literature due to the complexity of these models.

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The exponential and Weibull distributions are the most widely used life distributions in the ALT literature, the exponential distribution due to its simplicity and the Weibull distribution due to its versatility. While ALT models on numerous other distributions have been researched, many of these distributions are mostly overlooked in the Bayesian literature. The log-normal and gamma distributions are widely used in standard reliability analysis (see, for example, [26]), but these distributions are less common in the Bayesian ALT setup and have not been considered for multiple-stressor ALT experiments. Some of the main contributions to the Bayesian literature on single-stressor ALT models using the log-normal distribution or gamma distribution include [4, 14, 13, 9].

In this paper, Bayesian dual-stress ALT models using the log-normal and gamma distributions are developed. Motivations for the Bayesian paradigm include advances in Markov chain Monte Carlo (MCMC) methods that enable Bayesian inference for mathematically intractable posteriors, the small sample sizes typically observed in ALT experiments, and the capacity for incorporating expert knowledge from reliability engineers. The generalized Eyring relationship is used as the TTF, which accommodates a thermal stressor, a non-thermal stressor, and the interaction between these stressors. The models are applied to a real dataset, where the robustness given different flat prior choices is assessed, model comparison metrics are computed, and estimates of the predictive reliability and mean time to failure (MTTF) are computed. A simulation study is also performed to investigate the accuracy of parameter estimates obtained via the MCMC simulations.

The paper is organized as follows. The ALT setup, models for the log-normal and gamma life distributions, and prior specifications are defined in Section 2. In Section 3, the use of the models is demonstrated in an application to an ALT dataset for some high-reliability device. The log-concavity of the full conditional posteriors is assessed in the appendix to identify suitable MCMC methods to employ. Posterior samples are obtained via MCMC methods to perform inference and the robustness of the models is also assessed. A simulation study is conducted in Section 4, with a focus on newly proposed Bayesian log-normal and gamma ALT models. Lastly, concluding remarks are provided in Section 5.

## 2. The generalized Eyring ALT models

Consider an accelerated test with two stressors, one thermal and one non-thermal. The generalized Eyring relationship can be used to extrapolate the life characteristics of the items under consideration, by taking into account the effect on the time to failure of the thermal stressor, non-thermal stressor, as well as the interaction between these stressors. Let there be $k$ distinct accelerated levels of the stressors, given by $\{T_i, S_i\}$, $i = 1, \ldots, k$, where $T_i$, $i = 1, \ldots, k$, are the accelerated levels of the thermal stressor and $S_i$, $i = 1, \ldots, k$, are the accelerated levels of the non-thermal stressor. A constant stress loading with no censoring is assumed, i.e., throughout the ALT experiment, each item is exposed to the constant application of a specific accelerated level of the stressors until failure.

Suppose that $\lambda_i$ is some life measure, then the generalized Eyring relationship incorporating the two stressors is given by

$$\lambda_i = \frac{1}{T_i} \exp\left(\theta_1 + \frac{\theta_2}{T_i} + \theta_3 V_i + \frac{\theta_4 V_i}{T_i}\right),$$

where $\theta_1, \theta_2, \theta_3$, and $\theta_4$ are unknown parameters, and $V_i$ is a function of the $i$th level of the non-thermal stressor. The parameters $\theta_1$, $\theta_2$, $\theta_3$, and $\theta_4$ are characteristics of the specific physical or chemical process involved in the failure mode [5]. The generalized Eyring relationship can also be extended to include additional non-thermal stressors and their interactions with the thermal stressor.

A common assumption in the literature is that the scale parameter of a life distribution is dependent on the accelerated stressors, whereas the shape parameter is not (see, for example, [12, 18, 27, 25]). This reparameterization of the scale parameter aligns with the established understanding of the physics of degradation and failure mechanisms. The scale parameter of the life distribution should reflect how the expected lifetimes of items are shortened due to increased stresses. Furthermore, the functional reparameterization allows for a smooth extrapolation to the normal operating conditions.

Suppose that $n_i$ items are tested at each of the $k$ distinct stressor levels, where a total of $n = \sum_{i=1}^{k} n_i$ items are tested in the ALT experiment. Denoting the failure time of the $j$th item subjected to the $i$th level of the stressors by $x_{ij}$, $j = 1, \ldots, n_i$, $i = 1, \ldots, k$, the likelihood function of some life distribution is given by

$$L\left(\underline{x}\,|\Omega\right) = \prod_{i=1}^{k}\prod_{j=1}^{n_i} f\left(x_{ij}\,|\Omega\right),$$

where $f(\cdot)$ is the probability density function (PDF) of the life distribution and $\Omega = \{\omega_1, \omega_2, ..., \omega_p\}$ is the set of parameters associated with the life distribution.

In the Bayesian ALT paradigm, it is often difficult to specify a joint prior for the model parameters due to uncertainties on parameter dependencies. It is common to assume that the priors of the parameters $\omega_1, \omega_2, ..., \omega_p$ are independent (see, for example, [23, 27]). The independence assumption simplifies prior elicitation and allows for flexibility in the choice of priors for individual model parameters. Furthermore, Bayesian approximation methods such as MCMC often have to be used in ALT, where independent priors enable easier implementations and more efficient posterior updating. The joint prior can be written as

$$\pi\left(\Omega\right) = \pi(\omega_1)\pi(\omega_2)\cdots\pi(\omega_p),$$

where the joint posterior, up to proportionality, is then given by

$$\pi\left(\Omega\,|\underline{x}\right) \propto L\left(\underline{x}\,|\Omega\right)\pi\left(\Omega\right).$$

With the ALT setup described, the likelihood functions of the log-normal and gamma distributions under the generalized Eyring relationship can be defined.

### 2.1. Log-normal model

Let $X$ be a continuous random variable that follows a log-normal distribution with parameters $\mu$ and $\sigma^2$ ($\mu \in \mathbb{R}$, $\sigma^2 > 0$). The PDF of $X$ is

$$f\left(x\,\middle|\mu, \sigma^2\right) = \frac{1}{\sqrt{2\pi}\sigma x}\exp\left(-\frac{1}{2\sigma^2}\left(\ln\left(x\right) - \mu\right)^2\right), \qquad x > 0. \tag{1}$$

Assuming the log-normal distribution, the parameter $\mu$ depends on the level of the stressors, whereas $\sigma^2$ does not [17, 19]. The reparameterization of $\mu$ given by the generalized Eyring relationship is

$$\mu_i = -\ln(T_i) + \theta_1 + \frac{\theta_2}{T_i} + \theta_3 V_i + \frac{\theta_4 V_i}{T_i}. \tag{2}$$

For the lifetime of the $j$th item subjected to the $i$th level of the stressors, it follows from (1) and (2) that the log-normal PDF can be written as

$$f\left(x_{ij}\,\middle|\theta_1, \theta_2, \theta_3, \theta_4, \sigma^2\right) = \frac{1}{\sqrt{2\pi}\sigma x_{ij}}\exp\left(-\frac{1}{2\sigma^2}\left(\ln\left(x_{ij}\right) + \ln\left(T_i\right) - \theta_1 - \frac{\theta_2}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i}\right)^2\right).$$

The likelihood function of the generalized Eyring-log-normal (GELN) model is then given by

$$L\left(\underline{x}\,\middle|\theta_1, \theta_2, \theta_3, \theta_4, \sigma^2\right) = \left(2\pi\sigma^2\right)^{-\frac{n}{2}}\left(\prod_{i=1}^{k}\prod_{j=1}^{n_i}\frac{1}{x_{ij}}\right) \times \exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^{k}\sum_{j=1}^{n_i}\left(\ln\left(x_{ij}\right) + \ln\left(T_i\right) - \theta_1 - \frac{\theta_2}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i}\right)^2\right).$$

Table 1. Prior specification in terms of some parameter $\omega$.

| Prior | Domain | Hyperparameters | PDF |
|---|---|---|---|
| $\omega \sim$ Uniform $(a, b)$ | $\omega \in [a, b]$ | $b > a \geq 0$ | $\pi_1(\omega) = \frac{1}{b-a}$ |
| $\omega \sim$ Gamma $(a, b)$ | $\omega \in (0, \infty)$ | $a, b > 0$ | $\pi_2(\omega) = \frac{\omega^{a-1}}{\Gamma(a)b^a} \exp\left(-\frac{\omega}{b}\right)$ |
| $\omega \sim$ TN $\left(a, b^2\right)$ | $\omega \in [0, \infty)$ | $a \in \mathbb{R}, b^2 \geq 0$ | $\pi_3(\omega) = \frac{\frac{1}{b}\phi\left(\frac{\omega-a}{b}\right)}{1-\Phi\left(-\frac{\mu}{b}\right)}$ |

### 2.2. Gamma model

Let $X$ be a continuous random variable that follows a gamma distribution with shape parameter $\alpha$ and scale parameter $\beta$ ($\alpha > 0$, $\beta > 0$). The PDF of $X$ is

$$f(x \,|\alpha, \beta) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} \exp\left(-\frac{x}{\beta}\right), \qquad x > 0, \tag{3}$$

where $\Gamma(\cdot)$ denotes the gamma function. The reparameterization, using the generalized Eyring relationship, of the scale parameter $\beta$ is

$$\beta_i = \frac{1}{T_i} \exp\left(\theta_1 + \frac{\theta_2}{T_i} + \theta_3 V_i + \frac{\theta_4 V_i}{T_i}\right). \tag{4}$$

From (3) and (4), the gamma PDF for the lifetime of the $j$th item subjected to the $i$th level of the stressors is given by

$$f(x_{ij} \,|\alpha, \theta_1, \theta_2, \theta_3, \theta_4) = \frac{1}{\Gamma(\alpha)} \left(T_i \exp\left(-\theta_1 - \frac{\theta_2}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i}\right)\right)^\alpha \times x_{ij}^{\alpha-1} \exp\left(-x_{ij} T_i \exp\left(-\theta_1 - \frac{\theta_2}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i}\right)\right).$$

The likelihood function of the generalized Eyring-gamma (GEG) model is then given by

$$L(\underline{x} \,|\alpha, \theta_1, \theta_2, \theta_3, \theta_4) = (\Gamma(\alpha))^{-n} \left\{\prod_{i=1}^{k} \prod_{j=1}^{n_i} \left(T_i \exp\left(-\theta_1 - \frac{\theta_2}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i}\right)\right)^\alpha \times x_{ij}^{\alpha-1} \exp\left(-x_{ij} T_i \exp\left(-\theta_1 - \frac{\theta_2}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i}\right)\right)\right\}.$$

### 2.3. Priors

Several priors defined on the positive domain are considered for the model parameters of the GELN and GEG models. The priors considered are the uniform, gamma, and truncated normal distributions. These priors offer flexibility in terms of incorporating subjective knowledge into the models. However, the robustness of the models under flat specifications of these priors is investigated in this paper. The parameterizations of the priors are defined in Table 1. Note that $\phi(\cdot)$ and $\Phi(\cdot)$ denote the PDF and CDF, respectively, of the standard normal distribution.

In the application that follows, while parameter estimation and model comparison metrics are the main objectives, the robustness of the GELN and GEG models is also assessed under flat specifications of the priors. Since no subjective prior knowledge is typically available in ALT experiments, flat priors can be constructed by allowing for a sufficiently wide domain for the model parameters via suitable choices of the hyperparameters in Table 1. In order to allow for a fair comparison between the use of the different priors, the hyperparameters are chosen such that the prior variance of each prior is 10,000,000. This is achieved by setting $a = 0$ for the uniform

Table 2. Model specification via prior assignment.

| Model | Prior Assignment |
|---|---|
| $\text{GELN}_1$ | $\theta_1, \theta_2, \theta_3, \theta_4, \sigma^2 \sim \text{Uniform}(a, b)$ |
| $\text{GELN}_2$ | $\theta_1, \theta_2, \theta_3, \theta_4, \sigma^2 \sim \text{Gamma}(a, b)$ |
| $\text{GELN}_3$ | $\theta_1, \theta_2, \theta_3, \theta_4, \sigma^2 \sim \text{TN}(a, b^2)$ |
| $\text{GEG}_1$ | $\alpha, \theta_1, \theta_2, \theta_3, \theta_4 \sim \text{Uniform}(a, b)$ |
| $\text{GEG}_2$ | $\alpha, \theta_1, \theta_2, \theta_3, \theta_4 \sim \text{Gamma}(a, b)$ |
| $\text{GEG}_3$ | $\alpha, \theta_1, \theta_2, \theta_3, \theta_4 \sim \text{TN}(a, b^2)$ |

prior, $a = 1$ for the gamma prior, $a = 0$ for the truncated normal prior, and then solving the prior variance for the value of $b$ for each prior.

Assigning each prior to all parameters of the GELN and GEG models, the six candidate ALT models given is Table 2 are considered. It is of course straightforward to consider mixtures of these priors or varying hyperparameters for the model parameters as well.

The log-concavity of the GELN and GEG likelihood functions, as well as that of the priors, is assessed in the appendix. Considering the GELN and GEG models given in Table 2, some of the resulting full conditional posteriors are not log-concave. More specifically, the full conditional posteriors of $\sigma^2$ in the GELN models are not log-concave. For these non-log-concave densities, more advanced MCMC methods are required to generate posterior samples for inference (see the appendix for further discussions).

## 3. Application

A real ALT dataset, based on the failure times (in hours) of some durable device, is used to demonstrate the use of the GELN and GEG models. The test data are given in Table 3 and consists of 21 devices, where the accelerated stressors are temperature (measured in Kelvin) and relative humidity. The normal operating conditions of the devices are a temperature of $T_u = 313$K and a relative humidity of $S_u = 0.5$. There are three levels of the accelerated stressors in the experiment and the ALT experiment is terminated after all devices fail.

It is known that failures in the devices occur mainly due to decreased adhesion between layers of the material caused by high temperatures, or due to moisture entering the devices through openings created by decreased adhesion. It has also been noted that the devices are less likely to fail earlier when exposed to high air moisture levels at low temperatures, or when exposed to high temperatures with little moisture in the air. This makes the generalized Eyring relationship an ideal choice for the TTF, since it takes into account the effects of the thermal stressor, non-thermal stressor, and interaction between these stressors.

In this application, the Bayesian data analysis software OpenBUGS is used for the MCMC simulations to generate posterior samples. The convergence of the MCMC chains for the models is first investigated. For each model, three Markov chains are initialized with different starting values and 50,000 samples are generated. The trace plots show that good mixing and stationarity of the chains are achieved after a few thousand iterations. The modified Gelman-Rubin statistic, proposed by [2], indicates stable convergence of the chains before 10,000 iterations. Lastly, the Monte Carlo standard error of each parameter becomes sufficiently small compared to its mean parameter estimate. Considering the fast convergence of the models, a single Markov chain is initialized for each model, with a burn-in of 50,000 iterations and 150,000 retained iterations, to generate posterior samples on which to base inference.

Table 3. Failure times for 21 devices under accelerated stressors.

| # | Temperature | Humidity | Failure time | # | Temperature | Humidity | Failure time |
|---|---|---|---|---|---|---|---|
| 1 | 333 | 0.9 | 521 | 12 | 353 | 0.8 | 504 |
| 2 | 333 | 0.9 | 561 | 13 | 353 | 0.9 | 115 |
| 3 | 333 | 0.9 | 575 | 14 | 353 | 0.9 | 119 |
| 4 | 333 | 0.9 | 599 | 15 | 353 | 0.9 | 150 |
| 5 | 333 | 0.9 | 609 | 16 | 353 | 0.9 | 152 |
| 6 | 333 | 0.9 | 684 | 17 | 353 | 0.9 | 153 |
| 7 | 333 | 0.9 | 709 | 18 | 353 | 0.9 | 155 |
| 8 | 333 | 0.9 | 713 | 19 | 353 | 0.9 | 156 |
| 9 | 353 | 0.8 | 345 | 20 | 353 | 0.9 | 164 |
| 10 | 353 | 0.8 | 357 | 21 | 353 | 0.9 | 199 |
| 11 | 353 | 0.8 | 439 | | | | |

The summary statistics for the marginal posteriors of the models under consideration are presented in Table 4. It is clear that both the GELN and GEG models are robust under different choices of flat priors, since very similar summary statistics are produced for these two models, respectively. However, this result holds for the given prior specifications, where the use of restrictive subjective priors may significantly influence the posterior samples generated.

A commonly used measure for model comparison in the Bayesian ALT setup is the deviance information criterion (DIC) proposed by [24]. The DIC assesses the fit of a model to the data, but also penalizes the model for complexity in terms of overparameterization. [24] show that the DIC and the Akaike information criterion (AIC) are approximately equivalent for models with very weak prior information. That is, the guidelines for the AIC in [3] can be used to decide whether the models considered in this application have substantially different DIC values. Given the DIC values of the GELN and GEG models in Table 5, all models used in this application have substantial support and should be considered for making inferences (given the set of candidate models being compared). The DIC values indicate that the GELN models fit the data slightly better than the GEG models and very stable DIC values are observed when using the different flat priors.

The fit of the models is further compared by calculating the mean absolute error (MAE) between the predictive reliability and empirical reliability at the accelerated stress levels. Given the ALT setup described in Section 2, this MAE can be calculated as

$$\text{MAE}\left(\hat{\Omega}\right) = \frac{1}{n}\sum_{i=1}^{k}\sum_{j=1}^{n_i}\left|\hat{R}_{n_i}\left(x_{i(j)}\right) - R\left(x_{i(j)}\left|\hat{\Omega}\right.\right)\right|, \tag{5}$$

where $\hat{R}_{n_i}(x_{i(j)})$ denotes the empirical reliability and $R(x_{i(j)}|\hat{\Omega})$ denotes the predictive reliability given some parameter estimates $\hat{\Omega} = \{\hat{\omega}_1, \hat{\omega}_2, ..., \hat{\omega}_p\}$, both at the $j$th ordered failure time, $x_{i(1)} \leq x_{i(2)} \leq \cdots \leq x_{i(n_i)}$, under the $i$th accelerated level of the stressors. Table 5 presents the results for the MAE using, respectively for each candidate model, the posterior means as parameter estimates, which are the Bayes estimates under a squared error loss. Comparable conclusions regarding model comparison can be made using the MAE, where comparing the GELN models against the GEG models, there is not a significant difference between the MAE values. Again, the GELN models have a slightly better fit to the data and consistent MAE values are observed over the different flat priors.

Bayes factors can also be used for model comparison, but this requires the approximation of the marginal likelihood when working with mathematically intractable posteriors. [21] discuss several methods for approximating the marginal likelihood, with specific reference to complex models in the Bayesian ALT setup.

The posterior samples generated via the MCMC simulations can now be used to extrapolate the predictive reliability under the normal operating conditions. The predictive reliability under the normal operating conditions

Table 4. Summary statistics for the GELN and GEG models.

| Model | Parameter | Mean | Standard Deviation | 2.5th Percentile | Median | 97.5th Percentile |
|---|---|---|---|---|---|---|
| $GELN_1$ | $\theta_1$ | 1.0601 | 0.9880 | 0.0277 | 0.7545 | 3.7040 |
| | $\theta_2$ | 3.8649 | 1.0115 | 1.2220 | 4.1200 | 5.1650 |
| | $\theta_3$ | 1.7191 | 1.3711 | 0.0599 | 1.4060 | 5.0710 |
| | $\theta_4$ | 2.1296 | 1.5507 | 0.0896 | 1.8480 | 5.7620 |
| | $\sigma^2$ | 0.3745 | 0.1543 | 0.1796 | 0.3413 | 0.7621 |
| $GELN_2$ | $\theta_1$ | 1.1123 | 1.0392 | 0.0293 | 0.7820 | 3.9160 |
| | $\theta_2$ | 3.8136 | 1.0580 | 1.0390 | 4.0890 | 5.1610 |
| | $\theta_3$ | 1.7335 | 1.3865 | 0.0600 | 1.4100 | 5.1070 |
| | $\theta_4$ | 2.1175 | 1.5516 | 0.0881 | 1.8250 | 5.7530 |
| | $\sigma^2$ | 0.3765 | 0.1552 | 0.1803 | 0.3430 | 0.7670 |
| $GELN_3$ | $\theta_1$ | 1.0923 | 1.0343 | 0.0274 | 0.7633 | 3.9230 |
| | $\theta_2$ | 3.8411 | 1.0539 | 1.0260 | 4.1250 | 5.1570 |
| | $\theta_3$ | 1.7268 | 1.3650 | 0.0596 | 1.4240 | 5.0640 |
| | $\theta_4$ | 2.0831 | 1.5163 | 0.0868 | 1.8110 | 5.6620 |
| | $\sigma^2$ | 0.3746 | 0.1540 | 0.1802 | 0.3413 | 0.7649 |
| $GEG_1$ | $\alpha$ | 3.3907 | 1.0011 | 1.7160 | 3.2960 | 5.6040 |
| | $\theta_1$ | 0.9544 | 0.8836 | 0.0267 | 0.6847 | 3.3520 |
| | $\theta_2$ | 3.0863 | 0.8552 | 0.8821 | 3.2580 | 4.3290 |
| | $\theta_3$ | 1.4715 | 1.2163 | 0.0485 | 1.1790 | 4.5340 |
| | $\theta_4$ | 1.7481 | 1.3841 | 0.0631 | 1.4300 | 5.1380 |
| $GEG_2$ | $\alpha$ | 3.4095 | 1.0232 | 1.7230 | 3.3050 | 5.7100 |
| | $\theta_1$ | 0.9508 | 0.8759 | 0.0260 | 0.6850 | 3.3070 |
| | $\theta_2$ | 3.0884 | 0.8465 | 0.9121 | 3.2620 | 4.3320 |
| | $\theta_3$ | 1.4586 | 1.2272 | 0.0471 | 1.1510 | 4.5530 |
| | $\theta_4$ | 1.7402 | 1.3621 | 0.0636 | 1.4400 | 5.0700 |
| $GEG_3$ | $\alpha$ | 3.3907 | 1.0011 | 1.7160 | 3.2960 | 5.6040 |
| | $\theta_1$ | 0.9544 | 0.8836 | 0.0267 | 0.6847 | 3.3520 |
| | $\theta_2$ | 3.0863 | 0.8552 | 0.8821 | 3.2580 | 4.3290 |
| | $\theta_3$ | 1.4715 | 1.2163 | 0.0485 | 1.1790 | 4.5340 |
| | $\theta_4$ | 1.7481 | 1.3841 | 0.0631 | 1.4300 | 5.1380 |

Table 5. Model comparison metrics for the candidate models.

| | Models | | | | | |
|---|---|---|---|---|---|---|
| | $GELN_1$ | $GELN_2$ | $GELN_3$ | $GEG_1$ | $GEG_2$ | $GEG_3$ |
| **DIC** | 278.7 | 278.8 | 278.8 | 280.4 | 280.6 | 280.4 |
| **MAE** | 0.3084 | 0.3085 | 0.3085 | 0.3324 | 0.3327 | 0.3324 |

is given by

$$R(x_u\,|\underline{x}) = \int R(x_u\,|\Omega)\,\pi(\Omega\,|\underline{x})\,d\Omega, \tag{6}$$

where $R(x_u\,|\Omega)$ is the reliability function at time $x_u$ under the normal operating stress levels $T_u$ and $S_u$, and $\Omega$ is the set of parameters associated with the reliability function. $R(x_u\,|\underline{x})$ can be estimated as follows:

1. Sample a value for each parameter in $\Omega$ from the posterior.

2. Repeat Step 1 many times, say $M$ times, to generate the posterior samples $\Omega^{(m)} = \left\{\omega_1^{(m)}, \omega_2^{(m)}, ..., \omega_p^{(m)}\right\}$, $m = 1, ..., M$.
3. Estimate the integral in (6) by the Monte Carlo average

$$R\left(x_u\left|\underline{x}\right.\right) \approx \frac{1}{M}\sum_{m=1}^{M} R\left(x_u\left|\Omega^{(m)}\right.\right),$$

which is the expected reliability at time $x_u$ under the normal operating conditions.
4. Repeat Step 3 for a range of values of $x_u$ to obtain predictive reliability estimates, under the normal operating conditions, for a range of lifetimes.

The predictive reliability of the GELN and GEG models, for $0 \le x_u \le 50000$ under the normal operating conditions of $T_u = 313$K and $S_u = 0.5$, is displayed in Figure 1. While there is a notable difference in the predictive reliability between the GELN and GEG models, each model produces robust results under the different flat priors considered. Given that both model comparison metrics indicate that the GELN models have a slightly better fit than the GEG models, more confidence might be placed on the estimates produced by the GELN models.

The B10 life, which is the lifetime by which only 10% of population is expected to have failed, is a typical measure used in reliability analysis to determine warranty periods or the maximum service life before the replacement of products, components, or systems (see, for example, [6, 10]). The predictive B10 life under the normal operating conditions is around 2,230 hours for the GELN models and around 1,300 hours for the GEG models. When considering a warranty on this device, the manufacturer would typically set a warranty period that is significantly shorter than the B10 life in order to limit warranty replacements. As an example, suppose that this specific device is used on average for 4 hours per day and that the B10 life using the GEG model is considered. The B10 life translates to 325 use days, meaning that a manufacturer might set only a 6-month warranty so that substantially fewer than 10% of the devices incur warranty replacements.

Since the devices may function under stressors varying slightly from the normal operating conditions, it is important to investigate the sensitivity of the devices under slightly accelerated stressors. To assess this sensitivity, let us consider estimates of the MTTF under the normal operating conditions versus slightly accelerated stressors. Table 6 provides the MTTF estimates for the $\text{GELN}_1$ and $\text{GEG}_1$ models, given temperatures in the range (313K, 318K) and relative humidities in the range (0.50, 0.55). The results are displayed only for these two models, since the GELN and GEG models are robust with regards to the choice the different flat priors. It is clear that a deviation from the normal operating conditions can affect the MTTF to a great extent, where substantially earlier failures may occur due to the interactive impact when both stressors are accelerated.

The MTTF results under slightly accelerated stressors can also be used to inform design margins for planning future accelerated tests on this particular device. The degree of acceleration can provide an indication of the accelerated stress levels at which more devices can be tested, given a budgeted time for testing, to obtain additional failure data for reliability analysis.

## 4. Simulated data study

In this simulation study, an ALT experiment using two stressors, temperature and relative humidity, is considered. The newly defined GELN and GEG models are further explored through the use of simulated datasets. The generalized Eyring-Weibull (GEW) and generalized Eyring-Birnbaum-Saunders (GEBS) models, introduced in [22] and [20], respectively, are also included in this study. The normal operating conditions are assumed to be a temperature of $T_u = 325$K and a relative humidity of $S_u = 0.3$.

Six accelerated stress levels are considered, which are the distinct temperature-humidity pairs $\{T_i, S_i\} = \{(350, 0.5); (350, 0.7); (375, 0.5); (375, 0.7); (400, 0.5); (400, 0.7)\}$. Equal sample sizes are considered per stress level, specifically $n_i = \{5, 10, 15, 25, 50, 100\}$, $i = 1, ..., k$. Two parameterization cases are considered. Case 1: The GELN model with parameters $\theta_1 = 4,\ \theta_2 = 2,\ \theta_3 = 1,\ \theta_4 = 2,\ \sigma^2 = 0.25$ is the true model. Case 2: The GEG model with parameters $\alpha = 2,\ \theta_1 = 7,\ \theta_2 = 2,\ \theta_3 = 1,\ \theta_4 = 1$ is the true model. These parameter values

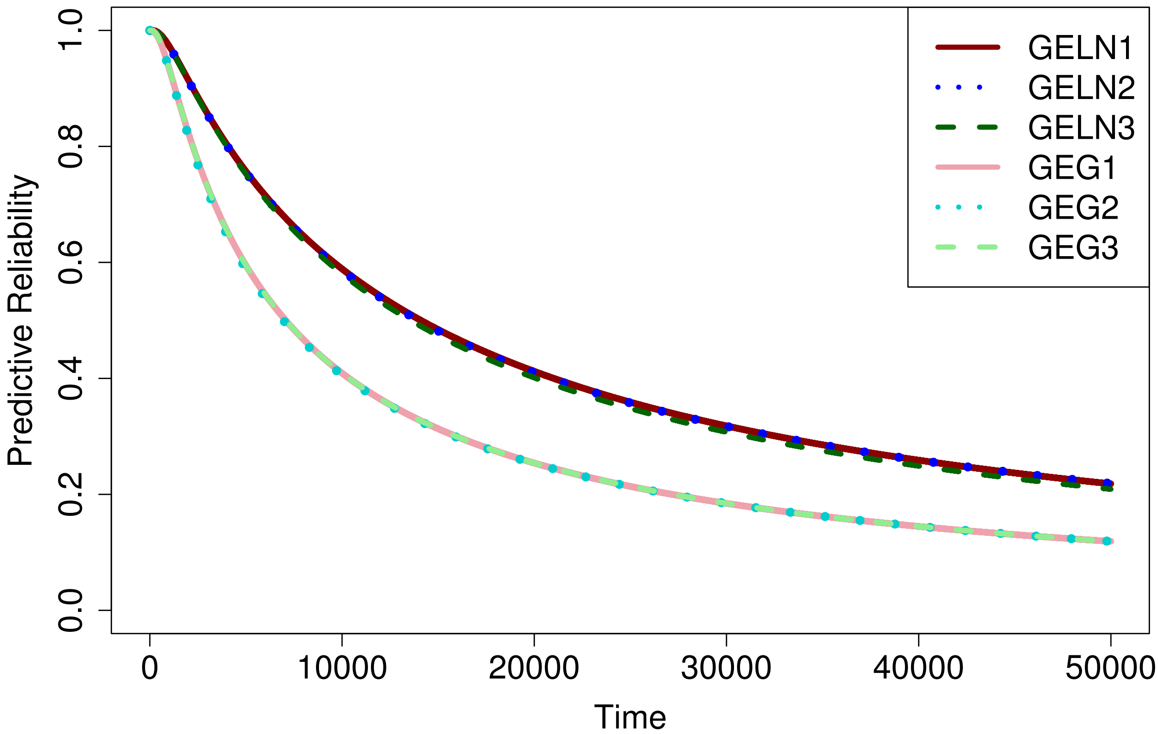


Figure 1. Predictive reliability of the GELN and GEG models under the normal operating conditions.

Table 6. MTTF under slightly accelerated stressors.

| Model | Temperature | Relative Humidity | | | | | |
|---|---|---|---|---|---|---|---|
| | | 0.50 | 0.51 | 0.52 | 0.53 | 0.54 | 0.55 |
| $GELN_1$ | 313 | 18907 | 16808 | 14975 | 13372 | 11965 | 10729 |
| | 314 | 18446 | 16401 | 14616 | 13054 | 11683 | 10478 |
| | 315 | 17998 | 16006 | 14267 | 12745 | 11409 | 10234 |
| | 316 | 17564 | 15623 | 13929 | 12445 | 11143 | 9997 |
| | 317 | 17142 | 15252 | 13600 | 12154 | 10885 | 9768 |
| | 318 | 16733 | 14891 | 13281 | 11872 | 10634 | 9545 |
| $GEG_1$ | 313 | 10091 | 9145 | 8305 | 7555 | 6885 | 6285 |
| | 314 | 9886 | 8961 | 8139 | 7405 | 6750 | 6163 |
| | 315 | 9686 | 8781 | 7977 | 7259 | 6618 | 6043 |
| | 316 | 9491 | 8606 | 7819 | 7117 | 6489 | 5927 |
| | 317 | 9301 | 8436 | 7665 | 6978 | 6364 | 5813 |
| | 318 | 9116 | 8269 | 7516 | 6843 | 6241 | 5702 |

are arbitrary and the study can be repeated for any other parameter values, or data can be simulated from any of the other ALT models. A single ALT dataset is generated for each sample size, respectively for each case, from the specified ALT model. Using these generated datasets, the GELN, GEG, GEW, and GEBS models are implemented in OpenBUGS to generate posterior samples for inference. A flat gamma prior, $\text{Gamma}(1, 0.0001)$, is used for all

Table 7. MAE comparison for the candidate models.

| Case | $n_i$ | **Models** | | | |
|---|---|---|---|---|---|
| | | GELN | GEG | GEW | GEBS |
| Case 1 (GELN) | 5 | 0.1326 | 0.1445 | 0.1451 | **0.1309** |
| | 10 | 0.0659 | 0.0751 | 0.0741 | **0.0656** |
| | 15 | **0.0560** | 0.0677 | 0.0758 | 0.0569 |
| | 25 | **0.0542** | 0.0625 | 0.0708 | 0.0549 |
| | 50 | **0.0465** | 0.0557 | 0.0653 | 0.0477 |
| | 100 | **0.0258** | 0.0328 | 0.0485 | 0.0267 |
| Case 2 (GEG) | 5 | 0.1636 | 0.1794 | 0.1702 | **0.1581** |
| | 10 | **0.1224** | 0.1285 | 0.1241 | 0.1225 |
| | 15 | 0.0737 | 0.0830 | 0.0824 | **0.0708** |
| | 25 | 0.0561 | **0.0524** | 0.0536 | 0.0682 |
| | 50 | 0.0407 | **0.0291** | 0.0302 | 0.0527 |
| | 100 | 0.0317 | **0.0279** | 0.0298 | 0.0435 |

models throughout the simulation study. Similar to the real ALT application in Section 3, a single Markov chain is initialized for each model, with a burn-in of 50,000 iterations and 150,000 retained iterations.

First, the fit of the four ALT models is compared for the two cases using the MAE defined in (7). Only the MAE is considered in the simulation study, due to it being a more interpretable fit measure which considers the discrepancy between the empirical reliability and predictive reliability at the accelerated stress levels. [27] state that life distributions commonly used to model fatigue data can fit the central region relatively well, where all the candidate models may be reasonable in terms of goodness-of-fit for small sample sizes. Furthermore, both the Weibull and Birnbaum-Saunders distributions are relatively flexible in terms of their shape, meaning that deviations from the true model may be expected for small sample sizes.

Table 7 displays the MAE results for both cases over the different sample sizes, using the posterior means of the parameters for each respective model as parameter estimates. The smallest MAE for each sample size is highlighted. For Case 1, it can be noted that the GELN and GEBS models have very similar MAE values, with the GELN model exhibiting the slightly better fit for sample sizes larger than 15 per stress level. For Case 2, the GEBS and GELN models fit the simulated datasets better for small sample sizes, whereas the GEG model shows the better fit for sample sizes larger than 25 per stress level.

Parameter estimation for the GELN and GEG models is further investigated by comparing the true parameter values to estimates obtained from the posterior samples generated via MCMC simulations. The discrepancy between the true values and estimates can be measured in terms of the relative absolute deviation (RAD) for each parameter. For a parameter $\omega$, the RAD can be computed as

$$\text{RAD}\left(\hat{\omega}\right) = \left|\frac{\hat{\omega} - \omega}{\omega}\right|,$$

where $\hat{\omega}$ is an estimate for the parameter $\omega$. Table 8 shows the RAD for each parameter of the GELN and GEG models, respectively for Case 1 and Case 2, over the different sample sizes. Since the parameter deviations are measured in relative terms, the mean RADs over the five parameters of the models are also calculated for each sample size. It is clear that the mean RADs of both the GELN and GEG models generally decrease as the sample size increases. Overall, the parameters directly linked to the stressors, i.e., $\theta_2$, $\theta_3$, and $\theta_4$, seem to have higher RADs, indicating that they are more difficult to accurately estimate. Repeating this ALT experiment numerous times would provide summary information on the accuracy of the estimates over the sample sizes.

Table 8. RAD of the parameters for the GELN and GEG models.

| | Parameters for GELN model | | | | | |
|---|---|---|---|---|---|---|
| $n_i$ | $\sigma^2$ | $\theta_1$ | $\theta_2$ | $\theta_3$ | $\theta_4$ | **Mean** |
| 5 | 0.0450 | 0.2709 | 0.5014 | 0.6380 | 0.2416 | 0.3394 |
| 10 | 0.1928 | 0.1107 | 0.2672 | 0.4885 | 0.3087 | 0.2736 |
| 15 | 0.1593 | 0.0237 | 0.0520 | 0.5516 | 0.2175 | 0.2008 |
| 25 | 0.0344 | 0.0137 | 0.0331 | 0.3573 | 0.1308 | 0.1139 |
| 50 | 0.0985 | 0.0049 | 0.0934 | 0.2814 | 0.3065 | 0.1569 |
| 100 | 0.0170 | 0.0980 | 0.1681 | 0.3032 | 0.1011 | 0.1375 |
| | **Parameters for GEG model** | | | | | |
| $n_i$ | $\alpha$ | $\theta_1$ | $\theta_2$ | $\theta_3$ | $\theta_4$ | **Mean** |
| 5 | 0.2291 | 0.1569 | 0.3985 | 0.1377 | 0.1127 | 0.2070 |
| 10 | 0.1035 | 0.0441 | 0.2488 | 0.4763 | 0.1495 | 0.2044 |
| 15 | 0.0100 | 0.0969 | 0.2911 | 0.0801 | 0.1975 | 0.1351 |
| 25 | 0.0381 | 0.0460 | 0.1457 | 0.2320 | 0.0459 | 0.1015 |
| 50 | 0.0087 | 0.0844 | 0.1681 | 0.1076 | 0.2497 | 0.1237 |
| 100 | 0.0104 | 0.0701 | 0.2067 | 0.2114 | 0.0352 | 0.1068 |

## 5. Conclusions

In this paper, Bayesian dual-stress ALT models are proposed where lifetimes are assumed to follow either a log-normal distribution or gamma distribution. The generalized Eyring relationship is used as the TTF, which can be used to investigate the acceleration of life under one thermal stressor, one non-thermal stressor, and the interaction between these stressors. Several priors are considered to assess the robustness of the models. Due to the mathematically intractable posteriors, the log-concavity of the models is assessed and MCMC methods are employed to generate posterior samples on which to base inference.

The use of the GELN and GEG models is demonstrated in an application to a real dataset, with temperature and relative humidity as the accelerated stressors. Summary statistics on the marginal posteriors are provided, as well as estimates of the MTTF under slightly accelerated stressors and the expected reliability under the normal operating conditions. The model comparison metrics, summary statistics, and estimates show that both the GELN and GEG models are robust under different choices of flat priors. According to the DIC and MAE values, the GELN model has a slightly better fit to the data compared to the GEG model. The two models produce notably different estimates of the MTTF and predictive reliability, and both models show that the devices are sensitive to slightly accelerated stressors. The models are further investigated using simulated datasets from the GELN and GEG models. It is found that the GELN and GEG models have the best fit for these simulated datasets, respectively when the data is generated from the GELN and GEG models. The MCMC method also generally provides more accurate parameter estimates as the sample size increases.

The specification of the GELN and GEG models allows for a diverse range of subjective priors, should a reliability engineer want to incorporate prior expert knowledge. However, the use of subjective priors should to be handled with great care, since these complex models may be sensitive to the use of constrained or strong subjective priors.

## Statements

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## Appendix

In this appendix, the log-concavity of the full conditional posteriors of the GELN and GEG models is assessed to determine which MCMC techniques are appropriate to use. Due to the mathematically intractable posteriors of the models, MCMC methods that require specification of the full conditional posteriors up to at least proportionality only, are considered. Some of these MCMC methods, such as adaptive rejection sampling introduced by [7], can only sample from log-concave densities. In the case of non-log-concave densities, more advanced MCMC methods such as adaptive rejection Metropolis sampling [8] or slice sampling [15] can be used.

A twice differentiable function $f(x)$ is log-concave if the second derivative of $\ln(f(x))$ is non-positive on the domain of $x$ (see, for example, [1]), thus if

$$\frac{\partial^2 \ln\left(f\left(x\right)\right)}{\partial x^2} \leq 0 \qquad \forall x.$$

Since the product of two log-concave functions is again log-concave, it is sufficient to assess only the log-concavity of the likelihood functions with respect to each parameter as well as the log-concavity of the priors. From this, the log-concavity of the GELN and GEG models, using any of the specified priors, directly follows.

### *GELN model*

Let $\ell_1$ denote the log-likelihood function of the GELN model, that is

$$\ell_1 = \ln\left(\left(2\pi\sigma^2\right)^{-\frac{n}{2}}\left(\prod_{i=1}^{k}\prod_{j=1}^{n_i}\frac{1}{x_{ij}}\right)\right) - \frac{1}{2\sigma^2}\sum_{i=1}^{k}\sum_{j=1}^{n_i}\left(\ln\left(x_{ij}\right) + \ln\left(T_i\right) - \theta_1 - \frac{\theta_2}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i}\right)^2.$$

The second derivatives of $\ell_1$ with respect to $\theta_1$, $\theta_2$, $\theta_3$, $\theta_4$, and $\sigma^2$ are, respectively,

$$\begin{aligned}
\frac{\partial^2 \ell_1}{\partial \theta_1^2} &= -\frac{n}{\sigma^2}, \\
\frac{\partial^2 \ell_1}{\partial \theta_2^2} &= -\frac{1}{\sigma^2}\sum_{i=1}^{k}\frac{n_i}{T_i^2}, \\
\frac{\partial^2 \ell_1}{\partial \theta_3^2} &= -\frac{1}{\sigma^2}\sum_{i=1}^{k} n_i V_i^2, \\
\frac{\partial^2 \ell_1}{\partial \theta_4^2} &= -\frac{1}{\sigma^2}\sum_{i=1}^{k}\frac{n_i V_i^2}{T_i^2}, \\
\frac{\partial^2 \ell_1}{\partial \left(\sigma^2\right)^2} &= \frac{n}{2\left(\sigma^2\right)^2} - \frac{1}{\left(\sigma^2\right)^3}\sum_{i=1}^{k}\sum_{j=1}^{n_i}\left(\ln\left(x_{ij}\right) + \ln\left(T_i\right) - \theta_1 - \frac{\theta_2}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i}\right)^2.
\end{aligned}$$

Since $n, n_i > 0$ and $\sigma^2 > 0$, it is clear that the contributions of $L(\underline{x}|\theta_1, \theta_2, \theta_3, \theta_4, \sigma^2)$ to the full conditional posteriors of $\theta_1$, $\theta_2$, $\theta_3$, and $\theta_4$ are log-concave over their respective domains. However, the contribution of $L(\underline{x}|\theta_1, \theta_2, \theta_3, \theta_4, \sigma^2)$ to the full conditional posterior of $\sigma^2$ is only log-concave where

$$\sigma^2 \leq \frac{2}{n}\sum_{i=1}^{k}\sum_{j=1}^{n_i}\left(\ln\left(x_{ij}\right) + \ln\left(T_i\right) - \theta_1 - \frac{\theta_2}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i}\right)^2.$$

***GEG model***

Let $\ell_2$ denote the log-likelihood function of the GEG model, that is

$$\ell_2 = -n\ln\left(\Gamma\left(\alpha\right)\right) + \alpha\sum_{i=1}^{k}\sum_{j=1}^{n_i}\ln\left(T_i\right) - \alpha\sum_{i=1}^{k} n_i\left(\theta_1 + \frac{\theta_2}{T_i} + \theta_3 V_i + \frac{\theta_4 V_i}{T_i}\right)$$
$$+ \left(\alpha - 1\right)\sum_{i=1}^{k}\sum_{j=1}^{n_i}\ln\left(x_{ij}\right) - \sum_{i=1}^{k}\sum_{j=1}^{n_i}\left(x_{ij}T_i\exp\left(-\theta_1 - \frac{\theta_2}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i}\right)\right).$$

The second derivatives of $\ell_2$ with respect to $\theta_1$, $\theta_2$, $\theta_3$, $\theta_4$, and $\alpha$ are, respectively,

$$\frac{\partial^2 \ell_2}{\partial \theta_1^2} = -\sum_{i=1}^{k}\sum_{j=1}^{n_i}\left(x_{ij}T_i\exp\left(-\theta_1 - \frac{\theta_2}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i}\right)\right),$$
$$\frac{\partial^2 \ell_2}{\partial \theta_2^2} = -\sum_{i=1}^{k}\sum_{j=1}^{n_i}\left(\frac{x_{ij}}{T_i}\exp\left(-\theta_1 - \frac{\theta_2}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i}\right)\right),$$
$$\frac{\partial^2 \ell_2}{\partial \theta_3^2} = -\sum_{i=1}^{k}\sum_{j=1}^{n_i}\left(x_{ij}T_i V_i^2\exp\left(-\theta_1 - \frac{\theta_2}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i}\right)\right),$$
$$\frac{\partial^2 \ell_2}{\partial \theta_4^2} = -\sum_{i=1}^{k}\sum_{j=1}^{n_i}\left(\frac{x_{ij}V_i^2}{T_i}\exp\left(-\theta_1 - \frac{\theta_2}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i}\right)\right),$$
$$\frac{\partial^2 \ell_2}{\partial \alpha^2} = -n\psi_1\left(\alpha\right),$$

where $\psi_1(\alpha) = \partial^2/\partial\alpha^2(\ln(\Gamma(\alpha)))$ is the trigamma function. Since $n, T_i > 0$ and $\psi_1(\alpha) > 0$ for $\alpha > 0$, the contributions of $L(\underline{x}|\alpha, \theta_1, \theta_2, \theta_3, \theta_4)$ to the full conditional posteriors of $\theta_1$, $\theta_2$, $\theta_3$, $\theta_4$, and $\alpha$ are log-concave over their respective domains.

***Priors***

For a uniform prior between $a$ and $b$ on a parameter $\omega$, we have that

$$\frac{\partial^2 \ln\left(\pi_1\left(\omega\right)\right)}{\partial \omega^2} = 0,$$

so that $\pi_1(\omega)$ is log-concave over its domain.

If a parameter $\omega$ has a gamma prior with shape parameter $a$ and scale parameter $b$, then

$$\frac{\partial^2 \ln\left(\pi_2\left(\omega\right)\right)}{\partial \omega^2} = \frac{1-a}{\omega^2},$$

so that $\pi_2(\omega)$ is only log-concave over its domain if $a \geq 1$.

Suppose that a parameter $\omega$ has a truncated normal prior on $[0, \infty)$ with location parameter $a$ and scale parameter $b^2$. We then have that

$$\frac{\partial^2 \ln\left(\pi_3\left(\omega\right)\right)}{\partial \omega^2} = -\frac{1}{b^2},$$

so that $\pi_3(\omega)$ is log-concave over its domain.

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